On Carruth's Axioms for Natural Sums and Products[1]

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Abstract

In this paper three main results are presented: a bijection between natural sums and natural products, the completion of the axioms of Carruth for natural sums, and a new characterization of the natural sums in terms of Klaua's integral ordinals. After introducing some preliminary results, we present two lemmas and a proposition for the proof of the existence of a bijection between natural products and natural sums. Then we prove the incompleteness of Carruth's axioms by providing two counterexamples, and complete Carruth's axioms by adding a fifth axiom. Finally, we introduce a characterization of natural sums in terms of Klaua's integral ordinals and present two families of natural sums, which differ from Hessenberg's sum.

1 Introduction

The first natural sum and product of ordinals, the so-called *Hessenberg sum* and *Hessenberg product*, were introduced by Hessenberg in [5]. The novelty of these operations on the ordinals were their improved algebraic properties with respect to the initial natural product and sum on the natural numbers. Although Hessenberg's work was continued by Jacobsthal [6] by introducing the so-called *Jacobstal product*, Carruth [3] was the first to provide an axiomatisation for both the natural sums and products as well as showing that the Hessenberg sum is the smallest of all the natural sums and the highest of all the mixed sums. A further contribution characterising the Hessenberg sum as well as more general natural sums has been provided by Zuckerman [12].

In the recent literature, an infinitary version of the Hessenberg sum, were the Hessenberg sum is taken over all elements of a sequence of length $\alpha \geq \omega$, has been used in the work of Väänanen and Wang [10] and Wang [11] and studied extensively by Lipparini in [8, 9].

The suitable algebraic properties of these natural operations have been also used in the construction for other algebraic structures. Klaua [7] developed the so-called ring of the

[1]Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

integral ordinals and the field of the rational and real ordinals. Another field, the long real ordinals was also built and studied by Aspero and Tsaprounis [1], which is a complete ordered field (in contrast to Klaua's ring).

2 Preliminaries

We start presenting the usual representation of the ordinals in terms of polynomials, which was first developed by Cantor.

Theorem 2.1. *Any ordinal $\alpha > 0$ has an unique representation of the form*

$$\alpha = \sum_{i \leq m} \omega^{\alpha_i} \cdot a_i \quad \textit{with} \quad \alpha > \alpha_0 > \cdots > \alpha_m \geq 0,\ 1 \leq a_i < \omega,\ \textit{and}\ 0 \leq m < \omega\,.$$

This representation is called the Cantor normal form, denoted CNF.

The following is an auxiliary definition, which we introduce for the ease of the definition of natural sums; we call it the *common normal representation* with respect to two ordinals α and β.

Definition 2.2. *Let Ω be the class of ordinals and let $\alpha, \beta \in \Omega$ be two ordinals in* CNF, *i.e.,*

$$\alpha = \sum_{i \leq m} \omega^{\alpha_i} \cdot a_i \qquad \textit{and} \qquad \beta = \sum_{j \leq n} \omega^{\beta_j} \cdot b_j\,.$$

Furthermore, let $\{\xi_k : 0 \leq k \leq l\}$ be an enumeration of $\{\alpha_i : 0 \leq i \leq m\} \cup \{\beta_j : 0 \leq j \leq n\}$ such that $\xi_0 > \cdots > \xi_l \geq 0$.

Then, we refer to the common normal representation of α and β, denoted $\mathsf{CNR}(\alpha, \beta)$, *as:*

$$\alpha = \sum_{k=0}^{l} \omega^{\xi_k} \cdot \hat{a}_k \qquad \textit{and} \qquad \beta = \sum_{k=0}^{l} \omega^{\xi_k} \cdot \hat{b}_k$$

where

$$\hat{a}_k = \begin{cases} a_i & \textit{if } \xi_k = \alpha_i, \\ 0 & \textit{otherwise}, \end{cases} \qquad \textit{and} \qquad \hat{b}_k = \begin{cases} b_j & \textit{if } \xi_k = \beta_j, \\ 0 & \textit{otherwise.} \end{cases}$$

Notice that the for any two ordinals α, β as above, their common normal representation $\mathsf{CNR}(\alpha, \beta)$ exists. Besides, the common representation is also unique since $\{\xi_k : 0 \leq k \leq l\}$ is unique and the coefficients $\hat{a}_k$ and $\hat{b}_k$ are uniquely determined by the coefficients of Cantor's normal form of α and β, respectively.

2.1 Natural Sums and Products

Hessenberg's natural sum and product

We use the CNR and the CNF in order to define the so-called Hessenberg sum (see [5]), which is the first example of a natural sum and also the only example we could find in the literature.

