Chapter 4
Maximally Consistent Extensions

Throughout this chapter we require that all formulae are written in Polish notation and that the variables are among $v_0, v_1, v_2, \ldots$ Recall that by the PRENEX NORMAL FORM THEOREM 1.12 and by the VARIABLE SUBSTITUTION THEOREM 1.13, every formula can be transformed into an equivalent formula of the required form.

Maximally Consistent Theories

Let $\mathscr{L}$ be an arbitrary signature and let T be an $\mathscr{L}$-theory. We say that T is **maximally consistent** if T is consistent and for every $\mathscr{L}$-sentence σ we have *either* $\sigma \in \mathsf{T}$ *or* $\neg \operatorname{Con}(\mathsf{T} + \sigma)$. In other words, a consistent theory T is maximally consistent if no proper extension of T is consistent.

The following fact is just a reformulation of the definition.

FACT 4.1. *Let $\mathscr{L}$ be a signature and let T be a consistent $\mathscr{L}$-theory. Then T is maximally consistent iff for every $\mathscr{L}$-sentence σ, either $\sigma \in \mathsf{T}$ or $\mathsf{T} \vdash \neg\sigma$.*

Proof. By THEOREM 1.14.(c)&(d) we have:

$$\neg \operatorname{Con}(\mathsf{T} + \sigma) \quad \Longleftrightarrow \quad \mathsf{T} \vdash \neg\sigma$$

Hence, an $\mathscr{L}$-theory is maximally consistent *iff* for every $\mathscr{L}$-sentence σ, either $\sigma \in \mathsf{T}$ or $\mathsf{T} \vdash \neg\sigma$. ⊣

As a consequence of FACT 4.1 we get

LEMMA 4.2. *Let $\mathscr{L}$ be a signature and let T be a consistent $\mathscr{L}$-theory. Then T is maximally consistent iff for every $\mathscr{L}$-sentence σ, either $\sigma \in \mathsf{T}$ or $\neg\sigma \in \mathsf{T}$.*

Proof. We have to show that the following equivalence holds:

$$\forall\sigma\big(\sigma \in \mathsf{T} \ \textit{ or } \ \mathsf{T} \vdash \neg\sigma\big) \quad \Longleftrightarrow \quad \forall\sigma\big(\sigma \in \mathsf{T} \ \textit{ or } \ \neg\sigma \in \mathsf{T}\big)$$

($\Rightarrow$) Assume that for every $\mathscr{L}$-sentence σ we have $\sigma \in \mathsf{T}$ or $\mathsf{T} \vdash \neg\sigma$. If $\sigma \in \mathsf{T}$, then the implication obviously holds. If $\sigma \notin \mathsf{T}$, then $\mathsf{T} \vdash \neg\sigma$, and since T is consistent, this implies $\mathsf{T} \nvdash \sigma$. Now, by TAUTOLOGY (F.0), this implies $\mathsf{T} \nvdash \neg\neg\sigma$ and by our assumption we finally get $\neg\sigma \in \mathsf{T}$.

($\Leftarrow$) Assume that for every $\mathscr{L}$-sentence σ we have $\sigma \in \mathsf{T}$ or $\neg\sigma \in \mathsf{T}$. If $\sigma \in \mathsf{T}$, then the implication obviously holds. Now, if $\sigma \notin \mathsf{T}$, then by our assumption we have $\neg\sigma \in \mathsf{T}$, which obviously implies $\mathsf{T} \vdash \neg\sigma$. ⊣

Maximally consistent theories have similar features as complete theories: Recall that an $\mathscr{L}$-theory T is complete if for every $\mathscr{L}$-sentence σ we have *either* $\mathsf{T} \vdash \sigma$ *or* $\mathsf{T} \vdash \neg\sigma$.

As an immediate consequence of the definitions we get

FACT 4.3. *Let $\mathscr{L}$ be a signature, let T be a consistent $\mathscr{L}$-theory, and let $\mathbf{Th}(\mathsf{T})$ be the set of all $\mathscr{L}$-sentences which are provable from T.*

(a) *If T is complete, then $\mathbf{Th}(\mathsf{T})$ is maximally consistent.*

(b) *If T is maximally consistent, then $\mathbf{Th}(\mathsf{T})$ is equal to T.*

The next lemma gives a condition under which a theory can be extended to maximally consistent theory.

LEMMA 4.4. *If an $\mathscr{L}$-theory T has a model, then T has a maximally consistent extension.*

Proof. Let $\mathbf{M}$ be a model of the $\mathscr{L}$-theory T and let $\mathsf{T}_{\mathbf{M}}$ be the set of $\mathscr{L}$-sentences σ such that $\mathbf{M} \vDash \sigma$. Then $\mathsf{T}_{\mathbf{M}}$ is obviously a maximally consistent theory which contains T. ⊣

Later we shall see that every consistent theory has a model. For this, we first show how a consistent theory can be extended to a maximally consistent theory.

