Can the Continuum Hypothesis be settled ?

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outline

Galileo Paradox

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Galileo Again

Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities.

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Decartes *Principles of Philosophy, Part 1, XXVI*

Cantor

Definition

A set A is equinumerous with set B if the elements of A can be put in one to one correspondence with the elements of B

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The points of the line are equinumerous with the points in the plane, the points in the 3 dimensional space etc.

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Definition

A system S is said to be *infinite* when it is similar to a proper part of it self. Otherwise S is said to be a *finite* system. (**Dedekind**: *Was sind und was sollen die Zahalen, 64*)

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So I finally believe myself to have found the reason why the totality designated by (x) in my earlier letters cannot be correlated one-to-one with the totality designated by (n).

(*Cantor letter to Dedekind ,18 December 1873*)

The Continuum Hypothesis

Question (Cantor-1878)

Are infinite sets of reals only of two kinds: Those that are equinumerous with the integers (like rationals, algebraic numbers etc) and those that equinumerous with the whole sets (like the trancedentsals) ?

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Answer (Cantor's answer Continuum Hypothesis)

Every infinite set of reals is either countable or equinumerous with the whole real line

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The Continuum Hypothesis is equivalent to

Hypothesis

If F is a function from the reals onto an ordinal α then the cardinality of α is at most $\aleph_1$.

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Dieudonné-1976: *Beyond classical analysis there is an infinity of different mathematics and for the time being no definitive reason compels us to chose one rather than another*

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The pragmatic We have many options for set theory (hence for the foundation of Mathematics). We should pick the most useful, the most elegant etc.

The Gödelean conviction

Gödel-1947 *Cantor's conjecture must be either true or false and its undecidability from the axioms can only mean that these axioms do not contain a complete description of this reality and such a belief is by no means chimerical , since it is possible to point out ways in which a decision of the question might nevertheless be obtained*

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There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field and yielding such powerful methods for solving problems . . . that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory

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Theorem (Levy-Solovay 1967)

The continuum hypothesis is independent even if one adds to the axioms of Set Theory any of the accepted axioms of strong infinity.

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Gödel It is consistent to have an uncountable set which is the complement of an analytic set ("Co-Analytic set ") with no perfect subset.

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The large infinities have a "smoothing " effect on smaller infinite sets.

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There is an exact notion of a set of reals being "nice" -The notion of universally Baire.

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Conjecture

In the presence of large cardinals the largest cardinal onto which one can map the reals by universally Baire map is $\aleph_2$

If this conjecture is true then it can be used to argue for restricting hte possible values of the continuum to only two values: $\aleph_1$ and $\aleph_2$.

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There were many generalization of Martin axiom. Almost all forcing axioms stronger that the initial Martin Axiom imply $2^{\aleph_0} = \aleph_2$.

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The present inner models program attempts to construct **L** like models that will avoid this disadvantages of Gödel's **L**.
The attempt is to get what Woodin named "The ultimate **L**.
All **L**-like models satisfy the continuum hypothesis, so the success of this program may be an argument for accepting the Continuum Hypothesis.

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Is this an outrageous speculation?

A Physical Example

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If x, y, z are mutually orthogonal then one can not measure simultaneously any two of S_x, S_y, S_z. (The corresponding operators do not commute.) But the squares S_x^2, S_y^2, S_z^2 do commute and hence can be measured simultaneously . It is a result of QM that always

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The hidden variable assumption claims that the particle carry some predetermined values of S_x^2, S_y^2, S_y^2 such that this values are what we measure. The Kochen-Specker Theorem claims that this is impossible.

Theorem (Kochen-Specker)

There is no function S defined on the unit sphere of the 3-dimensional space S_2 *such that for every* $x \in S_2$ $S(x) = 1, 0$ *and such that for every* $x, y, z \in S_2$ *which are mutually orthogonal*

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Theorem (Pitowsky)

Assume the Continuum Hypothesis . Then there is a function S defined on S_2 getting only the values $0, 1$ and such that for every $x \in S_2$ the set of pairs (y, z) such that x, y, z are mutually orthogonal and such that

$$S(x) + S(y) + S(z) \neq 2$$

is countable.(call such a function "Pitowsky's function"

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Resolution of the Einstein-Podolsky-Rosen and Bell Paradoxes

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(Received 16 February 1982)

A model of spin-$\frac{1}{2}$ statistics that explains the observed frequencies on the basis of the validity of the principle of locality is proposed. The model is based on the observation that certain density conditions on the unit sphere correspond with the observed frequencies while the resulting expectation values violate Bell's inequality.

PACS numbers: 03.65.Bz

Bell[1] has observed that no hidden-variable theory satisfying a principle of locality can reproduce the quantum statistics of electron pairs in the singlet spin state. Bell's argument was simplified by Wigner[2] and put in its most general testable form by Clauser and Horne.[3] Various experiments[4] designed to test the locality principle have shown the observed frequencies to conform with quantum mechanics (i.e., to violate Bell's

that includes complete proofs and generalizations to other spin (angular momentum) states, as well as some predictions, will be published shortly.

