DERIVED REPRESENTATION SCHEMES AND SUPERSYMMETRIC GAUGE THEORY

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ABSTRACT. In these expanded lecture notes of the minicourse held at the workshop on "Homotopy algebras, deformation theory and quantization" at the Mathematical Research and Conference Center in Będlewo, the theory of derived representation schemes is reviewed with the aim to present the simplest instance of the relation to $N = 2$ supersymmetric gauge theory.

CONTENTS

1. Introduction

In these notes we review parts of the theory of derived representation schemes as developed in [3–6], with an emphasis on examples and with a view towards the relation to $N = 2$ supersymmetric gauge theory and an application [14] to the analytic properties of gauge theory partition functions.

One motivation for the theory of derived representation schemes is the approach to non-commutative geometry proposed by Kontsevich and Rosenberg [19]: to an associative algebra A (over $\mathbb{C}$, say) we can associate a sequence A_n of *commutative* algebras, namely the algebras of polynomial functions on the space of representations (algebra homomorphisms) $A \to \mathrm{End}(\mathbb{C}^n)$. The idea is that the non-commutative geometry of A should be encoded in the commutative geometry of the algebras A_n. Of course this idea has its limitations: for example there are associative algebras, such as algebras of differential operators which don't have finite dimensional representations. Even if they have "enough" representations, the representation schemes $\mathrm{Spec}(A_n)$ can be very singular. In this case it is natural to pass to the world of derived geometry, which in our case means that we replace our algebra by a suitable resolution, technically a cofibrant replacement, which is a differential graded algebra, and apply the functor $A \mapsto A_n$ to the resolution. We obtain a differential graded commutative algebra whose homology is called the n-th representation homology $H_\bullet(A, n)$. With the machinery of model categories one sees that this construction is independent (up to isomorphism) of the choice of resolution. Moreover one has that $H_0(A, n) = A_n$ and if the associative algebra is smooth (quasi-free) in the sense of Cuntz and Quillen [11], then all A_n are smooth and thus the representation schemes $\mathrm{Rep}_n^A = \mathrm{Spec}(A_n)$ are smooth. Also $H_i(A, n) = 0$ for $i \neq 0$, for smooth algebras (under mild finiteness assumptions), see [4, Theorem 21]. So, in a sense, higher representation homology groups give additional information on the non-commutative geometry of an associative algebra, measuring the non-smoothness.

One of the main non-trivial examples we will consider is the polynomial algebra in two variables $A = k[x, y]$ over a field k of characteristic 0. This algebra is not smooth as an associative algebra and indeed the representation schemes are the *commuting schemes* of pairs of commuting $n \times n$ matrices, known to be singular schemes for $n \geq 2$. For the representation homology we have a conjectural description of its GL_n-invariant part, see Conjecture 3.19, leading to new combinatorial identities (which can be proved). Another example, relevant to gauge theory, is obtained by replacing the relation $xy - yx = 0$, defining the commuting scheme, by $xy - yx + ij = 0$ with some additional variables i, j. The representations we are after are such that x and y are sent to $n \times n$ matrices and i, j to $n \times r$, $r \times n$ matrices, respectively. More precisely we are considering representations of a quiver with relations. These representations belong to the 0-level of the moment map in the description of framed instanton on $\mathbb{R}^4$ by Atiyah–Drinfeld–Hitchin–Manin: the moduli space $\mathcal{M}(n, r)$ of framed $U(r)$ instantons with instanton number n is the GIT quotient of the 0-level by the natural action of GL_n. It is a smooth algebraic variety with an action of $GL_r \times GL_2$, with GL_2 acting by linear transformations of x, y.

This establishes the connection to $N = 2$ supersymmetric gauge theory. Indeed it turns out that the character-valued Euler characteristic of the GL_n-invariants in the representation homology of the ADHM quiver of dimension (n, r) coincides with the instanton number n contribution $\mathbb{Z}_{5D}^{(n)}$ to the K-theoretic Nekrasov partition function of the $N = 2$ supersymmetric pure Yang–Mills theory with gauge group $U(r)$ in an Ω-background [28]. Mathematically $Z_{5D}^{(n)}$ may be defined as $\sum_i (-1)^i \operatorname{ch}_T H^i(\mathcal{M}(n, r), \mathcal{O})$ in terms of the characters of the sheaf cohomology with respect to action of the torus of diagonal matrices in $GL_r \times GL_2$. Replacing the structure sheaf $\mathcal{O}$ by other T-equivariant vector bundles yields partition functions of gauge theories with matter fields. They depend on the vector bundles through their classes in the equivariant K-theory. Since the cohomology groups have finite dimensional weight spaces $Z_{5D}^{(n)}$ and the partition function $Z_{5D} = \sum v^n Z_{5D}^{(n)}$ make sense as formal power series. Also they can be computed by the localization formula, see [24, 28]. Still the question of convergence is subtle and is part of our discussion below.

These lecture notes consist of three parts, roughly corresponding to the three lectures of the minicourse. Section 2 contains foundational material on representation schemes of associative algebras with examples and exercises. In Section 3 we introduce derived representation schemes and representation homology. Some basic results are quoted, in particular about the comparison of representation homology with more classical invariants. The example of the derived commuting scheme is described in more detail; in this case the Harish–Chandra isomorphism conjecture and its relation with constant term identities, which we use in the third part, is explained. In Section 4 we introduce Nekrasov partition functions in four and five dimensions and explain the relation to representation homology. The relation with generalized random matrix models and the application to the convergence of the partition function are sketched in this section.

Throughout these lectures, the ground field k is assumed to contain $\mathbb{Q}$ ($k = \mathbb{C}$ in Section 4),

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2. Representation schemes

Let Alg_k the category of unital associative algebras over the field k of characteristic zero. Its objects are associative algebras A with unit 1_A and a morphism $f \colon A \to B$ is a linear map such that, for all $a, b \in A$,

$$f(ab) = f(a)f(b), \quad f(1_A) = 1_B.$$

2.1. **Representations.** If V is a k-vector space then the space $\mathrm{End}_k(V)$ of k-linear maps $V \to V$ is an object of Alg_k. A *representation* of A on a vector space V is a morphism $\rho\colon A \to \mathrm{End}_k(V)$. The dimension of a representation is the dimension of the underlying vector space V. If $\dim V = n$ is finite then upon choosing a basis we can assume that $V = k^n$ and identify $\mathrm{End}_k(V)$ with the algebra $M_n(k)$ of $n \times n$ matrices with entries in k. The group $GL_n(k)$ of invertible matrices acts on the set of representation by change of basis. Representations related by this action are called equivalent.

We are interested in the sequence of representation spaces

$$\mathrm{Rep}_n^A(k) = \mathrm{Hom}_{\mathrm{Alg}_k}(A, M_n(k)), \quad n = 1, 2, 3, \dots,$$

and in the sets of equivalence classes $\mathrm{Rep}_n^A(k)/GL_n(k)$.

2.2. **Representation schemes.** The set $\mathrm{Rep}_n^A(k)$ is the set of k-rational points of an affine scheme over k, the n-th representation scheme Rep_n^A. Before giving the construction of this scheme, or rather of its coordinate ring $A_n = k[\mathrm{Rep}_n^A]$, we characterize it by a universal property, by giving its sets of B-points for any B. For a commutative unital algebra B let $M_n(B)$ be the algebra of matrices with entries in B and set

$$\mathrm{Rep}_n^A(B) = \mathrm{Hom}_{\mathrm{Alg}_k}(A, M_n(B)),$$

the set of n-dimensional matrix representations of A with coefficients in B. A morphism $f\colon B \to B'$ of unital commutative algebras induces a morphism of associative algebras $M_n(f)\colon M_n(B) \to M_n(B')$ (act on each matrix entry) and a map

$$f_* = M_n(f) \circ \text{—}\colon \mathrm{Rep}_n^A(B) \to \mathrm{Rep}_n^A(B'),$$

such that $(f \circ g)_* = f_* \circ g_*$ for any composition of morphisms f, g of commutative unital algebras. In other words Rep_n^A is a covariant functor

$$\mathrm{Rep}_n^A\colon \mathrm{ComAlg}_k \to \mathrm{Set} \tag{2.1}$$

from the category of unital commutative algebras to the category of sets.

Definition 2.1. Let n be a natural number and A a unital associative algebra. A pair (B, π) consisting of a commutative algebra B and representation $\pi\colon A \to M_n(B)$ with coefficients in B is called *universal* if for any n-dimensional representation $\rho\colon A \to M_n(B')$ there is a unique morphism $f\colon B \to B'$ so that $\rho = f_*\pi$.

Proposition 2.2. *Let $A \in \mathrm{Alg}_k$ and $n \in \{1, 2, \dots\}$. Then there is a universal representation (A_n, π_n). It is unique in the sense that for any other universal (A'_n, π'_n), there exists a unique isomorphism $f\colon A'_n \to A_n$ such that $\pi'_n = \pi_n \circ f$.*

The uniqueness is standard for this sort of definitions and is left to the reader. The existence is proven below. The universal property of π_n implies that GL_n acts by automorphisms on A_n. Let $A_n^{GL_n} = \{a \in A\colon g \cdot a = a \text{ for all } g \in GL_n\}$ be the algebra of invariants.

Definition 2.3. The *n-th representation scheme of A* is $\mathrm{Rep}_n^A = \mathrm{Spec}(A_n)$. The *$n$-th character scheme of A* is $\mathrm{Rep}_n^A/\!/GL_n = \mathrm{Spec}(A_n^{GL_n})$.

We will mostly consider the coordinates rings A_n and $A_n^{GL_n}$ rather than the schemes themselves.

Remark 2.4. It may be useful to interpret the result geometrically: suppose that $B = \mathcal{O}(X)$ is the space of regular functions on X. Then a representation of A with coefficients in B is a family of representations with parameter space X. The statement is that every such family is the pullback of a universal family parametrized by Rep_n^A by a map $X \to \mathrm{Rep}_n^A$.

