More Configurations on Elliptic Curves

Dedicated to the memory of Branko Grünbaum

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Abstract

We construct elliptic $(3r_s, sr_3)$ configurations for all integers $r \geq s \geq 1$. This solves an open problem of Branko Grünbaum. The configurations which we build have mirror symmetry and even D_3 symmetry if r is a multiple of 3. Moreover, the configurations are dynamic in the sense that the points can be moved along the elliptic curve in such a way that all line incidences are preserved.

1 Introduction

1.1 Elliptic configurations

The study of configurations has a long and rich history. We refer to Grünbaum [3] as a main reference, and the bibliography in [4] for an overview of newer developments. To fix the notation, let $p, l, \pi, \lambda \in \mathbb{N}$. Then a (p_λ, l_π) *configuration* is a set of p points and l lines in the projective plane such that each point is incident to λ lines and each line is incident to π points. If $p = l$ and consequently $\lambda = \pi$, we just write (p_λ) instead of (p_λ, l_π).

Of particular interest are *elliptic configurations*, i.e., configurations whose points lie on an elliptic curve or, more specifically, in the torsion group of an elliptic curve. A long standing open problem is the question of Grünbaum [3, Section 4.8, Open problem 4]: For which integers r are there elliptic $(3r_4, 4r_3)$ configurations? Notice that for $r = 3$, the Hesse configuration $(9_4, 12_3)$ can be realized in the complex projective plane as the set of inflection points of an elliptic curve, but it has no realization with straight lines in the Euclidean or projective plane because of the Sylvester-Gallai theorem. On the other hand, examples of elliptic $(12_4, 16_3)$ configurations can be found in Grünbaum [3, p. 249], Coxeter [1, p. 440], and Feld [2] (where one can find also an example of an elliptic $(36_7, 84_3)$ configuration. For an elliptic $(24_6, 48_3)$ configuration see [7]. In [8], Metelka identified 8 elliptic $(12_4, 16_3)$ configurations.

Recently, some progress was reported in [4], where elliptic $((p-1)_3)$ configurations are constructed for every prime $p > 7$. Moreover there are $(3r_4, 4r_3)$ configurations whenever $3r = p - 1$ for some prime $p > 7$, and for every $k \geq 2$ there is an elliptic $(9k_4, 12k_3)$ configuration with D_3 symmetry (the symmetry group of an equilateral triangle).

In the present article we generalize the ideas in [4] and show that elliptic $(3r_4, 4r_3)$ configurations exist for all $r \geq 4$ and that one can even construct elliptic $(3r_s, sr_3)$ configurations for any $s \in \mathbb{N}$ and $r \geq s$. We offer constructions with D_1 symmetry (mirror symmetry), and, if r is a multiple of 3, D_3 symmetry. A particularly pleasing property is that the configurations are dynamic in the sense that one can move the points of the configurations along the elliptic curve in such a way, that all line incidences and rotational symmetries are preserved. Thus, Grünbaum's question is completely answered.

1.2 Elliptic curves

Recall that an elliptic curve in Weierstrass normal form

$$\Gamma_{g_2,g_3}:\ y^2 = 4x^3 - g_2 x - g_3$$

in $\mathbb{C}^2$ can be parametrized by the Weierstrass function $\wp(z) := \wp(z, g_2, g_3)$ via

$$\gamma : \mathbb{C} \to \bar{\mathbb{C}}^2, \quad z \mapsto \big(\wp(z), \wp'(z)\big).$$

For real g_2, g_3, the Weierstrass function has a real period ω_1 and an imaginary period ω_2. Then, depending on the parameters g_2, g_3, the curve Γ_{g_2,g_3} in $\mathbb{R}^2$ consists of one or two connected components. We will call the unbounded component *odd branch*, and the bounded component *even branch*. The odd branch is parametrized by

$$\gamma_{\text{odd}} : (0, \omega_1) \to \mathbb{R}^2, \quad t \mapsto \big(\wp(t), \wp'(t)\big),$$

and the even branch, if it exists, by

$$\gamma_{\text{even}}[0, \omega_1) \to \mathbb{R}^2, \quad t \mapsto \big(\wp(t + \omega_2/2), \wp'(t + \omega_2/2)\big)$$

(see Coxeter [1, p. 441]). The projective version of the curve Γ_{g_2,g_3} carries the group operation $\oplus$ of an elliptic curve with the neutral element $\mathscr{O} = (0, 1, 0)$ which corresponds to $\gamma(0)$. The group operation is compatible with the parametrization:

$$\gamma(z_1) \oplus \gamma(z_2) = \gamma(z_1 + z_2) \tag{1}$$

(see [9, Chapter VI, §3] for details).