Let α and β be ordinals in CNF, i.e.,

$$\alpha = \sum_{i \leq m} \omega^{\alpha_i} \cdot a_i \qquad \text{and} \qquad \beta = \sum_{j \leq n} \omega^{\beta_j} \cdot b_j \,.$$

Furthermore, let $\mathsf{CNR}(\alpha, \beta)$ be $\alpha = \sum_{k \leq l} \omega^{\xi_k} \cdot m_k$ and $\beta = \sum_{k \leq l} \omega^{\xi_k} \cdot n_k$, respectively. Then we define the so-called Hessenberg sum, denoted #, of α and β as:

$$\alpha \,\#\, \beta := \sum_{k \leq l} \omega^{\xi_k} \cdot (m_k + n_k) \tag{1}$$

Now, we can define the product as the closed operation $*$ by stipulating:

$$\alpha * \beta = \begin{cases} 0 & \text{if } \alpha = 0 \text{ or } \beta = 0, \\ \displaystyle\sum_{\substack{i \leq m \\ j \leq n}} \omega^{\alpha_i \# \beta_j} \cdot a_i \, b_j & \text{otherwise.} \end{cases}$$

where Σ denotes the Hessenberg sum. Along the rest of the paper, we will always denote by Σ the Hessenberg sum over a sequence of ordinals.

Carruth's axioms for natural sums and products

From the definition of Hessenberg's natural sum and product, Carruth built his axioms for natural operations in 1941 (see [3]).

Carruth's Axioms for Natural Sums 2.3. *A natural sum is a closed binary operation $+$ on the class of ordinals which satisfies the following conditions (for all ordinals $\alpha, \beta, \gamma, \delta$):*

Axiom S1. $\alpha + \beta = \beta + \alpha$

Axiom S2. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

Axiom S3. $\alpha + 0 = 0 + \alpha = \alpha$

Axiom S4. $\alpha + \delta > \beta + \delta$ *iff* $\alpha > \beta$

Carruth's Axioms for Natural Products 2.4. *A natural product is defined as a closed operation $\cdot$ on the class of ordinals, such that for the Hessenberg sum #, the following conditions are satisfied (for all ordinals $\alpha, \beta, \gamma, \delta$):*

Axiom P1. $\alpha \cdot \beta = \beta \cdot \alpha$

Axiom P2. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

Axiom P3. $\alpha \cdot 1 = \alpha$

Axiom P4. $\alpha \cdot \delta > \beta \cdot \delta$ *iff* $\alpha > \beta$

Axiom P5. $(\alpha \cdot \beta) \,\#\, (\alpha \cdot \delta) = \alpha \cdot (\beta \,\#\, \delta)$

Axiom P6. $\omega^\alpha \cdot \omega^\beta = \omega^{\sigma(\alpha,\beta)}$, *where $\sigma(\alpha, \beta)$ is an ordinal depending on α and β.*

3 Bijection between natural sums and natural products

The first lemma guarantees that a natural sum can always be deduced from any natural product.

Lemma 3.1. *If the operation "$\cdot$" is a natural product and the function $\sigma : \Omega \times \Omega \to \Omega$ is such that for every $\alpha, \beta \in \Omega$, $\omega^\alpha \cdot \omega^\beta = \omega^{\sigma(\alpha,\beta)}$, then $\sigma(\alpha, \beta)$ is a natural sum.*

Proof. First notice that the binary operation $\sigma : \Omega \times \Omega \longrightarrow \Omega$, $(\alpha, \beta) \mapsto \sigma(\alpha, \beta)$ is well-defined by stipulating

$$\sigma(\alpha, \beta) = \delta \iff \omega^\alpha \cdot \omega^\beta = \omega^\delta.$$

We check now that σ meets the properties of a natural sum.

1. Commutativity follows straightforward from Axiom P1.
2. If α, β and δ are ordinals, by Axiom P2 we have

$$\omega^\alpha \cdot (\omega^\beta \cdot \omega^\delta) = (\omega^\alpha \cdot \omega^\beta) \cdot \omega^\delta$$
$$\omega^{\sigma(\alpha,\sigma(\beta,\delta))} = \omega^{\sigma(\sigma(\alpha,\beta),\delta)}$$

   Hence, we have $\sigma\big(\alpha, \sigma(\beta, \delta)\big) = \sigma\big(\sigma(\alpha, \beta), \delta\big)$, which shows that σ is associative.

3. By Axiom P3 and since $\omega^0 = 1$, we have $\omega^{\sigma(\alpha,0)} = \omega^\alpha \cdot \omega^0 = \omega^\alpha$, and hence then $\sigma(\alpha, 0) = \alpha$.
4. For any three ordinals α, β, δ, the following are equivalent:

$$\sigma(\alpha, \delta) > \sigma(\beta, \delta)$$
$$\omega^{\sigma(\alpha,\delta)} > \omega^{\sigma(\beta,\delta)}$$
$$\omega^\alpha \cdot \omega^\delta > \omega^\beta \cdot \omega^\delta$$
$$\omega^\alpha > \omega^\beta$$
$$\alpha > \beta$$

   where the second and fourth equivalence follows from the strict monotonicity of the power function.

□

Below, we relax our notation slightly and allow that ω^{α_k} is written after the integer a_k.

Lemma 3.2. *Let "$\cdot$" denote any natural product. If α and β are two ordinals with CNF $\alpha = \sum_{k\le m} a_k\, \omega^{\alpha_k}$, $\beta = \sum_{j\le n} b_j\, \omega^{\beta_j}$, then*

$$\alpha \cdot \beta = \sum_{k\le m, j\le n} a_k b_j\, \omega^{\sigma(\alpha_k,\beta_j)} \tag{2}$$

where σ is the natural sum induced by the product "$\cdot$" as in Lemma 3.1.

