Building Loops with classical Lego Train Tracks

Lorenz Halbeisen

Department of Mathematics, ETH Zentrum, Rämistrasse 101, 8092 Zürich, Switzerland

lorenz.halbeisen@math.ethz.ch

Norbert Hungerbühler

Department of Mathematics, ETH Zentrum, Rämistrasse 101, 8092 Zürich, Switzerland

norbert.hungerbuehler@math.ethz.ch

key-words: irrationality, independence over a field, Morse index

2020 Mathematics Subject Classification: **11J72**

1 The geometry of classical Lego Train Tracks

In January 2023, Werner Durandi posed the following competition task to his students at the Kollegium St. Fidelis: *Is it possible to build a loop with classical Lego™ train tracks which starts and ends exactly at a switch, and in case this is not possible, what would be the best solution with the minimum deviation?* See Figure 1.

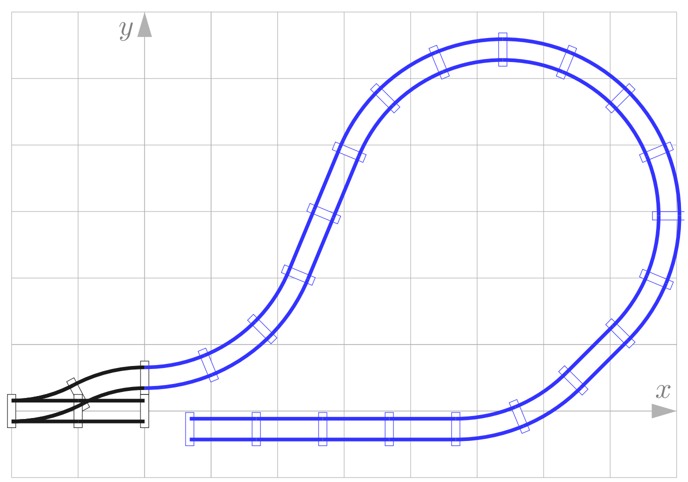

Figure 1: Is there a Lego loop which closes exactly?

In Section 2 we prove that there is no such *Durandi loop* which closes exaclty. Furthermore, in Section 3 we provide a few loops that start and end at a switch which are not perfect, but are sufficiently precise for practice. Finally, in Section 4 we show that the deviation can actually be made arbitrarily small.

Before we can show that there are no Durandi loops, we have to specify the geometry of the classical blue Lego train tracks, which were on the market until 1980 — it is worth mentioning that the modern Lego train tracks have a different geometry, in particular the switches.

Figure 2 shows the geometry of the straight and the curved tracks, and of the switches.

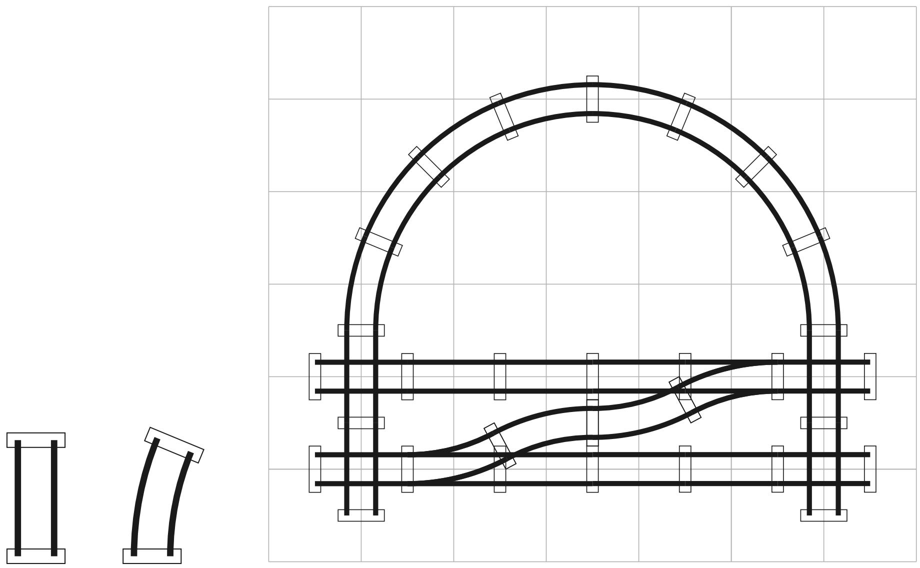

Figure 2: A straight track piece has the unit length 1. A full circle built form 16 curved track pieces has diameter 5, and the two parallel tracks of a switch have distance $\frac{1}{2}$.