Every regular cubic curve can be brought, by a projective transformation, into the D_3 symmetric normal form

$$\Gamma_{D_3}: \; x^3 - 3xy^2 - 3(b-3)(x^2+y^2) + 4b^2(b-9) \;=\; 0$$

with $b \in \mathbb{R} \setminus \{1\}$ (see [4]). Depending on the parameter b, the curve Γ_{D_3} consists only of the odd branch or of both, the odd and the even branch (see Figure 1). One of the three symmetry axes is the x-axis, and the rotational C_3 symmetry is with respect to the origin.

The configurations which we want to build will be constructed with the help of the arithmetic structure of $\Gamma_{g2,g3}$ and Γ_{D_3}, respectively. Let $Q := \gamma\big(\frac{q\omega_1}{n}\big)$ for some integer parameter q with $(q,n) = 1$. By (1), Q has order n and generates a group G_n which is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. To fix the notation, let

$$P_u := u * Q = \underbrace{Q \oplus \ldots \oplus Q}_{u \text{ times}} = \gamma\Big(\frac{uq\omega_1}{n}\Big),$$

and we can identify the point P_u on the curve with $u \in \mathbb{Z}/n\mathbb{Z}$. If we take, in addition, the points $P'_u := \gamma\big(\frac{uq\omega_1}{n} + \frac{\omega_2}{2}\big)$, we obtain a group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$: The points P_u correspond to $\{0\} \times \mathbb{Z}/n\mathbb{Z}$, and the points P'_u to $\{1\} \times \mathbb{Z}/n\mathbb{Z}$. In particular, we have

$$\begin{aligned} P'_{u_1} \oplus P'_{u_2} = \gamma\Big(\frac{qu_1\omega_1}{n} + \frac{\omega_2}{2}\Big) \oplus \gamma\Big(\frac{qu_2\omega_1}{n} + \frac{\omega_2}{2}\Big) = \\ = \gamma\Big(\frac{q(u_1+u_2)\omega_1}{n} + \omega_2\Big) = \gamma\Big(\frac{q(u_1+u_2)\omega_1}{n}\Big) = P_{u_1+u_2}. \end{aligned}$$

1.3 Notation

We identify points P_u on the odd branch of the curve directly with the corresponding value $(0,u)$ in $\{0\} \times \mathbb{Z}/n\mathbb{Z}$, and similarly, we identify $(1,u)$ with points P'_u on the even branch. We will also call the former *odd points* and the latter *even points*. When speaking of any point u (without specifying the branch), we mean a point on any component of the curve or the corresponding element in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Finally, instead of writing $\oplus$ we just write $+$ from now on.

If the points u, v, w are collinear, then the corresponding line is denoted by $[u,v,w]$. Three distinct points u, v, w on the elliptic curve are collinear if and only if their sum (interpreted as sum on the elliptic curve or in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$) is zero, and we just write $u + v + w = 0$.

Our configurations will only consist of lines that contain 3 points and hence are of the form $[(0,u),(0,v),(0,w)]$ or $[(0,u),(1,v),(1,w)]$ (where the order of points could be changed). We call the former *0-lines* and the latter 1-*lines*.

1.4 Operations on lines

We introduce three operations on lines: A conjugation, a rotation, and a translation.

If $[u, v, w]$ is a line, then $-[u, v, w] := [-u, -v, -w]$ is the *conjugate line* of $[u, v, w]$. In the context of a configuration on a Γ_{D_3} curve, the line and the conjugate line are mirror symmetric with respect to the x-axis. Hence, an elliptic configuration on a Γ_{D_3} curve is mirror symmetric with respect to the x-axis, if each of its conjugate lines is an element of the configuration.

Let $k \in \mathbb{Z}$. For any point u in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ we define

$$\rho_k(u) := u + (0, k).$$

If $[u, v, w]$ is a line with points in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3k\mathbb{Z}$ for some positive integer k, we can define the *rotated line*

$$\rho_k([u, v, w]) := [\rho_k(u), \rho_k(v), \rho_k(w)].$$

Note that this indeed yields a line, since $\rho_k(u) + \rho_k(v) + \rho_k(w) = u + v + w + (0, 3k)$. In the context of a configuration on a Γ_{D_3} curve, this rotation corresponds to a rotation with angle $2\pi/3$ about the origin. So, an elliptic configuration on a Γ_{D_3} curve has rotational C_3 symmetry with respect to the origin, if each of its rotated lines is an element of the configuration.

