Some implications of Ramsey Choice for families of n-element sets

Lorenz Halbeisen
Department of Mathematics, ETH Zentrum, Rämistrasse 101, 8092 Zürich, Switzerland
lorenz.halbeisen@math.ethz.ch

Salome Schumacher
Department of Mathematics, ETH Zentrum, Rämistrasse 101, 8092 Zürich, Switzerland
salome_schumacher@hotmail.com

key-words: Axiom of Choice, weak forms of the Axiom of Choice, Ramsey Choice, Partial Choice for infinite families of n-element sets, Permutation models, Pincus' transfer theorems

2020 Mathematics Subject Classification: **03E25** 03E35

Abstract

For $n \in \omega$, the weak choice principle RC_n is defined as follows:

For every infinite set X there is an infinite subset $Y \subseteq X$ with a choice function on $[Y]^n := \{z \subseteq Y : |z| = n\}$.

The choice principle C_n^- states the following:

For every infinite family of n-element sets, there is an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a choice function.

The choice principles LOC_n^- and WOC_n^- are the same as C_n^-, but we assume that the family $\mathcal{F}$ is linearly orderable (for LOC_n^-) or well-orderable (for WOC_n^-).

In the first part of this paper, for $m, n \in \omega$ we will give a full characterization of when the implication $\mathrm{RC}_m \Rightarrow \mathrm{WOC}_n^-$ holds in ZF. We will prove the independence results by using suitable Fraenkel-Mostowski permutation models. In the second part, we will show some generalizations. In particular, we will show that $\mathrm{RC}_5 \Rightarrow \mathrm{LOC}_5^-$ and that $\mathrm{RC}_6 \Rightarrow \mathrm{C}_3^-$, answering two open questions from Halbeisen and Tachtsis [4]. Furthermore, we will show that $\mathrm{RC}_6 \Rightarrow \mathrm{C}_9^-$ and that $\mathrm{RC}_7 \Rightarrow \mathrm{LOC}_7^-$.

1 Definitions and Terminology

The notation we use is standard and follows that of [5]. Now we list some definitions that shall be used in the sequel:

Definition 1.1. *Let n be an arbitrary positive natural number.*

1. C_n^- *states that every infinite family $\mathcal{F}$ of sets of size n has an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a choice function.*
2. LOC_n^- *states that every infinite, linearly orderable family $\mathcal{F}$ of sets of size n has an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a choice function.*
3. WOC_n^- *states that every infinite, well-orderable family $\mathcal{F}$ of sets of size n has an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a choice function.*

4. RC_n *states that every infinite set* X *has an infinite subset* $Y \subseteq X$ *such that the set*

$$[Y]^n = \{z \subseteq Y : |z| = n\}$$

has a choice function.

5. *Let* $\mathcal{F}$ *be an infinite family of* n*-element sets. A Kinna-Wagner selection function of* $\mathcal{F}$ *is a function* f *with* $\mathrm{dom}(f) = \mathcal{F}$ *such that for all* $p \in \mathcal{F}$, $\emptyset \neq f(p) \subsetneq p$.

6. KW_n^- *states that every infinite family* $\mathcal{F}$ *of sets of size* n *has an infinite subset* $\mathcal{G} \subseteq \mathcal{F}$ *with a Kinna-Wagner selection function.*

7. LOKW_n^- *states that every infinite, linearly orderable family* $\mathcal{F}$ *of sets of size* n *has an infinite subset* $\mathcal{G} \subseteq \mathcal{F}$ *with a Kinna-Wagner selection function.*

In 1995, Montenegro proved in [6] that $\mathrm{RC}_n \Rightarrow \mathrm{C}_n^-$ for all $n \in \{2, 3, 4\}$. It is still unknown whether this implication holds for any $n \geq 5$. In 2017, Halbeisen and Tachtsis found interesting results concerning the implications $\mathrm{RC}_m \Rightarrow \mathrm{C}_n^-$ and $\mathrm{RC}_m \Rightarrow \mathrm{RC}_n$ for $m, n \in \omega \setminus \{0, 1\}$ (see [4]). Among other results they proved that the following statements are consistent with ZF or provable in ZF, respectively:

