Magic Sets

Lorenz Halbeisen*, Marc Lischka†, Salome Schumacher‡

key-words: magic sets, symmetric functions, Continuum Hypothesis, Martin's Axiom, sets of range uniqueness

2010 Mathematics Subject Classification: **03E20** 26A99 03E50 03E35

Abstract

In this paper we study magic sets for certain families $\mathcal{H} \subseteq {}^{\mathbb{R}}\mathbb{R}$ which are subsets $M \subseteq \mathbb{R}$ such that for all functions $f, g \in \mathcal{H}$ we have that $g[M] \subseteq f[M] \Rightarrow f = g$. Specifically we are interested in magic sets for the family $\mathcal{G}$ of all continuous functions that are not constant on any open subset of $\mathbb{R}$. We will show that these magic sets are stable in the following sense: Adding and removing a countable set does not destroy the property of being a magic set. Moreover, if the union of less than $\mathfrak{c}$ meager sets is still meager, we can also add and remove sets of cardinality less than $\mathfrak{c}$ without destroying the magic set.

Then we will enlarge the family $\mathcal{G}$ to a family $\mathcal{F}$ by replacing the continuity with symmetry and assuming that the functions are locally bounded. A function $f : \mathbb{R} \to \mathbb{R}$ is symmetric iff for every $x \in \mathbb{R}$ we have that $\lim_{h \downarrow 0} \frac{1}{2}\left(f(x+h) + f(x-h)\right) = f(x)$. For this family of functions we will construct $2^{\mathfrak{c}}$ pairwise different magic sets which cannot be destroyed by adding and removing a set of cardinality less than $\mathfrak{c}$. We will see that under the continuum hypothesis magic sets and these more stable magic sets for the family $\mathcal{F}$ are the same. We shall also see that the assumption of local boundedness cannot be omitted. Finally, we will prove that for the existence of a magic set for the family $\mathcal{F}$ it is sufficient to assume that the union of less than $\mathfrak{c}$ meager sets is still meager. So for example Martin's axiom for σ-centered partial orders implies the existence of a magic set.

1 Introduction

In 1993, Berarducci and Dikranjan proved [1, Theorem 8.5] that under the continuum hypothesis (CH) there exists a magic set for the family of all nowhere constant, continuous (n.c.c.) functions. In other words there is a set $M \subseteq \mathbb{R}$ such that for all functions $f, g \in C(\mathbb{R}, \mathbb{R})$ which are not constant on any open subset $U \subseteq \mathbb{R}$

$$g[M] \subseteq f[M] \Rightarrow f = g. \tag{1}$$

The existence of such a set is not provable in ZFC, as shown in [3, Example 5.17] and in [5]. In this paper we will show that we can weaken the requirements of [1, Theorem 8.5] by replacing CH by the assumption that the union of less than $\mathfrak{c}$ meager sets is meager, i.e. $\operatorname{add}(\mathcal{M}) = \mathfrak{c}$. Note that for example Martin's axiom for σ-centered partial orders implies $\operatorname{add}(\mathcal{M}) = \mathfrak{c}$. A proof of this fact can be found in [7, Chapter II, Theorem

*Department of Mathematics, ETH Zürich, Switzerland; lorenz.halbeisen@math.ethz.ch
†Department of Mathematics, ETH Zürich, Switzerland; marc.lischka@math.ethz.ch
‡Department of Mathematics, ETH Zürich, Switzerland; salome.schumacher@math.ethz.ch

2.20]. Moreover, we will enlarge the family of all n.c.c. functions to the set $\mathcal{F}$ of all symmetric, locally bounded, nowhere constant (s.l.b.n.c.) functions, where a function is called symmetric iff for every $x \in \mathbb{R}$ we have that

$$\lim_{h \downarrow 0} \frac{f(x+h)+f(x-h)}{2} = f(x).$$

In $\mathcal{F}$ there are functions with discontinuities. Based on Berarducci's and Dikranjan's proof we will construct $2^{\mathfrak{c}}$ pairwise different sets which are not only magic for the set $\mathcal{F}$ but also stay magic when we remove a set $Y_0 \in [\mathbb{R}]^{<\mathfrak{c}}$ and add a set $Y_1 \in [\mathbb{R}]^{<\mathfrak{c}}$ to M. Such a set will be called strongly magic for the family $\mathcal{F}$.

In 1999, Burke and Ciesielski proved the following Theorem:

Theorem 1.1. *([3, Theorem 5.10]) Every magic set M for the family of all n.c.c. is nowhere meager. I.e. $M \cap U$ is not meager for every non-empty open set $U \subseteq \mathbb{R}$.*

Using this result we will be able to show that every magic set for the family $\mathcal{F}$ contains $2^{\mathfrak{c}}$ pairwise different strongly magic sets.

In Section 3 we will prove that there does not exist a set of range uniqueness for the family of all symmetric, nowhere constant functions. A set of range uniqueness for a family $\mathcal{H} \subseteq {}^{\mathbb{R}}\mathbb{R}$ is a set $M \subseteq \mathbb{R}$ such that for all $f, g \in \mathcal{H}$

$$g[M] = f[M] \Rightarrow f = g. \tag{2}$$

Note that (2) is weaker than the corresponding condition (1) in the definition of a magic set. So the assumption that the functions in $\mathcal{F}$ are locally bounded is necessary for the existence of a magic set.

Finally, in Section 4 we will show that if $\operatorname{add}(\mathcal{M}) = \mathfrak{c}$, a magic set for the family $\mathcal{G}$ of all n.c.c. functions cannot be destroyed by adding and removing sets of cardinality less than $\mathfrak{c}$. So if $\operatorname{add}(\mathcal{M}) = \mathfrak{c}$, magic sets and strongly magic sets are the same for the family $\mathcal{G}$. Moreover, we will see that we cannot destroy a magic set M for the family $\mathcal{G}$ by removing a meager set but we can destroy M by adding a meager set. Finally, we prove that we can remove and add a countable set to a magic set for the family $\mathcal{F}$ or $\mathcal{G}$ and the resulting set is still magic.