If $S_t = \gamma_{\text{odd}}(t)$ is a point on the odd branch of an elliptic curve and ε is an arbitrary real number, then the *translated point* S_t^ε is defined by $S_t^\varepsilon := \gamma_{\text{odd}}(t - 2\varepsilon)$. Similarly, if $S'_t = \gamma_{\text{even}}(t)$ is a point on the even branch, the translated point is defined by $S'^{\varepsilon}_t := \gamma_{\text{even}}(t + \varepsilon)$. If the three points $S_{t_1}, S'_{t_2}, S'_{t_3}$ form a 1-line, then the translated 1-line is given by the points $S^\varepsilon_{t_1}, S'^\varepsilon_{t_2}, S'^\varepsilon_{t_3}$. Indeed these three points are on a line, since

$$\begin{aligned} S^\varepsilon_{t_1} \oplus S'^\varepsilon_{t_2} \oplus S'^\varepsilon_{t_3} = \gamma(t_1 - 2\varepsilon) \oplus \gamma\Big(t_2 + \varepsilon + \frac{\omega_2}{2}\Big) \oplus \gamma\Big(t_3 + \varepsilon + \frac{\omega_2}{2}\Big) = \\ = \gamma(t_1 + t_2 + t_3 + \omega_2) = S_{t_1} \oplus S'_{t_2} \oplus S'_{t_3} = 0. \end{aligned}$$

2 Simple elliptic $(3r_s, sr_3)$ configurations

Definition 1. *Let r and s be positive integers. A simple elliptic $(3r_s, sr_3)$ configuration has r points on the odd branch and $2r$ points on the even branch.*

The reason for the name "simple" is that a simple configuration only consists of 1-lines: Indeed, if there are l_0 0-lines and l_1 1-lines, then the number of odd points is $3l_0 + l_1$ while the number of even points is $2l_1$. So, for a simple configuration we have

$$2l_1 = 2(3l_0 + l_1) \Rightarrow l_0 = 0.$$

We will also see that simple configurations are given by quite simple constructions.

Definition 2. *An elliptic configuration is called dynamic, if one can move the points along the curve in such a way, that all line incidences are preserved.*

In particular, an elliptic configuration is dynamic if every point of the curve is a point of a configuration of the same type.

Lemma 3. *An elliptic configuration which only consists of* 1*-lines is dynamic.*

Proof. We can apply for an arbitrary real ε the corresponding translation as defined in Section 1.4 simultaneously to all points of the configuration. Then, every 1-line is translated to the 1-line through the translated points. *q.e.d.*

Below, we will prove the following result.

Theorem 4. *For all $r \geq s \geq 1$ there exists a simple $(3r_s, sr_3)$ configuration with D_3 symmetry if $r \equiv 0 \pmod 3$ and D_1 symmetry otherwise. These configurations are dynamic. The dynamic versions of D_3 symmetric configurations have rotational C_3 symmetry.*

We first consider the case when $s = 4$.

2.1 Construction of simple $(3r_4, 4r_3)$ configurations

We start by constructing elliptic $(3r_4, 4r_3)$ configurations for $r \geq 4$. If $3 \mid r$, then these configurations will have D_3 symmetry.

Let $r \geq 4$. The following points are understood to be in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2r\mathbb{Z}$. The set of configuration points is $\mathcal{P} := \mathcal{P}_0 \cup \mathcal{P}_1$, where

$$\mathcal{P}_0 := \big\{(0,1), (0,3), (0,5), \ldots, (0, 2r-1)\big\},$$
$$\mathcal{P}_1 := \big\{(1,0), (1,1), (1,2), \ldots, (1, 2r-1)\big\}.$$

$\mathcal{P}$ contains $r + 2r = 3r$ points as needed. As mentioned previously, we only use 1-lines. Define $\mathcal{L}_1$ to be the set of the following lines:

$$[a_0, b_0, c_0] := [(0,-1), (1,0), (1,1)]$$
$$[a_{i+1}, b_{i+1}, c_{i+1}] := [a_i - (0,2), b_i + (0,1), c_i + (0,1)] \quad \text{for } i \in \{0, \ldots, 2r-2\}$$

and let $\mathcal{L}_2$ contain the lines

$$[d_0, e_0, f_0] := [(0,-3), (1,0), (1,3)]$$
$$[d_{i+1}, e_{i+1}, f_{i+1}] := [d_i - (0,2), e_i + (0,1), f_i + (0,1)] \quad \text{for } i \in \{0, \ldots, 2r-2\}.$$

Observe that

$$[a_i, b_i, c_i] = [(0, -1 - 2i), (1, i), (1, 1 + i)],$$
$$[d_i, e_i, f_i] = [(0, -3 - 2i), (1, i), (1, 3 + i)],$$

where all numbers are to be read modulo $2r$. We claim that $\mathcal{L}_1 \cup \mathcal{L}_2$ yields appropriate lines for a $(3r_4, 4r_3)$ configuration. This can be seen by considering the following points:

- The elements of $\mathcal{L}_1 \cup \mathcal{L}_2$ define 1-lines in $\mathcal{P}$: All a_i and d_i are odd points and lie in $\mathcal{P}$. All b_i, c_i, e_i, f_i are even points and lie in $\mathcal{P}$. Furthermore, $a_i + b_i + c_i = 0 = d_i + e_i + f_i$ and no two points of a line are equal.
- All lines are different: If $a_i = a_j$ for $i \neq j$, then the sets $\{b_i, c_i\}$ and $\{b_j, c_j\}$ are different. So, all lines in $\mathcal{L}_1$ are different. Similarly, all lines in $\mathcal{L}_2$ are different. Moreover, the lines in $\mathcal{L}_1$ are different from those in $\mathcal{L}_2$ since the differences of the even points on a line are $(0, 1)$ and $(0, 3)$ respectively. Note that for $r = 3$ we would obtain two equal lines since then $[d_0, e_0, f_0] = [(0, -3), (1, 0), (1, 3)] = [d_3, f_3, e_3]$.
- Each point in $\mathcal{P}_0$ occurs twice in the set of the a_i and twice in the set of the d_i. And the lines of both $\mathcal{L}_1$ and $\mathcal{L}_2$ contain each point in $\mathcal{P}_1$ twice.