Proof. Notice first that the CNF is the Hessenberg natural sum of the terms $a_k\omega^{\alpha_k}$ and $b_j\omega^{\beta_j}$, respectively. By the distributive property of the natural product, we have:

$$\alpha\cdot\beta = \left(\sum_{k\le m} a_k\omega^{\alpha_k}\right)\cdot\left(\sum_{j\le n} b_j\omega^{\beta_j}\right) = \sum_{k\le m, j\le n} a_k b_j \omega^{\sigma(\alpha_k,\beta_j)}$$

□

Proposition 3.3. *Let $\lambda : \Omega\times\Omega \mapsto \Omega$ be a natural sum. Then the binary operation "$\cdot$" on $\Omega\times\Omega$ defined by stipulating*

$$\alpha\cdot\beta := \sum_{k\le m, j\le n} a_k b_j \omega^{\lambda(\alpha_k,\beta_j)} \tag{3}$$

is a natural product.

Proof. $\mathsf{Axiom\ P1}$, $\mathsf{Axiom\ P2}$, $\mathsf{Axiom\ P3}$, and $\mathsf{Axiom\ P6}$ are immediate from the definition of the product "$\cdot$". To see that "$\cdot$" satisfies $\mathsf{Axiom\ P4}$, assume that $\alpha\cdot\delta > \beta\cdot\delta$. Now, choose the CNR of α,β and the CNF for δ:

$$\alpha = \sum_{i\le r} a_i\omega^{\xi_i}, \qquad \beta = \sum_{i\le r} b_i\omega^{\xi_i}, \qquad \delta = \sum_{k\le m} d_k\omega^{\delta_k}.$$

Then, the following are equivalent:

$$\begin{aligned}
&\alpha\cdot\delta > \beta\cdot\delta \\
\iff &\sum_{i\le r, k\le m} a_i d_k \omega^{\lambda(\xi_i,\delta_k)} > \sum_{i\le r, k\le m} b_i d_k \omega^{\lambda(\xi_i,\delta_k)} \\
\iff &\alpha > \beta
\end{aligned}$$

The first equivalence follows from the definition of the product. The second equivalence follows from the next argument. By the first equivalence, $a_0d_0 \ge b_0d_0$ and hence either $a_0 > b_0$, in which case $\alpha > \beta$, or $a_0 = b_0$. In the latter case, we define $\alpha_1 := \sum_{1\le i\le m} a_i\omega^{\xi_i}$ and $\beta_1 := \sum_{1\le i\le m} b_i\omega^{\xi_i}$. The first equivalence together with $a_0 = b_0$ implies $\alpha_1\cdot\delta \ge \beta_1\cdot\delta$.

We further iterate this reasoning for a general $l \le r$, given that $\alpha_l\cdot\delta > \beta_l\cdot\delta$ and $a_i = b_i$, for any $i \le l$, where

$$\alpha_l := \sum_{l\le i\le r} a_i\omega^{\xi_i}, \quad \beta_l := \sum_{l\le i\le r} b_i\omega^{\xi_i}.$$

We obtain then $a_{l+1} \ge b_{l+1}$. If $a_{l+1} > b_{l+1}$, then $\alpha > \beta$. Otherwise, $a_{l+1} = b_{l+1}$. Hence, since $\alpha\cdot\delta \ge \beta\cdot\delta$ and $a_i = b_i$ for any $i \le l$, $\alpha_{l+1}\cdot\delta \ge \beta_{l+1}\cdot\delta$.

If there is no $i < r$ such that $a_i > b_i$, this iterating process ends after finite r steps and so we have defined the finite sequences $\{\alpha_l\}_{l\le r}, \{\beta_l\}_{l\le r}$ and shown that $a_i = b_i$ for any $i \le r$, which implies:

$$\alpha\cdot\delta = \sum_{i\le r, k\le m} a_i d_k \omega^{\lambda(\xi_i,\delta_k)} = \sum_{i\le r, k\le m} b_i d_k \omega^{\lambda(\xi_i,\delta_k)} = \beta\cdot\delta$$

Such equality is a contradiction since $\alpha \cdot \delta > \beta \cdot \delta$. Thus, there exists $l_0 \leq r$, such that $a_i = b_i$, for any $i \leq l_0$ and $a_{l_0} > b_{l_0}$. In turn, this implies $\alpha_{l_0} > \beta_{l_0}$ and $\alpha_l > \beta_l$ for any $l \leq l_0$. In particular, $\alpha > \beta$.