(α) If $m, n \in \omega \setminus \{0, 1\}$ are such that there is a prime p with $p \nmid m$ and $p \mid n$, then

$$\mathrm{RC}_m \not\Rightarrow \mathrm{RC}_n \text{ and } \mathrm{RC}_m \not\Rightarrow \mathrm{C}_n^- .$$

(β) $\mathrm{RC}_5 \not\Rightarrow \mathrm{LOC}_2^-$ and $\mathrm{RC}_5 \not\Rightarrow \mathrm{LOC}_3^-$.

(γ) For every $n \in \omega \setminus \{0, 1\}$ we have that $\mathrm{C}_n^- \Rightarrow \mathrm{LOC}_n^- \Rightarrow \mathrm{WOC}_n^-$ but none of these implications is reversible.

(δ) For every $n \in \omega \setminus \{0, 1\}$ the implication $\mathrm{RC}_{2n} \Rightarrow \mathrm{LOKW}_n^-$ holds. In particular we have that $\mathrm{RC}_6 \Rightarrow \mathrm{LOC}_3^-$ (notice that $\mathrm{LOKW}_3^- \Leftrightarrow \mathrm{LOC}_3^-$).

In Section 2 of this paper, we will give a full characterization of when the implication $\mathrm{RC}_n \Rightarrow \mathrm{WOC}_m^-$ (for $n, m \in \omega \setminus \{0, 1\}$) is provable in ZF. To be more precise, it will be shown (see Theorem 2.10) that for every $m, n \in \omega \setminus \{0, 1\}$, $\mathrm{RC}_m \Rightarrow \mathrm{WOC}_n^-$ is provable in ZF if an only if the following condition holds: Whenever we can write n in the form

$$n = \sum_{i<k} a_i p_i,$$

where $p_0, \ldots, p_{k-1}$ are prime numbers and $a_0, \ldots, a_{k-1} \in \omega \setminus \{0\}$, then we find $b_0, \ldots, b_{k-1} \in \omega$ with

$$m = \sum_{i<k} b_i p_i.$$

In order to prove the independence of this implication with ZF, we shall use permutation models (see [5] for an introduction to permutation models and to models of ZFA). With Pincus' transfer theorems (see [7]), we are able to transfer the results obtained in ZFA to ZF. Furthermore, Theorem 2.10 gives us the following three special cases:

1. For all $n \in \omega$ we have that $\mathrm{RC}_n \Rightarrow \mathrm{WOC}_n^-$ (see Corollary 2.3).
2. Let p be a prime number, $m \in \omega \setminus \{0\}$ and $n \in \omega \setminus \{0, 1\}$. Then
$$\mathrm{RC}_{p^m} \Rightarrow \mathrm{WOC}_n^-$$
if and only if $n \mid p^m$ or $p = 2$, $m = 1$ and $n = 4$ (see Corollary 2.12).
3. If $\mathrm{RC}_m \not\Rightarrow \mathrm{WOC}_n^-$, we also have that $\mathrm{RC}_m \not\Rightarrow \mathrm{RC}_n^-$ and $\mathrm{RC}_m \not\Rightarrow \mathrm{C}_n^-$ (see Corollary 2.11). This generalizes Halbeisens and Tachtsis' result (α).

In Section 3, we will give some insights into the question what happens when we weaken the assumption that our family of n-element sets is well-ordered. We will prove that $\mathrm{RC}_6 \Rightarrow \mathrm{C}_n^-$ for $n \in \{3, 9\}$ and that $\mathrm{RC}_n \Rightarrow \mathrm{LOC}_n^-$ for $n \in \{5, 7\}$.