Concerning the symmetry of the configuration, observe the following: For $i \in \mathbb{Z}$, we have that

$$-[a_i, b_i, c_i] = [(0, 2i + 1), (1, -i), (1, -i - 1)] = [a_{-i-1}, c_{-i-1}, b_{-i-1}]$$

taking all indices modulo $2r$. Therefore the lines in $\mathcal{L}_1$ form a D_1 symmetry. Similarly, this can be seen to be true for the lines in $\mathcal{L}_2$, where $-[d_i, e_i, f_i] \equiv [d_{-i-3}, f_{-i-3}, e_{-i-3}]$ for all $i \in \mathbb{Z}$.

Furthermore, we even obtain a D_3 symmetry for these configurations if r is a multiple of 3. To see this, let $n := \frac{2r}{3}$ and notice that since $n \equiv -2n \pmod{2r}$ we have

$$\rho_n[a_i, b_i, c_i] = [(0, n - 2i - 1), (1, i + n), (1, i + n + 1)] = [a_{i+n}, b_{i+n}, c_{i+n}],$$
$$\rho_n[d_i, e_i, f_i] = [(0, n - 2i - 3), (1, i + n), (1, i + n + 3)] = [d_{i+n}, e_{i+n}, f_{i+n}].$$

Since the configuration is simple, it follows from Lemma 3 that it is dynamic. Also observe that if three points on Γ_{D_3} are vertices of an equilateral triangle centered at the origin, then the translated points have also this property. So, when we translate a D_3 symmetric elliptic configuration on a Γ_{D_3} curve, there results an elliptic configuration with rotational C_3 symmetry.

This completes the proof of Theorem 4 for $s = 4$. Figure 1 shows the smallest simple D_3 symmetric configuration according to this construction, while Figure 2 is a translated version of it. Figure 3 shows a simple D_1 symmetric $(33_4, 44_3)$ configuration.

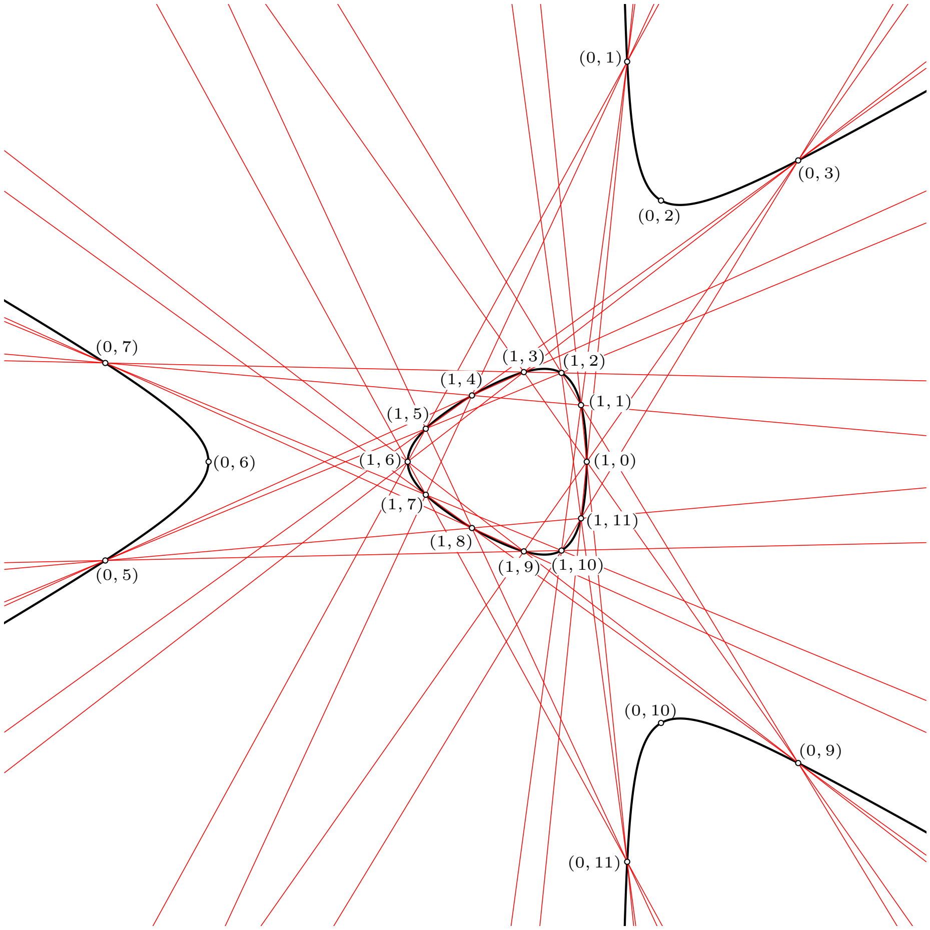

Figure 1: Simple D_3-symmetric $(18_4, 24_3)$ configuration, with parameter $q = 1$.