To see that "$\cdot$" also satisfies Axiom P5, we assume that the ordinals α, β, δ have the same representations as above. Thus, we have:

$$\begin{aligned}
\delta \cdot (\alpha \# \beta) &= \sum_{k \leq m} d_k \omega^{\delta_k} \cdot (\alpha \# \beta) \\
&= \sum_{k \leq m} d_k \omega^{\delta_k} \cdot \sum_{i \leq r} (a_i + b_i) \omega^{\xi_i} \\
&= \sum_{k \leq m, i \leq r} a_i d_k \omega^{\lambda(\xi_i, \delta_k)} \# \sum_{k \leq m, i \leq r} b_i d_k \omega^{\lambda(\xi_i, \delta_k)} \\
&= \alpha \cdot \delta \# \beta \cdot \delta
\end{aligned}$$

□

The previous results allow us to prove the following theorem.

Theorem 3.4. *There is a bijection between the class of all natural sums and the class of all natural products.*

Proof. Let $\mathcal{S}$, $\mathcal{P}$ be the classes of all natural sums and all natural products, respectively, and let $\Gamma : \mathcal{S} \longrightarrow \mathcal{P}$ be the function which maps any natural sum σ to a natural product as in Lemma 3.2.

Γ is well-defined by Proposition 3.3, because for any $\sigma \in \mathcal{S}$, $\Gamma(\sigma)$ is a natural product. If there were $\sigma_1, \sigma_2 \in \mathcal{S}$ such that $\sigma_1(\alpha, \beta) = \sigma_2(\alpha, \beta) = \sum_{k \leq m, j \leq n} a_k b_j \omega^{\sigma(\alpha_k, \beta_j)}$ for any $\alpha, \beta \in \Omega$ then $\delta_1 = \delta_2$ by the definition of a natural product.

Furthermore, the function Γ is injective by Lemma 3.1. The surjectivity is assured by Lemmas 3.1 and 3.2, since each natural product in $\mathcal{P}$ defines a unique natural sum in $\mathcal{S}$. □

4 The need for a new Axiom

Natural sums, as any type of ordinal sum, were initially defined as those sums which preserve the properties of Peano's sum on $\mathbb{N}$, extending such sum to all ordinals.

Below, we present two examples that show how Carruth's axioms for natural sums (2.3) allow the existence of sums that do not preserve Peano's sum on $\mathbb{N}$. We argue that hence the axioms for natural sums are incomplete and provide an additional axiom that excludes such degenerate sums — completing the axioms for natural sums.

Example 4.1. Let α, β be two ordinals. Then, we define the sum by setting

$$\begin{aligned}
&\alpha + 0 = 0 + \alpha := \alpha \\
&\alpha + \beta = \beta + \alpha := \alpha \# \beta \# 1, \text{ if } \alpha, \beta \neq 0
\end{aligned}$$

where $\#$ denotes Hessenberg's sum. It is easy to see that the defined sum satisfies Axiom S1–Axiom S4.

However, this is a sum where the successor ordinal α^+ of α is not $\alpha + 1$. In particular, $n + 1 = n \# 2$ for any $n \in \mathbb{N} - \{0\}$.

Example 4.2. If α, β are two different ordinals, we define the sum by setting.

$$\begin{aligned} \alpha + 0 = 0 + \alpha &:= \alpha \\ 1 + 1 &:= 2 \\ \alpha + 1 = 1 + \alpha &:= \alpha \# 2 && \text{if } \alpha \geq 2 \\ \alpha + \beta = \beta + \alpha &:= \alpha \# \beta \# 2 && \text{if } \alpha \geq 2 \end{aligned}$$

The sum is well-defined and Axiom S1 and Axiom S3 follow immediately from the definition.

For associativity, let α, β, γ be any three ordinals. Consider the following three cases:

- If $\alpha, \beta, \gamma \geq 2$, then
$$\alpha + (\beta + \gamma) = \alpha + (\beta \# \gamma \# 2) = \alpha \# \beta \# \gamma + 4 = (\alpha \# \beta \# 2) + \gamma = (\alpha + \beta) + \gamma.$$
- If $\alpha, \beta \geq 2$ and $\gamma = 1$, then
$$\alpha + (\beta + 1) = \alpha + (\beta \# 2) = \alpha \# \beta \# 4 = (\alpha \# \beta \# 2) + 1 = (\alpha + \beta) + \gamma.$$
- If $\alpha \geq 2$ and $\beta = \gamma = 1$, then
$$(\alpha + \beta) + \gamma = \alpha + 1 \# 2 = \alpha \# 4 = \alpha + 2 = \alpha + (1 + 1).$$

In any other case, at least one of the ordinals α, β, γ is equal to 0 and associativity holds.

For Axiom S4, we pursue a similar proof by cases. Since in the definition above, the sum has been established by giving a definition of the form $\alpha + \beta = \alpha \# \beta \# R(\alpha, \beta)$ with $R(\alpha, \beta)$ some ordinal that depends on α, β, we have:

$$\begin{gathered} \alpha + \delta > \beta + \delta \\ \alpha \# \delta \# R(\alpha, \delta) > \beta \# \delta \# R(\beta, \delta) \\ R(\alpha, \delta) \geq R(\beta, \delta) \end{gathered}$$

This is fulfilled in our definition, since for $\alpha, \beta \geq 2$ we have

$$R(\alpha, \beta) > R(\alpha, 1) \geq R(1, 1) = R(\alpha, 0)$$

and therefore, Axiom S4 holds.

