Dr Halbeisen

Modules 110PMA003 & 110PMA107

Department of Pure Mathematics

Solutions, Week 1

The pdf-file you may download from
http://www.math.berkeley.edu/∼halbeis/4students/zero.html

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1. (a) First we expand $(a-b)(a+b)$:
$(a-b)(a+b) = (a^2 \underbrace{+ab - ba}_{=0} - b^2) = (a^2 - b^2)$.

Now, $(a-b)(a+b)(a^2+b^2) = (a^2-b^2)(a^2+b^2) = (a^4 \underbrace{+a^2b^2 - b^2a^2}_{=0} - b^4) = (a^4 - b^4)$.

[8]

(b) First we write it down in a nicer way such that it is better to read:
$x2xy\big(x^2(y3-z) - 2x(-y+z)\big) = 2x^2y\big(x^2(3y-z) - 2x(z-y)\big)$.

Now, we expand one term after the other:
$x^2(3y-z) = x^23y - x^2z = 3x^2y - x^2z$; $2x(z-y) = 2xz - 2xy$

So, $\big(x^2(3y-z) - 2x(z-y)\big) = (3x^2y - x^2z) - (2xz - 2xy) = 3x^2y - x^2z - 2xz + 2xy$, and finally we get

$$2x^2y\big(x^2(3y-z) - 2x(z-y)\big) = \underbrace{2x^2y(3x^2y - x^2z - 2xz + 2xy)}_{6x^4y^2 - 2x^4yz - 4x^3yz + 4x^3y^2}$$

[8]

(c) First we write it down in a nicer way such that it is better to read:
$b^2a\big(d(-a^3+3b) + a(4b^2 - ada) + b(4ab - 3d)\big) = ab^2\big(d(3b - a^3) + a(4b^2 - a^2d) + b(4ab - 3d)\big)$.

Now, we expand one term after the other:
$d(3b - a^3) = 3bd - a^3d$; $a(4b^2 - a^2d) = 4ab^2 - a^3d$; $b(4ab - 3d) = 4ab^2 - 3bd$.

So, $\big(d(3b-a^3) + a(4b^2 - a^2d) + b(4ab - 3d)\big) = (3bd - a^3d) + (4ab^2 - a^3d) + (4ab^2 - 3bd) = \underbrace{3bd - 3bd}_{=0}\underbrace{+4ab^2 + 4ab^2}_{=8ab^2}\underbrace{-a^3d - a^3d}_{=-2a^3d} = 8ab^2 - 2a^3d$, and finally we get

$$b^2a\big(d(-a^3+3b) + a(4b^2 - ada) + b(4ab - 3d)\big) = \underbrace{ab^2(8ab^2 - 2a^3d)}_{8a^2b^4 - 2a^4b^2d}$$

[8]

2. (a)

$$\begin{aligned} xw^2+2 &= 8y && -2 \text{ (on both sides)} \\ xw^2 &= 8y-2 && \text{divide by } x \text{ (on both sides)} \\ w^2 &= \tfrac{8y-2}{x} && \text{take the square root (on both sides)} \\ w &= \sqrt{\tfrac{8y-2}{x}} \end{aligned}$$

[8]

(b)

$$\begin{aligned} x(2-w) &= 6w+y && \text{expand l.s. (left side)} \\ 2x-xw &= 6w+y && \mid -6w \\ 2x-xw-6w &= y && \text{rearrange l.s.} \\ 2x+(-x-6)w &= y && \mid -2x \\ (-x-6)w &= y-2x && \mid : (-x-6) \\ w &= \tfrac{y-2x}{-x-6} && \text{rearrange r.s. (right side)} \\ w &= \tfrac{2x-y}{x+6} \end{aligned}$$

[8]

(c)

$$\begin{aligned} \tfrac{y}{w} &= \tfrac{w}{x^3y} && \mid \cdot w,\ \cdot(x^3y),\ \sqrt{\ } \\ \sqrt{x^3y^2} &= \sqrt{w^2} && \text{rearrange both sides} \\ y\sqrt{x^3} &= w \end{aligned}$$