2 The existence of $2^{\mathfrak{c}}$ pairwise different strongly magic sets

Definition 2.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is called symmetric iff for all $x \in \mathbb{R}$*

$$\lim_{h \downarrow 0} \frac{f(x+h)+f(x-h)}{2} = f(x).$$

Remark 2.2. *The set of all symmetric functions is closed under multiplication with scalars and addition.*

Remark 2.3. *The function*

$$\begin{aligned} \xi : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0; \end{cases} \end{aligned}$$

is symmetric and bounded but neither the left- nor the right-sided limit exists at $x = 0$.

Definition 2.4. *A function $f : \mathbb{R} \to \mathbb{R}$ is locally bounded iff for every $x \in \mathbb{R}$ there is a neighborhood U of x such that $f|_U$ is bounded.*

Lemma 2.5. *Let g and h be two symmetric, locally bounded functions in ${}^{\mathbb{R}}\mathbb{R}$ and let $D \subseteq \mathbb{R}$ be a dense set. Then*

$$g|_D = h|_D \iff g = h.$$

Proof. Let D be a dense subset of $\mathbb{R}$ and assume that there are two different symmetric, locally bounded functions g and h such that $g|_D = h|_D$. Let $\hat{f} := g - h$ and choose an $\alpha \in \{-1, 1\}$ such that there is an $x_0 \in \mathbb{R}$ with $\alpha \hat{f}(x_0) > 0$. Note that the function $f = \alpha \hat{f}$ is symmetric and locally bounded. Let $I \subseteq \mathbb{R}$ be an arbitrary open interval containing x_0. By induction we will show that for every positive integer m there is a $z_m \in I$ such that $f(z_m) > \frac{m}{2} f(x_0)$.

$m = 1$: Choose $z_1 := x_0 \in I$.
$m \mapsto m+1$: Let $z_m \in I$ such that $f(z_m) > \frac{m}{2} f(x_0)$. Choose a sequence $(h_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_{\geq 0}$ such that $\lim_{n\to\infty} h_n = 0$ and $z_m + h_n \in D$ for all $n \in \mathbb{N}$. Since f is symmetric we have that

$$\frac{m}{2} f(x_0) < f(z_m) = \frac{1}{2} \lim_{n\to\infty} (f(z_m + h_n) + f(z_m - h_n)) = \frac{1}{2} \lim_{n\to\infty} f(z_m - h_n).$$

So there is an $n_0 \in \mathbb{N}$ such that $f(z_m - h_{n_0}) > m f(x_0) \geq \frac{m+1}{2} f(x_0)$. Define $z_{m+1} := z_m - h_{n_0}$.

We proved that f is not locally bounded. This is a contradiction. ⊣

Corollary 2.6. *The family of all symmetric, locally bounded functions in ${}^{\mathbb{R}}\mathbb{R}$ has cardinality $\mathfrak{c}$.*

Proof. Use Lemma 2.5 with $D = \mathbb{Q}$. ⊣

Corollary 2.7. *Let f and g be symmetric, locally bounded functions. If $f \neq g$, there exists a non-empty interval I such that $f(x) \neq g(x)$ for all $x \in I$.*

Proof. Assume that for all open, non-empty intervals $I \subseteq \mathbb{R}$ there is an $x \in I$ such that $f(x) = g(x)$. I.e. there is a dense subset $D \subseteq \mathbb{R}$ with

$$f|_D = g|_D.$$

Therefore by Lemma 2.5 we have that $f = g$. ⊣

Definition 2.8. *We define $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ to be the set of all symmetric, locally bounded, nowhere constant (s.l.b.n.c.) functions.*

Lemma 2.9. *Let $f \in \mathcal{F}$. Then every fiber of f is nowhere dense. I.e. the set $f^{-1}(\{x\})$ is nowhere dense for every $x \in \mathbb{R}$.*

Proof. Let $f \in \mathcal{F}$ and assume that there is an $x \in \mathbb{R}$ and a non-empty interval $(a, b) \subseteq \mathbb{R}$ such that

$$(a, b) \setminus \overline{f^{-1}(\{x\})} = \emptyset.$$

Similarly to Lemma 2.5 we can show that then $f|_{(a,b)}$ is identically equal to x. But f is nowhere constant. This is a contradiction. ⊣

Definition 2.10. *Let $\mathcal{H} \subseteq {}^{\mathbb{R}}\mathbb{R}$. A set $M \subseteq \mathbb{R}$ is a strongly magic set for the family $\mathcal{H}$ if and only if for all $f, g \in \mathcal{H}$ and all subsets Y_0 and Y_1 of $\mathbb{R}$ with cardinality less than $|M|$ we have that*

$$g[(M \setminus Y_0) \cup Y_1] \subseteq f[(M \setminus Y_0) \cup Y_1] \Rightarrow f = g.$$

Remark 2.11. *Note that there is a model of ZFC in which there exists a magic set and all these magic sets are of cardinality less than $\mathfrak{c}$. (See [4, p.22].)*

By modifying the proof of [1, Theorem 8.1 and Theorem 8.5] we can prove the following Theorem:

Theorem 2.12. *Let $\mathrm{add}(\mathcal{M}) = \mathfrak{c}$ and let $N \subseteq \mathbb{R}$ be a nowhere meager set. Then N contains $2^{\mathfrak{c}} = |\mathcal{P}(\mathbb{R})|$ pairwise different strongly magic sets for the family $\mathcal{F}$.*

Proof. It suffices to prove that N contains $2^{\mathfrak{c}}$ pairwise different magic sets to which we can add and remove a set of cardinality less than $\mathfrak{c}$ without destroying them. Let $\mathcal{C} := \{\langle f, g\rangle \in \mathcal{F} \times \mathcal{F} \mid f \neq g\}$. Note that by Corollary 2.6 we have that $|\mathcal{C}| = \mathfrak{c}$. For every function $\gamma \in {}^{\mathfrak{c}}2$ we construct a magic set $M_\gamma \subseteq N$ by transfinite induction. Let $\alpha \in \mathfrak{c}$ and assume that we have already constructed $m^0_\beta, m^1_\beta \in \mathbb{R}$ for every $\beta \in \alpha$. By Lemma 2.7 there is a non-empty open interval $I_\alpha \subseteq \mathbb{R}$ such that

$$f_\alpha(x) \neq g_\alpha(x) \text{ for every } x \in I_\alpha.$$

Divide I_α into two disjoint intervals I^0_α and I^1_α with non-empty interior. Now we choose $m^0_\alpha \in \mathbb{R}$ and $m^1_\alpha \in \mathbb{R}$ such that the following conditions are satisfied:

$(1)_\alpha$ $m^0_\alpha \in I^0_\alpha \cap N$ and $m^1_\alpha \in I^1_\alpha \cap N$;

$(2)_\alpha$ $m^\delta_\alpha \notin \bigcup_{\beta\in\alpha} f^{-1}_\beta(\{g_\beta(m^0_\beta)\}) \cup \bigcup_{\beta\in\alpha} f^{-1}_\beta(\{g_\beta(m^1_\beta)\}) =: A$ for every $\delta \in 2$;

$(3)_\alpha$ $m^\delta_\alpha \notin \bigcup_{\beta\in\alpha} g^{-1}_\alpha(\{f_\alpha(m^0_\beta)\}) \cup \bigcup_{\beta\in\alpha} g^{-1}_\alpha(\{f_\alpha(m^1_\beta)\}) =: B$ for every $\delta \in 2$;

$(4)_\alpha$ $m^\delta_\alpha \notin \bigcup_{\beta\in\alpha} g^{-1}_\alpha(\{g_\alpha(m^0_\beta)\}) \cup \bigcup_{\beta\in\alpha} g^{-1}_\alpha(\{g_\alpha(m^1_\beta)\}) =: C$ for every $\delta \in 2$ and

$(5)_\alpha$ $m^\delta_\alpha \notin \bigcup_{\beta\in\alpha} \{m^0_\beta\} \cup \bigcup_{\beta\in\alpha} \{m^1_\beta\} =: D$ for every $\delta \in 2$.

Note that m^0_α and m^1_α exist since A, B, C and D are all unions of less than $\mathfrak{c}$ meager sets by Lemma 2.9. Since add($\mathcal{M}$)=$\mathfrak{c}$ we have that A, B, C and D are meager. So $(N \cap I^\delta_\alpha) \setminus (A \cup B \cup C \cup D)$ is non-empty for every $\delta \in 2$ because N is nowhere meager.

For every $\gamma \in {}^{\mathfrak{c}}2$ let

$$M_\gamma := \{m^{\gamma(\alpha)}_\alpha \mid \alpha \in \mathfrak{c}\} \subseteq N.$$

Note that by construction $M_\gamma \neq M_{\gamma'}$ for every $\gamma, \gamma' \in {}^{\mathfrak{c}}2$ with $\gamma \neq \gamma'$. So there are $2^{\mathfrak{c}} = |{}^{\mathfrak{c}}2|$ pairwise different sets M_γ. Now let $\gamma \in {}^{\mathfrak{c}}2$ and let Y_0 and Y_1 be two subsets of $\mathbb{R}$ with cardinality less than $\mathfrak{c}$.

Claim 1: *For every $\alpha \in \mathfrak{c}$ we have that $g_\alpha\left(m^{\gamma(\alpha)}_\alpha\right) \notin f_\alpha[M_\gamma]$.*

Proof of Claim 1. Let $\alpha \in \mathfrak{c}$ and suppose for a contradiction that there is an $m^{\gamma(\beta)}_\beta \in M_\gamma$ such that

$$g_\alpha\left(m^{\gamma(\alpha)}_\alpha\right) = f_\alpha\left(m^{\gamma(\beta)}_\beta\right) \tag{3}$$

There are three cases:
Case 1: $\alpha = \beta$
Then $g_\alpha\left(m^{\gamma(\alpha)}_\alpha\right) = f_\alpha\left(m^{\gamma(\alpha)}_\alpha\right)$. So $m^{\gamma(\alpha)}_\alpha \notin I_\alpha$. This is a contradiction to $(1)_\alpha$.
Case 2: $\alpha \in \beta$
By (3) we have that $m^{\gamma(\beta)}_\beta \in f^{-1}_\alpha\left(\left\{g_\alpha\left(m^{\gamma(\alpha)}_\alpha\right)\right\}\right)$. This is a contradiction to $(2)_\beta$.
Case 3: $\beta \in \alpha$
By (3) we have that $m^{\gamma(\alpha)}_\alpha \in g^{-1}_\alpha\left(\left\{f_\alpha\left(m^{\gamma(\beta)}_\beta\right)\right\}\right)$. This is a contradiction to $(3)_\alpha$.
Therefore, $g_\alpha\left(m^{\gamma(\alpha)}_\alpha\right) \notin f_\alpha[M_\gamma]$. ⊣

Claim 2: *For every $\alpha \in \mathfrak{c}$ we have that $g_\alpha[(M_\gamma \setminus Y_0) \cup Y_1] \not\subseteq f_\alpha[(M_\gamma \setminus Y_0) \cup Y_1]$.*

Proof of Claim 2. Let $\alpha \in \mathfrak{c}$. For every $k \in \mathbb{R} \setminus \{0\}$ there is a unique $\epsilon_k \in \mathfrak{c}$ such that

$$\langle kf_\alpha, kg_\alpha\rangle = \langle f_{\epsilon_k}, g_{\epsilon_k}\rangle.$$

Then we define $m_k := m_{\epsilon_k}^{\gamma(\epsilon_k)}$. Let $k, l \in \mathbb{R} \setminus \{0\}$ with $k \neq l$. Without loss of generality assume that $\epsilon_k \in \epsilon_l$. Then by $(4)_{\epsilon_l}$

$$(lg_\alpha)(m_l) \neq (lg_\alpha)(m_k).$$

Therefore, $g_\alpha(m_l) \neq g_\alpha(m_k)$. So the set $\{g_\alpha(m_k) \mid k \in \mathbb{R} \setminus \{0\}\}$ has cardinality $\mathfrak{c}$ and there is an $l \in \mathbb{R} \setminus \{0\}$ with

$$g_\alpha(m_l) \in \{g_\alpha(m_k) \mid k \in \mathbb{R} \setminus \{0\}\} \setminus (f_\alpha[Y_1] \cup g_\alpha[Y_0])$$

since $|f_\alpha[Y_1]| \leq |Y_1| < \mathfrak{c}$ and $|g_\alpha[Y_0]| \leq |Y_0| < \mathfrak{c}$. Note that by Claim 1 $g_\alpha(m_l) \notin f_\alpha[M_\gamma]$ for every $k \in \mathbb{R} \setminus \{0\}$. So

$$g_\alpha(m_l) \notin f_\alpha[M_\gamma \setminus Y_0] \cup f_\alpha[Y_1] = f_\alpha[(M_\gamma \setminus Y_0) \cup Y_1].$$