These examples show that such axioms open up the concept of natural sum (of ordinals) which do not preserve the standard sum on $\mathbb{N}$. Following the initial definitions given by Hessenberg and Jakobstal, in order to preserve the ordinal sum on $\mathbb{N}$, we propose the following axiom — due to obvious reasons, we call it Successor Axiom.

Successor Axiom. *For all natural sums* $\lambda, \gamma : \Omega \times \Omega \to \Omega$ *and* $\alpha \in \Omega$*, it holds*

$$\lambda(\alpha, 1) = \gamma(\alpha, 1)$$

Proposition 4.3. *If* $\oplus$ *is a natural sum with respect to Carruth's Axioms and for any ordinal* α *we have* $\alpha \oplus 1 = \alpha \# 1$*, then, for any limit ordinal* η *and for any* $m \in \mathbb{N}$ *we have:*

$$\eta \oplus m = \eta \# m$$

In particular, $m \oplus n = m + n$ *for any* $m, n \in \mathbb{N}$ *and "+" denoting the standard sum.*

Proof. Let η be any limit ordinal. Then we have $\eta \oplus 1 = \eta \# 1$. If we assume $\eta \oplus n = \eta \# n$, then $\eta \oplus n \oplus 1 = \eta \# n \# 1$, so, the proof follows by induction. □

Before we prove the next result, let us first briefly introduce the so-called Presburger Arithmetic:

Presburger Arithmetic. *The language of the Presburger arithmetic contains the elements* 0 *and* 1*, and the binary operation "+" such that the following axioms are fulfilled for any* $x, y \in \mathbb{N}$*:*

- PA_1 *For each* $x \in \mathbb{N}$ *we have* $0 \neq x + 1$
- PA_2 *If* $x + 1 = y + 1$ *then* $x = y$.
- PA_3 $x + 0 = x$
- PA_4 $x + (y + 1) = (x + y) + 1$
- PA_5 *For any first-order formula* φ *in the language of Presburger arithmetic with a free variable* x*:*

$$\Big(\varphi(0) \wedge \forall x\big(\varphi(x) \to \varphi(x + 1)\big)\Big) \to \forall y\, \varphi(y)$$

Interestingly, we see that the inclusion of the Successor Axiom is crucial for natural sums to become Presburger sums.

Proposition 4.4. *Let + denote any natural sum following Carruth's axioms*

(1) *If the sum "+" satisfies additionally the* Successor Axiom*, then it builds a Presburger arithmetic.*

(2) *Carruth's Axioms for natural sums are not sufficient to obtain the Presburger arithmetic.*

Proof. (1) We prove each axioms of Presburger Arithmetic separately.

PA_1: If $0 = x + 1$ for some $x \in \mathbb{N}$, then by Axiom S4, $x + 1 < 1$ implies $x < 0$.

PA_2: This axiom follows again from Axiom S4, since $x > y$ implies $x + 1 > y + 1$.

PA_3 & PA_4: These axioms follow straightforward from Axiom S3 and Axiom S2, respectively.

PA_4: Notice that for any $y \in \mathbb{N} \setminus \{0\}$ there exists an $\alpha \in \mathbb{N}$ such that $y = \alpha \# 1 = \alpha + 1$. Thus, if for any first-order formula φ in the Presburger Arithmetic we have $\varphi(0)$ and for any x we have $\varphi(x) \to \varphi(x+1)$, then for any $y \in \mathbb{N}$ we have $\varphi(y)$ (notice that any y is a successor except for $y = 0$).

(2) We consider the natural sum $\oplus$ given in Example 4.2, i.e., without the Successor Axiom. With this sum, 1 is not the successor of any element, since for any $z \in \mathbb{N}$, $z \oplus 1 > 1$. Hence, for some first-order formula φ,

$$\varphi(0) \wedge \forall x \big(\varphi(x) \to \varphi(x+1)\big)$$

does not imply $\forall y \varphi(y)$, since $\varphi(y)$ might be false for $y = 1$. □

5 Characterization of natural sums

In this section, we would like to give a different approach to the natural sums. For this, we will make use of the results of Dieter Klaua [7], who extended the class of the ordinals with Hessenberg's sum and product to the ring G of the so-called integral ordinals. We will see that embedding the ordinals into G can be beneficial for generating natural sums, since ordinals can then be subtracted.

5.1 The ring of integral ordinals

We next provide a slightly more general result, based on the work of Klaua, allowing for the construction of any rings with the natural products instead for only those rings that extend the Hessenberg product. We start by introducing the equivalence relation upon we build the elements of the ring G. Since the ordinals play the same role in the ring G as the natural numbers play in the ring $\mathbb{Z}$, we call them *natural ordinals*. The class of all natural ordinals (i.e., the class Ω), is denoted $\mathbb{N}_\Omega$.