[8]

(d)

$$\begin{aligned} \sqrt{y^3} &= \sqrt[3]{\tfrac{xw}{x+w}}+1 && \mid -1,\ ^3,\ \cdot(x+w), \text{ expand l.s.} \\ x(y^{\frac{3}{2}}-1)^3+w(y^{\frac{3}{2}}-1)^3 &= xw && \mid -w(y^{\frac{3}{2}}-1)^3, \text{ rearrange r.s.} \\ x(y^{\frac{3}{2}}-1)^3 &= w\big(x-(y^{\frac{3}{2}}-1)^3\big) && \mid : \big(x-(y^{\frac{3}{2}}-1)^3\big) \\ \frac{x\left(y^{\frac{3}{2}}-1\right)^3}{x-\left(y^{\frac{3}{2}}-1\right)^3} &= w \end{aligned}$$

[8]

3. (a) $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^{2} = 2$

On the other hand, $\sqrt{2} < 2$, and therefore, $\sqrt{2}^{\sqrt{2}} < \sqrt{2}^{2} = 2$, which implies $\sqrt{2}^{\left(\sqrt{2}^{\sqrt{2}}\right)} < \sqrt{2}^{2} = 2$. So,

$$\sqrt{2}^{\left(\sqrt{2}^{\sqrt{2}}\right)} < \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$$

[8]

(b) 10^{10} is 10 billion, which is a "1 with ten 0's", so, $\left(10^{10}\right)^{10}$ is a "1 with hundred 0's". On the other hand, $10^{(10^{10})}$ is a 1 with 10 billion 0's", so, $10^{(10^{10})}$ is much bigger than $\left(10^{10}\right)^{10}$. [8]

(c) The equation $x^{(x^x)} = \left(x^x\right)^x$ holds for $x = 1$ and for $x = 2$. [8]

4. (a) First note that $\sqrt[7]{a} = a^{\frac{1}{7}}$ and that $\sqrt{a}^{3} = a^{\frac{3}{2}}$. Thus, $\sqrt[7]{a}\,(\sqrt{a})^3 = a^{\frac{1}{7}+\frac{3}{2}} = a^{\frac{23}{14}}$. Further, $\sqrt[28]{x^3} = x^{\frac{3}{28}}$ and thus we get the equation

$$\begin{aligned} a^{\frac{23}{14}} &= x^{\frac{3}{28}} && \mid\ ^{28} \\ \underbrace{\left(a^{\frac{23}{14}}\right)^{28}}_{=a^{46}} &= \underbrace{\left(x^{\frac{23}{14}}\right)^{28}}_{=x^{3}} && \mid \sqrt[3]{\ } \\ \underbrace{\left(a^{46}\right)^{\frac{1}{3}}}_{=a^{\frac{46}{3}}} &= \underbrace{\left(x^{3}\right)^{\frac{1}{3}}}_{=x} \end{aligned}$$

and so, $x = \sqrt[3]{a^{46}}$. [10]

(b) First note that $\sqrt[7]{\frac{a}{x^2}} = \frac{\sqrt[7]{a}}{\sqrt[7]{x^2}} = \frac{a^{\frac{1}{7}}}{x^{\frac{2}{7}}} = x^{-\frac{2}{7}}a^{\frac{1}{7}}$, so $x^{\frac{3}{7}}\sqrt[7]{\frac{a}{x^2}} = \left(x^{\frac{3}{7}}x^{-\frac{2}{7}}\right)a^{\frac{1}{7}} = x^{\frac{1}{7}}a^{\frac{1}{7}} = (xa)^{\frac{1}{7}}$. Further, $\sqrt[7]{7a^2} = \left(7a^2\right)^{\frac{1}{7}}$. Thus, we get the equation

$$\begin{aligned} (xa)^{\frac{1}{7}} &= \left(7a^2\right)^{\frac{1}{7}} && \mid\ ^{7} \\ xa &= 7a^2 && \mid : a \\ x &= 7a \end{aligned}$$

[10]