If we normalize the length of a straight track to 1, then the diameter of a circle is 5 and the distance between the two parallel tracks of a switch is $\frac{1}{2}$.

We divide the curved tracks into 16 different types as indicated in Figure 3.

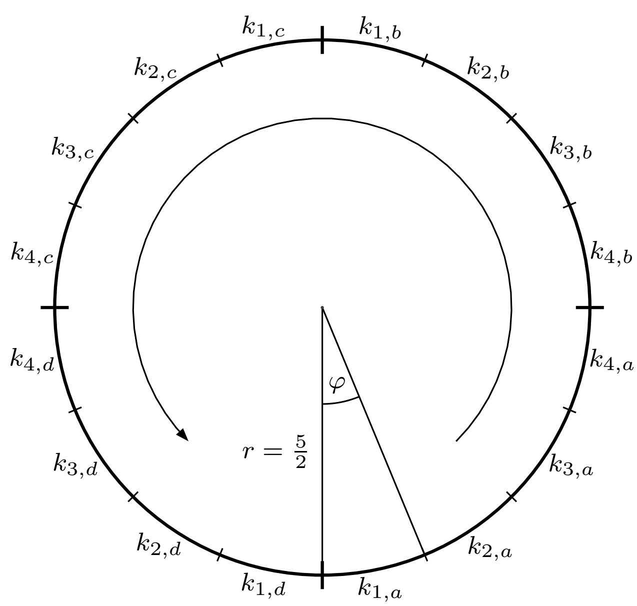

Figure 3: Circle segments, $\varphi = \frac{\pi}{8}$.

In the following two Figures 4 and 5 we define the x and y increments of the different types of curved and straight tracks.

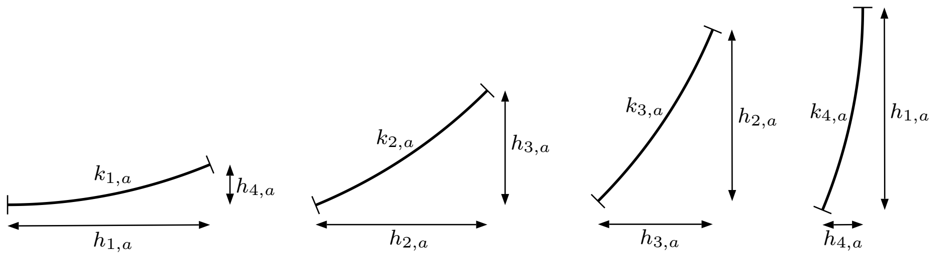

Figure 4: x and y increments of the circle segments.

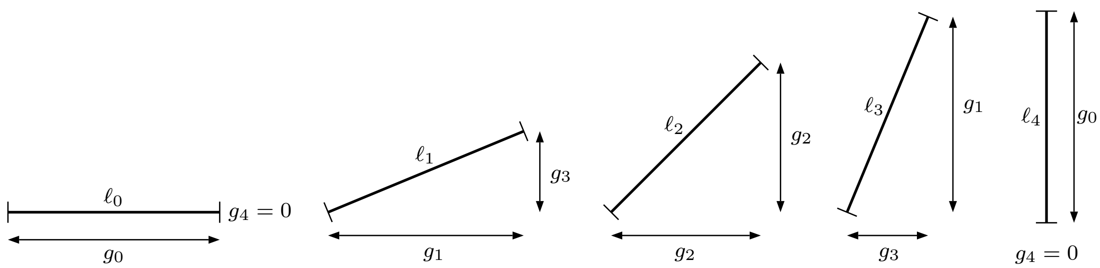

Figure 5: x and y increments of the straight segments in the direction of angles $k\varphi$, $k = 0, \dots, 4$.

If we travel along a Durandi loop, our x and y coordinates change as we travel along a single curved or straight track by exactly the amounts g_j or $h_{j,t}$. The increment is positive or negative depending on the type of the track and the direction of travel. The corresponding values can easily be calculated concretely if we use that for $\varphi = \frac{\pi}{8}$ we have $\sin(\varphi) = \frac{1}{2}\sqrt{2-\sqrt{2}}$ and $\cos(\varphi) = \frac{1}{2}\sqrt{2+\sqrt{2}}$. We obtain:

$$\begin{aligned} h_{1,a} &= \frac{5}{4}\left(\sqrt{2-\sqrt{2}}\right) & \qquad h_{2,a} &= \frac{5}{4}\left(\sqrt{2}-\sqrt{2-\sqrt{2}}\right) \\ h_{3,a} &= \frac{5}{4}\left(-\sqrt{2}+\sqrt{2+\sqrt{2}}\right) & \qquad h_{4,a} &= \frac{5}{4}\left(2-\sqrt{2+\sqrt{2}}\right) \end{aligned}$$