2.2 Simple $(3r_s, sr_3)$ configurations for arbitrary $r \geq s \geq 1$

Now, we generalize the construction of the previous section to $(3r_s, sr_3)$ configurations where $r \geq s \geq 1$. For the proof we have to distinguish two cases which we consider separately in the following subsections. The first subsection covers the case $r \geq s \geq 1$ with r odd or $r = s$ even, the second subsections treats the case $r > s \geq 1$ with r even.

2.2.1 r odd or $r = s$ even

Let $r \geq s \geq 1$ with r odd or $r = s$ even. We use the same set of points $\mathcal{P} := \mathcal{P}_0 \cup \mathcal{P}_1$ as in Section 2.1. For $1 \leq j \leq \lceil \frac{s-1}{2} \rceil =: \bar{s}$ let $\mathcal{L}^j$ be the set consisting of the following lines:

$$[a_0^j, b_0^j, c_0^j] := [(0, 1 - 2j), (1, 0), (1, 2j - 1)],$$

$$[a_{i+1}^j, b_{i+1}^j, c_{i+1}^j] := [a_i^j - (0, 2), b_i^j + (0, 1), c_i^j + (0, 1)] \quad \text{for } i \in \{0, \dots, 2r - 2\},$$

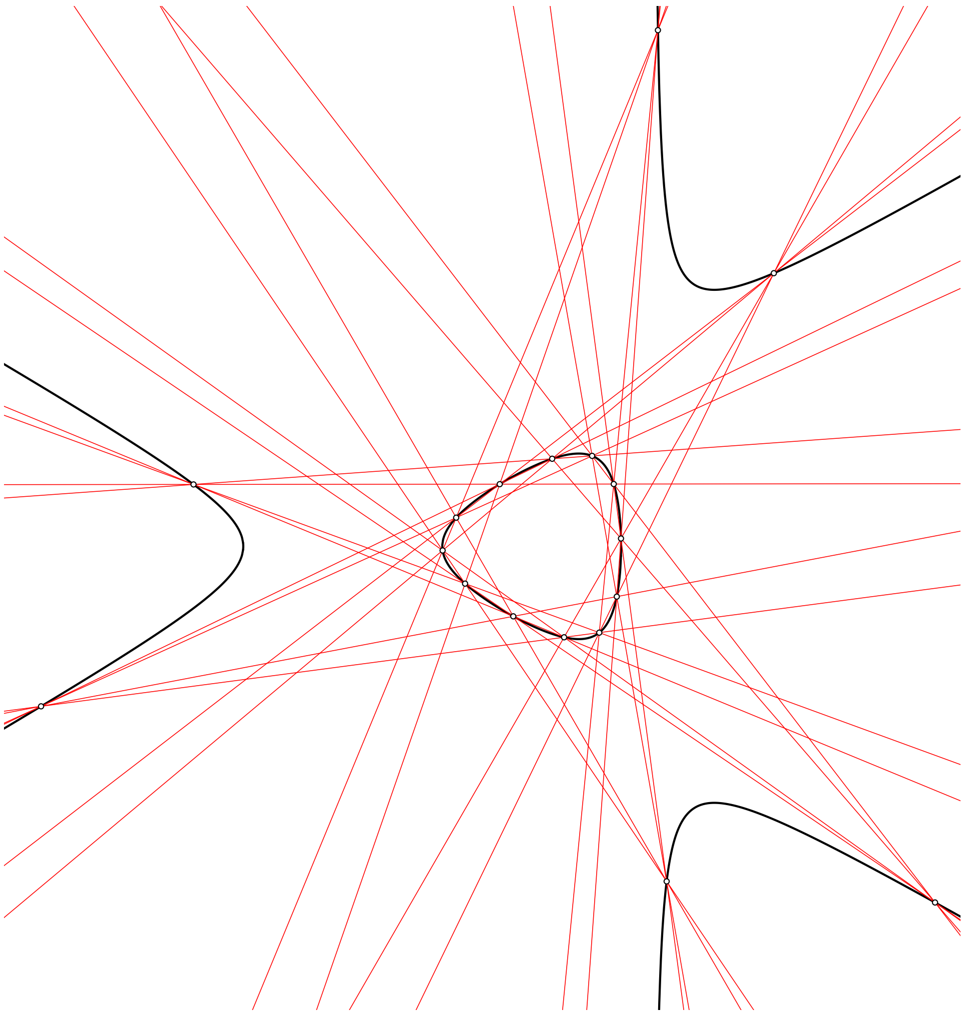

Figure 2: A translated version of the simple D_3-symmetric $(18_4, 24_3)$ configuration in Figure 1.