Given some natural ordinals $\gamma_1, \delta_1, \gamma_2, \delta_2 \in \mathbb{N}_\Omega$, we define the following equivalence relation:

$$(\gamma_1, \delta_1) \sim_G (\gamma_2, \delta_2) \text{ iff } \gamma_1 \# \delta_2 = \gamma_2 \# \delta_1 \tag{4}$$

where $\#$ is the Hessenberg sum. For natural ordinals $\gamma, \delta \in \mathbb{N}_\Omega$, let $G(\gamma, \delta)$ be the class of all pairs $(\gamma', \delta') \in \mathbb{N}_\Omega \times \mathbb{N}_\Omega$ such that $(\gamma, \delta) \sim_G (\gamma', \delta')$. Each equivalence class $G(\gamma, \delta)$ is an *integral ordinal* and the class of integral ordinals will be denoted by G. Now, from each integral ordinal $G(\gamma, \delta)$ we choose a representative $\alpha \in G(\gamma, \delta)$ and define $[\alpha] := G(\gamma, \delta)$. Furthermore, the class of integral ordinals $[\alpha]$ where $\alpha = (\gamma, \delta)$ and $\gamma \geq \delta$ is denoted by G_Ω. Notice that the elements of G_Ω correspond to the elements of $\mathbb{N}_\Omega$.

Now, we define a ring structure on the class G:

Definition 5.1. *For $i = 1, 2$ let $\alpha_i = (\gamma_i, \delta_i)$.*

1. *The order relation $\leq_G$ on G is defined by stipulating*

$$[\alpha_1] \leq_G [\alpha_2] \;\Leftrightarrow\; \gamma_1 \# \delta_2 \leq \gamma_2 \# \delta_1 \,. \tag{5}$$

2. *The sum* $[\alpha_1]+[\alpha_2]$ *is defined by stipulating*

$$[\alpha_1]+[\alpha_2] := G(\gamma_1 \# \gamma_2, \delta_1 \# \delta_2)\,.$$

With respect to the sum "+" we have the neutral element $0_G = G(0,0)$*, and for each integral ordinal* $[\alpha] = G(\gamma,\delta)$ *we have that* $[-\alpha] := G(\delta,\gamma)$ *is the inverse of* $[\alpha]$*, i.e.,* $[-\alpha]+[\alpha] = 0_G$*.*

3. *The product* $[\alpha_1]\cdot[\alpha_2]$ *is defined by stipulating*

$$[\alpha_1]\cdot[\alpha_2] := G\big((\gamma_1 * \gamma_2) \# (\delta_1 * \delta_2),\, (\gamma_1 * \delta_2) \# (\gamma_2 * \delta_1)\big)\,. \tag{6}$$

where $*$ *is any natural product. With respect to the product "$\cdot$" we have the neutral element* $1_G := G(1,0)$*.*

For the sake of simplicity we identify integral ordinals $[\alpha]$ with α. As an immediate consequence of the definitions we have the following results analogous to the ones obtained in [7].

Proposition 5.2. *1. The relation* $\leq_G$ *is well-defined and a total order.*

2. *The integral sum is well-defined and has the following properties:*

   (a) $\alpha + 0 = 0 + \alpha = \alpha$

   (b) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

   (c) $\alpha + \beta = \beta + \alpha$

   (d) If $\alpha_1 < \alpha_2$, then $\alpha_1 + \beta < \alpha_2 + \beta$, for any integral ordinal β.

   (e) $\alpha + (-\alpha) = 0$.

3. *The integral product is well-defined and has the following properties:*

   (a) $\alpha \cdot \beta = \beta \cdot \alpha$

   (b) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

   (c) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$

   (d) $\beta > 0$ and $\alpha_1 < \alpha_2 \Rightarrow \alpha_1 \cdot \beta < \alpha_2 \cdot \beta$

   (e) $\beta < 0$ and $\alpha_1 < \alpha_2 \Rightarrow \alpha_1 \cdot \beta > \alpha_2 \cdot \beta$

   (f) $\alpha \cdot 1 = \alpha$

   (g) $\alpha \cdot 0 = 0$

Properties 2.(a)-(e) imply that $(G, +, 0_G)$ has a group structure due to the commutativity, associativity, existence of an identity element and cancellability of the sum. In addition, properties 3.(a)-(g) guarantee the structure of a commutative ring for $(G, +, \cdot, 0_G, 1_G)$. As a further consequence of Proposition 5.2 and analogously as in [7] we obtain the following

Corollary 5.3. *The structure* $(G, +, \cdot, 0_G, 1_G)$ *is an integral domain.*

Proof. Let $[\alpha_1], [\alpha_2] \in G$ such that $[\alpha_i] = G(\beta_i, \gamma_i)$ for $\beta_i, \gamma_i \in \Omega$ and $i = 1, 2$. If we assume $[\alpha_1] \cdot [\alpha_2] = [0]$, then we obtain the following identities, which are equivalent:

$$(\beta_1 * \beta_2) \# (\gamma_1 * \gamma_2) = (\beta_1 * \gamma_2) \# (\beta_2 * \gamma_1) \tag{7}$$

$$G(\beta_1, 0) \cdot G(\beta_2, 0) + G(\gamma_1, 0) \cdot G(\gamma_2, 0) = G(\beta_1, 0) \cdot G(\gamma_2, 0) + G(\beta_1, 0) \cdot G(\gamma_1, 0) \tag{8}$$

$$G(\beta_1, 0) \cdot (G(\beta_2, 0) + G(0, \gamma_2)) = G(\gamma_1, 0) \cdot (G(\beta_2, 0) + G(0, \gamma_2)) \tag{9}$$

