1.5 Complex numbers: $\sqrt{-1}$

We introduce a new number "i" and define $i^2 := -1$, thus, $i = \sqrt{-1}$.

Let $\mathbb{C} = \big\{(a+ib) : a, b \in \mathbb{R}\big\}$ be the set of **complex numbers**.

Before we define how to add, subtract, multiply and divide complex numbers, we give introduce some notation.

Let $z = (a+ib) \in \mathbb{C}$ be a complex number. The *real number* a is called the **real part** of z and is denoted by $\mathrm{Re}(z)$, so, $\mathrm{Re}(z) = a$; the *real number* b is called the **imaginary part** of z and is denoted by $\mathrm{Im}(z)$, so, $\mathrm{Im}(z) = b$.

Let again $z = (a+ib) \in \mathbb{C}$ be a complex number. If the imaginary part of z is 0, so, if $b = 0$, then we write just a instead of $(a+i0)$. Thus, we consider real numbers as complex numbers with imaginary part equals to 0, which implies that each real numbers is also a complex number and therefore, $\mathbb{R} \subseteq \mathbb{C}$. On the other hand, since there is no real number r such that $r^2 = -1$, not every complex number is a real number.

We can represent complex numbers on a 2-dimensional diagram, called **Argand diagram** (or **Gaussian plane**). An Argand diagram is a *Cartesian coordinate system* (also called *rectangular coordinate system*) where one axis is called the **real axis** and the other one is called the **imaginary axis**.

For a complex number $z = (a+ib)$, we define $|z| := \sqrt{a^2+b^2}$ and call $|z|$ the **modulus** of z. The modulus of a complex number is the same as the **absolute value** $|r|$ of a real number r (where $|r| = r$ for $r \geq 0$, and $|r| = -r$ for $r \leq 0$).

If $z = (a+ib)$ and $b < 0$, then we write $z = (a - i|b|)$ rather than $z = (a+ib)$. For example we write $(3 - i2)$ rather than $\big(3 + i(-2)\big)$.

Addition in $\mathbb{C}$

Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$ be two complex numbers, then

$$z_1 + z_2 := \big((a_1 + a_2) + i(b_1 + b_2)\big) .$$

In particular, $(a+ib) + (0+i0) = (a+ib)$, thus, 0 is still neutral with respect to addition. Further we have $(a+ib) + (-a-ib) = (0+i0) = 0$, so, $(-a-ib)$ is the inverse element (with respect to addition) of $(a+ib)$, and therefore, we also have subtraction in $\mathbb{C}$, defined as follows:

$$z_1 - z_2 := \big((a_1 - a_2) + i(b_1 - b_2)\big) .$$

Multiplication in $\mathbb{C}$

Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$ be two complex numbers, then

$$z_1 \cdot z_2 := \big((a_1 \cdot a_2 - b_1 \cdot b_2) + i(a_1 \cdot b_2 + b_1 \cdot a_2)\big) .$$

To get this, just expand $(a_1+ib_1)\cdot(a_2+ib_2)$ and remember that $i^2 = 1$. In particular, $(a + ib) \cdot (1 + i0) = (a + ib)$, thus, 1 is still neutral with respect to multiplication. Further we have

$$(a + ib) \cdot \big(\tfrac{a}{a^2+b^2} - i\tfrac{b}{a^2+b^2}\big) = (1 + i0) = 1 ,$$

thus, $\big(\tfrac{a}{a^2+b^2} - i\tfrac{b}{a^2+b^2}\big)$ is the inverse element (with respect to multiplication) of $(a+ib)$, and therefore, we also have division in $\mathbb{C}$, defined as follows (for $z_2 \neq 0$):

$$\frac{z_1}{z_2} := \bigg(\Big(\frac{a_1 \cdot a_2 + b_1 \cdot b_2}{a_2^2 + b_2^2}\Big) + i\Big(\frac{b_1 \cdot a_2 - a_1 \cdot b_2}{a_2^2 + b_2^2}\Big)\bigg) .$$

Conjugates

For a complex number $z = (a+ib)$, we define $\overline{z} := (a-ib)$ and call $\overline{z}$ the **conjugate** of z. In the Argand diagram, the conjugate of a complex number z is just the reflection of z on the real axis, therefore, $\overline{(\overline{z})} = z$ (the conjugate of the conjugate of z is equal to z). Further, if $\mathrm{Im}(z) = 0$, then $\overline{z} = z$ (if r is a real number, then $\overline{r} = r$).