2.3. **Construction of A_n.** We are ready to give a proof of Prop. 2.2, which we do by exhibiting an algebra A_n given A and n: A_n is the commutative algebra with generators a_{ij}, one for each $a \in A$ and $1 \leq i, j \leq n$ subject to the relations

$$1_{ij} = \delta_{ij}1, \quad (\lambda a + \mu b)_{ij} = \lambda a_{ij} + \mu b_{ij}, \quad (ab)_{ij} = \sum_{l=1}^{n} a_{il} b_{lj},$$

for all $\lambda, \mu \in k$, $a, b \in A$, $1 \leq i, j \leq n$. The universal representation is $\pi_n \colon a \mapsto (a_{ij})_{1 \leq i,j \leq n}$ and is clearly a representation. If $\rho \colon A \mapsto M_n(B)$ is a representation then the matrix entries $\rho_{ij}(a) \in B$ of $\rho(a)$ obey the same relations as the a_{ij}, implying that the morphism $f \colon A_n \to B$ with $f(a_{ij}) = \rho_{ij}(a)$ is well defined and obeys $f_* \pi_n = \rho$. If $\tilde{f}_* \pi_n = \rho$ for some other morphism $\tilde{f}$ then $\tilde{f}(a_{ij}) = \rho_{ij}(a)$ and $\tilde{f}$ has the same values on generators of A_n as f and is thus equal to f.

2.4. **The representation functor as an adjoint functor.** Recall that a pair of functors $F \colon C \to D$, $G \colon D \to C$ between categories C, D is called an adjoint pair if there is a family of bijections

$$\varphi_{x,y} \colon \mathrm{Hom}_D(F(x), y) \to \mathrm{Hom}_C(x, G(y))$$

which is natural in both x and y (i.e., $\varphi_{—,—}$ is a natural transformation between the two functors on $C^{\mathrm{op}} \times D$). In this case we write $F \colon C \leftrightarrows D : G$ and say that F is left adjoint to G or that G is right adjoint to F.

We apply this notion to the functor $B \mapsto M_n(B)$ on the category ComAlg_k of commutative algebras over k, sending B to the associative algebra of $n \times n$ matrices with entries in B.

Theorem 2.5. *The functor $M_n \colon \mathrm{ComAlg}_k \to \mathrm{Alg}_k$ has a left adjoint functor $(—)_n$:*

$$(—)_n \colon \mathrm{Alg}_k \leftrightarrows \mathrm{ComAlg}_k : M_n$$

In other words we have isomorphisms

$$\mathrm{Hom}_{\mathrm{ComAlg}_k}(A_n, B) \to \mathrm{Hom}_{\mathrm{Alg}_k}(A, M_n(B)) \tag{2.2}$$

that are natural in A and B.

In particular the functor (2.1) is corepresented by A_n and $A \mapsto A_n$ is a functor from associative algebras to commutative algebras. It is a standard fact that Theorem 2.5 follows from Prop. 2.2: given a universal representation (A_n, π_n), the map 2.2 sends $f \in \mathrm{Hom}_{\mathrm{ComAlg}_k}(A_n, B)$ to $M_n(f) \circ \pi_n$. The universal property of π_n is the statement that this map is an isomorphism.

2.5. **Examples.**

Example 2.6. Let $A = k[x]$ be the algebra of polynomials in a variable x with coefficients in k. Then for any $n \times n$ matrix $X \in M_n(B)$ there is a unique representation ρ such that $\rho(x) = X$. Thus Rep_n^A is an affine space of dimension n^2 and $A_n = k[x_{ij}, i, j = 1, \dots n]$ is a polynomial algebra in n^2 variables, the matrix entries x_{ij}. The invariant subalgebra A^{GL_n} is generated by the coefficients c_i of the characteristic polynomial (of $-X$)

$$\det(t1 + X) = t^n + \sum_{i=1}^{n} c_i(X)t^{n-i}$$

of $X = (x_{ij})$, which are algebraically independent. Thus

$$A_n^{GL_n} \simeq k[c_1, \dots, c_n]$$

Example 2.7. The polynomial algebra $A = k\langle x_1, \dots, x_m\rangle$ in non-commutative variables $x_1, \dots, x_m$ is the algebra of formal linear combinations of words $x_{i_1} \cdots x_{i_k}$. It is the free algebra with generators $x_1, \dots, x_m$, namely morphisms $A \to B$ in Alg_k are in one-to-one correspondence with maps $\{x_1, \dots, x_m\} \to B$. Thus Rep_n^A is an affine space of dimension mn^2 and A_n is the polynomial algebra in matrix entries x_{aij}, $a = 1, \dots, m$, $1 \le i, j \le n$. The algebra of invariants is generated by the traces

$$\mathrm{tr}(X_{a_1} \cdots X_{a_k}), \quad 1 \le k \le n^2, \quad 1 \le a_i \le m$$

of the matrices $X_a = (x_{aij})$, see [35], Ch. 8. These invariants are not algebraically independent in general: for example if $m = 2$ and $n = 2$, $A_n^{GL_n}$ is generated by five algebraically independent invariants: $\mathrm{tr}(X_i^j)$ $i, j = 1, 2$, and $\mathrm{tr}(X_1X_2)$.

Example 2.8. Let $A = k[x, y]$ be the polynomial algebra in two (commuting) variables x, y. Then a representation $A \mapsto M_n(B)$ is the same thing as a pair (X, Y) of commuting matrices and $A_n = k[x_{ij}, y_{ij}, i, j = 1, \dots, n]/I$ where I is the ideal generated by

$$\sum_{l=1}^{n} x_{il}y_{lj} - y_{il}x_{lj}, \quad i, j = 1, \dots, n.$$

The corresponding scheme Rep_n^A is called the commuting scheme.

Example 2.9. More generally let A be a finitely presented algebra. So A is the quotient of a free algebra $F = k\langle x_1, \dots, x_m\rangle$ by the two-sided ideal generated by relations $r_i \in F$, $i = 1, \dots, p$. Let $R_n = k[X_1, \dots, X_m]$ denote the polynomial ring in mn^2 variables $X_i = (x_{iab})_{a,b=1}^n$, viewed as entries of m matrices $X_1, \dots, X_m$. The universal representation for the free algebra is the morphism $\pi_n \colon F \to M_n(R_n)$ sending x_i to X_i. Then $A_n = k[X_1, \dots, X_m]/I$ where I is the ideal generated by the matrix entries of the matrix-valued polynomials $\pi_n(r_i)$, $i = 1, \dots, p$. In fact this construction works for any finitely generated algebra: the ideal I is still finitely generated by the Hilbert basis theorem.

Example 2.10. An example with empty representation schemes: the algebra of polynomial differential operators $A = k\langle x, D\rangle/(Dx - xD - 1)$ does not have any finite dimensional representations since for any $\rho \colon A \to M_n(k)$, we have $\rho(D)\rho(x) - \rho(x)\rho(D) = 1$ which is impossible since the left-hand side has trace zero while the trace of the right-hand side is n.

2.6. **Relative representation scheme.** It will be useful to consider a slight generalization of representation schemes. Let A be a unital associative algebra over k and $j \colon S \to A$ a morphism of unital associative algebras. Then for any representation $\rho_S \colon S \to \mathrm{End}(V)$ we have a scheme $\mathrm{Rep}_V^{S\backslash A}$, called the representation scheme relative to (S, V) whose k-points are morphisms $\rho \colon A \to \mathrm{End}(V)$ such that $\rho \circ j = \rho_S$. If we take $S = k, V = k^n$ and ρ_S sending 1_A to the identity matrix we recover the definition of representation scheme. The main motivation for us comes from representations of quivers which we review next.

2.7. **Path algebras of quivers.** Let $Q = (Q_0, Q_1, h, t)$ be a quiver (or directed graph). Thus Q consists of a set Q_0 of vertices, Q_1 of arrows and two maps $h, t\colon Q_1 \to Q_0$ (head, tail of an arrow). We say that $a \in Q_1$ goes from i to j and write $i \xrightarrow{a} j$ if $h(a) = j$ and $t(a) = i$. A path on Q from $i \in Q_0$ to $j \in Q_0$ is a finite sequence $(i = i_0, e_1, i_1 \dots, e_\ell, i_\ell = j)$ such that $i_k \in Q_0$, $e_k \in Q_1$ and $h(e_k) = i_k = t(e_{k-1})$ for all $k = 1, \dots, \ell$. Denote by $P_{i,j}$ the set of paths from i to j and $P = \sqcup P_{i,j}$ the set of all paths. There is an obvious associative concatenation map $P_{i,j} \times P_{j,k} \to P_{k,j}$. The *path algebra* kQ of Q is the vector space with basis P. The product of basis elements is the concatenation if defined and zero otherwise. The paths $p_i = (i)$ of length 0 are orthogonal idempotents in kQ:

$$p_i p_j = \delta_{ij} p_i, \quad i, j \in Q_0.$$

If Q_0 is finite, which we will always assume, then kQ is a unital algebra with unit $1 = \sum_{i \in Q_0} p_i$ and the idempotent p_i span a unital subalgebra S.

Let $(V_i)_{i \in Q_0}$ be a collection of vector spaces labeled by the vertices of a quiver Q. A representation of Q on (V_i) is simply an assignment of a linear map $\rho(e)\colon V_{t(e)} \to V_{h(e)}$ for each edge $e \in Q_1$. Equivalently it is a representation of the path algebra kQ on $V = \oplus_{i \in Q_0} V_i$ such that $\rho(p_i)$ is the projection onto V_i for all idempotents p_i. In other words we have a representation of kQ relative to the representation V of the subalgebra S of idempotents.

A *quiver with relations* is a quiver with a set of formal linear combinations r_k of paths in P_{i_k, j_k} with the same endpoints. The path algebra of a quiver with relations r_k is the quotient kQ/I of the path algebra of the quiver by the two sided ideal I generated by the relations r_k.

Example 2.11. Let Q be the quiver with one vertex and m arrow $x_1, \dots, x_m$. The the path algebra of Q is the free algebra $k\langle x_1, \dots, x_m\rangle$. Relations are just elements of this free algebra and the path algebra of a quiver with relations with underlying quiver Q is a finitely generated algebra with a presentation by generators and relations.

Let S be as above the algebra generated by the idempotents p_i, $i \in Q_0$ and ρ_S be the representation of S sending p_i to the projection onto V_i in a direct sum $V = \oplus_{i \in Q_0} V_i$. Then we have a morphism $S \to kQ/I$ and a representation of the quiver Q with relations is a representation on V whose pull-back to S is ρ_S.

2.8. **Smooth algebras.** A notion of smoothness for associative algebras was introduced by Cuntz and Quillen [11] under the name of quasi-free algebras. One definition, which we adopt, is via the lifting property for square zero ideals. It is the associative version of a characterization of smoothness for affine schemes by Grothendieck (Chapter 0 in EGA IV).

An ideal $I \subset B$ of an algebra $B \in \mathrm{Alg}_k$ is called a square zero ideal if $I^2 = 0$, i.e., if $ab = 0$ for every pair $a, b \in I$.