Furthermore, for the straight tracks we get:

$$g_0 = 1 \qquad g_1 = \frac{\sqrt{2+\sqrt{2}}}{2} \qquad g_2 = \frac{\sqrt{2}}{2} \qquad g_3 = \frac{\sqrt{2-\sqrt{2}}}{2}$$

The following fact, which allows us to write $h_{1,a}, h_{2,a}, h_{3,a}, h_{4,a}$ in terms of g_0, g_1, g_2, g_3, can be easily verified:

Fact 1.

$$h_{1,a} = \tfrac{5}{2}\, g_3, \quad h_{2,a} = \tfrac{5}{2}\,(g_2 - g_3), \quad h_{3,a} = \tfrac{5}{2}\,(g_1 - g_2), \quad h_{4,a} = \tfrac{5}{2}\,(g_0 - g_1)\,.$$

2 There are no Durandi loops

Before we show that there are no Durandi loops, we prove the following two auxiliary results.

Lemma 2. *Let V be the set*

$$\{q \cdot g_0 + r \cdot g_1 + s \cdot g_2 + t \cdot g_3 : \; q, r, s, t \in \mathbb{Q}\}.$$

Then $(V, +)$ is a 4-dimensional vector space over $\mathbb{Q}$ with basis $\{g_0, g_1, g_2, g_3\}$.

Proof. It is obvious that V is a vector space over $\mathbb{Q}$. So, it remains to show that the real numbers g_0, g_1, g_2, g_3 are linearly independent over $\mathbb{Q}$.

First we show that g_0, g_1, g_2 are linearly independent over $\mathbb{Q}$: For this, assume towards a contradiction that there are $\tilde{p}, \tilde{q}, \tilde{r} \in \mathbb{Q}$ with $(\tilde{p}, \tilde{q}, \tilde{r}) \neq (0,0,0)$, such that

$$\tilde{p} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} + \tilde{q} \cdot \frac{\sqrt{2}}{2} = \tilde{r}.$$

By multiplying by the product of the denominators of $\frac{\tilde{p}}{2}$, $\frac{\tilde{q}}{2}$ and $\tilde{r}$, we find $p, q, r \in \mathbb{Z}$ such that

$$p\sqrt{2+\sqrt{2}} + q\sqrt{2} = r.$$

This leads to the following sequence of equations:

$$\begin{aligned} p\sqrt{2+\sqrt{2}} &= r - q\sqrt{2} \\ p^2\,(2+\sqrt{2}) &= r^2 - 2\,rq\sqrt{2} + 2\,q^2 \\ \sqrt{2}\,(p^2 + 2\,rq) &= r^2 + 2\,q^2 - 2\,p^2 \end{aligned}$$

Since $\sqrt{2}$ is irrational, the last equation holds only in the case when

$$p^2 + 2\,rq = 0 \qquad \text{and} \qquad r^2 + 2\,q^2 - 2\,p^2 = 0.$$

This gives us $r^2 + 4\,rq + 2\,q^2 = 0$, and therefore

$$r = -2\,q \pm q\sqrt{2},$$

which shows that $r \notin \mathbb{Z}$.

Now, we show that g_3 is linearly independent of g_0, g_1, g_2 over $\mathbb{Q}$: For this, assume towards a contradiction that there are $p, q, r \in \mathbb{Q}$ such that

$$p\sqrt{2+\sqrt{2}} + q\sqrt{2} = r + \sqrt{2-\sqrt{2}}.$$

Notice that

$$(\sqrt{2}-1)\sqrt{2+\sqrt{2}} = \sqrt{2-\sqrt{2}}$$

which can be easily verified. Similarly as above, we get the following sequence of equations:

$$\begin{aligned} p\sqrt{2+\sqrt{2}} + q\sqrt{2} &= r + \left(\sqrt{2}-1\right)\sqrt{2+\sqrt{2}} \\ \sqrt{2+\sqrt{2}} &= \frac{r - q\sqrt{2}}{1 + p - \sqrt{2}} \\ \sqrt{2+\sqrt{2}} &= \frac{r - q\sqrt{2}}{1 + p - \sqrt{2}} \cdot \frac{1 + p + \sqrt{2}}{1 + p + \sqrt{2}} \\ \sqrt{2+\sqrt{2}} &= \frac{pr + r - 2\,q + \sqrt{2}\,(r - pq - q)}{p^2 + 2\,p - 1} \end{aligned}$$