A simple calculation shows that for any complex number z we have

$$\mathrm{Re}(z)\frac{z + \overline{z}}{2} \qquad \textit{and} \qquad \mathrm{Im}(z)\frac{z + \overline{z}}{2i} .$$

Further, it is also quite easy to see that

$$\overline{z_1 + z_2 + \ldots + z_n} = \overline{z_1} + \overline{z_2} + \ldots \overline{z_n}$$

and

$$\overline{z_1 \cdot z_2 \cdot \ldots \cdot z_n} = \overline{z_1} \cdot \overline{z_2} \cdot \ldots \cdot \overline{z_n} .$$

Thus, the conjugate of a sum is the sum of the conjugates and the conjugate of a product is the product of the conjugates. Further we get $\overline{-z} = -\overline{z}$ and $\overline{\big(\tfrac{1}{z}\big)} = \tfrac{1}{\overline{z}}$, thus, the conjugate of the inverse (w.r.t. addition and multiplication respectively) is the inverse of the conjugate.

If $z = (a+ib)$, then $z \cdot \overline{z} = (a+ib)\cdot(a-ib) = \big((a^2+b^2)+i0\big) = a^2+b^2$, and therefore, since $|z| = \sqrt{a^2+b^2}$, we have $z \cdot \overline{z} = |z|^2$. Hence, we get the following: If z_1 and $z_2 \neq 0$ are complex numbers, then

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \underbrace{\frac{\overline{z_2}}{\overline{z_2}}}_{=1} = \frac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}} = \frac{z_1 \cdot \overline{z_2}}{|z_2|^2},$$

which is in fact the same as above.

The unit circle in $\mathbb{C}$

Let $S_1 := \{z \in \mathbb{C} : |z| = 1\}$ be the **unit circle** in $\mathbb{C}$.

For a $z_0 \in S_1$, let the **argument** of z_0, denoted by $\arg(z_0)$, be the *length of the arc from* 1 *counterclockwise to* z_0 *of the unit circle*. Counterclockwise is also called *positive direction*, and hence, clockwise is also called *negative direction*.

Remark: We may go around several times, if we like, until we stop at z_0. Thus, $\arg(z_0)$ is *not unique*!

The argument of a z_0 on the unit circle is in fact an **angle** and we will denote angles by Greek letters like $\alpha, \beta, \varphi, \psi$.

For a non-zero complex number $z \in \mathbb{C}$, let

$$\arg(z) := \arg \underbrace{\left(\frac{z}{|z|}\right)}_{\in S_1},$$

and let $\arg(0) := 0$.

Since the whole unit circle is of length 2π, the angle φ is the same as the angle $\varphi \pm k2\pi$ (where $k \in \mathbb{N}$). However, we usually consider an angle φ as a non-negative real number less than 2π (this means $0 \leq \varphi < 2\pi$).

For any non-zero complex number $z \in \mathbb{C}$, there is a *unique* $z_0 \in \mathbb{C}$, namely $z_0 = \frac{z}{|z|}$, such that $z = r \cdot z_0$ for some positive $r \in \mathbb{R}$ (in fact, $r = |z|$). If $z = 0$, then we can write $z = 0 \cdot z_0$, which is also in this form, but the $z_0 \in S_1$ is no longer unique.

For $z_0 \in S_1$, let $\cos\big(\arg(z_0)\big) := \mathrm{Re}(z_0)$ and $\sin\big(\arg(z_0)\big) := \mathrm{Im}(z_0)$. (Note that this corresponds to the usual definition of *cosine* and *sine*!) Hence, each complex number $z \in \mathbb{C}$ can be written in the form $z = r \cdot \big(\cos(\varphi) + i\sin(\varphi)\big)$, where $r = |z|$ and $\varphi = \arg(z)$.