Definition 2.12. An associative algebra $A \in \mathrm{Alg}_k$ is *formally smooth* if it has the following lifting property for square zero ideals $I \subset B$ in an arbitrary $B \in \mathrm{Alg}_k$: every map $A \to B/I$ factors as $A \to B \to B/I$, where the second map is the canonical projection. An associative algebra is called *smooth* if it is finitely generated and formally smooth.

The main examples of smooth associative algebras are free associative algebras: they have the lifting property for all ideals, not just square zero ideals.

The main non-examples are the polynomial algebras in $m \geq 2$ variables or more generally the coordinate rings of affine schemes of dimension ≥ 2. In particular a smooth commutative algebra is usually not smooth as an associative algebra: the commutative notion of formal smoothness requires the lifting property only for commutative B.

Theorem 2.13. *Representation schemes of smooth algebras are smooth.*

The proof is left as an exercise (see Exercise 6).

2.9. **Exercises.**

(1) Show that $A_1 = A/A[A,A]A$, the quotient of A be the two-sided ideal generated by commutators $[a,b] = ab - ba$, $a, b \in A$.
(2) Show that if $1 \in [A,A]$ then $A_n = 0$ for all n. Show that this is the case for the algebra of differential operators in m variables with polynomial coefficients. Hint: show that $\mathrm{Rep}_n^A(B)$ is empty except if $B = 0$ using the fact that the trace of an $n \times n$ matrix vanishes on commutators.
(3) Show that $A \mapsto A_n$ is a functor.
(4) An *idempotent* in an algebra is an element e such that $e^2 = e$.
    (a) Show that if $f\colon A \mapsto B/I$ is a map to a quotient by a square zero ideal and $e \in A$ is an idempotent then there exists an idempotent $E \in B$ such that $f(E) = e + I$.
    (b) Show that the direct sum $A \oplus B$ of smooth algebras is smooth.
    (c) Show that path algebras of quivers are smooth.
(5) Let k be algebraically closed and $f(x) \in k[x]$ a nonzero polynomial. Show that $k[x]/f(x)k[x]$ is smooth if and only if the polynomial $f(x)$ has only simple zeros.
(6) Use the adjunction of Theorem 2.5 to show that if A is smooth then A_n is smooth for all n.

3. Derived representation schemes

3.1. **Differential graded algebras.** A $\mathbb{Z}$ grading of a k-vector space A is a decomposition as a direct sum

$$A = \oplus_{i \in \mathbb{Z}} A_i$$

The summands A_i are called homogeneous components and we say that $a \in A_i$ is homogeneous and has *degree* i or $|a| = i$. A vector space with a $\mathbb{Z}$-grading is called a $\mathbb{Z}$-graded vector space. Morphisms of $\mathbb{Z}$-graded vector spaces are degree preserving linear maps. Ordinary vector spaces can be viewed as $\mathbb{Z}$-graded vector spaces with $A_i = 0$ for $i \neq 0$. They form a full subcategory of the category of $\mathbb{Z}$-graded vector spaces.

A $\mathbb{Z}$*-graded algebra* is a unital associative algebra A with a $\mathbb{Z}$-grading such that $A_i \cdot A_j \subset A_{i+j}$. The unit element 1 necessarily belongs to A_0, which is a subalgebra.

A *differential* on a $\mathbb{Z}$-graded vector space A is a linear map $d\colon A \to A$ such that

(1) $d(A_i) \subset A_{i-1}$ (d has degree -1),
(2) $d \circ d = 0$.

A *chain complex over* k is a $\mathbb{Z}$-graded vector space with a differential. A morphism of chain complexes, also called a *chain map*, is a morphism of $\mathbb{Z}$-graded vector spaces commuting with the differentials.

The *homology* of a chain complex (C, d) is the graded vector space $H_\bullet(C, d) = \oplus H_i(C, d)$, with

$$H_i(C, d) = \mathrm{Ker}(d\colon C_i \to C_{i-1})/\mathrm{Im}(d\colon C_{i+1} \to C_i).$$

This is well-defined since the space of *boundaries* $B(d) = \mathrm{Im}(d)$ is contained in the space of *cycles* $Z(d) = \mathrm{Ker}(d)$ because $d^2 = 0$. Also a chain map $f\colon C \to C'$ sends cycles to cycles and boundaries to boundaries and thus induces a map $H_\bullet(f)\colon H_\bullet(C) \to H_\bullet(C')$ such that $H_\bullet(fg) = H_\bullet(f)H_\bullet(g)$. In other words we have a homology functor $H_\bullet$ from the category of chain complexes to the category of graded vector spaces.

Definition 3.1. A *differential graded algebra* (dga) over k is a $\mathbb{Z}$-graded algebra A with a differential d which is a *graded derivation*:

$$d(ab) = d(a)b + (-1)^{|a|}ad(b),$$

for all homogeneous $a, b \in A$. A morphism of differential graded algebras is a morphism of algebras which is also a chain map.

An immediate consequence of the definition is that the homology of a dga is a $\mathbb{Z}$-graded algebra. Indeed the cycles form a $\mathbb{Z}$-graded algebra and the boundaries form an ideal in the algebra of cycles. Also a morphism $f\colon A \to B$ induces an algebra morphism $H_\bullet(f)\colon H_\bullet(A) \to H_\bullet(B)$.

Definition 3.2. A dga is *commutative* if

$$ab = (-1)^{|a||b|}ba$$

for all homogeneous a, b. Morphisms of commutative dgas are morphisms of dgas.

Definition 3.3. A morphism of dgas or of commutative dgas is called a *quasi-isomorphism* if it induces an isomorphism in homology.

We denote by DGA_k the category of (unital) differential graded algebras over k and by CDGA_k the full subcategory of commutative dgas.

Definition 3.4. The tensor product of dgas $A \otimes B$ is the dgas with homogeneous components

$$(A \otimes B)_i = \oplus_{j+l=i} A_j \otimes B_l,$$

product

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|}aa' \otimes bb',$$

and differential

$$d(a \otimes b) = da \otimes b + (-1)^{|a|}a \otimes db.$$

3.2. **Free algebras.** Free algebras are important examples of graded algebras. Let $V^{\otimes m} = V \otimes \cdots \otimes V$ (m factors) denote the m-fold tensor product of a vector space V with itself.

The *tensor algebra* of a graded vector space $V = \oplus V_i$ is

$$T(V) = k \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$$

with homogeneous components

$$T(V)_i = \oplus_k \oplus_{i_1+\cdots+i_k=i} V_{i_1} \otimes \cdots \otimes V_{i_k}.$$

The product is defined by concatenation:

$$(a_1 \otimes \cdots \otimes a_n)(b_1 \otimes \cdots \otimes b_m) = a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m.$$

It is the free graded algebra generated by V (T is left adjoint of the forgetful functor $\mathrm{DGA}_k \to \mathrm{Vect}_k$ to vector spaces).

If $(x_i)_{i\in I}$ is a homogeneous basis of V then $T(V)$ has a homogeneous basis of *words*

$$x_{i_1}x_{i_2}\cdots x_{i_m}, \quad m \geq 0, i_s \in I,$$

including the empty word 1.

The symmetric group S_l, which is the group of permutations of l letters, acts on $V^{\otimes l}$ with the Koszul sign rule: the transposition s_i of i and $i+1$ acts via

$$s_{ij}(\cdots \otimes v_i \otimes v_{i+1} \otimes \cdots) = (-1)^{|v_i||v_{i+1}|} \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots .$$

Definition 3.5. The *symmetric algebra* of a graded vector space $V = \oplus V_i$ is the direct sum of coinvariants

$$\mathrm{Sym}(V) = \bigoplus_{m=0}^{\infty} \mathrm{Sym}^m(V), \quad \mathrm{Sym}^m(V) = (V^{\otimes m})_{S_m},$$

with the understanding that $\mathrm{Sym}^0(V) = k$. In other words, $\mathrm{Sym}(V)$ is the quotient of $T(V)$ by the two-sided ideal generated by $vw - (-1)^{|v||w|}wv$ for all homogeneous $v, w \in V$. It is the free graded commutative algebra generated by V.

If $(x_i)_{i\in I}$ is a homogeneous basis of V then we denote $\mathrm{Sym}(V)$ also $k[x_i, i \in I]$. It has a basis of monomials $\prod_{i\in I} x_i^{n_i}$ where $n_i = 0$ except for finitely many i and $n_i \in \{0, 1\}$ for variables x_i of odd degree.

If V is a chain complex then $T(V)$ and $\mathrm{Sym}(V)$ are differential graded algebras. The differential is uniquely determined by the condition that the inclusion of generators $V \to T(V)$, $V \to \mathrm{Sym}(V)$ are chain maps. The algebras $T(V)$, $\mathrm{Sym}(V)$ are the free differential graded (commutative) algebras generated by the chain complex V. We will need to consider more generally *semi-free* differential graded algebras, whose underlying graded algebras are free but whose differential is not assumed to be induced by a differential on V.

3.3. **Representation schemes of differential graded algebras.** The generalization of the functors $(—)_n$ to dgas is straightforward: if B is a commutative dga then $M_n(B)$ with differential acting on each matrix entry is a dga. A representation of a dga A with coefficients in B is a morphism $A \to M_n(B)$ of dgas. There is a universal representation $A \to M_n(A_n)$ given by the same construction as in Section 2.3. This gives:

Proposition 3.6. *There is a pair of adjoint functors*

$$(—)_n \colon \mathrm{DGA}_k \leftrightarrows \mathrm{CDGA}_k : M_n$$

3.4. **Representation homology.** A dga over k is called *semi-free* if it is free as a graded algebra, i.e., it is isomorphic as a graded algebra (forgetting the differential) to the tensor algebra of a graded vector space. A (non-negatively graded) semi-free resolution of an algebra A, viewed as a dga concentrated in degree 0, is a quasi-isomorphism $QA \to A$, where QA is semi-free and concentrated in non-negative degrees.

Definition 3.7. The nth *representation homology* $H_\bullet(A,n)$ of an algebra $A \in \mathrm{Alg}_k$ is the homology

$$H_\bullet(A,n) = H_\bullet((QA)_n)$$

of the n-th representation algebra of a semi-free resolution of A.

In Section 4 we will need a generalization to the relative case $S \to A$. A resolution is taken in the category of differential graded algebras over S, resulting in a homology $H_\bullet(S\backslash A, V)$ see [5] for details.

For Definition 3.7 to make sense we need the existence of semi-free resolutions and that the result is independent of the choice of resolution.