By assumption, for $i = 1, 2$ we have $G(\beta_i, 0) \gtrless G(\gamma_i, 0)$, since otherwise, $\beta_i = \gamma_i$ which implies $[\alpha_i] = 0$. Hence, if $G(\beta_1, 0) \gtrless G(\gamma_1, 0)$ by Property 3.(f) of Proposition 5.2, we obtain that $G(\beta_1, 0) \cdot (G(\beta_2, 0) + G(0, \gamma_2)) \gtrless G(\gamma_1, 0) \cdot (G(\beta_2, 0) + G(0, \gamma_2))$, arriving at a contradiction with identity (7). □

Last, we give a key result whose proof can be found in [7]:

Proposition 5.4. *There is an embedding of the natural ordinals in the ring G of the integral ordinals.*

5.2 Main results

Now we are able to introduce the characterization of natural sums. We will make use of the following auxiliary

Lemma 5.5. *If $\oplus$ denotes any natural sum on the natural ordinals, then*

$$\alpha \oplus \beta = \alpha \# \beta \# R(\alpha, \beta)$$

where $\#$ denotes Hessenberg's sum and $R(\alpha, \beta)$ is an integral ordinal (depending on α and β).

Proof. Since $R(\alpha, \beta)$ is arbitrary, this is just a consequence of Proposition 5.4 with $R(\alpha, \beta) = G(\alpha \oplus \beta, \alpha \# \beta)$. □

We will characterize now the function $R : \Omega \times \Omega \to G$, $(\alpha, \beta) \mapsto G(\alpha \oplus \beta, \alpha \# \beta)$. In particular, we shall present axioms for the integral ordinal $R(\alpha, \beta)$ for natural sums.

The function $R : \Omega \times \Omega \to G$ satisfies the following conditions:

(1) $R(\alpha, \beta) = R(\beta, \alpha)$: Since $\oplus$ is commutative, we have

$$\alpha \# \beta \# R(\alpha, \beta) = \alpha \oplus \beta = \beta \oplus \alpha = \beta \# \alpha \# R(\beta, \alpha).$$

(2) For any ordinals α, β, γ we have

$$R(\alpha, \beta) \# R\big(\alpha \# \beta \# R(\alpha, \beta), \gamma\big) = R(\beta, \gamma) \# R\big(\alpha, \beta \# \gamma \# R(\beta, \gamma)\big).$$

Since $\oplus$ is associative, we have $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$. So, on the one hand we have

$$(\alpha \oplus \beta) \# \gamma \# R(\alpha \oplus \beta, \gamma) \;=\; \alpha \# \beta \# \gamma \# R(\alpha, \beta) \# R\big(\alpha \# \beta \# R(\alpha, \beta), \gamma\big)$$

and on the other hand we have

$$\alpha \oplus \big(\beta \# \gamma \# R(\beta,\gamma)\big) \;=\; \alpha \# \beta \# \gamma \# R(\beta,\gamma) \# R\big(\alpha, \beta \# \gamma \# R(\beta,\gamma)\big).$$

(3) For all ordinals α we have $R(\alpha, 0) = 0$, which follows from $G(\alpha,\alpha) = 0_G$ and Definition 5.1.

(4) For any ordinals α, β, δ with $\alpha > \beta$ we have $R(\alpha,\delta) - R(\beta,\delta) > \beta - \alpha$: The following statements are equivalent:

$$\begin{aligned} \alpha &> \beta \\ \alpha \oplus \delta &> \beta \oplus \delta \\ \alpha \# \delta \# R(\alpha,\delta) &> \beta \# \delta \# R(\beta,\delta) \\ \alpha \# R(\alpha,\delta) &> \beta \# R(\beta,\delta) \\ R(\alpha,\delta) - R(\beta,\delta) &> \beta - \alpha \end{aligned}$$

So, since $\oplus$ satisfies Axiom S4, we have $R(\alpha,\delta) - R(\beta,\delta) > \beta - \alpha$

(5) If the natural sum $\oplus$ satisfies the Successor Axiom, then for all ordinals α and all $n \in \mathbb{N}$ we have $R(\alpha, n) = 0$: Since $\alpha \oplus n = \alpha \# n$, we have

$$\alpha \# n = \alpha \oplus n = \alpha \# n \# R(\alpha, n)$$

which implies $R(\alpha, n) = 0$.