By Pythagoras, for each angle φ we have $\cos(\varphi)^2 + \sin(\varphi)^2 = 1$ (this is because for any $z_0 = (a + ib) \in S_1$ with $\arg(z_0) = \varphi$ we have $|z_0| = \sqrt{a^2 + b^2} = 1$, and by definition we have $a = \cos(\varphi)$ and $b = \sin(\varphi)$). Further, for any φ we easily get

$$\begin{array}{rclcrcl} \cos(\varphi + 2\pi) & = & \cos(\varphi), & \qquad & \cos(-\varphi) & = & \cos(\varphi)\,, \\ \sin(\varphi + 2\pi) & = & \sin(\varphi), & & \sin(-\varphi) & = & -\sin(\varphi)\,. \end{array}$$

$e^{i\varphi}$:

Above we defined $e^z := \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Now, let us replace z by $i\varphi$:

$$\begin{aligned} e^{i\varphi} &= \left(\frac{(i\varphi)^0}{0!} + \frac{(i\varphi)^1}{1!} + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \frac{(i\varphi)^4}{4!} + \frac{(i\varphi)^5}{5!} + \dots\right) \\ &= \left(1 + i\varphi + i^2\frac{\varphi^2}{2!} + i^3\frac{\varphi^3}{3!} + i^4\frac{\varphi^4}{4!} + i^5\frac{\varphi^5}{5!} + \dots\right) \\ &= \left(1 + i\varphi - \frac{\varphi^2}{2!} - i\frac{\varphi^3}{3!} + \frac{\varphi^4}{4!} + i\frac{\varphi^5}{5!} - - + + \dots\right) \\ &= \left(1 + \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} \pm \dots\right) + i\left(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} \pm \dots\right) \end{aligned}$$

Hence, $\mathrm{Re}\left(e^{i\varphi}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{2n}}{(2n)!}$, and $\mathrm{Im}\left(e^{i\varphi}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{2n+1}}{(2n+1)!}$.

If we replace $i\varphi$ by $-i\varphi$ in the formula $e^{i\varphi} = \sum_{n=0}^{\infty} \frac{(i\varphi)^n}{n!}$, we see that $\overline{e^{i\varphi}} = e^{-i\varphi}$. Thus, $\left|e^{i\varphi}\right| = \sqrt{e^{i\varphi} \cdot \overline{e^{i\varphi}}} = \sqrt{e^{i\varphi} \cdot e^{-i\varphi}} = \sqrt{e^{i\varphi - i\varphi}} = \sqrt{e^0} = \sqrt{1} = 1$, which implies that for any φ we get $e^{i\varphi} \in S_1$. In other words, for any φ, the complex number $e^{i\varphi}$ lies on the unit circle.

Further one can show that for any φ, $\arg\left(e^{i\varphi}\right) = \varphi$, which implies $e^{i\varphi} = \big(\cos(\varphi) + i\sin(\varphi)\big)$. In other words, $\cos(\varphi) = \mathrm{Re}\left(e^{i\varphi}\right)$ and $\sin(\varphi) = \mathrm{Im}\left(e^{i\varphi}\right)$, and therefore, as a consequence we get

$$\begin{aligned} \cos(\varphi) &= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{2n}}{(2n)!} \\ \sin(\varphi) &= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{2n+1}}{(2n+1)!} \end{aligned}$$

Therefore, each $z \in \mathbb{C}$ can be written in the form $z = re^{i\varphi}$, where $r = |z|$ and $\varphi = \arg(z)$.

Transformations:

- $(a + ib) \rightsquigarrow re^{i\varphi}$:
  $r = \sqrt{a^2 + b^2}$, and φ is such that $\cos(\varphi) = \dfrac{a}{r}$ *and* $\sin(\varphi) = \dfrac{b}{r}$.
- $re^{i\varphi} \rightsquigarrow (a + ib)$:
  $a = r \cdot \cos(\varphi)$ and $b = r \cdot \sin(\varphi)$.

Multiplication and division:

Let z_1 and z_2 be two complex numbers and let $r = |z_1|$, $\varphi = \arg(z_1)$, $s = |z_2|$, $\psi = \arg(z_2)$. Thus, $z_1 = r \cdot e^{i\varphi}$ and $z_2 = s \cdot e^{i\psi}$.

Multiplication: Let z_1 and z_2 as above.

$$z_1 \cdot z_2 = r \cdot e^{i\varphi} \cdot s \cdot e^{i\psi} = (r \cdot s) \cdot (e^{i\varphi} \cdot e^{i\psi}) = (r \cdot s) \cdot e^{i\varphi + i\psi} = (r \cdot s) \cdot e^{i(\varphi+\psi)} .$$

Therefore, $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ and $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$.