Let us first consider the existence question in a simple, apparently innocent, example, the two-dimensional algebra $A = k[x]/(x^2)$ of dual numbers. The general case follows the same pattern. As a first approximation to a semi-free resolution we adjoin a variable p of degree 1 that "kills the relations";

$$R = k\langle x, p\rangle, \quad dp = x^2, \quad dx = 0.$$

Then $Z_0(R) = k[x]$ and $B_0(R) = \mathrm{span}(d(x^m p x^\ell)) = x^2 k[x]$, and $H_0(R) = A$ as desired. However $H_1(R) \neq 0$ as it contains the class of $px - xp$ which cannot possibly be a boundary as all boundaries have degree at least 2 in x. So we add a new variable killing this cocycle;

$$R' = k\langle x, p, p'\rangle, \quad dp' = px - xp.$$

Now the strategy is clear: we recursively add new variables of higher and higher degree to kill cocycles degree by degree and we get a sequence of algebras

$$R \subset R' \subset R'' \subset \cdots$$

whose homology in degree zero is A and whose homology in positive degree vanishes up to higher and higher degree. The direct limit (union) of this sequence is a resolution QA.

Exercise. Let $A = k[x]/(x^2)$. Show that $QA = k\langle p_0, p_1, p_2, \dots\rangle$ with $\deg p_i = i$, differential defined by $dp_0 = 0$ and

$$dp_i = \sum_{k=0}^{i-1} (-1)^k p_{i-1-k} p_k, \quad i \geq 1,$$

and map $QA \to A$ sending p_0 to x and p_i to 0 for $i > 0$, is a semi-free resolution of A.

Proposition 3.8. *The representation homology is independent of the choice of semi-free resolution up to isomorphism of graded commutative algebras.*

3.5. **Derived representation schemes.** Proposition 3.8 is a consequence of a stronger statement holding on the level of homotopy categories. We formulate it for simplicity in the absolute case and refer to [5] for the relative case of an algebra homomorphism $S \to A$.

Theorem 3.9 (Berest, Khatchatryan, Ramadoss [5])**.** *The functor $A \mapsto A_n$ has a total left derived functor*

$$L(-)_n \colon \mathrm{Ho}(\mathrm{DGA}_k) \to \mathrm{Ho}(\mathrm{CDGA}_k)$$

between homotopy categories.

We proceed to give a short explanation of the terminology used in the statement of this theorem. The setting is Quillen's theory of model categories [36, 37]. See [13, 16, 17] for accounts of this theory and the appendices to [5] for a summary and the applications to categories of differential graded algebras.

The homotopy categories in Theorem 3.9 are localizations with respect to the class of quasi-isomorphisms. Recall that a *localization* $D = C[W^{-1}]$ of a category C with respect to a class W of morphisms is a pair (D, γ) consisting of a category D and a functor $\gamma\colon C \to D$ sending morphisms in W to isomorphisms and such that for any other pair (D', γ') with this property, there is a unique functor $f\colon D \to D'$ such that $\gamma' = f \circ \gamma$. Given a localization (D, γ) and a functor $f\colon C \to E$, a *left derived functor* $g = Lf$ of f is a pair (g, t) consisting of a functor $g\colon D \to E$ and a natural transformation $t\colon g \circ \gamma \to f$ such that for any other pair (g', t') there exists a unique natural transformation $s\colon g' \to g$ such that t' is the composition $g' \circ \gamma' \to g \circ \gamma \to f$ of the natural transformations $s \circ \gamma$ and t. As usual with these definitions through universal properties, localizations and derived functors, if they exist, they are unique in the appropriate sense.

What makes it possible to work with these abstract definitions is Quillen's theory of model categories. These are categories with finite limits and colimits, equipped with three distinguished classes of morphisms, called weak equivalences, fibrations and cofibrations, obeying a system of axioms borrowed from homotopy theory of topological spaces, which provide the prototypical example. The categories $\mathrm{DGA}_k, \mathrm{CDGA}_k$ are model categories such that weak equivalence are quasi-isomorphisms and fibrations are epimorphisms in each degree.

The homotopy category $\mathrm{Ho}(C)$ of a model category C is the localization $(C[W^{-1}], \gamma_C)$ with respect to the class W of weak equivalences and a *total left derived functor* Lf of a functor $f\colon C \to D$ between model categories is a left derived functor of the composition $\gamma_D \circ f\colon C \to \mathrm{Ho}(D)$.

To get a more explicit description of homotopy categories and derived functors we need to involve fibrations and cofibrations. A *fibrant object* of a model category is an object A so that the map $A \to *$ to the terminal object is a fibration. Similarly a *cofibrant object* A is such that the map $\varnothing \to A$ from the initial object is a cofibration. It follows from the axioms that for every object A of a model category there is a cofibrant object QA and a weak equivalence $QA \to A$ which is also a fibration. Such an object QA is called a cofibrant replacement of A and is the generalization of a projective resolution in homological algebra.

The homotopy category of a model category C such that all objects are fibrant, such as $\mathrm{DGA}_k, \mathrm{CDGA}_k$, may be realized concretely as the category whose objects are the objects in C and whose morphisms $A \to B$ are homotopy classes $\pi(QA, QB)$ of morphisms between cofibrant replacements. Homotopies between morphisms are defined in model categories. For model categories of differential graded algebras they can be described explicitly in terms of the algebraic de Rham algebra $\Omega = k[t] \oplus k[t]dt$ (with the convention that the differential $d\colon t \mapsto dt$ has degree -1). Namely, a homotopy between morphisms f and $g\colon A \to B$ with A cofibrant is a morphism $A \to B \otimes \Omega$ such that $h(0) = f$ and $h(1) = g$ (the evaluation $h(a)$ at a means setting $t = a$ and $dt = 0$).

The functor $L(—)_n$ applied to an associative algebra A is the coordinate ring of the *nth derived representation scheme* $\mathrm{DRep}_n A$. A semi-free differential graded algebra QA with a surjective morphism $QA \to A$ is an example of a cofibrant replacement of A and the coordinate ring $\mathcal{O}(\mathrm{DRep}_n A)$ is realized as the commutative differential graded algebra $(QA)_n$. It is shown in [5, Section 2.3.6] that for a suitable choice of cofibrant replacement the derived representation scheme is isomorphic to the *derived space of actions*, a derived scheme previously introduced by Ciocan-Fontanine and Kapranov, see [10, Section 3.3]

3.6. **Comparison maps.** In this section we introduce two algebras that map to and from the invariant part of representation homology. The *trace map* maps the symmetric algebra of the cyclic homology to the representation homology and the *Harish–Chandra homomorphism* maps the invariant part of the representation homology to the symmetric algebra of the first representation homology. These constructions allow us to compare the representation homology with more classical invariants.

Definition 3.10. The *character* of a representation $\rho\colon A \to M_n(B)$ with coefficients in a commutative algebra B is the map $A \to B$ given by the trace (sum of diagonal matrix entries)

$$\chi_\rho(a) = \mathrm{tr}\,\rho(a).$$

The two basic properties of characters are easy to prove:

Proposition 3.11.

(1) *χ_ρ vanishes on commutators $[a, b] = ab - ba$.*
(2) *If ρ and ρ' are equivalent then $\chi_\rho = \chi_{\rho'}$.*

Let $[A, A]$ be the vector subspace of A spanned by commutators. Then $\chi_\rho(a)$, as a function of ρ is a GL_n-invariant function on Rep_n^A. We thus obtain a map $\mathrm{Tr_n}\colon A/[A, A] \to A_n^{GL_n}$. The formal definition is:

Definition 3.12. The *trace map*

$$\mathrm{Tr}_n\colon A/[A, A] \to A_n^{GL_n}$$

is the character of the universal representation:

$$\mathrm{Tr}_n(a) = \mathrm{tr}\,\pi_n(a) = \sum_{i=1}^{n} a_{ii}$$

Example 3.13. Let $A = k[x]$. Then $A/[A, A] = A$, $A_n = k[x_{ij}, 1 \le i, j \le n]$. Let $X = (x_{ij}) \in M_n(A_n)$.

$$\mathrm{Tr}_n(x^p) = \mathrm{tr}(X^p) = \sum_{i_1,\ldots,i_p=1}^{n} x_{i_1,i_2} x_{i_2,i_3} \cdots x_{i_p,i_1}.$$

More general invariant functions can be obtained by taking linear combinations of products of characters. This yields an algebra homomorphism, extending Tr_n

$$\mathrm{Tr}_n\colon \mathrm{Sym}(A/[A, A]) \to A_n^{GL_n}. \tag{3.1}$$

Theorem 3.14 (Procesi 1976 [34])**.** *The trace map 3.1 is surjective.*

For $k[x]$ this means that the functions on $n \times n$ matrices

$$\mathrm{tr}(X^{p_1}) \cdots \mathrm{tr}(X^{p_m}), \qquad X = (x_{ij}),$$

span the space of GL_n invariant polynomials in the matrix entries, cf. Example 2.7.

Now $A/[A,A] = HC_0(A)$ is the degree zero cyclic homology and the trace map extends [5] naturally to a map

$$\mathrm{Tr}_n\colon HC_\bullet(A) \to H_\bullet(A,n).$$

We get a comparison map from cyclic homology to representation homology. It is shown in [6] that while a naive extension of Procesi's theorem does not hold, one gets a stabilization results for augmented algebras. In the case of augmented algebras there are GL_n-equivariant maps $A_{n+1} \to A_n$, $H_\bullet(A,n+1) \to H_\bullet(A,n)$ so one has an inverse limit $H_\bullet(A,\infty)$ which has a Hopf algebra structure. Then the trace map defines an isomorphism of Hopf algebras $\mathrm{Sym}\,\overline{HC}_\bullet(A) \to H_\bullet(A,\infty)^{GL_\infty}$, where $\overline{HC}_\bullet(A)$ is the reduced cyclic homology of A. See [6] for details and more precise statements.

The other comparison map is with the derived version of the "commutativisation" of an associative algebra A, i.e. the commutative algebra $A/A[A,A]A$. As we have seen in Exercise 1 it coincides with algebra A_1 of functions on the first representation scheme which is the degree zero part of the first representation homology $H_\bullet(A,1)$.