But $g_\alpha(m_l) \in g_\alpha[M_\gamma] \setminus g_\alpha[Y_0]$ and therefore $g_\alpha(m_l) \in g_\alpha[(M_\gamma \setminus Y_0) \cup Y_1]$. ⊣

So for every $\gamma \in {}^{\mathfrak{c}}2$ the set $M_\gamma \subseteq N$ is strongly magic. ⊣

Corollary 2.13. *If* $\mathrm{add}(\mathcal{M}) = \mathfrak{c}$ *there are* $2^{\mathfrak{c}} = |\mathcal{P}(\mathbb{R})|$ *pairwise different strongly magic sets for the family* $\mathcal{F}$.

Proof. Use Theorem 2.12 with $N = \mathbb{R}$. ⊣

Corollary 2.14. *Let* $\mathrm{add}(\mathcal{M}) = \mathfrak{c}$ *and let* $M \subseteq \mathbb{R}$ *be a magic set for the family* $\mathcal{F}$. *Then* M *contains* $2^{\mathfrak{c}} = |\mathcal{P}(\mathbb{R})|$ *pairwise different strongly magic sets.*

Proof. Let $M \subseteq \mathbb{R}$ be a magic set. By Theorem 1.1 the set M is nowhere meager. Now apply Theorem 2.12 with $N = M$. ⊣

3 There is no magic set for all nowhere constant, symmetric functions

Definition 3.1. *([6, p. 384]) Let* $\mathcal{H} \subseteq {}^{\mathbb{R}}\mathbb{R}$. *A set* $M \subseteq \mathbb{R}$ *is a set of range uniqueness for the family* $\mathcal{H}$ *if and only if for all* $f, g \in \mathcal{H}$ *we have that*

$$f[M] = g[M] \Rightarrow f = g.$$

Remark 3.2. *Every magic set is a set of range uniqueness.*

Definition 3.3. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is additive if for all* $x, y \in \mathbb{R}$

$$f(x + y) = f(x) + f(y).$$

Lemma 3.4. *There exists no set of range uniqueness for the family of all nowhere constant, additive functions.*

Proof. Let B be a Hamel basis of $\mathbb{R}$. I.e. every $x \in \mathbb{R}$ has a unique representation of the form

$$x = \sum_{b \in B} a_b(x) b,$$

where $a_b(x) \in \mathbb{Q}$ for every $b \in B$ and $a_b(x) = 0$ for all but finitely many $b \in B$. Assume that there is a set of range uniqueness $M \subseteq \mathbb{R}$ for the family of all additive functions. For every $b \in B$ we define

$$A_b := \{x \in \mathbb{R} \mid a_b(x) \neq 0\}.$$

Claim: For every $b \in B$ we have that

$$A_b \cap M \neq \emptyset.$$

Proof of the Claim. Assume that there is a $b_0 \in B$ such that

$$A_{b_0} \cap M = \emptyset.$$

We define two nowhere constant, additive functions f and g as follows: Let $f(b_0) := 1$, let $g(b_0) := -1$ and $f(b) = g(b) = 0$ for all $b \in B \setminus \{b_0\}$. We define

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \sum_{b \in B} a_b(x) f(b) \end{aligned} \qquad \text{and} \qquad \begin{aligned} g : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \sum_{b \in B} a_b(x) g(b). \end{aligned}$$

Note that $f \neq g$ and

$$(f(x) \neq 0 \iff x \in A_{b_0}) \wedge (g(x) \neq 0 \iff x \in A_{b_0}).$$

But since $A_{b_0} \cap M = \emptyset$ we have that

$$f[M] = g[M] = \{0\}.$$

This is a contradiction. $\dashv$

Now we define two nowhere constant, additive functions f and g such that $f[M] = g[M] = \mathbb{Q}$. Let $\mathbb{Q} = \{q_n \mid n \in \mathbb{N}\}$ and

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \sum_{b \in B} a_b(x) f(b), \end{aligned}$$

where we define $f(b) \in \mathbb{R}$ for every $b \in B$ by induction on n:

$n = 0$: Let $b_0 \in B$ and choose an $x_0 \in A_{b_0} \cap M$. We define $f(b_0) := \frac{q_0}{a_{b_0}(x_0)} \in \mathbb{Q}$ and $f(b) := 0$ for all $b \in B \setminus \{b_0\}$ with $a_b(x_0) \neq 0$. Then $f(x_0) = a_{b_0}(x_0) f(b_0) = q_0$. Moreover, let

$$B_0 := B \setminus \{b \in B \mid a_b(x_0) \neq 0\}.$$

$n - 1 \mapsto n$: Let $b_n \in B_{n-1}$ and choose an $x_n \in A_{b_n} \cap M$. Note that by construction $x_n \notin \{x_k \mid k < n\}$. We define

$$f(b_n) := \frac{q_n - \sum_{k<n} a_{b_k}(x_n) f(b_k)}{a_{b_n}(x_n)}$$

and $f(b) := 0$ for all $b \in B_{n-1} \setminus \{b_n\}$ with $a_b(x_n) \neq 0$. Then $f(x_n) = q_n$. Moreover let $B_n := B_{n-1} \setminus \{b \in B \mid a_b(x_n) \neq 0\}$.

For all $b \in B \setminus \bigcup_{n \in \mathbb{N}} B_n$ let $f(b) := 0$. Note that f is nowhere constant and $f[M] = \mathbb{Q}$. Moreover, let

$$\begin{aligned} g : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{1}{2} f(x). \end{aligned}$$

Then $f[M] = g[M] = \mathbb{Q}$ but $f \neq g$. This is a contradiction. $\dashv$

Since every additive function is symmetric, the following corollary follows directly from Lemma 3.4:

COROLLARY 3.5. *There exists no set of range uniqueness for the family of all nowhere constant, symmetric functions.*

4 Magic sets and strongly magic sets

In this section let $\mathcal{G} \subseteq {}^{\mathbb{R}}\mathbb{R}$ be the family of all nowhere constant, continuous (n.c.c.) functions and let $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ be the family of all symmetric, locally bounded, nowhere constant (s.l.b.n.c.) functions.