Universal List of Sentences

Let $\mathscr{L}$ be an arbitrary but fixed countable signature, where by "countable" we mean that the symbols in $\mathscr{L}$ can be listed in a FINITE or POTENTIALLY INFINITE list $L_{\mathscr{L}}$.

First, we encode the symbols of $\mathscr{L}$ corresponding to the order in which they appear in the list $L_{\mathscr{L}}$: The first symbol is encoded with "2", the second with "22", the third with "222", and so on. For every symbol $\zeta \in L_{\mathscr{L}}$ let $\#\zeta$ denote the code of ζ. So, the code of a symbol of $\mathscr{L}$ is just a sequence of 2's.

Furthermore, we encode the logical symbols as follows:

Symbol ζ	Code $\#\zeta$
$=$	11
$\neg$	1111
$\wedge$	111111
$\vee$	11111111
$\rightarrow$	1111111111
$\exists$	111111111111
$\forall$	11111111111111
v_0	1
v_1	111
$\vdots$	$\vdots$
v_n	$\underbrace{1111 \;\ldots\; 11111}_{(2n+1)\ 1\text{'s}}$

In the next step, we encode strings of symbols: Let $\bar{\zeta} \equiv \zeta_1\zeta_2\zeta_3 \ldots \zeta_n$ be a finite string of symbols, then

$$\#\bar{\zeta} := \#\zeta_1 0 \#\zeta_2 0 \#\zeta_3 \ldots 0 \#\zeta_n$$

For a string $\#\zeta$ (*i.e.*, a string of 0's, 1's, and 2's) let $|\#\zeta|$ be the length of $\#\zeta$ (*i.e.*, the number of 0's, 1's, and 2's which appear in $\#\zeta$).

Now, we order the codes of strings of symbols by their length and lexicographically, where $0 < 1 < 2$. If, with respect to this ordering, $\#\zeta_1$ is less than $\#\zeta_2$, we write $\zeta_1 \prec \zeta_2$.

Finally, let $\Lambda_{\mathscr{L}} = [\sigma_1, \sigma_2, \ldots]$ be the potentially infinite list of all $\mathscr{L}$-sentences, ordered by "$\prec$" (*i.e.*, $\sigma_i \prec \sigma_j$ *iff* $i < j$). We call $\Lambda_{\mathscr{L}}$ the **universal list of $\mathscr{L}$-sentences**.

Lindenbaum's Lemma

In this section we show that every consistent set of $\mathscr{L}$-sentences T can be extended to a maximally consistent set of $\mathscr{L}$-sentences $\overline{\mathsf{T}}$. Since the universal list of $\mathscr{L}$-sentences contains all possible $\mathscr{L}$-sentences, every set of $\mathscr{L}$-sentences can be can be listed in a (finite or potentially infinite) list. So, we do not have to assume that the (possibly infinite) set of $\mathscr{L}$-sentences T is completed and definite.

LINDENBAUM'S LEMMA 4.5. *Let $\mathscr{L}$ be a countable signature and let T be a consistent set of $\mathscr{L}$-sentences. Furthermore, let σ_0 be an $\mathscr{L}$-sentences which cannot* *be proved from* T, *i.e.*, $\mathsf{T} \nvdash \sigma_0$. *Then there exists a maximally consistent set* $\overline{\mathsf{T}}$ *of* $\mathscr{L}$*-sentences which contains* $\neg\sigma_0$ *as well as all the sentences of* T.

Proof. Let $\Lambda_{\mathscr{L}} = [\sigma_1, \sigma_2, \ldots]$ be the universal list of $\mathscr{L}$-sentences. First we extend $\Lambda_{\mathscr{L}}$ with the $\mathscr{L}$-sentence $\neg\sigma_0$; let $\Lambda^0_{\mathscr{L}} = [\neg\sigma_0, \sigma_1, \sigma_2, \ldots]$.

Now, we go through the list $\Lambda^0_{\mathscr{L}}$ and define step by step a list $\overline{\mathsf{T}}$ of $\mathscr{L}$-sentences: For this, we define T_0 as the empty list, *i.e.*, $T_0 := [\,]$. If T_n is already defined, then

$$T_{n+1} := \begin{cases} T_n + [\sigma_n] & \text{if } \operatorname{Con}(T + T_n + \sigma_n), \\ T_n & \text{otherwise.} \end{cases}$$

Let $\overline{\mathsf{T}} = [\sigma_{i_0}, \sigma_{i_1}, \ldots]$ be the resulting list, *i.e.*, $\overline{\mathsf{T}}$ is the union of all the T_n's.