The direct sum of representations $\rho_i\colon A \mapsto M_{n_i}(B)$ of dimension n_i, $i = 1,\dots,m$ of $A \in \mathrm{Alg}_k$ is the representation $\rho = \rho_1 \oplus \cdots \oplus \rho_m$ of dimension $n = \sum n_i$ given by the block diagonal matrices

$$\rho(a) = \begin{pmatrix} \rho_1(a) & & & \\ & \rho_2(a) & & \\ & & \ddots & \\ & & & \rho_m(a) \end{pmatrix}.$$

This defines a map (restriction to direct sum representations)

$$\varphi\colon A_n \to A_{n_1} \otimes \cdots \otimes A_{n_m}.$$

Indeed we have a map $\oplus\colon \prod_{i=1}^m \mathrm{Rep}^A_{n_i}(B) \to \mathrm{Rep}^A_n(B)$ which defines a map

$$\prod_{i=1}^m \mathrm{Hom}(A_{n_i}, B) \simeq \mathrm{Hom}(\otimes_{i=1}^n A_{n_i}, B) \to \mathrm{Hom}(A_n, B).$$

Now take $B = \otimes_{i=1}^n A_{n_i}$ (the coproduct in CDGA_k): the image of the identity map is φ.

We will be mostly concerned with the special case where all $n_i = 1$. In the description of Section 2.3 the map $A_n \to A_1^{\otimes n}$ is given by

$$a_{ij} \mapsto a_i \delta_{ij}$$

where $a_i = 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$ with a in the ith factor. It induces a morphism of algebras, called the Harish-Chandra homomorphism

$$A_n^{GL_n} \to (A_1^{\otimes n})^{S_n}$$

Example 3.15. Let $A = k[x]$. Then $A_n = k[X] = k[x_{ij}, i,j = 1,\dots,n]$ and $A_1^{\otimes n} = k[x_1,\dots,x_n]$. The algebra $A_n^{GL_n}$ consists of conjugation invariant polynomial functions on $n \times n$-matrices which is a polynomial algebra $A_n = k[c_1,\dots,c_n]$

in the coefficients of the characteristic polynomial, see Example 2.6. The Harish-Chandra homomorphism $k[c_1, \ldots, c_n] \to k[x_1, \ldots, x_n]^{S_n}$ is the restriction to diagonal matrices and sends c_r to the elementary symmetric function

$$\sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}.$$

It is well-known to be an isomorphism.

The Harish-Chandra homomorphism extends to a homomorphism of differential graded algebras and induces an algebra homomorphism

$$H_\bullet(A, n)^{GL_n} \to (H_\bullet(A, 1)^{\otimes n})^{S_n}$$

called the derived Harish-Chandra homomorphism. It restricts to the Harish-Chandra homomorphism in degree 0. In the previous example the representation homology is concentrated in degree 0 and thus the derived Harish-Chandra homomorphism is an isomorphism. In the next section we discuss a less trivial example.

3.7. **Derived commuting schemes.** Here we return to Example 2.8. We view the algebra $k[x, y]$ as an associative algebra: it is the quotient of the free algebra $k\langle x, y\rangle$ by the relation $xy - yx = 0$. Following the strategy of Section 3.4 to construct a semi-free resolution and adjoin a variable θ of degree 1 with differential

$$d\theta = xy - yx.$$

It turns out that we do not need to adjoin other variables to get a resolution. Namely the differential graded algebra $QA = k\langle x, y, \theta\rangle$ with this differential has vanishing homology in negative degree so that the map $QA \to A$ defined by setting θ to 0 is a quasi-isomorphism.

The representation homology is the homology of the commutative differential graded algebra $C_\bullet(A, n) = k[X, Y, \Theta] = k[x_{ij}, y_{ij}, \theta_{ij}, i, j = 1, \ldots, n]$, with differential

$$dx_{ij} = dy_{ij} = 0, \quad d\theta_{ij} = \sum_{l=1}^{n} (x_{il} y_{lj} - y_{il} x_{lj}).$$

The homology of this complex seems to be hard to compute except for $n = 1$ in which case the differential vanishes so that

$$H_\bullet(A, 1) = k[x, y, \theta] = k[x, y] \oplus k[x, y]\theta.$$

The derived Harish–Chandra homomorphism is then the map

$$H_\bullet(k[x, y], n)^{GL_n} \to k[x_1, \ldots, x_n, y_1, \ldots, y_n, \theta_1, \ldots, \theta_n]^{S_n} \tag{3.2}$$

induced by the map of differential graded algebras

$$x_{ij} \mapsto x_i \delta_{ij}, \quad y_{ij} \mapsto y_i \delta_{ij}, \quad \theta_{ij} \mapsto \theta_i \delta_{ij}.$$

Conjecture 3.16. *The derived Harish-Chandra homomorphism 3.2 is an isomorphism.*

The conjecture has a combinatorial identity (which can actually be proved) as an interesting consequence. It is obtained by comparing character-valued Euler characteristics. The complex QA is infinite dimensional but decomposes as a direct sum of subcomplexes QA_m with fixed weight $m \in \mathbb{Z}^2_{\geq 0}$ for the action of $(k^\times)^2$ so that

x_{ij} has weight $(1,0)$, y_{ij} has weight $(0,1)$ and θ_{ij} has weight $(1,1)$. The homology and its invariant part decompose accordingly into finite dimensional weight spaces:

$$H_\bullet(A,n) = \oplus_{m\in\mathbb{Z}^2_{\geq 0}} H_\bullet(A,n)_m,$$
$$H_\bullet(A,n)^{GL_n} = \oplus_{m\in\mathbb{Z}^2_{\geq 0}} H_\bullet(A,n)_m^{GL_n}.$$

We can then define the character-valued Euler characteristic as the generating function of the Euler characteristics of the weight subcomplexes. We are interested in the invariant part, so we set

$$\begin{aligned}\chi(A,n) &= \sum_{m\in\mathbb{Z}^2_{\geq 0}} \sum_{i\geq 0} (-1)^i \dim H_i(A,n)_m^{GL_n} q_1^{m_1} q_2^{m_2}\\ &= \sum_{m\in\mathbb{Z}^2_{\geq 0}} \sum_{i\geq 0} (-1)^i \dim C_i(A,n)_m^{GL_n} q_1^{m_1} q_2^{m_2}.\end{aligned}$$

Here we used that the Euler characteristic of the homology of a finite dimensional complex is the same as the Euler characteristic of the complex and that taking invariants for a reductive group such as GL_n commutes with passing to homology.

Recall that the dimension of the GL_n-invariant subspace of a finite dimensional representation of GL_n is given by Weyl's formula

$$\dim V = \frac{1}{n!}\,\mathrm{CT}\left(\mathrm{ch}_A V \prod_{i\neq j}\left(1-\frac{z_i}{z_j}\right)\right)$$

in terms of the character

$$\mathrm{ch}_A V = \sum_{\mu\in\mathbb{Z}^n} \dim V_\mu z_1^{\mu_1}\cdots z_n^{\mu_n},$$

where $V = \oplus V_\mu$ is the decomposition of V into weight spaces for the action of the torus $A = (k^\times)^n$ of diagonal matrices in GL_n. Here

$$\mathrm{CT}\colon \mathbb{Z}[z_1^{\pm 1},\dots,z_n^{\pm 1}] \to \mathbb{Z}$$

is the constant term (coefficient of $z_1^0\cdots z_n^0$) of a Laurent polynomial. Applying Weyl's formula to $C_\bullet(A,n)_m$ gives

$$\chi(A,n) = \frac{1}{n!}\,\mathrm{CT} \sum_{m\in\mathbb{Z}^2_{\geq 0}} \sum_{d\geq 0} (-1)^d\, \mathrm{ch}_A(C_d(A,n)_m) \prod_{i\neq j}\left(1-\frac{z_i}{z_j}\right) q_1^{m_1} q_2^{m_2}.$$

where CT is extended to a map $\mathbb{Z}[z_1^{\pm 1},\cdots,z_n^{\pm 1}][[q_1,q_2]] \to \mathbb{Z}[[q_1,q_2]]$ by acting on coefficients. The characters are easy to calculate explicitly in terms of geometric series since the weight spaces are spanned by monomials. We get

$$\chi(A,n) = \frac{1}{n!}\,\mathrm{CT} \prod_{i,j=1}^{n}{}' \frac{(1-q_1q_2z_i/z_j)(1-z_i/z_j)}{(1-q_1z_i/z_j)(1-q_2z_i/z_j)}. \tag{3.3}$$

The notation $\prod'$ indicates that one should omit the factors $1-z_i/z_j$ with $i=j$.

This is to compared with the Euler characteristic of

$$(H_\bullet(A,1)^{\otimes n})^{S_n} = k[x_i,y_i,\theta_i, i=1,\dots,n]^{S_n}.$$

This Euler characteristic can be computed by enumerating S_n-orbits of monomials. The answer can be written in many different ways, see [3]. Here is one formulation, which will be useful to us to estimate gauge theory partition functions. Another, mentioned below in Conjecture 3.19, leads to generalizations for reductive Lie groups replacing GL_n.

Lemma 3.17. *Let $E_n = k[x_i, y_i, \theta_i, i = 1, \dots, n]^{S_n}$ be the algebra of S_n-invariants of the free graded commutative algebra with generators x_i, y_i of degree 0 and θ_i of degree 1. Let $(E_n)_{d,m}$ be the weight m components of the homogeneous component of degree d. Let further*

$$a_n(q_1, q_2) = \sum_{m\in\mathbb{Z}^2_{\geq 0}} \sum_{d=0}^{n} (-1)^d \dim(E_n)_{d,m} q_1^{m_1} q_2^{m_2}.$$

be the character-valued Euler characteristic of $(H_\bullet(A, 1)^{\otimes n})^{S_n}$. Then the formal power series $a_n(q_1, q_2)$ converges to a holomorphic function for $|q_1| < 1, |q_2| < 1$ and is the coefficient of v^n of the Taylor expansion at 0 of

$$\sum_{n=0}^{\infty} v^n a_n(q_1, q_2) = \frac{1 - v}{\prod_{n=1}^{\infty}(1 - vq_1^n)(1 - vq_2^n)} = \exp\left(\sum_{n=1}^{\infty} \frac{1 - q_1^n q_2^n}{(1 - q_1^n)(1 - q_2^n)} \frac{v^n}{n}\right).$$

Let $(q; q)_n = \prod_{j=1}^{n}(1 - q^j)$. Using the product formula for the q-exponential function

$$\frac{1}{\prod_{j=0}^{\infty}(1 - vq^n)} = \sum_{j=0}^{\infty} \frac{v^n}{(q; q)_n}$$

we get the more explicit formula

$$a_n(q_1, q_2) = \sum_{j=0}^{n} \frac{q_2^j}{(q_1; q_1)_j (q_2, q_2)_{n-j}} \tag{3.4}$$

The comparison of (3.4) with (3.3) leads to an identity of formal power series in q_1, q_2 (and of holomorphic functions on the neighbourhood of 0).