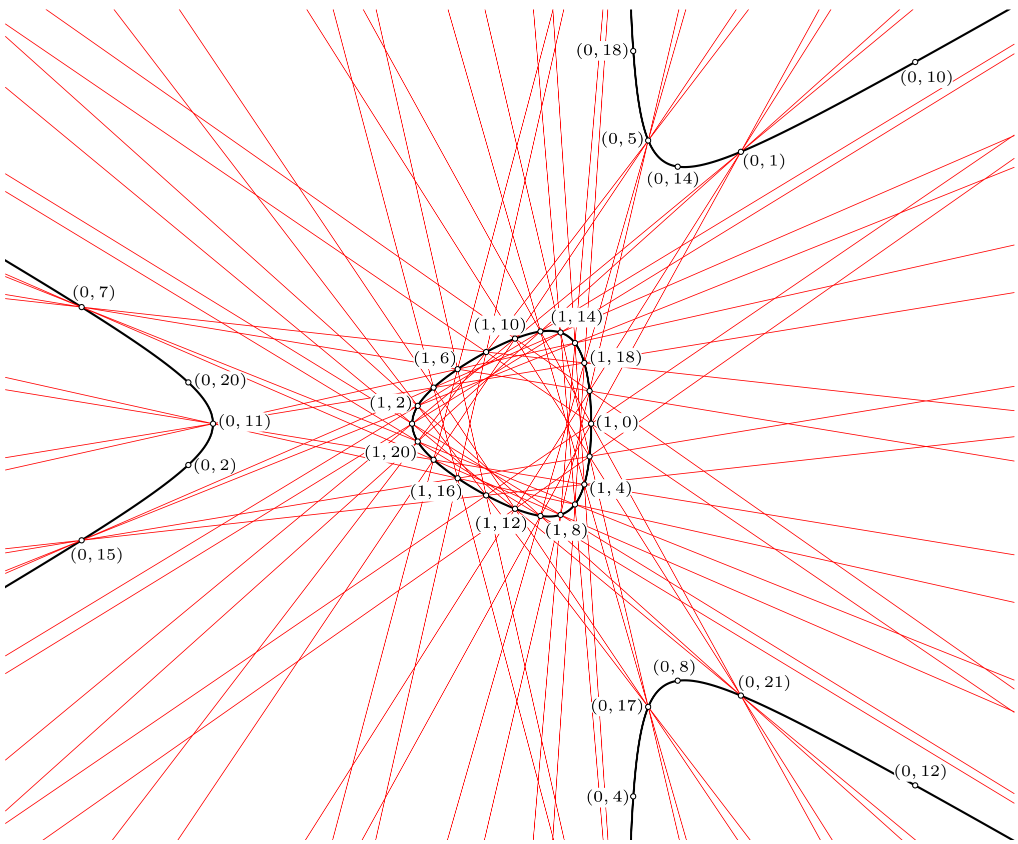

Figure 3: Simple D_1-symmetric $(33_4, 44_3)$ configuration, with parameter $q = 5$.

and $\mathcal{L}^*$ consist of

$$[d_0, e_0, f_0] := [(0, -r), (1, 0), (1, r)]$$
$$[d_{i+1}, e_{i+1}, f_{i+1}] := [d_i - (0, 2), e_i + (0, 1), f_i + (0, 1)] \quad \text{for } i \in \{0, \ldots, r-2\}.$$

Observe that

$$[a_i^j, b_i^j, c_i^j] = [(0, 1-2j-2i), (1, i), (1, 2j+i-1)],$$
$$[d_i, e_i, f_i] = [(0, -r-2i), (1, i), (1, r+i)].$$

When s is even, we claim that $\mathcal{L} := \bigcup_{j=1}^{\bar{s}} \mathcal{L}^j$ yields appropriate lines for a $(3r_s, sr_3)$ configuration, and if s is odd, then $\mathcal{L} := \mathcal{L}^* \cup \bigcup_{j=1}^{\bar{s}} \mathcal{L}^j$ does. To see this, we check the following:

- The elements of $\mathcal{L}$ define 1-lines in $\mathcal{P}$: All a_i^j and d_i are odd and lie in $\mathcal{P}$ and so do the even points b_i^j, c_i^j, e_i^j and f_i^j. Furthermore, $a_i^j + b_i^j + c_i^j = 0 = d_i + e_i + f_i$ for all i, j, and no two points of a line are equal.
- All lines are different: For different j, the difference between b_i^j and c_i^j is different, and different from the difference between e_i and f_i. For fixed j all sets $\{b_i^j, c_i^j\}$ are different. The same can be checked for the sets $\{e_i, f_i\}$.
- If s is odd, each point in $\mathcal{P}_0$ occurs $s-1$ times in the set of the a_i^j and once in the set of the d_i. The lines of all $\mathcal{L}^j$ and $\mathcal{L}^*$ together contain each point in $\mathcal{P}_1$ exactly s times. Similarly, when s is even, each point in $\mathcal{P}$ is contained in s lines.