2 On the implication $\mathrm{RC}_m \Rightarrow \mathrm{WOC}_n^-$

2.1 When is $\mathrm{RC}_m \Rightarrow \mathrm{WOC}_n^-$ provable in ZF?

In this section, we will characterise the values m and n for which the implication $\mathrm{RC}_m \Rightarrow \mathrm{WOC}_n^-$ is provable in ZF. However, before we state and prove the main result of this section, we introduce some notation and prove an auxiliary result.

Two finite partitions $\{x_i : 0 \leq i \leq l\}$ and $\{y_j : 0 \leq j \leq k\}$ of sets of the same cardinality are of the *same type*, if $l = k$ and for each $0 \leq i \leq l$ we have $|x_i| = |y_i|$.

Let k be a positive integer and let $n = \sum_{i<k} a_i p_i$, where $p_0, \dots, p_{k-1}$ are prime numbers and $a_0, \dots, a_{k-1} \in \omega \setminus \{0\}$. Furthermore, for an infinite, well-ordered set λ, let $\mathcal{F} = \{F_\alpha : \alpha \in \lambda\}$ be an infinite family of pairwise disjoint n-element sets, where for each $\alpha \in \lambda$, F_α is partitioned into sets $F_{\alpha,i}$ $(i < k)$, where $|F_{\alpha,i}| = a_i p_i$, i.e.,
$$F_\alpha = \bigcup_{i<k} F_{\alpha,i} \quad \text{and} \quad F_{\alpha,i} \cap F_{\alpha,i'} = \emptyset \text{ whenever } i \neq i'.$$

In particular, for any $\alpha, \alpha' \in \lambda$, the partitions $\{F_{\alpha,i} : i < k\}$ and $\{F_{\alpha',i} : i < k\}$ are of the same type.

For $\alpha \in \lambda$ we say that $d \subseteq F_\alpha$ *diagonalises* F_α if for all $i < k$, $|F_{\alpha,i} \cap d| = 1$. Let
$$\mathcal{D}_\alpha := \{d \subseteq F_\alpha : d \text{ diagonalises } F_\alpha\}$$
and for each $\alpha \in \lambda$ let D_α be a non-empty subset of $\mathcal{D}_\alpha$ such that for any $\alpha, \alpha' \in \lambda$ we have $|D_\alpha| = |D_{\alpha'}|$.

Finally, for some positive integer $t \geq 1$ and some prime number p, for each $\alpha \in \lambda$ let $\{D^p_{\alpha,j} : j < t\}$ be a partition of $[D_\alpha]^p$ such that for any $\alpha, \alpha' \in \lambda$, the partitions $\{D^p_{\alpha,j} : j < t\}$ and $\{D^p_{\alpha',j} : j < t\}$ are of the same type.

Lemma 2.1. *Let* $n = \sum_{i<k} a_i p_i$, $\mathcal{F} = \{F_\alpha : \alpha \in \lambda\}$, $F_\alpha = \bigcup\{F_{\alpha,i} : i < k\}$, D_α, *and* $\{D^p_{\alpha,j} : j < t\}$ *be as above. Furthermore, let* $p := p_{i_0}$ *for some* $p_{i_0} \in \{p_0, \ldots, p_{k-1}\}$, *and assume that for some integer* $l \geq 0$ *there is a choice function*

$$h\colon \left[\bigcup_{\alpha\in\lambda} D_\alpha\right]^{l+p} \to \bigcup_{\alpha\in\lambda} D_\alpha .$$

Then there is an infinite subset $\lambda' \subseteq \lambda$ *such that we are in at least one of the following cases:*

(a) There is a choice function

$$h' : \left[\bigcup_{\alpha\in\lambda'} D_\alpha\right]^{l} \to \bigcup_{\alpha\in\lambda'} D_\alpha .$$

(b) We can simultaneously refine the partitions on $\{F_\alpha : \alpha \in \lambda'\}$ *to partitions of the same type (and extend accordingly the corresponding sets* D_α*).*

(c) We can simultaneously refine the partitions on $\{[D_\alpha]^p : \alpha \in \lambda'\}$ *to partitions of the same type.*

(d) For each $\alpha \in \lambda'$ *we can choose a non-empty proper subset* D'_α *of* D_α, i.e.,

$$\emptyset \neq D'_\alpha \subsetneq D_\alpha,$$

such that for all $\alpha, \beta \in \lambda'$ *we have* $|D'_\alpha| = |D'_\beta|$.