Proposition 5.6. *The following conditions are necessary and sufficient for $R : \Omega \times \Omega \to G$, such that $\oplus : \Omega \times \Omega \to \Omega$, where $\alpha \oplus \beta = \alpha \# \beta \# R(\alpha,\beta)$, is a natural sum which satisfies Carruth's and the Successor Axiom:*

(a) *For any ordinals α, β we have $R(\alpha,\beta) = R(\beta,\alpha)$.*

(b) *For any ordinals α, β, γ we have*

$$R(\alpha,\beta) \# R\big(\alpha \# \beta \# R(\alpha,\beta), \gamma\big) = R(\beta,\gamma) \# R\big(\alpha, \beta \# \gamma \# R(\beta,\gamma)\big).$$

(c) *For any ordinal α and any $n \in \mathbb{N}$ we have $R(\alpha, n) = 0$.*

(d) *For any ordinals α, β, δ with $\alpha > \beta$ we have $R(\alpha,\delta) - R(\beta,\delta) > \beta - \alpha$.*

Proof. Above we have seen that the conditions are necessary, So, it remains to show that the conditions are also sufficient.

(a) $R(\alpha,\beta) = R(\beta,\alpha)$ implies that $\oplus$ is commutative.

(b) $R(\alpha,\beta) \# R\big(\alpha \# \beta \# R(\alpha,\beta), \gamma\big) = R(\beta,\gamma) \# R\big(\alpha, \beta \# \gamma \# R(\beta,\gamma)\big)$ implies that $\oplus$ is associative.

(c) $R(\alpha, n) = 0$ implies that $\oplus$ satisfies Axiom S3 and the Successor Axiom.

(d) If $\alpha > \beta$, then $R(\alpha,\delta) - R(\beta,\delta) > \beta - \alpha$ implies that $\oplus$ satisfies Axiom S4.

$\square$

We now present a simple example for a natural sum obtained by using the characterization above.

Example 5.7. Let $R : \Omega \times \Omega \to G$ be defined by stipulating

$$R(\alpha, \beta) := \begin{cases} 0 & \text{if } \alpha \in \mathbb{N} \text{ or } \beta \in \mathbb{N}, \\ \omega & \text{otherwise.} \end{cases}$$

Then $\alpha \oplus \beta := \alpha \# \beta \# R(\alpha, \beta)$ is a natural sum which satisfies the *Successor Axiom*:

To see this, by Proposition 5.6 it is enough to show that $R(\alpha, \beta)$ satisfies (a)–(d): Conditions (a), (c), (d) are easily verified. For condition (b) we consider the following four cases:

- $R(\alpha, \beta) = \omega = R\big(\alpha \# \beta \# R(\alpha, \beta), \gamma\big)$: In this case we have that none of α, β, γ belongs to $\mathbb{N}$, which implies that $R(\beta, \gamma) = \omega = R\big(\alpha, \beta \# \gamma \# R(\beta, \gamma)\big)$, and therefore we get

$$R(\alpha, \beta) \# R\big(\alpha \# \beta \# R(\alpha, \beta), \gamma\big) \;=\; \omega \# \omega \;=\; R(\beta, \gamma) \# R\big(\alpha, \beta \# \gamma \# R(\beta, \gamma)\big).$$

- $R(\alpha, \beta) = 0$ and $R\big(\alpha \# \beta \# R(\alpha, \beta), \gamma\big) = \omega$: In this case we have that either $\alpha \in \mathbb{N}$ or $\beta \in \mathbb{N}$ and that $\gamma \notin \mathbb{N}$. If $\alpha \in \mathbb{N}$ and $\beta \notin \mathbb{N}$, then $R(\beta, \gamma) = \omega$ and $R\big(\alpha, \beta \# \gamma \# R(\beta, \gamma)\big) = 0$, and if $\alpha \notin \mathbb{N}$ and $\beta \in \mathbb{N}$, then $R(\beta, \gamma) = 0$ and $R\big(\alpha, \beta \# \gamma \# R(\beta, \gamma)\big) = \omega$, and therefore we get

$$R(\alpha, \beta) \# R\big(\alpha \# \beta \# R(\alpha, \beta), \gamma\big) \;=\; \omega \;=\; R(\beta, \gamma) \# R\big(\alpha, \beta \# \gamma \# R(\beta, \gamma)\big).$$

- $R(\alpha, \beta) = \omega$ and $R\big(\alpha \# \beta \# R(\alpha, \beta), \gamma\big) = 0$: This case is similar to the previous case.

- $R(\alpha, \beta) = 0 = R\big(\alpha \# \beta \# R(\alpha, \beta), \gamma\big)$: In this case we have that either $\alpha \in \mathbb{N}$ or $\beta \in \mathbb{N}$ and $\gamma \in \mathbb{N}$. Since $\gamma \in \mathbb{N}$, we have $R(\beta, \gamma) = 0$, and since either α or $\beta \# \gamma$ belongs to $\mathbb{N}$, we have $R\big(\alpha, \beta \# \gamma \# R(\beta, \gamma)\big) = 0$, and therefore we get

$$R(\alpha, \beta) \# R\big(\alpha \# \beta \# R(\alpha, \beta), \gamma\big) \;=\; 0 \;=\; R(\beta, \gamma) \# R\big(\alpha, \beta \# \gamma \# R(\beta, \gamma)\big).$$

Example 5.7 shows that there are natural sums satisfying the Successor Axiom which are different from the Hessenberg sum.

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