Modules 110PMA003 & 110PMA107

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Lecture Notes by Lorenz Halbeisen

1 Number systems and their features

1.1 Natural numbers: addition

We could just say that the **natural numbers** is the set $\{0, 1, 2, 3, \dots\}$, denoted by $\mathbb{N}$, but this is not a definition. (What is the meaning of "$\dots$"; and what about the sequence " , |, ||, |||, ||||, ...", which can easily be identified with the set $\mathbb{N}$?)

A more mathematical definition of the natural numbers is the following:

(a) $\diamond$ is a natural number ($\diamond$ is just a symbol, you can also take any other symbol)

(b) If n is a natural number (n is also just a symbol), then $s(n)$—the "successor of n—is also a natural number.

(c) If n is a natural number, then *either* $n = 0$ *or* there exists a natural number m such that $n = s(m)$.

(d) If n and m are natural numbers and $s(m) = s(n)$, then $n = m$.

What kind of "objects" (this means, what kind of natural numbers) we get with this construction?

By (a) and (b), we get that $\diamond$, $s(\diamond)$, $s\big(s(\diamond)\big)$, $\dots$, $s\Big(s\big(\dots s\big(s(\diamond)\big)\dots\big)\Big), \dots$ are all natural numbers.

Are all these things different?

Yes! Assume towards a contradiction that two of the above objects are equal, say $s\Big(s\big(s(\diamond)\big)\Big) = s\Bigg(s\bigg(s\Big(s\big(s(\diamond)\big)\Big)\bigg)\Bigg)$. In order to use (d), put $n = s\big(s(\diamond)\big)$ and $m = s\bigg(s\Big(s\big(s(\diamond)\big)\Big)\bigg)$. Now, by (d) we get $n = m$, and thus $s\big(s(\diamond)\big) = s\bigg(s\Big(s\big(s(\diamond)\big)\Big)\bigg)$. Again by (d) we get $s(\diamond) = s\Big(s\big(s(\diamond)\big)\Big)$ and finally we end up with

$$\diamond = s\big(s(\diamond)\big)\,. \qquad (*)$$

Put $m = s(\diamond)$, then by $(*)$ we have $\diamond = s(m)$ and therefore by (c) we get $\diamond \neq \diamond$, which is obviously a contradiction.

Addition

The **addition** "+" of two natural numbers is defined as follows:

(i) For any natural number n we define $n + \diamond := n$ (the symbol $\diamond$ is called the **neutral element** with respect to addition)

(ii) For natural numbers n and m we define $n + s(m) := s(n + m)$.

Example: $s\big(s(\diamond)\big) + s\Big(s\big(s(\diamond)\big)\Big)$? By putting $n = s\big(s(\diamond)\big)$ and $m = s\big(s(\diamond)\big)$, by (ii) we get $n + s(m) = s(n + m)$, hence, $s\big(s(\diamond)\big) + s\Big(s\big(s(\diamond)\big)\Big) = s\Big(s\big(s(\diamond)\big) + s\big(s(\diamond)\big)\Big)$. Again by (ii) we get $s\big(s(\diamond)\big)+s\big(s(\diamond)\big) = s\Big(s\big(s(\diamond)\big)+s(\diamond)\Big)$, and therefore, $s\Big(s\big(s(\diamond)\big)+ s\big(s(\diamond)\big)\Big) = s\bigg(s\Big(s\big(s(\diamond)\big) + s(\diamond)\Big)\bigg)$. Again by (ii) we get $s\bigg(s\Big(s\big(s(\diamond)\big) + s(\diamond)\Big)\bigg) = s\Bigg(s\bigg(s\Big(s\big(s(\diamond)\big) + \diamond\Big)\bigg)\Bigg)$, and using (i), we finally get

$$s\big(s(\diamond)\big) + s\Big(s\big(s(\diamond)\big)\Big) = s\Bigg(s\bigg(s\Big(s\big(s(\diamond)\big)\Big)\bigg)\Bigg).$$

Usually, one identifies $\diamond$ with 0 (called "zero"), $s(\diamond)$ with 1 (called "one"), $s\big(s(\diamond)\big)$ with 2 (called "two"), ..., and $s(n)$ with $n + 1$. So, 0 becomes the neutral element with respect to addition.

Natural numbers as sets

We can define natural numbers as sets of natural numbers as follows:

(α) $0 := \emptyset$ ($\emptyset$ is the empty set)

(β) $n + 1 := n \cup \{n\}$ (where $\{n\}$ is the set containing just the single element n)

In this notation, for example the number 3 is $\Big\{\emptyset, \{\emptyset\}, \big\{\emptyset, \{\emptyset\}\big\}\Big\}$, which is the same as $\{0, 1, 2\}$. In general, the natural number n is the set of all natural numbers which are smaller than n, so in particular, since there is no natural number smaller than 0, we get $0 = \emptyset$.