In 1981 Diamond, Pomerance and Rubel proved in [6, Theorem 0] that we can remove and add finitely many points to a set of range uniqueness for the family of all entire functions without destroying it. By slightly modifying their proof, we see that we can add and remove finitely many points to a magic set for the family $\mathcal{G}$ and the resulting set is still magic. In this section we will generalize this result. We will prove that if $\mathrm{add}(\mathcal{M}) = \mathfrak{c}$, strongly magic sets and magic sets are the same for the family $\mathcal{G}$ but that we cannot add arbitrary meager sets to a magic set for the family $\mathcal{G}$ without destroying it. Moreover, we will prove that a magic set for the family $\mathcal{F}$ cannot be destroyed by adding and removing countable sets.

LEMMA 4.1. *([3, Lemma 5.3]) Let $f \in {}^{\mathbb{R}}\mathbb{R}$ be a continuous function and let $g \in {}^{\mathbb{R}}\mathbb{R}$ be a n.c.c. function. Then $g \circ f$ is a n.c.c. function.*

Proof. See [3, Lemma 5.3]. ⊣

LEMMA 4.2 (Cantor's Theorem on countable dense orders). *Let $I, J \subseteq \mathbb{R}$ be open intervals and let $A \subseteq I$ and $B \subseteq J$ be countable dense sets such that both A and B have neither a maximum nor a minimum. Then there is a monotonically increasing homeomorphism*

$$\xi : I \to J$$

such that $\xi[A] = B$. Moreover, there is a monotonically decreasing homeomorphism $\eta : I \to J$ such that $\eta[A] = B$.

DEFINITION 4.3. *([2]) An uncountable, closed subset of $\mathbb{R}$ without isolated points is called a perfect set.*

DEFINITION 4.4. *([3, p.2], [2]) An s_0-set is a set $S \subseteq \mathbb{R}$ with the property that for every perfect set $P \subseteq \mathbb{R}$ there is a perfect set $Q \subseteq P$ such that $Q \cap S = \emptyset$. S is a strong s_0-set, if $f[S]$ is an s_0-set in $\mathbb{R}$ for every $f \in C(\mathbb{R}, \mathbb{R})$.*

LEMMA 4.5. *([3, Theorem 5.6(5)]) Every set of range uniqueness is a strong s_0-set.*

Proof. See [3, Theorem 5.6(5)]. ⊣

REMARK 4.6. *([3, p.13]) Every set of cardinality less than $\mathfrak{c}$ is a strong s_0-set.*

LEMMA 4.7. *([3, Lemma 5.9]) Let $M \subseteq \mathbb{R}$ be a meager strong s_0-set and let $x, y \in \mathbb{R}$ such that $x \neq y$. Then there is a n.c.c. function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\eta[M]$ is countable and $\eta(x) \neq \eta(y)$.*

Proof. The proof is similar to the proof of [3, Lemma 5.9]. ⊣

PROPOSITION 4.8. *Assume that* $\mathrm{add}(\mathcal{M}) = \mathfrak{c}$. *Let $M \subseteq \mathbb{R}$ be a magic set for the family $\mathcal{G}$ and let $D \subseteq \mathbb{R}$ be a set of cardinality less than $\mathfrak{c}$. Then $M \cup D$ is still a magic set for the family $\mathcal{G}$.*

Proof. Let $M \subseteq \mathbb{R}$ be a magic set for the family $\mathcal{G}$, let $D \subseteq \mathbb{R}$ be a set of cardinality less than $\mathfrak{c}$ and assume that $M \cup D$ is not a magic set for the family $\mathcal{G}$. So there are functions $f, g \in \mathcal{G}$ such that

$$g[M] \not\subseteq f[M] \quad \text{and} \quad g[M \cup D] \subseteq f[M \cup D]. \tag{4}$$

Since $\mathrm{add}(\mathcal{M}) = \mathfrak{c}$ and by Remark 4.6, both $f[D]$ and $g[D]$ are meager strong s_0-sets and therefore, $f[D] \cup g[D]$ is a meager strong s_0-set. By Lemma 4.7 there is a n.c.c. function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\eta(f[D] \cup g[D])$ is countable and such that $\eta \circ f \neq \eta \circ g$. Note that by Lemma 4.1, $\tilde{f} := \eta \circ f \in \mathcal{G}$ and $\tilde{g} := \eta \circ g \in \mathcal{G}$.

Since $\tilde{f}$ and $\tilde{g}$ are n.c.c. functions, there is an interval $[a,b] = H \subseteq \mathbb{R}$ with non-empty interior and endpoints in M such that

$$\tilde{f}[H] \cap \tilde{g}[H] = \emptyset.$$

Then there is a n.c.c. function $\zeta : \mathbb{R} \to \mathbb{R}$ such that

(i) $\zeta[\tilde{f}[H]] \subseteq \left[\frac{1}{2}, 1\right]$ and $\zeta[\tilde{g}[H]] \subseteq \left[0, \frac{1}{2}\right]$;

(ii) $\zeta(\tilde{f}(a)) = \zeta(\tilde{f}(b)) = \zeta(\tilde{g}(a)) = \zeta(\tilde{g}(b)) = \frac{1}{2}$;

(iii) $\zeta[\tilde{f}[\mathbb{R}]] \subseteq [0,1]$ and $\zeta[\tilde{g}[\mathbb{R}]] \subseteq [0,1]$.