Theorem 3.18. *The constant term identity*

$$\frac{1}{n!} \operatorname{CT} {\prod_{i,j=1}^{n}}' \frac{(1 - q_1 q_2 z_i/z_j)(1 - z_i/z_j)}{(1 - q_1 z_i/z_j)(1 - q_2 z_i/z_j)} = \sum_{j=0}^{n} \frac{q_2^j}{(q_1; q_1)_j (q_2, q_2)_{n-j}}$$

holds in $\mathbb{Z}[[q_1, q_2]]$, where the prime means that the factors $(1 - z_i/z_j)$ with $i = j$ are omitted.

We derived this formula by comparing Euler characteristics from the conjecture that the Harish-Chandra map is an isomorphism. However Theorem 3.18 can be deduced replacing $k[x, y]$ by the quantum plane $k\langle x, y\rangle/(xy - \zeta yx)$ with ζ not a root of unity. In this case the representation homology is concentrated in degree 0 and the Harish-Chandra isomorphism conjecture can be proved, see [3]. The Euler characteristics are insensitive to ζ since the chain complexes are modules independent of ζ, the only dependence is in the differential.

We end the discussion with a conjectural generalization of this identity [3]. Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$ and adjoint representation $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$. The Haar measure dg on G is normalized so that $\int_G dg = 1$. Let $T \subset G$ be a maximal torus. The Weyl group $W = N(T)/T$ acts on the Lie algebra $\mathfrak{h}$ of T.

Conjecture 3.19. [3] *For any $q_1, q_2 \in \mathbb{C}$ so that $|q_1|, |q_2| < 1$,*

$$\int_G \frac{\det(1 - q_1 q_2 \operatorname{Ad}(g))}{\det(1 - q_1 \operatorname{Ad}(g)) \det(1 - q_2 \operatorname{Ad}(g))} dg = \frac{1}{|W|} \sum_{w \in W} \frac{\det(1 - q_1 q_2 w)}{\det(1 - q_1 w) \det(1 - q_2 w)}.$$

where the determinants are defined by the natural action of W on $\mathfrak{h}$.

This conjecture is proved for $G = U(n)$ since it is a rephrasing of the identity of Theorem 3.18. It was checked for $U(n), SU(n)$, for classical groups in the stable range and up to second order in the Taylor expansion in powers of q_1, q_2 for arbitrary G in [3], and for B_2, G_2 [32]. Note that the left-hand side can be written as a constant term via Weyl's integration formula [7, Ch. IX, §9]. The set of roots $R \subset \mathfrak{h}^*$ spans the root lattice Q. Let $\mathbb{Z}Q$ be the group ring of Q. We use the customary notation e^α to denote the element of $\mathbb{Z}Q$ corresponding to $\alpha \in Q$. The constant term $\mathrm{CT}\colon \mathbb{Z}Q \to \mathbb{Z}$ is the linear map such that $e^\alpha \mapsto \delta_{\alpha,0}$. It extends to a linear map $\mathbb{Z}Q[[q_1, q_2]] \to \mathbb{Z}[[q_1, q_2]]$ defined by acting on coefficients. The conjecture may be written as ($r = \dim \mathfrak{h}$)

$$\frac{(1 - q_1 q_2)^r}{(1 - q_1)^r (1 - q_2)^r} \mathrm{CT} \prod_{\alpha \in R} \frac{(1 - q_1 q_2 e^\alpha)(1 - e^\alpha)}{(1 - q_1 e^\alpha)(1 - q_2 e^\alpha)} = \sum_{w \in W} \frac{\det(1 - q_1 q_2 w)}{\det(1 - q_1 w) \det(1 - q_2 w)}.$$

The conjecture is a consequence of a more general conjecture. A representation of $k[x, y]$ in $n \times n$ matrices is the same as a Lie algebra homomorphism $\mathfrak{a} \to \mathfrak{gl}_n$ from the two-dimensional abelian Lie algebra $\mathfrak{a}$ with basis x, y. A Lie algebra version of derived representation schemes exists, see [2,3], and gives a representation homology $H_\bullet(\mathfrak{a}, \mathfrak{g})$ for reductive Lie algebras $\mathfrak{g}$ such that $H_0(\mathfrak{a}, \mathfrak{g})$ is the coordinate ring of the scheme of Lie algebra homomorphisms $\mathfrak{a} \to \mathfrak{g}$. The adjoint action of $\mathfrak{g}$ extends to the representation homology and one has a Harish-Chandra homomomorphism $H_\bullet(\mathfrak{a}, \mathfrak{g})^{\operatorname{ad} \mathfrak{g}} \to H_\bullet(\mathfrak{a}, \mathfrak{h})^W$, which is conjectured to be an isomorphism for $\mathfrak{a}$ two-dimensional abelian and $\mathfrak{g}$ reductive with Cartan subalgebra $\mathfrak{h}$ and Weyl group W. It is checked for classical Lie algebras in the stable range where the rank tends to infinity.

The following statement, mildly supporting the conjecture, is left as an exercise.

Exercise. Prove Conjecture 3.19 for $q_2 = 0$ in the following way. Recall that the Chevalley theorem, see [8, Ch. V, §5] states that for any semisimple Lie algebra $\mathfrak{g}$ of rank r the algebra $\mathrm{Sym}(\mathfrak{g}^*)^{\mathfrak{g}}$ of invariants in the symmetric algebra for the coadjoint action of $\mathfrak{g}$ is isomorphic by the canonical restriction map $\mathfrak{g}^* \to \mathfrak{h}^*$ to the algebra of Weyl group invariant functions $\mathrm{Sym}(\mathfrak{h}^*)^W$ on a Cartan subalgebra.

Let $w \in W$ act on $\mathfrak{h}$ with eigenvalues $\lambda_1, \dots, \lambda_r$. Show that the induced action of w on $\mathrm{Sym}^d \mathfrak{h}^*$ has eigenvalues $\lambda_1^{n_1} \cdots \lambda_r^{n_r}$ with $n_i \in \mathbb{Z}_{\geq 0}$, $\sum n_i = d$. Deduce that the generating function $\sum_{d=0}^\infty q^d \operatorname{tr}(w|_{\mathrm{Sym}^d(\mathfrak{h}^*)})$ of characters of w is $\det(1 - qw)^{-1}$. Show with the Chevalley isomorphism that

$$\int_G \frac{1}{\det(1 - q \operatorname{Ad}(g))} dg = \frac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1 - qw)}.$$

Moreover $\mathrm{Sym}(\mathfrak{h}^*)^W$ is a free commutative algebra with generators $I_{d_1}, \dots, I_{d_r}$ with $I_{d_i} \in \mathrm{Sym}^d(\mathfrak{h}^*)$ homogeneous of degree d_i. Show that

$$\int_G \frac{1}{\det(1 - q \operatorname{Ad}(g))} dg = \prod_{i=1}^r \frac{1}{1 - q^{d_i}}.$$

or, as a constant term identity,

$$\frac{1}{|W|}\,\mathrm{CT}\prod_{\alpha\in R}\frac{1-e^\alpha}{1-qe^\alpha}=\prod_{i=1}^r\frac{1-q}{1-q^{d_i}}. \tag{3.5}$$

Remark 3.20. The Macdonald constant term conjecture [20], proved for any root systems R by Cherednik [9],

$$\frac{1}{|W|}\,\mathrm{CT}\prod_{n=0}^\infty\prod_{\alpha\in R}\frac{1-q^ne^\alpha}{1-q^nte^\alpha}=\prod_{n=0}^\infty\prod_{i=1}^r\frac{(1-q^nt)(1-q^{n+1}t^{d_i-1})}{(1-q^{n+1})(1-q^nt^{d_i})}, \tag{3.6}$$

for root systems of rank r, is a two variable version of the identity

$$\int_G\det(1-q\,\mathrm{Ad}(\mathfrak{g}))dg=\prod_{i=1}^r(1-q^{2d_i-1})$$

to which it reduces for $t=q^2$ (Macdonald's original conjecture [20, Conjecture 3.1] is for $t=q^k$, $k\in\mathbb{Z}_{>0}$ when the products reduce to finite products). The latter identity follows in the same way as in the exercise from the Hopf–Koszul–Samelson theorem stating that the invariants $(\bigwedge\mathfrak{g})^{\mathfrak{g}}$ in the exterior algebra of a semisimple Lie algebra $\mathfrak{g}$ is the exterior algebra in generators of degree $2d_i-1$, $i=1,\dots,r$. So our conjecture is an "even" version of the Macdonald constant term identity and the Macdonald identity itself can be understood in terms of representation homology of a super Lie algebra, see [3]. Note that the Macdonald identity (3.6) also reduces to (3.5) as $q\to 0$.

4. $N=2$ supersymmetric gauge theory

In this lecture we describe the connection of representation homology with $N=2$ supersymmetric gauge theory in the simplest case of pure super Yang–Mills theory, and its application to analytic properties of the partition function. The ground field is taken to be $k=\mathbb{C}$.

4.1. **Instanton partition function of $N=2$ Yang–Mills theory.** The starting point of our discussion is Nekrasov's instanton partition function which is the contribution of instantons to the full partition function of the gauge theory in the Ω-background.

The Nekrasov instanton partition function of $N=2$ supersymmetric Yang–Mills theory with gauge group $U(r)$ on $\mathbb{R}^4$ in the Ω-background with parameters ϵ_1,ϵ_2 is given as a sum over r-tuples $\vec{Y}=(Y_i)_{i=1}^r$ of Young diagrams of total size $|\vec{Y}|$:

$$Z_{4D}(\epsilon_1,\epsilon_2,a,\mathfrak{q},\lambda)=\sum_{\vec{Y}}\mathfrak{q}^{|\vec{Y}|}\prod_{\alpha,\beta=1}^r\prod_{b\in Y_\alpha}\frac{1}{E_{\alpha\beta}(b)(\epsilon_1+\epsilon_2-E_{\alpha\beta}(b))},$$
$$E_{\alpha\beta}(b)=a_\alpha-a_\beta-l_{Y_\beta}(b)\epsilon_1+(a_{Y_\alpha}(b)+1)\epsilon_2.$$

In this formula

- The Coulomb parameters $a=(a_1,\dots,a_r)\in\sqrt{-1}\mathbb{R}^n$ parametrize boundary conditions of scalar fields in the vector multiplets.
- The Ω-background parameters ϵ_1,ϵ_2 are equivariant parameters for the action of $U(1)^2$ on $\mathbb{R}^4=\mathbb{C}^2$.