To see the D_1 symmetry, observe that

$$-[a_i^j, b_i^j, c_i^j] = [(0, 2i+2j-1), (1, -i), (1, 1-i-2j)] = [a_{1-2j-i}^j, c_{1-2j-i}^j, b_{1-2j-i}^j]$$

where indices are taken modulo $2r$. Hence the lines in $\mathcal{L}^j$ are D_1 symmetric. Similarly, we have $-[d_i, e_i, f_i] = [d_{r-i}, f_{r-i}, e_{r-i}]$.

To see that D_3 symmetry occurs when r is a multiple of 3, let $n := \frac{2r}{3}$ and notice that

$$\rho_n[a_i^j, b_i^j, c_i^j] = [(0, 1-2j-2i+n), (1, i+n), (1, i+n+2j-1)] = [a_{i+n}^j, b_{i+n}^j, c_{i+n}^j]$$

and

$$\rho_n[d_i, e_i, f_i] = [(0, -r-2i-2n), (1, i+n), (1, i+n+r)] = [d_{i+n}, e_{i+n}, f_{i+n}].$$

Finally, since also in the present construction only 1-lines are used, the $(3r_s, sr_3)$ configurations are dynamic.

2.2.2 r even and $r > s \geq 1$

Let $r > s \geq 1$ with r even and define

$$\mathcal{P}_0 := \{(0,0),(0,2),\dots,(0,2r-2)\} \quad \text{and} \quad \mathcal{P}_1 := \{(1,0),(1,1),\dots,(1,2r-1)\}$$

and $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$. For $1 \leq j \leq \lceil \frac{s-1}{2} \rceil =: \bar{s}$ let $\mathcal{L}^j$ be the set of the following lines:

$$\begin{aligned}
[a_0^j, b_0^j, c_0^j] &:= [(0,-2j),(1,0),(1,2j)],\\
[a_{i+1}^j, b_{i+1}^j, c_{i+1}^j] &:= [a_i^j - (0,2), b_i^j + (0,1), c_i^j + (0,1)] \quad \text{for } i \in \{0,\dots,2r-2\},
\end{aligned}$$

and $\mathcal{L}^*$ consist of the lines

$$\begin{aligned}
[d_0, e_0, f_0] &:= [(0,-r),(1,0),(1,r)],\\
[d_{i+1}, e_{i+1}, f_{i+1}] &:= [d_i - (0,2), e_i + (0,1), f_i + (0,1)] \quad \text{for } i \in \{0,\dots,r-2\}.
\end{aligned}$$

Observe that

$$\begin{aligned}
[a_i^j, b_i^j, c_i^j] &= [(0,-2j-2i),(1,i),(1,2j+i)]\\
[d_i, e_i, f_i] &= [(0,-r-2i),(1,i),(1,r+i)].
\end{aligned}$$

It can be checked very similarly as in previous section that $\mathcal{L} := \bigcup_{j=1}^{\bar{s}} \mathcal{L}^j$ if s is even, and $\mathcal{L} := \mathcal{L}^* \cup \bigcup_{j=1}^{\bar{s}} \mathcal{L}^j$ if s is odd, yield appropriate lines for a $(3r_s, rs_3)$-configuration. Again we obtain D_1 symmetry for all such configurations and D_3 symmetry for $r \equiv 0 \pmod 3$.

Note that here we use the point $(0,0)$ at infinity in the D_1 symmetric case, and the points $(0,0),(0,\frac{2r}{3})$ and $(0,\frac{4r}{3})$ at infinity in the D_3 symmetric case. Since the configuration is dynamic, we can still obtain a D_3 symmetric configuration with all points finite if we choose $\varepsilon = \frac{3\omega_1}{4r}$ for the translation. Figure 4 shows such a situation for $r = 12, s = 5$ and parameter $q = 5$.