Note that $\zeta \circ \tilde{f} \neq \zeta \circ \tilde{g}$ and $(\zeta \circ \tilde{g})[M \cup D] \subseteq (\zeta \circ \tilde{f})[M \cup D]$. Let $c, d \in (a,b)$ such that $c < d$. Let $h_0 : [a,c] \to \left[0, \frac{1}{2}\right]$ be a strictly monotonically decreasing function with $h_0(a) = \frac{1}{2}$ and $h_0(c) = -1$ and let $h_2 : [d,b] \to \left[\frac{1}{2}, 2\right]$ be a strictly monotonically decreasing function with $h_2(d) = 2$ and $h_2(b) = \frac{1}{2}$. Define $F := ((\zeta \circ \tilde{f})[D] \cup (\zeta \circ \tilde{g})[D] \cup \mathbb{Q}) \cap (-1, 2)$. Note that $(\zeta \circ \tilde{f})[D] \cup (\zeta \circ \tilde{g})[D] \subseteq F$ by property (iii) and that F is a countable, dense subset of $(-1,2)$ that has neither a maximum nor a minimum. Let $N \subseteq M$ be a countable dense subset of (c,d). By Lemma 4.2 there is a monotonically increasing homeomorphism

$$h_1 : (c,d) \to (-1,2)$$

such that $h_1[N] = F$. We define

$$\begin{aligned} f_0 : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} (\zeta \circ \tilde{f})(x) & \text{if } x \notin H; \\ h_0(x) & \text{if } x \in [a,c]; \\ h_1(x) & \text{if } x \in (c,d); \\ h_2(x) & \text{if } x \in [d,b]; \end{cases} \end{aligned}$$

and

$$\begin{aligned} g_0 : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} (\zeta \circ \tilde{g})(x) & \text{if } x \in H; \\ f_0(x) & \text{otherwise.} \end{cases} \end{aligned}$$

Note that by construction $f_0 \neq g_0$ and $f_0, g_0 \in \mathcal{G}$. So since M is a magic set for the family $\mathcal{G}$

$$g_0[M] \nsubseteq f_0[M]. \tag{5}$$

By construction we have that

$$g_0[(M \cup D) \cap H] = (\zeta \circ \tilde{g})[(M \cup D) \cap H] \subseteq (\zeta \circ \tilde{f})[(M \cup D) \setminus \operatorname{int}(H)] = f_0[(M \cup D) \setminus \operatorname{int}(H)].$$

Therefore,

$$\begin{aligned} g_0[M] &= g_0[M \cap H] \cup g_0[M \setminus H] = g_0[M \cap H] \cup f_0[M \setminus H] \subseteq g_0[M \cap H] \cup f_0[M] \\ &\subseteq g_0[(M \cup D) \cap H] \cup f_0[M] \subseteq f_0[(M \cup D) \setminus \operatorname{int}(H)] \cup f_0[M] \\ &\subseteq f_0[M \setminus \operatorname{int}(H)] \cup f_0[D \setminus \operatorname{int}(H)] \cup f_0[M] \subseteq f_0[M] \cup (\zeta \circ \tilde{f})[D] \subseteq f_0[M]. \end{aligned}$$

This is a contradiction to (5) and therefore, $M \cup D$ is a magic set for the family $\mathcal{G}$. ⊣

Remark 4.9. *Note that Diamond, Pomerance and Rubel have already used the idea to compose f and g with a suitable function in their proof of [6, Theorem 0].*

Proposition 4.10. *Assume that* $\operatorname{add}(\mathcal{M}) = \mathfrak{c}$. *Let* $M \subseteq \mathbb{R}$ *be a magic set for the family* $\mathcal{G}$. *Let* $D \subseteq \mathbb{R}$ *be a set of cardinality less than* $\mathfrak{c}$. *Then* $M \setminus D$ *is still a magic set for the family* $\mathcal{G}$.

Proof. Let $M \subseteq \mathbb{R}$ be a magic set for the family $\mathcal{G}$, let $D \subseteq \mathbb{R}$ be a set of cardinality less than $\mathfrak{c}$ and assume that $M \setminus D$ is not a magic set for the family $\mathcal{G}$. Without loss of generality we assume that $D \subseteq M$. Then there are functions $f, g \in \mathcal{G}$ such that

$$g[M] \not\subseteq f[M] \text{ and } g[M \setminus D] \subseteq f[M \setminus D].$$

Let $g_0, f_0, \tilde{f}, \tilde{g} \in \mathcal{G}$ and ζ be the function we get when we do the same construction as in the proof of Proposition 4.8. Since $f_0 \neq g_0$ and M is a magic set for the family $\mathcal{G}$

$$g_0[M] \not\subseteq f_0[M]. \tag{6}$$

Moreover, we have that

$$(\zeta \circ \tilde{g})[M \setminus D] \subseteq (\zeta \circ \tilde{f})[M \setminus D].$$

Therefore,

$$\begin{aligned} g_0[M] &= g_0[(M \setminus D) \cap H] \cup g_0[(M \setminus D) \setminus H] \cup g_0[D \cap H] \cup g_0[D \setminus H] \\ &= (\zeta \circ \tilde{g})[(M \setminus D) \cap H] \cup (\zeta \circ \tilde{f})[(M \setminus D) \setminus H] \cup (\zeta \circ \tilde{g})[D \cap H] \cup (\zeta \circ \tilde{f})[D \setminus H] \\ &\subseteq (\zeta \circ \tilde{f})[(M \setminus D) \setminus \operatorname{int}(H)] \cup f_0[M] \cup (\zeta \circ \tilde{g})[D] \cup (\zeta \circ \tilde{f})[D] \\ &\subseteq f_0[M] \end{aligned}$$

This is a contradiction to (6) and therefore, $M \setminus D$ is a magic set for the family $\mathcal{G}$. ⊣

Remark 4.11. *Since every Cantor set is nowhere dense and not a strong* s_0*-set, by Lemma 4.5 we cannot add arbitrary meager sets to a magic set without destroying it.*

Corollary 4.12. *If* $\operatorname{add}(\mathcal{M}) = \mathfrak{c}$, *every magic set for the family* $\mathcal{G}$ *is a strongly magic set.*

Proof. We can apply Proposition 4.8 and Proposition 4.10. ⊣

Corollary 4.13. *Let* M *be a magic set for the family* $\mathcal{G}$ *and let* $D \subseteq \mathbb{R}$ *be a countable set. Then* $M \setminus D$ *and* $M \cup D$ *are both magic sets for the family* $\mathcal{G}$.