Thus, we have

$$\sqrt{2+\sqrt{2}} \quad = \quad s + t\sqrt{2} \qquad \text{for some } s, t \in \mathbb{Q},$$

which is a contradiction to the linear independence of 1, $\sqrt{2}$, and $\sqrt{2+\sqrt{2}}$. ∎

We will now introduce a modified version of the Morse index for parameterized plane curves. The original literal definition of the Morse index for a closed plane curve holds no more information than the number of its local minima, which is equal to the number of its local maxima (see, e.g., [1, §2]). By a slight modification, which also takes into account the direction of the parameterization, we obtain an index that provides more information about the curve, which will be needed later in the proof of the main Theorem 9.

Definition 3. *Let $\gamma : [a,b] \to \mathbb{R}^2$, $t \mapsto \big(\gamma_1(t), \gamma_2(t)\big)$, be a parametrized C^1 curve with the following properties:*

- *The curve is regular, i.e., $\dot\gamma(t) \neq 0$ for all t.*
- *The set $\mathrm{crit}(\gamma) := \{t \in [a,b] : \dot\gamma_2(t) = 0\}$ consist of finitely many critical points or intervals.*
- *If $[a,b] \subset \mathrm{crit}(\gamma)$ is a point ($a = b$) or a maximal interval ($a < b$) in $\mathrm{crit}(\gamma)$, then γ is C^2 on $(a-\varepsilon, a)$ and $(b, b+\varepsilon)$ for some $\varepsilon > 0$ and $\ddot\gamma_2 \neq 0$ on both of these intervals.*

Let $c := [a,b]$ as above. Then we associate to c the Morse index $+1$ *if $\dot\gamma_1 < 0$ in c and $\ddot\gamma_2 < 0$ on both sides of c or if $\dot\gamma_1 > 0$ in c and $\ddot\gamma_2 > 0$ on both sides of c. We associate to c the* Morse index -1 *if $\dot\gamma_1 > 0$ in c and $\ddot\gamma_2 < 0$ on both sides of c or if $\dot\gamma_1 < 0$ in c and $\ddot\gamma_2 > 0$ on both sides of c. We associate the* Morse index 0 *to c if $\ddot\gamma_2$ changes sign when passing through c (see Figure 6).*

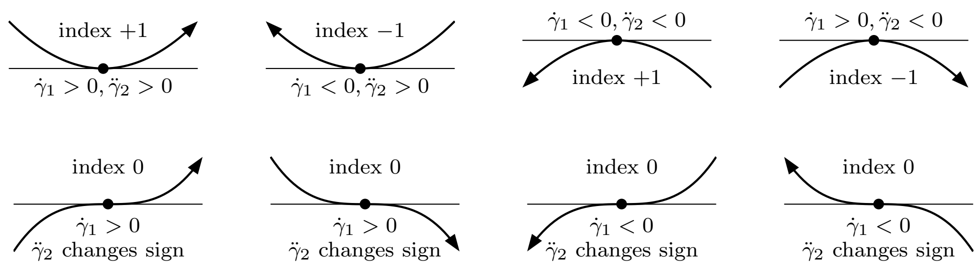

Figure 6: Morse index for parametrized curves.

In the proof of Theorem 9 we will use the following property of the Morse index from Definition 3.

Proposition 4. *The sum of the Morse indices of a closed curve γ is even.*

Proof. In a first step, we deform the curve continuously in such a way that all critical intervals shrink to a point without changing the sum of the indices (see Figure 7).

In a second step, we can remove all critical points of index 0 as indicated in Figure 8. Again, this does not change the sum of the indices.

Now, observe that if we travel along the curve in the direction which is given by the parametrization, then local maxima and local minima necessarily alternate. This can happen in only two essentially different ways, as indicated in Figure 9. In the first situation, the sum of the two corresponding indices is 0, in the other case it is ± 2, depending on the direction of travel. Therefore the total sum must be an even number. ∎

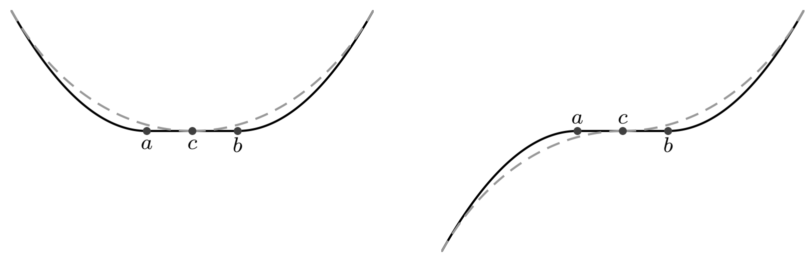

Figure 7: Shrink a critical interval $[a, b]$ to a point c: Replace the black original curve by the modified dashed curve.