3 Closing remarks and open problems

For a fixed curve Γ_{D_3} and fixed numbers $r \geq s \geq 1$ there is still some freedom in our construction of a $(3r_s, sr_3)$ configuration: We can choose the parameter $\varepsilon \in \mathbb{R}$ (see Section 1.4) and the parameter $q \in \mathbb{N}$ (see Section 1.2). However the resulting configurations are combinatorially isomorphic: There is a bijection of the points which preserves the line incidences. But choosing a different ε leads in general to configurations which are geometrically non-isomorphic: there is no projective map which maps

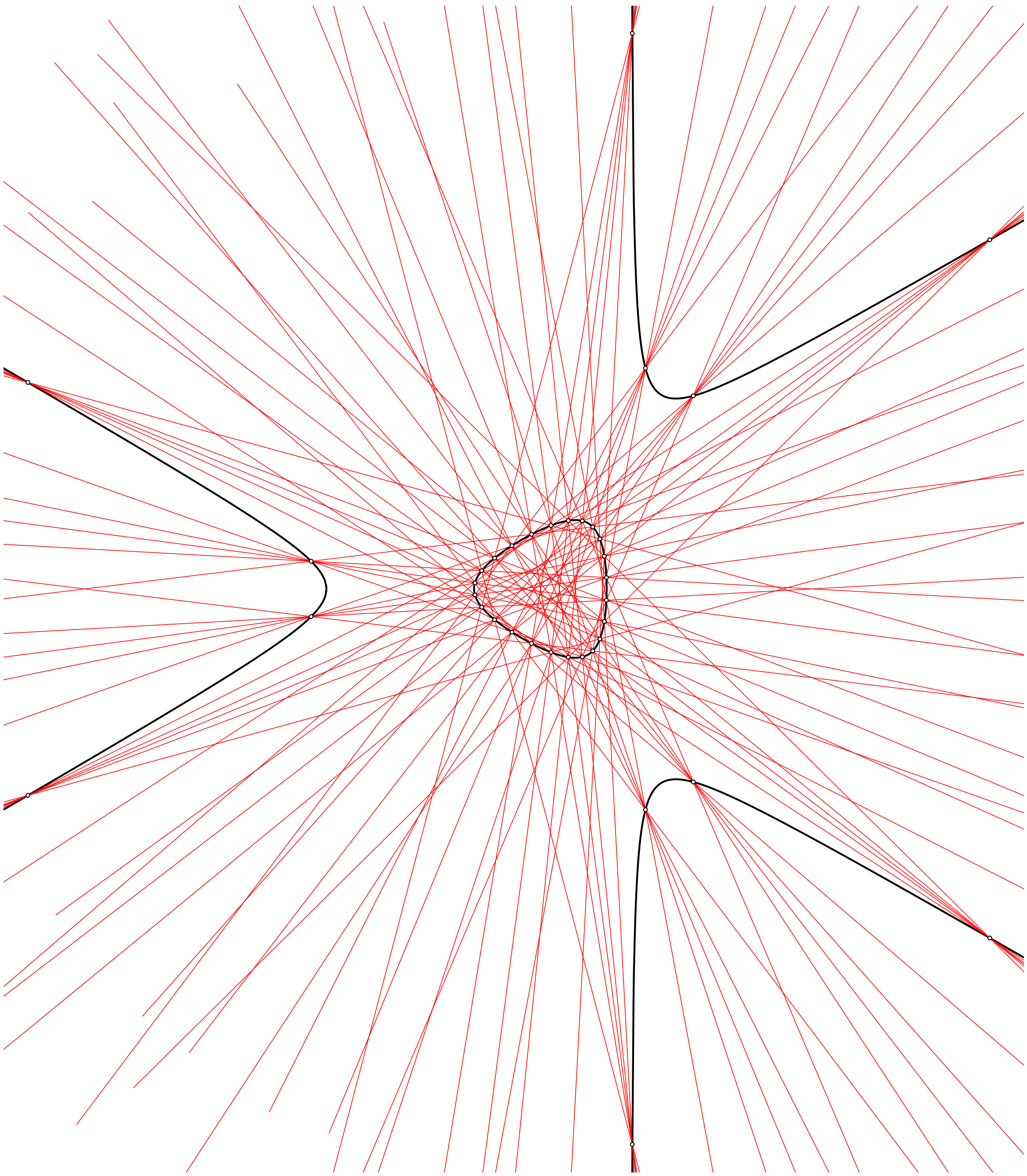

Figure 4: A D_3 symmetric $(36_5, 60_3)$ configuration, with parameter $q = 5$.

one configuration to the other. Indeed, the configuration in Figure 4 with $\varepsilon = \frac{3\omega_1}{4r}$ cannot be geometrically isomorphic to the corresponding configuration with $\varepsilon = 0$, since the latter occupies the three inflection points of the curve (at infinity), while the former does not. Also the choice of different values for the parameter q leads in general to geometrically non-isomorphic configurations, since a projective map preserves the order of the points on both branches of the curve, while different values of q lead in general to a different order of the points (compare Figure 1 and Figure 3).

We conclude with two open problems.

- Does every $(3r_s, sr_3)$ configuration of the combinatorial type constructed in Section 2 have its points necessarily on a cubic curve?
- Which curves other than cubic curves carry dynamic configurations?

Observe that the miraculous chains of Poncelet polygons introduced in [5] and [6] are examples of dynamic configurations carried by conics. So far, only a (9_3) configuration consisting of three Poncelet triangles, and a (24_3) configuration consisting of six Poncelet quadrilaterals are known.

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