Notice that the construction only works if we assume the LAW OF EXCLUDED MIDDLE: Even in the case when we cannot decide whether $T + T_n + \sigma_n$ is consistent or not, we assume, from a metamathematical point of view, that *either* $T + T_n + \sigma_n$ is consistent *or* $T + T_n + \sigma_n$ is inconsistent (and neither both nor none).

CLAIM. $\overline{\mathsf{T}}$ *is a maximally consistent set of* $\mathscr{L}$*-sentences which contains* $\neg\sigma_0$ *as well as all the sentences of* T.

Proof of Claim. First we show that $\neg\sigma_0$ belongs to $\overline{\mathsf{T}}$, then we show that $\mathsf{T} + \overline{\mathsf{T}}$ is consistent (which implies that $\overline{\mathsf{T}}$ is consistent), in a third step we show that $\overline{\mathsf{T}}$ contains T, and finally we show that for every $\mathscr{L}$-sentence σ we have either $\sigma \in \overline{\mathsf{T}}$ or $\neg \operatorname{Con}(\overline{\mathsf{T}} + \sigma)$.

$\neg\sigma_0$ *belongs to* $\overline{\mathsf{T}}$: Since $\mathsf{T} \nvdash \sigma_0$, by PROPOSITION 1.14.(c) we have $\operatorname{Con}(\mathsf{T} + \neg\sigma_0)$, and since $T_0 = [\,]$, we also have $\operatorname{Con}(\mathsf{T} + T_0 + \neg\sigma_0)$. Thus, $\neg\sigma_0 \in T_1$ (in fact $T_1 = [\neg\sigma_0]$) which shows that $\neg\sigma_0 \in \overline{\mathsf{T}}$.

$\mathsf{T} + \overline{\mathsf{T}}$ *is consistent*: By the COMPACTNESS THEOREM 1.15 it is enough to show that every finite subset of $\mathsf{T} + \overline{\mathsf{T}}$ is consistent. So, let $\mathsf{T}' + T_k$ be a finite subset of $\mathsf{T} + \overline{\mathsf{T}}$, where T' is a finite subset of T and T_k is some finite initial segment of the list $\overline{\mathsf{T}}$. Notice that since $\mathsf{T} + \neg\sigma_0$ is consistent, also $\mathsf{T}' + \neg\sigma_0$ is consistent. If $T_k = [\,]$ or $T_k = [\neg\sigma_0]$, this implies that also $\mathsf{T}' + T_k$ is consistent. Otherwise, $T_k = [\ldots, \sigma_n]$ for some σ_n in $\Lambda^0_{\mathscr{L}}$, which implies that $T_k = T_n + [\sigma_n]$. Now, by construction we get $\operatorname{Con}(\mathsf{T} + T_n + \sigma_n)$, which implies the consistency of $\mathsf{T}' + T_k$.

$\overline{\mathsf{T}}$ *contains all sentences of* T: For every $\sigma \in \mathsf{T}$ there is a $\sigma_n \in \Lambda^0_{\mathscr{L}}$ such that $\sigma \equiv \sigma_n$. By $\operatorname{Con}(\mathsf{T} + T_n + \sigma_n)$ we get $\sigma_n \in T_{n+1}$, hence, $\sigma_n \in \overline{\mathsf{T}}$ and therefore $\sigma \in \overline{\mathsf{T}}$.

For every σ, *either* $\sigma \in \overline{\mathsf{T}}$ *or* $\neg \operatorname{Con}(\overline{\mathsf{T}} + \sigma)$: For every $\mathscr{L}$-sentence σ there is a $\sigma_n \in \Lambda^0_{\mathscr{L}}$ such that $\sigma \equiv \sigma_n$. By the law of excluded middle, we have *either* $\operatorname{Con}(\mathsf{T} + T_n + \sigma_n)$, which implies $\sigma_n \in T_{n+1}$ and therefore $\sigma \in \overline{\mathsf{T}}$, *or* $\neg \operatorname{Con}(\mathsf{T} + T_n + \sigma_n)$, which implies $\neg \operatorname{Con}(\overline{\mathsf{T}} + \sigma_n)$, *i.e.*, $\neg \operatorname{Con}(\overline{\mathsf{T}} + \sigma)$. $\dashv_{\text{Claim}}$

Thus, the list $\overline{\mathsf{T}}$ has all the required properties, which completes the proof. $\dashv$

The following fact summarises the main properties of $\overline{\mathsf{T}}$.