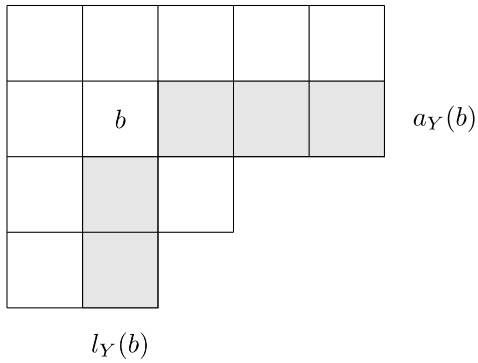

FIGURE 1. Arm and leg length of a box b of a Young diagram Y. In this example $a_Y(b) = 3$, $l_Y(b) = 2$.

- $a_Y(b), l_Y(b)$ denote the arm and leg length of the box $b \in Y$. The arm and leg lengths of a box b in a Young diagram Y are the number of boxes to its right and below it, respectively, see Figure 1.

The intuitive definition of the instanton partition function Z_{4D} of a gauge theory in 4 dimensions with gauge group $U(r)$ should be a count of instantons (anti-selfdual connections) with proper behaviour at infinity up to gauge equivalence. Instantons come with a topological invariant called the instanton number and $Z_{4D} = \sum_{n=0}^{\infty} q^n Z_{4D}^{(n)}$ with q related to the coupling constant and $Z_{4D}^{(n)}$ the contribution of instantons of instanton number n. So we try and write

$$Z_{4D}^{(n)} = \int_{\mathcal{M}(n,r)} 1 \tag{4.1}$$

where $\mathcal{M}(n,r)$ is the moduli space of framed instantons of instanton number n, which is best viewed mathematically as the moduli space of torsion free sheaves on $\mathbb{CP}^2$ of rank r, second Chern class n and with fixed trivialization at infinity. The way Nekrasov makes sense of this expression is to consider 1 as an equivariant cohomology class of the action of $U(1)\times U(1)$ on $\mathbb{CP}^2$, which induces a $U(1)\times U(1)$ action on $\mathcal{M}(n,r)$. From the present point of view it is better to consider the partition function as the limit of the partition function of a 5-dimensional theory.

4.2. **Five-dimensional supersymmetric theory.** Z_{4D} is the limit of the instanton partition function on $\mathbb{R}^4 \times S^1_\lambda$ as the radius λ of the circle tends to 0:

$$Z_{5\mathrm{D}}(q_1,q_2,u,v) = \sum_{\vec{Y}} v^{|\vec{Y}|} \prod_{\alpha,\beta=1}^{r} \prod_{b\in Y_\alpha} \frac{1}{(1-F_{\alpha\beta}(b))(1-q_1q_2F_{\alpha\beta}(b)^{-1})}, \tag{4.2}$$
$$F_{\alpha\beta}(b) = u_\alpha u_\beta^{-1} q_1^{-l_{Y_\beta}(b)} q_2^{a_{Y_\alpha}(b)+1}$$

The partition function Z_{4D} is obtained by setting

$$q_i = e^{-\lambda\epsilon_i}, \quad u_\alpha = e^{-\lambda a_\alpha}, \quad v = \mathfrak{q}\lambda^{2r} e^{-\lambda r(\epsilon_1+\epsilon_2)/2},$$

and taking the limit of each summand as $\lambda \to 0$.

In the five-dimensional case equivariant cohomology is replaced by equivariant K-theory and the integration is replaced by pushforward of the map to a point, which may be expressed in terms of sheaf cohomology and can be computed by the localization formula.

We proceed to explain the steps of this construction, starting from the Atiyah–Drinfeld–Hitchin–Manin (ADHM) description of $\mathcal{M}(n,r)$ in terms of linear algebra data.

4.3. **ADHM equations.** In the ADHM description, $\mathcal{M}(n,r)$ is a Hamiltonian quotient

$$\mathcal{M}(n,r) = T^*(M_{n,n} \times M_{n,r})//_{\mu}\, GL_n$$

under the symplectic action of GL_n by conjugation on the space of $n \times n$ matrices $M_{n,n}$ and by left multiplication on $n \times r$ matrices $M_{n,r}$. It is a smooth algebraic variety of dimension $2nr$. The Hamiltonian quotient is the Geometric Invariant Theory (GIT) quotient $\mu^{-1}(0)/_{\mathrm{GIT}}\, GL_n$ for the moment map

$$\mu\colon T^*(M_{n,n} \times M_{n,r}) \to M_{n,n}, \quad \mu(X,I,Y,J) = [X,Y] + IJ.$$

Here X, I are coordinates on $M_{n,n} \times M_{n,r}$ and the dual coordinates Y, J are on the dual vector space, which we identify with $M_{n,n} \times M_{r,n}$ via the trace pairing. GIT means that we take the GL_n-orbits of the four-tuples $(X,I,Y,J) \in \mu^{-1}(0)$ obeying the stability condition: there is no non-trivial proper subspace of $\mathbb{C}^n$ containing $I(\mathbb{C}^r)$ that is invariant under X and Y.

We refer to [12] for an explanation of the relation to torson free sheaves and instantons and the hyperkähler structure on $\mathcal{M}(n,r)$. The variety $\mathcal{M}(n,r)$ is a special case of a Nakajima quiver variety, see [23], which is the natural context for our discussion.

4.4. **Torus action.** The torus $T = U(1)^2 \times U(1)^r$ acts on $\mathcal{M}(n,r)$: $(t_1,t_2) \in U(1)^2$ acts by $(X,I,Y,J) \mapsto (t_1X, t_1t_2I, t_2Y, J)$ and the action $U(1)^r$ is the restriction to diagonal matrices of the action of $U(r)$ given by $g \cdot (X,I,Y,J) \mapsto (X, Ig^{-1}, Y, gJ)$. This action induces an action on the sheaf cohomology groups $H^i(X,\mathcal{O})$ with coefficient in the structure sheaf, which decomposes into a direct sum of finite dimensional weight spaces for T.

4.5. **Instanton count in the five-dimensional theory and equivariant K-theory.** The contribution $Z^{(n)}_{5\mathrm{D}}$ of instanton number n to the instanton partition function $Z_{5\mathrm{D}} = \sum_{n\geq 0} v^n Z^{(n)}_{5\mathrm{D}}$ is by definition

$$Z^{(n)}_{5\mathrm{D}} = \sum_{i=0}^{2nr} (-1)^i \mathrm{ch}_T H^i(\mathcal{M}(n,r),\mathcal{O}),$$

which can be understood as the push-forward of the class of the trivial bundle in T-equivariant theory [24–26, 28]. It may be understood in more physical terms as a Witten index, see [27]. The cohomology groups $H^i(\mathcal{M}(n,r),\mathcal{O})$ have finite dimensional weight spaces for the action of $U(1)^2$ and only positive weights appear. Thus, in terms of the equivariant parameters u_i of $U(1)^r$ and q_1, q_2 of $U(1)^2$,

$$Z^{(n)}_{5\mathrm{D}} \in \mathbb{Z}[u_1^{\pm 1}, \dots, u_r^{\pm 1}][[q_1,q_2]]$$

which lies in a completion of the K-theory $K_T(\mathrm{pt})$ of a point. The action of T on $\mathcal{M}(n,r)$ has isolated fixed points labeled by r-tuples of Young diagrams of total size r. The localization formula gives then the answer (4.2), see [26, 28].

4.6. **Gauge theory on S^4, AGT correspondence.** The Nekrasov partition function appears in a variety of contexts in physics and mathematics. We mention here a few.

- The square of the absolute value of the Nekrasov partition function $|Z_{4D}|^2$ (or $|Z_{5\mathrm{D}}|^2$) appears in the integrand over the Coulomb parameters of the partition function of $N = 2$ supersymmetric gauge theory on S^4 (or $S^4 \times S^1$) with ellipsoidal metric with half-axes ϵ_1, ϵ_2 [31,33].
- Partition functions for gauge theory with matter fields are obtained by replacing the trivial bundle by suitable vector bundles (or their Chern characters in the $4D$ case).
- By the AGT (Alday–Gaiotto–Tachikawa) correspondence [1], Nekrasov instanton partition functions (with suitable matter fields) are related to conformal blocks of Liouville or Toda theories, or their q-deformations for the theory in five dimensions. The pure Yang–Mills gauge theory case we consider here can be obtained in a confluent limit, see [15], in which conformal blocks degenerate to square norms of Whittaker vectors.

4.7. **A special case of the AGT correspondence in 5 dimensions: Gaiotto states.** The AGT correspondence is far from being understood mathematically in the general case. One degenerate limit has a relatively clear meaning in representation theory. It concerns the representation theory of the deformed Virasoro algebra [38], which is the associative algebra with topological generators T_n, $n \in \mathbb{Z}$ with quadratic relations

$$[T_n, T_m] = -\sum_{l=1}^{\infty} r_l (T_{n-l}T_{m+l} - T_{m-l}T_{n+l}) - \frac{(1-q_1)(1-q_2)}{1-q_1q_2}(q_1^n q_2^n - q_1^{-n} q_2^{-n})\delta_{m+n,0},$$

$$\sum_{l\geq 0} r_l x^l = \exp \sum_{n\geq 1} \frac{(1-q_1^n)(1-q_2^n)}{1+q_1^n q_2^n}\frac{x^n}{n}.$$

In a suitable limit $q_1, q_2 \to 1$ we recover the Virasoro algebra, and much of the representation theory of the Virasoro algebra applies to the deformed case. The relevant objects for our discussion are Whittaker vectors in Verma modules.

The Verma module M_h with highest weight $h \in \mathbb{C}$ is generated by a vector $|h\rangle$ with relations $T_n|h\rangle = \delta_{n,0}h|h\rangle$, for $n \geq 0$. It has a grading $M_h = \oplus_{n=0}^{\infty} M_{h,n}$ by eigenspaces of T_0 to the eigenvalues $h + n$. The decomposition into eigenspaces is orthogonal for the Shapovalov bilinear form on M_h, which is the unique bilinear form such that $S(|h\rangle, |h\rangle) = 1$ and $S(T_n x, y) = S(x, T_{-n}y)$.

A Gaiotto state (or Whittaker vector) [15] is a formal power series $|G\rangle = \sum_{n=0}^{\infty} \xi^n |G_n\rangle$ with coefficients $|G_n\rangle \in M_{h,n}$ such that

$$T_1|G\rangle = \xi|G\rangle, \quad T_j|G\rangle = 0, \quad j \geq 2,$$

and with the normalization condition $|G_0\rangle = |h\rangle$.

The normed squared $S(|G\rangle, |G\rangle)$ is a degenerate limit of a conformal block and coincides by the AGT correspondence to the Nekrasov partition functions of the $N = 2$ supersymmetric Yang–Mills theory.

