1.3 Rational numbers: a so-called field

To each non-zero integer $q \neq 0$ we introduce an **inverse element** $\frac{1}{q}$ and define

$$q \cdot \frac{1}{q} := 1\,.$$

Since 1 is the neutral element with respect to multiplication, we obviously have $\frac{1}{1} = 1$ (like $(-0) = 0$) and we can identify each $q \in \mathbb{Z}$ with $\frac{q}{1}$.

Further, for non-zero integers $q, s \in \mathbb{Z}$, we define:

(a) $\frac{p}{q} \cdot \frac{r}{s} := \frac{p \cdot r}{q \cdot s}$ (this defines the multiplication of two fractions)

(b) $\frac{p}{q} + \frac{r}{s} := \frac{p \cdot s + r \cdot q}{q \cdot s}$ (this defines the addition of two fractions)

(c) $\frac{p}{q} = \frac{r}{s}$ if, and only if, $p \cdot s = r \cdot q$ (this defines when two fractions are equal)

For non-zero integers $q, s \in \mathbb{Z}$, similar as above we could prove the following:

- $1 \cdot \frac{p}{q} = \frac{p}{q}$ (this says that 1 is still neutral w.r.t. the multiplication).
- $\dfrac{1}{\left(\frac{1}{q}\right)} = q$ (like $\big(-(x)\big) = x$).
- $\dfrac{\left(\frac{p}{q}\right)}{\left(\frac{r}{s}\right)} = \dfrac{p \cdot s}{q \cdot r}$.
- If $\frac{p}{q} \cdot \frac{r}{s} = 0$, then p or r (or both) must be 0.

We also write $p : q$ instead of $p \cdot \frac{1}{q}$, which is in fact the definition of the **division**.

Let $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{Z},\ q \neq 0\}$ be the set of **rational numbers**.

Remember: $\frac{p}{q} = \frac{r}{s} \iff p \cdot s = r \cdot q$, thus, for example 3 and $\frac{-51}{-17}$ are just different representations of the same(!) rational number.

Summary

For all $x, y, z \in \mathbb{Q}$ we have the following:

(a) $x + y = y + x \quad ; \quad x \cdot y = y \cdot x$ (commutative law)

(b) $(x + y) + z = x + (y + z) \quad ; \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$ (associative law)

(c) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ (distributive law)

(d) $x + 0 = x \quad ; \quad 1 \cdot x = x$ (neutral elements w.r.t. "+" and " · ")

(e) $x + (-x) = 0$ (x has an inverse w.r.t. "+")
$x \neq 0$, then $x \cdot \frac{1}{x} = 1$ ($x \neq 0$ has an inverse w.r.t. " · ")

Note that "+" and " · " are not symmetric (*e.g.*, distributive law, inverse elements).

A set together with two operations "+ and " · ", with two neutral elements 0 and 1, and with two inverse operations "−" and " : " so that all conditions (a)-(e) are satisfied, such a set is called a **field**. So, $(\mathbb{Q}, 0, +, -, 1, \cdot, :)$ is a field. In the following we will see two other fields, namely the set of real numbers and the set of complex numbers.