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Group Theory (Module 210PMA208)

Department of Pure Mathematics

Week 10

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46. Let G be a group. On Hw8.Q38 you showed that for every $x \in G$, the mapping $\varphi_x : G \to G$ defined by $\varphi_x(a) := xax^{-1}$ is an automorphism of G. Such an automorphism is called an inner automorphism of G. Let $\mathrm{Inn}(G)$ denote the set of all inner automorphisms of G, and let $Z(G)$ denote the centre of G. Further, define $\psi : G \to \mathrm{Aut}(G)$ by $\psi(x) := \varphi_x$.

    (a) Show that ψ is a homomorphism from G to $\mathrm{Aut}(G)$.

    (b) Using (a), show that $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$.

    (c) Use the First Isomorphism Theorem to show that $G/Z(G) \cong \mathrm{Inn}(G)$.

47. Show that for any $n \geq 2$, S_n is generated by $(1,2)$ and $(1,2,\dots,n)$.

48. Let π and ρ be the permutations

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 2 & 1 & 8 \end{pmatrix} \text{ and } \rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

    (a) Write π and ρ in terms of disjoint cycles.

    (b) Write π in terms of transpositions.

    (c) Write π as a product of the permutations $(1,2)$ and $(1,2,\dots,8)$.

49. Let π and ρ be as is Question 48.

    (a) Find the orders of π, ρ, $\pi\rho$ and $\pi^2\rho$.

    (b) Write π^{147}, ρ^{-1001}, $(\pi\rho)^{514}$, and $(\pi^2\rho)^{-333}$ in terms of disjoint cycles.

50. (a) List all 24 elements of S_4 in cycle notation.

    (b) Show that the subset $V = \big\{\iota,\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\big\}$ is a subgroup of S_4.

    (c) Show that V is a normal subgroup of S_4. (Use the fact that for any permutations π and ρ, π has the same cycle structure as $\rho\pi\rho^{-1}$.)