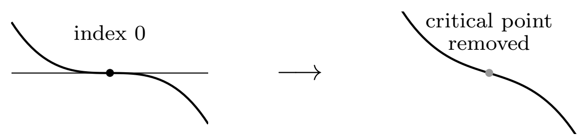

Figure 8: Removing critical points of index 0.

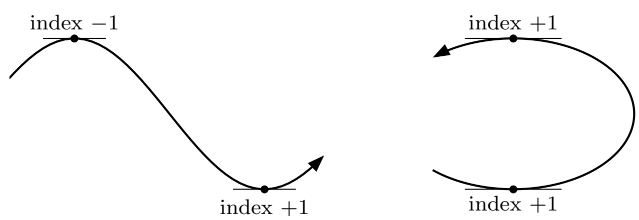

Figure 9: Alternating critical points.

Remark 5. *From the proof above it is clear that the index sum for a parametrized closed curve equals twice its turning number. However, we will not need this property.*

The fact that the sum of the Morse indices of a parametrized closed curve is even is crucial for the proof of the next lemma, which in turn is an essential brick in the proof of the main Theorem 9. In order to formulate and prove the next lemma, we have to introduce some notation.

When we travel by train along the Lego tracks, we may travel over 24 different types of tracks, namely $k_{1,a}, \ldots, k_{4,a}$, $k_{1,b}, \ldots, k_{4,b}$, $k_{1,c}, \ldots, k_{4,c}$, $k_{1,d}, \ldots, k_{4,d}$, $\ell_0, \ldots, \ell_4$, or mirrored versions of ℓ_1, ℓ_2, ℓ_3. If we travel along a track of type $k_{j,t}$ (where $j \in \{1, \ldots, 4\}$ and $t \in \{a, b, c, d\}$) in positive direction (i.e., counter-clockwise, see Figure 3) we denote it by $k^+_{j,t}$, otherwise, if we pass it in negative direction (i.e., clockwise) we denote it by $k^-_{j,t}$. Let $n^+_{j,t}$ be the number of curved tracks of type $k^+_{j,t}$, let $n^-_{j,t}$ be the number of curved tracks of type $k^-_{j,t}$, and let $n_{j,t} := n^+_{j,t} - n^-_{j,t}$.

Now we are ready to prove the following lemma, which gives a relationship between the four integers $n_{1,a}, n_{1,b}, n_{1,c}, n_{1,d}$ in a Durandi loop.

Lemma 6. *In a Durandi loop the sum* $n_{1,a} + n_{1,b} + n_{1,c} + n_{1,d}$ *equals* $2 \pmod 4$.

Proof. We close the Durandi loop by replacing the switch with a cap which we travel in clockwise direction. For this cap we place a $k^-_{1,d}$ track on the left of the lower end of the Dorandi loop and a $k^-_{1,c}$ track on the left of the upper end of the Dorandi loop. Then the $k^-_{1,d}$ and the $k^-_{1,c}$ track are connected with a tight non-standard curve. Let $\alpha = n'_{1,a} + n'_{1,b} + n'_{1,c} + n'_{1,d}$ be the corresponding sum in the closed loop. According to Proposition 4, the index sum in the closed loop is an even number $\beta = 2k$. Observe that $\alpha = 2\beta = 4k$: Indeed, a critical point of index 0 involves two curves $k^{\pm}_{1,a}, k^{\pm}_{1,b}, k^{\pm}_{1,c}, k^{\pm}_{1,d}$ of opposite sign, a critical point of index $+1$ involves two curves of positive type, and a critical point of index -1 involves two curves of negative type. But then, taking into account the two curves $k^-_{1,d}$ and a $k^-_{1,c}$ which we used to close the Durandi loop, we have that $n_{1,a} + n_{1,b} + n_{1,c} + n_{1,d} = 2\beta + 2 = 4k + 2$. ■

Remark 7. *From the proof of Lemma 6 and Remark 5 it follows that $n'_{1,a} + n'_{1,b} + n'_{1,c} + n'_{1,d}$ equals four times the turning number of a closed Lego loop.*

The following corollary is an immediate consequence of Lemma 6.