Proof. The proof is the same as the proofs of Proposition 4.8 and Proposition 4.10. ⊣

Lemma 4.14. *Let* f *be a s.l.b.n.c. function and let* $I \subseteq \mathbb{R}$ *be a non-empty, open interval such that* $f[I]$ *is bounded and there is a* $z < \sup_{x \in I} f(x)$ *in* $\mathbb{R}$ *with*

$$\forall x \in I\ (f(x) > z \Rightarrow f \text{ is not continuous at } x).$$

Then for every $\lambda > 0$ *we can find a non-empty, open interval* $J \subseteq I$ *such that*

$$f[J] \subseteq \left(\sup_{x \in I} f(x) - \lambda, \sup_{x \in I} f(x) \right].$$

Proof. Let z, f and I be as in the lemma and let $\lambda > 0$. Choose a sequence $(x_n)_{n \in \mathbb{N}} \subseteq I$ with

$$\lim_{n \to \infty} f(x_n) = \sup_{x \in I} f(x) \quad \text{and} \quad f(x_n) > z \text{ for all } n \in \mathbb{N}.$$

By assumption f is not continuous at every x_n. So for every $n \in \mathbb{N}$ we have that

$$E_n := \{\lambda_n > 0 \mid \forall \delta > 0 \exists x \in \mathbb{R}\ (|x - x_n| < \delta \wedge |f(x) - f(x_n)| \geq \lambda_n\} \neq \emptyset.$$

For every $n \in \mathbb{N}$ we define $\epsilon_n := \frac{\sup E_n}{2} > 0$.

Case 1: $\lim_{n\to\infty} \epsilon_n \neq 0$.

Choose a $\beta > 0$ such that
$$\forall N \in \mathbb{N} \exists n \geq N \ (\epsilon_n > \beta).$$
Let $m \in \mathbb{N}$ with $\epsilon_m > \beta$ and
$$0 \leq \sup_{x\in I} f(x) - f(x_m) < \frac{\beta}{23}.$$
By definition of ϵ_m there is a sequence $(y_n)_{n\in\mathbb{N}} \subseteq I$ converging to x_m such that for all $n \in \mathbb{N}$
$$|f(x_m) - f(y_n)| \geq \epsilon_m > \beta.$$
Note that if $f(x_m) \leq f(y_n)$ we have that $f(y_n) > \sup_{x\in I} f(x)$. This is a contradiction. Therefore,
$$f(x_m) - f(y_n) \geq \epsilon_m > \beta.$$
Without loss of generality we assume that $y_n < x_m$ for all $n \in \mathbb{N}$ and we define
$$h_n := x_m - y_n$$
for every $n \in \mathbb{N}$. Since f is symmetric, there is an $n \in \mathbb{N}$ such that
$$A := 2f(x_m) - f(x_m + h_n) - f(x_m - h_n) < \frac{\beta}{23} \quad \text{and} \quad x_m + h_n \in I.$$
Moreover we get that
$$A = 2f(x_m) - f(x_m + h_n) - f(y_n) > f(x_m) - f(x_m + h_n) + \beta > \sup_{x\in I} f(x) - f(x_m + h_n) + \frac{22}{23}\beta.$$
Therefore,
$$\sup_{x\in I} f(x) + \frac{21}{23}\beta < f(x_m + h_n).$$
This is a contradiction.

Case 2: $\lim_{n\to\infty} \epsilon_n = 0$

There is an $n_0 \in \mathbb{N}$ such that $0 < \epsilon_{n_0} \leq \frac{\lambda}{23}$. By definition of ϵ_{n_0} we have that there is a $\delta > 0$ such that
$$\forall x \in \mathbb{R} \ \left(|x - x_{n_0}| < \delta \Rightarrow |f(x) - f(x_{n_0})| < \frac{\lambda}{3} \right).$$
Define $J := (x_{n_0} - \delta, x_{n_0} + \delta) \cap I$ and note that by construction $f[J] \subseteq (\sup_{x\in I} f(x) - \lambda, \sup_{x\in I} f(x)]$. $\dashv$

Lemma 4.15. *Let f be a s.l.b.n.c. function and let $I \subseteq \mathbb{R}$ be a non-empty, open interval such that $f[I]$ is bounded and there is a $z > \inf_{x\in I} f(x)$ in $\mathbb{R}$ with*
$$\forall x \in I \ (f(x) < z \Rightarrow f \text{ is not continuous at } x).$$
Then for every $\lambda > 0$ we can find a non-empty, open interval $J \subseteq I$ such that
$$f[J] \subseteq \left[\inf_{x\in I} f(x), \inf_{x\in I} f(x) + \lambda\right).$$

Proof. The proof of this Lemma is similar to the proof of Lemma 4.14. ⊣

LEMMA 4.16. *Let $I \subseteq \mathbb{R}$ be a bounded, non-empty, open interval and let f and g be s.l.b.n.c. functions. Then there are non-empty, open intervals $J_n \subseteq \mathbb{R} \setminus I, n \in \mathbb{N}$, and $I_0 \subseteq I$ such that:*

(i) $\forall n \in \mathbb{N}\ (J_n \cap I_0) = \emptyset)$;

(ii) *for all natural numbers $n \neq m$ we have that* $\operatorname{dist}(J_n, J_m) \geq 1$ *and*

(iii) $\forall n \in \mathbb{N}\ (g[I_0] \cap f[J_n] = \emptyset)$.

Proof. Without loss of generality we can assume that $g|_I$ is bounded. Let $z := \frac{1}{2}\left(\sup_{x\in I} g(x) + \inf_{x\in I} g(x)\right)$. Then we are in at least one of the following two cases:

CASE 1: $\forall k \in \mathbb{N} \exists x \geq k\ (f(x) > z)$.

CASE 2: $\forall k \in \mathbb{N} \exists x \geq k\ (f(x) < z)$.