Division: Let z_1 and z_2 as above and assume $z_2 \neq 0$.

$$\frac{z_1}{z_2} = \frac{r \cdot e^{i\varphi}}{s \cdot e^{i\psi}} = \frac{r}{s} \cdot \frac{e^{i\varphi}}{e^{i\psi}} = \frac{r}{s} \cdot (e^{i\varphi} \cdot e^{-i\psi}) = \frac{r}{s} \cdot e^{i\varphi - i\psi} = \frac{r}{s} \cdot e^{i(\varphi-\psi)} .$$

Therefore, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

The equation $z^n = w$:

Let $w \in \mathbb{C}$ be a complex number and let $n \in \mathbb{N}$ be a positive natural number. Let $s = |w|$ and $\psi = \arg(w)$, so, $w = s \cdot e^{\psi}$. Assume $z^n = w$, what we can say about z? First, let us write z in the form $z = r \cdot e^{i\varphi}$, where $r = |z|$ and $\varphi = \arg(z)$. Now, what we get for z^n? Using the facts given above we get $z^n = \left(r \cdot e^{i\varphi}\right)^n = r^n \cdot (e^{i\varphi})^n = r^n \cdot e^{i(n\varphi)}$. Thus, if $z^n = w$, then $r^n \cdot e^{i(n\varphi)} = s \cdot e^{\psi}$, which implies $r^n = s$ and $n\varphi = \psi$. The first equation gives us

$$r = \sqrt[n]{s}$$

At a first glance, the second equation gives us $\varphi = \frac{\psi}{n}$, but because the angle $\psi = \psi + k \cdot 2\pi$ (for any $k \in \mathbb{N}$), we get in fact $\varphi = \frac{\psi + k \cdot 2\pi}{n} = \frac{\psi}{n} + \frac{k \cdot 2\pi}{n}$ (where $k \in \mathbb{N}$). Now, for $k \in \mathbb{N}$, let us define

$$\varphi_k := \frac{\psi}{n} + \frac{k \cdot 2\pi}{n}.$$

It is easy to see that for the angles φ_k we get $\varphi_n = \varphi_0$, $\varphi_{n+1} = \varphi_1$, $\varphi_{n+2} = \varphi_2$, and so on. Thus, even though we get infinitely many φ_k's, just n of these angles are distinct. Now, for $0 \leq k < n$, let

$$z_k := r \cdot e^{i\varphi_k}.$$

Since $r^n = s$ and (for all $k \in \mathbb{N}$) the angle $n\varphi_k$ is the same as the angle ψ, for all $k \in \mathbb{N}$ we get $(z_k)^n = s \cdot e^{i\psi} = w$, and finally, since just n of the angles φ_k are distinct, the complex numbers $z_0, z_1, \dots, z_{n-1}$ are the only solutions of the equation $z^n = w$. Since $|z_0| = |z_1| = \dots = |z_{n-1}| = r$, all the n solutions are on a circle with radius r. Further, since $\arg(z_{k+1}) - \arg(z_k) = \frac{2\pi}{n}$, the n solutions are equally distributed on the circle, and therefore, the n solutions form a regular n-gon.

Example: Let us find all solutions of $z^4 = -4$. In our notation, $-4 = w = s \cdot e^{i\psi}$ and therefore $s = 4$ and $\psi = \pi$. Now, the 4 solutions are $z_k = r \cdot e^{i\varphi_k}$, where $0 \leq k < 4$, $r = \sqrt[4]{s} = \sqrt[4]{4} = \sqrt{2}$ and $\varphi_k = \frac{\psi}{4} + \frac{k \cdot 2\pi}{4} = \frac{\pi}{4} + \frac{k \cdot \pi}{2}$. Thus,

$$\begin{array}{lclclcr}
z_0 & = & \sqrt{2} \cdot e^{i\frac{\pi}{4}} & & & = & (1+i), \\
z_1 & = & \sqrt{2} \cdot e^{i\frac{\pi}{4}+\frac{\pi}{2}} & = & \sqrt{2} \cdot e^{i\frac{3\pi}{4}} & = & (-1+i), \\
z_2 & = & \sqrt{2} \cdot e^{i\frac{\pi}{4}+\frac{2\pi}{2}} & = & \sqrt{2} \cdot e^{i\frac{5\pi}{4}} & = & (-1-i), \\
z_3 & = & \sqrt{2} \cdot e^{i\frac{\pi}{4}+\frac{3\pi}{2}} & = & \sqrt{2} \cdot e^{i\frac{7\pi}{4}} & = & (1-i).
\end{array}$$

Notice that the 4 solutions form a square on the circle with radius $\sqrt{2}$.

Now, try to find all solutions of the equation $z^8 = 16$ and write them in the form $(a + ib)$. (Your solutions should form an octagon, did you get it?)