We will see that the norm squared, which is a priori a formal power series in ξ has in fact a finite radius of convergence.

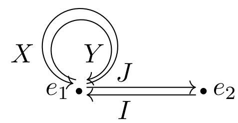

FIGURE 2. The quiver corresponding to the ADHM relations.

4.8. **Derived representation scheme for ADHM relations.** To relate gauge theory to representation homology we view the ADHM relations as representations of a quiver with relations.

Indeed, for the moment map μ defined in Section 4.3, $\mu^{-1}(0)$ is the relative representation scheme $\mathrm{Rep}_V^{S\backslash A}$ of the path algebra A of the quiver of Figure 2 on $V = \mathbb{C}^r \oplus \mathbb{C}^n$ with ADHM relations $XY - YX + IJ = 0$ on the generators. Here S is the algebra of idempotents, see 2.6, 2.7.

A cofibrant replacement QA of the path algebra with ADHM relation has an additional generator Θ of degree 1 whose differential enforces the relation on homology:

$$d\Theta = XY - YX + IJ, \quad dX = dY = dI = dJ = 0.$$

Then $H_\bullet(S\backslash A, V)$ is the homology of the free graded commutative algebra generated by matrix entries

$$\mathbb{C}[x_{\alpha\beta}, y_{\alpha\beta}, i_{\alpha\mu}, j_{\mu\beta}, \theta_{\alpha\beta} | \alpha, \beta = 1, \dots, n, \mu = 1, \dots, r]$$

with induced differential

$$d\theta_{\alpha\beta} = \sum_\gamma (x_{\alpha\gamma} y_{\gamma\beta} - y_{\alpha\gamma} x_{\gamma\beta}) + \sum_\mu i_{\alpha\mu} j_{\mu\beta}.$$

The torus $(\mathbb{C}^\times)^2$ acts by rescaling of X, Y. Also $GL_n \times GL_r$ acts on the representation homology. In particular we have an action of $T = U(1)^2 \times U(1)^r$ on the GL_n-invariants of representation homology $H_\bullet(S\backslash A, V)$ relative to the the subalgebra S generated by the idempotents.

The observation of [2] is that the character-valued Euler characteristic of the GL_n-invariants in representation homology of the ADHM quiver coincides with the contribution of instanton number n of the Nekrasov partition function on $\mathbb{R}^4 \times S^1$. Namely if we set

$$\chi_n = \chi(S\backslash A, V) = \sum_i (-1)^i \operatorname{ch}_T H_i(S\backslash A, V)^{GL_n}, \quad V = \mathbb{C}^n \oplus \mathbb{C}^r,$$

we observe that

$$\chi_n = Z_{5\mathrm{D}}^{(n)}.$$

Actually we arrive at the constant term formula (see Section 3.7 for the definition of CT)

$$\chi_n = \frac{1}{n!} \frac{(1-q_1q_2)^n}{(1-q_1)^n(1-q_2)^n}$$
$$\mathrm{CT} \prod_{j=1}^n \prod_{\alpha=1}^r \frac{1}{(1-u_\alpha/z_j)(1-q_1q_2z_j/u_\alpha)} \prod_{j\neq k} \frac{(z_j - z_k)(z_j - q_1q_2z_k)}{(z_j - q_1z_k)(z_j - q_2z_k)}$$

which is an alternative formula for $Z_{5\mathrm{D}}^{(n)}$. The constant term can be replaced by an integral over the torus $|z_i| = \rho$, $i = 1, \dots, n$ of the differential form obtained by multiplying by $\prod_j dz_j/(2\pi i z_j)$ (assuming $|q_j| < 1$, $|u_\alpha| = 1$). The radius ρ is such that $1 < \rho < |q_1 q_2|^{-1}$. Such integral formulas are known in the physics literature, see [21, 22]. The sum over r-tuples of partitions can be recovered by applying the residue theorem, see the appendix to [14] for a detailed proof.

Exercise. Prove this formula for χ_n (the calculation is very similar to the one in Section 3.7).

4.9. **Analytic properties of the partition function.** We now sketch a result on the convergence of the partition function $Z_{5\mathrm{D}}$ which is a priori a formal power series. The range of parameters of interest for gauge theory on $S^4 \times S^1$ with ellipsoid geometry is $\epsilon_1, \epsilon_2 > 0$ and $a_i \in \sqrt{-1}\mathbb{R}$. In the exponential variables $0 < q_1, q_2 < 1$, $|u_i| = 1$. In the AGT correspondence the central charge of Liouville theory is $c = 1 + 6(b + b^{-1})^2 \in (1, \infty)$, $\epsilon_1 = b$, $\epsilon_2 = b^{-1}$ and we may assume that $\operatorname{Re} \epsilon_i > 0$. There are two ranges of parameters relevant for the AGT correspondence: The strongly coupled Liouville theory with $1 < c < 25$, $\epsilon_1 = \bar{\epsilon}_2 \in S^1$, and the weakly coupled Liouville theory: $c > 25$, $\epsilon_1, \epsilon_2 > 0$. A relevant limiting case is the Nekrasov–Shatashvili limit $\epsilon_2 \to 0$ with fixed ϵ_1, and is connected to quantum integrable systems [29, 30].

In the exponential variables $q_j = \exp(-\lambda\epsilon_j)$ with $\lambda > 0$, the ranges are $q_1 = \bar{q}_2$ with $|q_j| < 1$ and $0 < q_1, q_2 < 1$, respectively.

Theorem 4.1. [14] *Let $|q_1|, |q_2| < 1$, $|u_\alpha| = 1$. Suppose that either $q_1 = \bar{q}_2$ or $q_1, q_2 \in \mathbb{R}_+$. Then the formal power series $Z_{5\mathrm{D}}(v)$ has convergence radius (at least) 1 and depends analytically on the parameters.*

Variants of this theorem for more general gauge theories, including with matter fields, can be proved by the same methods, see [14].

Corollary 4.2. *Under the assumptions of Theorem 4.1, the norm of the Gaiotto state for the deformed Virasoro algebra is analytic for $|\xi| < (q_1 q_2)^{1/2}$.*

The proof of the theorem amounts to an estimate of the asymptotic behaviour of the coefficients $Z^{(n)}_{5\mathrm{D}} = Z^{(n)}_{5\mathrm{D}}(q_1, q_2, \vec{u})$ of the formal power series $Z_{5\mathrm{D}}$: we need to show that $\limsup_{n\to\infty} (Z^{(n)}_{5\mathrm{D}})^{\frac{1}{n}} \le 1$. This can be done with techniques from unitary random matrix theory.

We write $Z^{(n)}_{5\mathrm{D}}$ as an expectation value

$$Z^{(n)}_{5\mathrm{D}} = \zeta_n E \left(\prod_{j=1}^{n} \prod_{\alpha=1}^{r} \frac{1}{(1 - u_\alpha/z_j)(1 - q_1 q_2 z_j/u_\alpha)} \right)$$

for a system of particles $z_1, \ldots, z_n$ on a circle of radius ρ with Boltzmann distribution

$$\frac{1}{\zeta_n} \exp\left(-\sum_{j<k} W(z_j/z_k) \right) \prod \frac{dz_i}{2\pi i z_i}$$

with the pair potential

$$W(z) = -\log \left| \frac{(1-z)(1-q_1 q_2 z)}{(1-q_1 z)(1-q_2 z)} \right|^2,$$

for z on the unit circle, which is repulsive at short distances, see Figure 3.

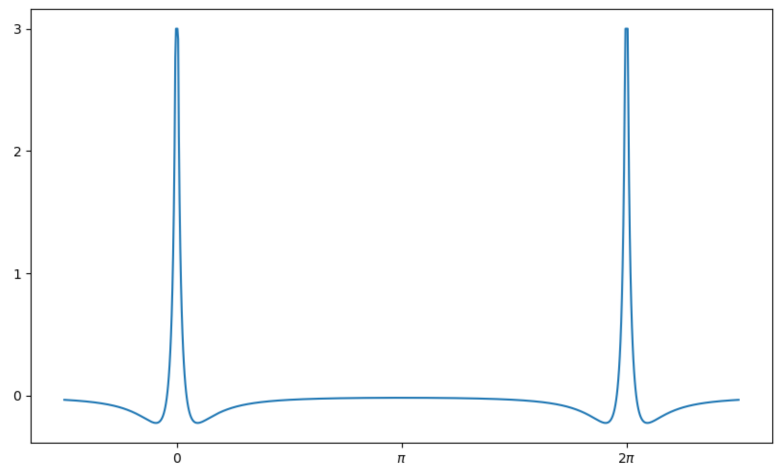

FIGURE 3. The pair potential $W(\exp(i\theta))$ for $q_1 = 0.7$ and $q_2 = 0.9$ as a function of the angle θ.

The estimate of $E(\cdots)$ uses techniques of potential theory adapted from random matrix theory, see [18]. One proves that for large n the particle configurations approach an equilibrium distribution on the circle, which is the solution to a variational problem. The essential observation [14] is that the pair potential W is positive definite in the sense that $\int_0^{2\pi} W(e^{i(\theta_1-\theta_2)})\rho(\theta_1)\rho(\theta_2)d\theta_1 d\theta_2 \geq 0$ for any continuous real-valued function ρ such that $\int_0^{2\pi}\rho(\theta)d\theta = 0$. It follows that the equilibrium distribution is the uniform distribution on the circle. The asymptotic behaviour of the integral is then calculated by evaluating the integrand on this distribution.

The normalization factor ζ_n ($Z^{(n)}_{\mathrm{5D}}$ at $r = 0$) is the character-valued Euler characteristic of the representation homology of $k[x, y]$ and can be computed explicitly as we saw in Section 3.7 with the result

$$\sum_{n=0}^{\infty} v^n \zeta_n = \exp\left(\sum_{n=1}^{\infty} \frac{1 - q_1^n q_2^n}{(1-q_1^n)(1-q_2^n)} \frac{v^n}{n}\right).$$

The right-hand side converges for $|v| < 1$ so we get that $\lim_{n\to\infty} |\zeta_n|^{\frac{1}{n}} = 1$.

4.10. **Open questions.**

- The formal limit $\lambda \to 0$ of the estimated radius of convergence converges to the expected radius of convergence in the 4D theory. However the convergence is not uniform and we cannot deduce a result on the analyticity of Z_{4D}.
- From the point of view of random matrices the equilibrium measure in the 4D theory is no longer uniform as the two-particle potential is attractive at intermediate distances. It would be interesting to describe this distribution.

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