The Lemma can be proved similarly in both cases, so we will assume that we are in Case 1. We can find non-empty, open intervals $\tilde{J}_n, n \in \mathbb{N}$, with properties (i), (ii) (with $\tilde{J}_n$ instead of J_n) and

$$\forall n \in \mathbb{N} \exists x \in \tilde{J}_n\ (f(x) > z).$$

Let $n \in \mathbb{N}$. We want to find a non-empty, open interval $J_n \subseteq \tilde{J}_n$ such that

$$f[J_n] \subseteq \left(z, \sup_{x\in\tilde{J}_n} f(x)\right].$$

If there is an $x \in \tilde{J}_n$ with $f(x) > z$ and such that f is continuous at x, we find such an interval by definition of continuity. Else we can find such an interval by Lemma 4.14. Analogously we can find an interval $I_0 \subseteq I$ such that

$$g[I_0] \subseteq \left[\inf_{x\in I} g(x), z\right).$$

By construction we have that for every $n \in \mathbb{N}$

$$g[I_0] \cap f[J_n] = \emptyset.$$

⊣

PROPOSITION 4.17. *Let $M \subseteq \mathbb{R}$ be a magic set for the family $\mathcal{F}$ of all s.l.b.n.c. functions and let $D \subseteq \mathbb{R}$ be a countable set. Then $M \cup D$ is still a magic set for the family $\mathcal{F}$.*

Proof. Let $M \subseteq \mathbb{R}$ be a magic set for the family $\mathcal{F}$ and let $D \subseteq \mathbb{R}$ be a countable set. Moreover, we assume that $M \cup D$ is not magic for the family $\mathcal{F}$. So there are functions $f, g \in \mathcal{F}$ such that

$$g[M] \not\subseteq f[M] \quad \text{and} \quad g[M \cup D] \subseteq f[M \cup D].$$

By Corollary 2.7 there is a non-empty, bounded, open interval $I \subseteq \mathbb{R}$ such that

$$\forall x \in I\ (g(x) \neq f(x)).$$

Moreover, choose intervals $(a, b) = I_0 \subseteq I$ and $J_n \subseteq \mathbb{R} \setminus I, n \in \mathbb{N}$ as described in Lemma 4.16. Since $\mathbb{R} \setminus (M \cup D)$ is dense in $\mathbb{R}$, we can assume that the endpoints of all these intervals are outside of $M \cup D$. Let

$f[D] \cup g[D] = \{d_n \mid n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ choose a $j_n \in J_n \cap M$. Let E be the set of all endpoints of the intervals I_0 and $J_n, n \in \mathbb{N}$. We define

$$\begin{aligned} f_1 : \mathbb{R} \setminus E &\to \mathbb{R} \\ x &\mapsto \begin{cases} f(x) + d_n - f(j_n) & \text{if } x \in J_n \text{ for an } n \in \mathbb{N}; \\ f(x) & \text{otherwise.} \end{cases} \end{aligned}$$

Moreover, we define

$$\begin{aligned} f_0 : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} f_1(x) & \text{if } x \in \mathbb{R} \setminus E; \\ \frac{1}{2}\left(\lim_{h \downarrow 0} f_1(x+h) + f_1(x-h)\right) & \text{otherwise;} \end{cases} \end{aligned}$$

and

$$\begin{aligned} g_0 : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} f_0(x) & \text{if } x \in \mathbb{R} \setminus \overline{I_0}; \\ g(x) & \text{if } x \in I_0; \\ \frac{1}{2}\left(\lim_{h \downarrow 0} f_0(x-h) + g(x+h)\right) & \text{if } x = a; \\ \frac{1}{2}\left(\lim_{h \downarrow 0} f_0(x+h) + g(x-h)\right) & \text{if } x = b. \end{cases} \end{aligned}$$

Note that $f_0, g_0 \in \mathcal{F}$ and $f_0 \neq g_0$. So since M is a magic set we have

$$g_0[M] \not\subseteq f_0[M].$$

By construction we have

$$g_0[(M \cup D) \cap I_0] = g[(M \cup D) \cap I_0] \subseteq f\left[(M \cup D) \setminus \bigcup_{n \in \mathbb{N}} \overline{J_n}\right] = f_0\left[(M \cup D) \setminus \bigcup_{n \in \mathbb{N}} \overline{J_n}\right].$$

Therefore,

$$\begin{aligned} g_0[M] &= g_0[M \cap I_0] \cup g_0[M \setminus I_0] = g_0[M \cap I_0] \cup f_0[M \setminus I_0] \subseteq g_0[M \cap I_0] \cup f_0[M] \\ &\subseteq g_0\left[(M \cup D) \cap I_0\right] \cup f_0[M] \subseteq f_0\left[(M \cup D) \setminus \bigcup_{n \in \mathbb{N}} \overline{J_n}\right] \cup f_0[M] \\ &= f_0\left[M \setminus \bigcup_{n \in \mathbb{N}} \overline{J_n}\right] \cup f_0\left[D \setminus \bigcup_{n \in \mathbb{N}} \overline{J_n}\right] \cup f_0[M] \subseteq f_0\left[D \setminus \bigcup_{n \in \mathbb{N}} \overline{J_n}\right] \cup f_0[M] = f\left[D \setminus \bigcup_{n \in \mathbb{N}} \overline{J_n}\right] \cup f_0[M] \\ &\subseteq f[D] \cup f_0[M] = f_0[M]. \end{aligned}$$

This is a contradiction. ⊣

Proposition 4.18. *Let $M \subseteq \mathbb{R}$ be a magic set for the family $\mathcal{F}$ and let $D \subseteq \mathbb{R}$ be a countable set. Then $M \setminus D$ is still a magic set for the family $\mathcal{F}$.*

Proof. The proof is similar to the proof of Proposition 4.17. ⊣

References

[1] Alessandro Berarducci and Dikran Dikranjan, *Uniformly approachable functions and spaces*, Rendiconti dell'Institutio di Matematica dell'Universita di Trieste. An International Journal of Mathematics, 25, pages 23-53, 1993.

[2] Jörg Brendle, *Strolling through paradise*, Fundamenta Mathematicae, 148, 1995.

[3] Maxim R. Burke and Krzysztof Ciesielski, *Sets on which measurable functions are determined by their range*, Canadian Journal of Mathematics, 49(6):1089-1116, 1997.

[4] Krzysztof Ciesielski and Janusz Pawlikowski, *The covering axiom, CPA: a combinatorial core of the iterated perfect set model*, Cambridge University Press, 2004.

[5] Krzysztof Ciesielski and Saharon Shelah, *A model with no magic sets*, The journal of symbolic logic, 64(4):1467-1490, 1999.

[6] Harold G. Diamond, Carl Pomerance and Lee Rubel, *Sets on which an entire function is determined by its range*, Mathematische Zeitschrift, 176:383-398, 1981.

[7] Kenneth Kunen, *Set Theory - An Introduction to Independence Proofs*, Elsevier Science Publishers B.V., 1980.