Corollary 8. *In a Durandi loop the sum $n_{1,a} + n_{1,b} - n_{1,c} - n_{1,d}$ is even.*

Now, we are ready to prove our main result.

Theorem 9. *There are no Durandi loops.*

Proof. Assume towards a contradiction that there is a Durandi loop starting from the upper track of a switch and ending at the lower track of the same switch.

After traveling along the loop, the difference in x direction is $\Delta x = 0$ and the difference in y direction is $\Delta y = -\frac{1}{2}$. Since, by Fact 1, we can express the values $h_{1,a}, h_{2,a}, h_{3,a}, h_{4,a}$ in terms of g_0, g_1, g_2, g_3, we can express Δx and Δy as a linear combination of g_0, g_1, g_2, g_3 with half-integer coefficients. For example, to compute Δx we have $h_{2,a} = \frac{5}{2}(g_2 - g_3) = -h_{2,b}$ and $h_{3,a} = \frac{5}{2}(g_1 - g_2) = -h_{3,b}$ with respect to the positive orientation in Figure 3. Hence, for instance,

$$n_{2,a} \cdot h_{2,a} + n_{2,b} \cdot h_{2,b} + n_{3,a} \cdot h_{3,a} + n_{3,b} \cdot h_{3,b} \;=\; g_2 \cdot \frac{5}{2}\left(n_{2,a} - n_{2,b} - n_{3,a} + n_{3,b}\right) + \dots$$

In particular, we find $p_i, q_i \in \mathbb{Q}$ for $i = 0, \dots, 3$ such that

$$\begin{aligned} p_0\, g_0 + p_1\, g_1 + p_2\, g_2 + p_3\, g_3 &= 0 \quad (= \Delta x), \\ q_0\, g_0 + q_1\, g_1 + q_2\, g_2 + q_3\, g_3 &= -\tfrac{1}{2} \quad (= \Delta y)\,. \end{aligned}$$

Now, since by Lemma 2 the numbers g_0, g_1, g_2, g_3 are linearly independent over $\mathbb{Q}$, we conclude that the only non-zero coefficient is $q_0 = -\frac{1}{2}$. Let m_4^+ denote the number of ℓ_4 tracks which we travel in positive y direction, m_4^- denote the number of ℓ_4 tracks which we travel in negative y direction, and $m_4 = m^+ - m^-$. Then, the coefficient q_0 of $g_0 = 1$ is given by

$$m_4 + \frac{5}{2}\big(n_{1,a} + n_{1,b} - n_{1,c} - n_{1,d}\big) = -\frac{1}{2}\,.$$

By Corollary 8, the sum $n_{1,a} + n_{1,b} - n_{1,c} - n_{1,d}$ is even, which implies that

$$m_4 + \frac{5}{2}\big(n_{1,a} + n_{1,b} - n_{1,c} - n_{1,d}\big) = m_4 + 5k \quad \text{for some integer } k.$$

Now, since m_4 is an integer, this implies that $m_4 + 5k$ is an integer, and therefore $m_4 + 5k \neq -\dfrac{1}{2}$. Hence, there are no Durandi loops. ■

3 Pseudo Durandi loops

In this section, we provide three loops that start and end approximately at the switch. Even though these loops are not perfect, they are sufficient for practice. In the last section we show that there is no best solution.

The first loop in Figure 10 is an asymmetric loop with the property that there is no left curve after a right curve and vice versa.

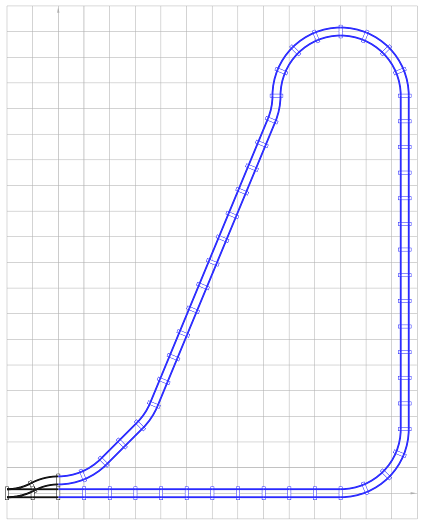

Figure 10:
$$\begin{aligned} \Delta x &= 0\,g_1 + 2\,g_2 + 12\,g_3 + 5 - 11 &\approx 0.006415 \\ \Delta y + \tfrac{1}{2} &= 0\,g_3 + 2\,g_2 + 12\,g_1 - 13 + \tfrac{1}{2} &\approx 0.000768 \end{aligned}$$

The second loop in Figure 11 is a symmetric loop which gives us $\Delta x = 0$.

