Department of Pure Mathematics

Module 110PMA207 – Linear Algebra

Assignment 10

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1. Let V be an n-dimensional vector space and let $f_1, \ldots, f_n$ be any linear functionals on V. Let $\{v_1, \ldots, v_n\}$ be a basis of V and for every i and every j where $1 \leq i, j \leq n$ let $a_{i,j} := f_i(v_j)$.

   (a) Let $x = \sum_{j=1}^{n} \lambda_j v_j$ be a vector in V. Show that

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

   (b) Show that if the column vectors of the matrix $(a_{i,j})_{1 \leq i,j \leq n}$ are linearly dependent, then there exists a non-zero vector $x \in V$ such that for all functionals f_i $(1 \leq i \leq n)$ we have $f_i(x) = 0$.

   (c) Show that the functionals $f_1, \ldots, f_n$ are linearly dependent if and only if there exists a non-zero vector x such that for all functionals f_i $(1 \leq i \leq n)$ we have $f_i(x) = 0$.

2. (a) Show that no matter what the numbers a, b, c, d, e, f are, the determinant of the matrix

$$\begin{pmatrix} a & b & 7 \\ c & d & 14 \\ e & f & -21 \end{pmatrix}$$

   is always a multiple of 7.

   (b) Let

$$A = \begin{pmatrix} 9 & 0 & 2 & 4 \\ 24 & 8 & 4 & 11 \\ 12 & 6 & 6 & 4 \\ 6 & 0 & 0 & 1 \end{pmatrix}$$

   Use the fact that 7 is a divisor of each of 902020, 2484055, 1266020, and 600005, to show that $\det(A)$ is a multiple of 7.

3. Find necessary and sufficient conditions for the real numbers a and b such that the vectors $v_1 = (a, b, 1)$, $v_2 = (0, 0, 1)$, and $v_3 = (b, a, 1)$ are linearly independent.

4. (a) Let

$$B = \begin{pmatrix} 1 & -1 & c \\ -1 & c & 1 \\ c & 1 & -1 \end{pmatrix}$$

   Find a real value for c such that $\det(B) = 0$.

   (b) For this real value c, find a non-zero vector $x \in \mathbb{R}^3$ such that $Bx = 0$.