Department of Pure Mathematics

Module 110PMA207 – Linear Algebra

Assignment 8

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1. Let $f, g \in V^*$ be two linear functionals, then the sum $(f+g)$ of f and g is again a mapping from V to $\mathbb{R}$, defined by $(f+g)(x) := f(x)+g(x)$ (for every $x \in V$).

   Show that the mapping $(f+g)$ is linear.

2. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$ and let f be the linear functional on $\mathbb{R}^2$ defined as follows:
$$f(e_1) = 3,\ \ f(e_2) = -1$$
   (a) Compute $f\big((2,-5)\big)$.

   (b) Find all vectors $x \in \mathbb{R}^2$ such that $f(x) = 0$.

   (c) Let $W = \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = f\big((x_1, x_2)\big)\Big\}$. Show that W is a 2-dimensional subspace of $\mathbb{R}^3$ and find a vector which is orthogonal to every vector in W.

3. Let $W = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : -x_1 + 2x_2 - 3x_3 = 0\big\}$.

   Find a non-trivial linear functional g on $\mathbb{R}^3$ such that
$$W = \big\{x : g(x) = 0\big\}\,.$$

4. Let
$$\begin{array}{rccc} \varphi: & V & \longrightarrow & V^{**} \\ & x & \longmapsto & x^{**} \end{array}$$
   where $x^{**}(y^*) := y^*(x)$ (for all $y^* \in V^*$).

   (a) Show that for all $x \in V$, all $y^* \in V^*$, and all $\lambda \in \mathbb{R}$ we have
$$\varphi(x)(\lambda y^*) = \lambda\varphi(x)(y^*)\,.$$
   (b) Show that for all $x \in V$ and all $\lambda \in \mathbb{R}$, the mapping $\varphi(\lambda x)$ is equal to the mapping $\lambda\varphi(x)$, by showing that for all $y^* \in V^*$ we have
$$\varphi(\lambda x)(y^*) = \lambda\varphi(x)(y^*)\,.$$
   (c) Show that φ is injective.