Let $S^{(2)}$ be the (surface of a) unit sphere in three-dimensional Euclidean space: $S^{(2)}=\{x \in E^{(3)} \mid |x|=1\}$. Define a spin function as any function, $s:S^{(2)}\to\{-\frac{1}{2},\frac{1}{2}\}$, which satisfies $s(-x) = -s(x)$. The purpose of the first part of this paper is to develop some *mathematical* constraints

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$\cap\, c(y,\theta)]$ is the (average) density of $\{x \mid s(x)=\frac{1}{2}\}$ in $c(y,\theta)$. We have the following:

Existence theorem.—There exists a spin function s such that for all $y \in S^{(2)}$ and all $0<\theta<\pi$ the set $\{x \mid s(x)=\frac{1}{2}\} \cap c(y,\theta)$ is m_θ measurable and

$$\frac{m_\theta[\{x \mid s(x)=\frac{1}{2}\} \cap c(y,\theta)]}{2\pi \sin\theta} = \begin{cases} \cos^2(\frac{1}{2}\theta) & \text{if} \quad s(y)=\frac{1}{2}, \\ \sin^2(\frac{1}{2}\theta) & \text{if} \quad s(y)=-\frac{1}{2}. \end{cases} \tag{1}$$

The complete proof of the theorem will be published separately. The existence theorem belongs to a family of "strange" or seemingly "paradoxical" results that one can prove in set theory. The proof involves transfinite induction on circles and is based on two observations. Firstly, that the intersection of two nonidentical circles contains at most two points and, secondly, that any subset of $c(y,\theta)$ whose coordinality is strictly less than the continuum is m_θ measurable and has m_θ measure zero. To ensure that the second premise is true, we have to assume the validity of the continuum hypothesis, or at least the validity of the (strictly) weaker Martin's axiom.[5] It is important to note that there exists no analytic expression or algorithm by which one can calculate the values of a spin function that satisfy Eq. (1) for the different directions. In fact, the set $\{x \mid s(x)=\frac{1}{2}\}$ turns out to be nonmeasurable in terms of the Lebesgue measure *on the sphere* and the existence theorem may turn out to be independent of the usual axioms of set theory. The proof of the theorem actually establishes the existence of infinite-

definite values everywhere on the sphere—our use of probabilities reflects our ignorance of these values.

I have interpreted formula (1) as an expression for conditional *probabilities*. A natural question to ask is whether we can find a probability space from which we get the values of (1) by conditionalization. In other words we are looking for a probability space such that for all $y \in S^{(2)}$ the event "spin up in the y direction" is defined and has probability $\frac{1}{2}$. Also we want that for all x and y the probability of the joint event "spin up in the x direction and spin up in the y direction" will be $\frac{1}{2}\cos^2(\frac{1}{2}\theta)$, where θ is the angle between x and y. With use of Bell's inequality one can prove that no such probability space exists.[6] [Roughly speaking the values $\frac{1}{2}\cos^2(\frac{1}{2}\theta)$ are incompatible with the additivity axiom for probability.] My way out of this problem is to interpret $\cos^2(\frac{1}{2}\theta)$ as the conditional expectation for "spin up" on a circle, given that the spin is up in the center of the circle. From this perspective Bell's theorem shows that

The existence of Pitowsky's models depend on the Set Theory

Theorem (Farah, M.)

The existence of Pitowsky's function is independent of Set theory. For instance we assume that the real line is real valued measurable, then there are no Pitowsky's function.

Another Potential Example

Definition

H is a separable Hilbert space. $B(H)$ is the algebra of bounded operators on H. $K(H)$ the ideal of compact operators. The Calkin Algebra of H is the quotient algebra $B(H)/K(H)$.

Theorem (Philips,Weaver,Farah)

The problem whether all automorphisms of the Calikin Algebra are inner is independent of ZFC. ("Inner" means induced by an isometry of the underlying Hilbert space.) In fact the Continuum hypothesis implies that there is an inner automorphism of the Calkin algebra which is not inner.

It is inconceivable that problems about Hilbert spaces similar to this problem could have a Physical meaning.

While being a wild shot it is not impossible that Scientific Theories will prefer one Set Theory over others because it makes the scientific theory simpler and more elegant. It may even be possible that in order to derive certain experimentally testable results one would have to prefer one Set Theory over others.

Conclusion

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The cost of ignoring it is a fall from the paradise:

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Conclusion

Independence is a fact of mathematical life. (Gödel's incompleteness theorem!)
The cost of ignoring it is a fall from the paradise:

Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können.
From the paradise, that Cantor created for us, no-one can expel us.
(*Hilbert (1926, p. 170), a lecture given in Münster to Mathematical Society of Westphalia on 4 June 1925*)

The monster of independence is unavoidable but it should be tamed

The monster of independence is unavoidable but it should be tamed
and it will be tamed !

Thank you for your attention!