FACT 4.6. *Let* $\mathsf{T}, \overline{\mathsf{T}}$, *and* σ_0 *be as above, and let* σ *and* σ' *be any* $\mathscr{L}$*-sentences.*

(a) $\neg\sigma_0 \in \overline{\mathsf{T}}$.

(b) *Either* $\sigma \in \overline{\mathsf{T}}$ *or* $\neg\sigma \in \overline{\mathsf{T}}$.

(c) *If* $\mathsf{T} \vdash \sigma$, *then* $\sigma \in \overline{\mathsf{T}}$.

(d) $\overline{\mathsf{T}} \vdash \sigma$ *iff* $\sigma \in \overline{\mathsf{T}}$.

(e) *If* $\sigma \Leftrightarrow \sigma'$, *then* $\sigma \in \overline{\mathsf{T}}$ *iff* $\sigma' \in \overline{\mathsf{T}}$.

Proof. (a) follows by construction of $\overline{\mathsf{T}}$.

Since $\overline{\mathsf{T}}$ is maximally consistent, (b) follows by LEMMA 4.2.

For (c), notice that $\mathsf{T} \vdash \sigma$ implies $\neg\,\mathrm{Con}(\mathsf{T} + \neg\sigma)$, hence $\neg\sigma \notin \overline{\mathsf{T}}$ and by (b) we get $\sigma \in \overline{\mathsf{T}}$.

For (d), let us first assume $\overline{\mathsf{T}} \vdash \sigma$. This implies $\mathrm{Con}(\overline{\mathsf{T}} + \sigma)$, hence $\mathrm{Con}(\mathsf{T} + \sigma)$, and by construction of $\overline{\mathsf{T}}$ we get $\sigma \in \overline{\mathsf{T}}$. On the other hand, if $\sigma \in \overline{\mathsf{T}}$, then we obviously have $\overline{\mathsf{T}} \vdash \sigma$.

For (e), recall that $\sigma \Leftrightarrow \sigma'$ is just an abbreviation for $\vdash \sigma \leftrightarrow \sigma'$. Thus, (e) follows immediately from (d). ⊣

Of course, this can work out only when the $\mathscr{L}$-sentences in $\overline{\mathsf{T}}$ "behave" like valid sentences in a model, which is indeed the case—as the following proposition shows.

PROPOSITION 4.7. *Let* $\overline{\mathsf{T}}$ *be as above, and let* $\sigma, \sigma_1, \sigma_2$ *be any* $\mathscr{L}$*-sentences.*

(a) $\neg\sigma \in \overline{\mathsf{T}} \iff$ NOT $\sigma \in \overline{\mathsf{T}}$

(b) $\wedge\sigma_1\sigma_2 \in \overline{\mathsf{T}} \iff \sigma_1 \in \overline{\mathsf{T}}$ AND $\sigma_2 \in \overline{\mathsf{T}}$

(c) $\vee\sigma_1\sigma_2 \in \overline{\mathsf{T}} \iff \sigma_1 \in \overline{\mathsf{T}}$ OR $\sigma_2 \in \overline{\mathsf{T}}$

(d) $\rightarrow\sigma_1\sigma_2 \in \overline{\mathsf{T}} \iff$ IF $\sigma_1 \in \overline{\mathsf{T}}$ THEN $\sigma_2 \in \overline{\mathsf{T}}$

Proof. (a) Follows immediately from FACT 4.6.(b).

(b) First notice that by FACT 4.6.(d), $\wedge\sigma_1\sigma_2 \in \overline{\mathsf{T}}$ *iff* $\overline{\mathsf{T}} \vdash \wedge\sigma_1\sigma_2$. Thus, by L_3 & L_4 and (MP) we get $\overline{\mathsf{T}} \vdash \sigma_1$ and $\overline{\mathsf{T}} \vdash \sigma_2$. Thus, by FACT 4.6.(d), we get $\sigma_1 \in \overline{\mathsf{T}}$ AND $\sigma_2 \in \overline{\mathsf{T}}$. On the other hand, if $\sigma_1 \in \overline{\mathsf{T}}$ AND $\sigma_2 \in \overline{\mathsf{T}}$, then, by FACT 4.6.(d), we get $\overline{\mathsf{T}} \vdash \sigma_1$ and $\overline{\mathsf{T}} \vdash \sigma_2$. Now, by TAUTOLOGY (B), this implies $\overline{\mathsf{T}} \vdash \wedge\sigma_1\sigma_2$, and by by FACT 4.6.(d) we finally get $\wedge\sigma_1\sigma_2 \in \overline{\mathsf{T}}$.

(c) & (d) follow from FACT 4.6.(e) and the fact that for each formula σ there is an equivalent formula σ' which contains neither "$\vee$" nor "$\rightarrow$" (see THEOREM **??**). ⊣