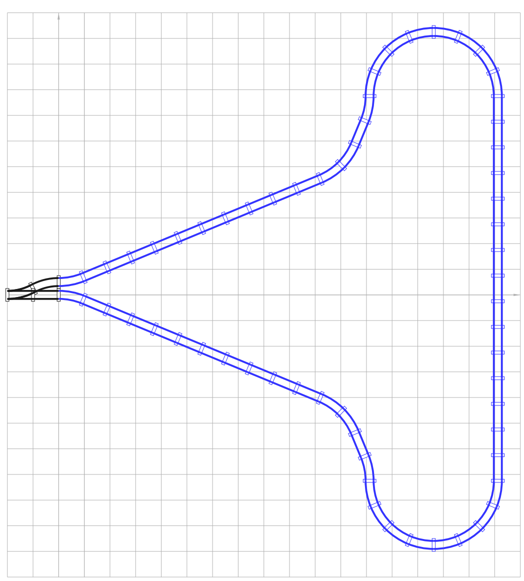

Figure 11: $\Delta x = 0$, $\Delta y + \frac{1}{2} = 2\left(10\,g_3 + 0\,g_2 + 1\,g_1 + \frac{5}{2} - \frac{15}{2}\right) + \frac{1}{2} \approx 0.001428$

The third loop in Figure 12 is essentially the symmetric form of the first loop.

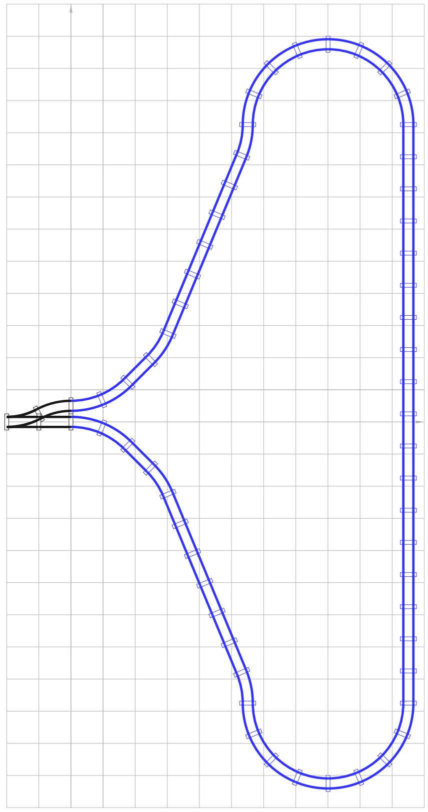

Figure 12: $\Delta x = 0, \quad \Delta y + \frac{1}{2} = 2\left(0\,g_3 + 1\,g_2 + 6\,g_1 + \frac{5}{2} - 9\right) + \frac{1}{2} \approx 0.000768$

4 A sequence of loops for which the deviation tends to zero

In order to see that there is no best solution, we consider symmetric loops of a certain type, namely symmetric loops which start with a left curve, followed by k straight tracks, 3 left curves, 8 right curves, some straight tracks, and then back in a symmetric way. Notice that for loops of this type, we have $\Delta x = 0$ and Δy depends only on the integer k. In fact, in order to obtain a good pseudo Durandi loop, we must have that $k \cdot g_1 \pmod 1$ is either close to $\frac{3}{4}$ or close to $\frac{1}{4}$. Now, by the famous result of Weyl [2], for every $\varepsilon > 0$ there are positive integers k, k' such that $|k \cdot g_1 \pmod 1) - \frac{3}{4}| < \varepsilon$ and $|k' \cdot g_1 \pmod 1) - \frac{1}{4}| < \varepsilon$, which implies that the value Δy can be made arbitrarily close to $-\dfrac{1}{2}$. Thus, there are pseudo Durandi loops with arbitrarily small deviation.

Figure 13 shows a pseudo Durandi loop of this type with $k = 49$ — better pseudo Durandi loops are obtained for $k = 130, 309, 488, 677, 2180, 3693, 5206, 6719, \ldots$

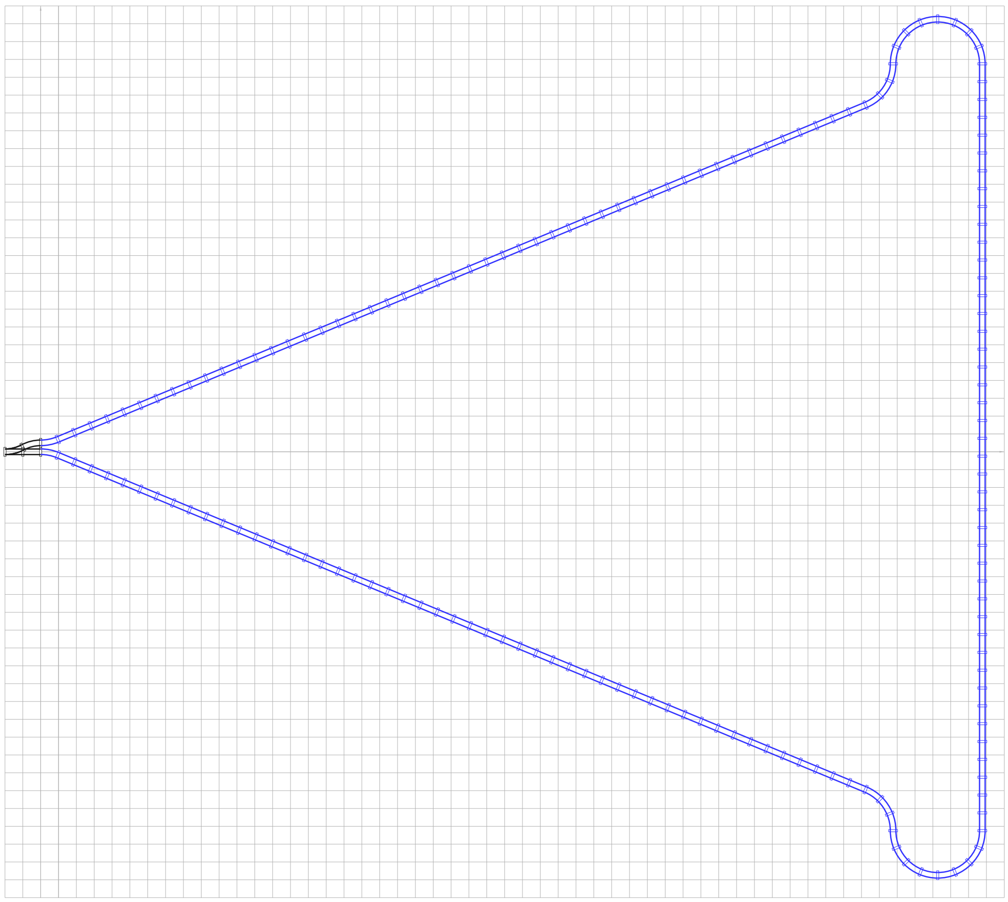

Figure 13: $\Delta x = 0, \quad \Delta y + \frac{1}{2} = 2\left(49\, g_1 + \frac{5}{2}\right) - 43 + \frac{1}{2} \approx 0.002976$

Acknowledgement

We would like to thank Werner Durandi for sharing with us his postscript-files of the Lego tracks. We are grateful to the referee for the valuable comments that helped improve this article.

References

[1] Milnor, J. (1963). *Morse theory.* Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J. https://doi.org/10.1515/9781400881802

[2] Weyl, H. (1916). Über die Gleichverteilung von Zahlen mod. Eins. *Math. Ann.*, 77(3):313–352. https://doi.org/10.1007/BF01475864

Summary. Playing with classical Lego train tracks it is natural to ask whether it is possible to build a loop which exactly starts and ends at a switch, and if this is not possible, what would be the best solution with the minimum deviation. It will be shown that there is no exact solution, but that a sequence of loops exists for which the deviation tends to zero.

LORENZ HALBEISEN (MR Author ID: 347678) received his Ph.D. in mathematics from ETH Zürich in 1994. He has been a lecturer at Queen's University Belfast and at the University of Zürich, and since 2022 he is professor at the ETH Zürich. He likes all kinds of puzzles and mathematical problems with a combinatorial flavor.

NORBERT HUNGERBÜHLER (MR Author ID: 326871) received his Ph.D. in mathematics from ETH Zürich in 1994. He has been a professor at UAB in Birmingham, Alabama, at the University of Fribourg, and since 2010 at ETH Zürich. He likes to combine different areas of mathematics and to look at familiar things from a new viewpoint.