Proof. Recall that for all $\alpha, \alpha' \in \lambda$ we have $|D_\alpha| = |D_{\alpha'}|$. Now, assume that there is a $j_0 < k$ such that for $n_{j_0} := a_{j_0} p_{j_0}$ and all $\alpha \in \lambda$ we have

$$n_{j_0} \nmid |D_\alpha| .$$

For all $\alpha \in \lambda$ and all $z \in F_\alpha$ define

$$\#z := \bigl|\bigl\{X \in D_\alpha : z \in X\bigr\}\bigr|.$$

Since $\sum_{z\in F_{\alpha,j_0}} \#z = |D_\alpha|$, $|F_{\alpha,j_0}| = n_{j_0}$ and $n_{j_0} \nmid |D_\alpha|$, it follows that

$$\emptyset \neq \bigl\{z \in F_{\alpha,j_0} : \forall z' \in F_{\alpha,j_0}(\#z \leq \#z')\bigr\} \subsetneq F_{\alpha,j_0}.$$

Therefore, we can simultaneously refine the partition on each F_α for $\alpha \in \lambda$. Moreover, notice that since n_{j_0} is finite, we find an infinite set $\lambda' \subseteq \lambda$ such that for each $\alpha \in \lambda'$, the block F_{α,j_0} is partitioned into two non-empty blocks F_{α,j_1} and F_{α,j_2} where for all $\alpha, \beta \in \lambda'$, $|F_{\alpha,j_1}| = |F_{\beta,j_1}|$ and $|F_{\alpha,j_2}| = |F_{\beta,j_2}|$. This shows that all the refined partitions are of the same type and we are in Case (b).

So, we can assume that for all $i < k$ and all $\alpha \in \lambda$ we have

$$n_i \mid |D_\alpha|$$

where $n_i := a_i p_i$.

We consider now the following four cases:

Case 1: There is a $Z_0 \in \big[\bigcup_{\alpha\in\lambda} D_\alpha\big]^l$ and an infinite subset $\lambda' \subseteq \lambda$ such that

$$\forall \alpha \in \lambda' \; \forall X \in [D_\alpha]^p \; \big(h(Z_0 \cup X) \in X\big).$$

By shrinking λ' if necessary, we may assume that $Z_0 \cap \bigcup_{\alpha\in\lambda'} D_\alpha = \emptyset$. For every $\alpha \in \lambda'$ and all $d \in D_\alpha$ define

$$\deg_\alpha(d) := \big|\big\{X \in [D_\alpha]^p : h(Z_0 \cup X) = d\big\}\big|.$$

Note that $\sum_{d\in D_\alpha} \deg_\alpha(d) = \big|[D_\alpha]^p\big| = \binom{|D_\alpha|}{p}$. Since $p = p_{i_0}$ and since $n_{i_0} \mid |D_\alpha|$, we have $p \mid |D_\alpha|$. Hence, it follows that $|D_\alpha| \nmid \binom{|D_\alpha|}{p}$. To see this, let $D := |D_\alpha|$ and notice that if $D = ap^s$ for some positive integers a, s where $p \nmid a$, then

$$\binom{D}{p} = \frac{ap^s \cdot (ap^s - 1) \cdot \, \ldots \, \cdot (ap^s - p + 1)}{1 \cdot 2 \cdot \, \ldots \, \cdot p} = \frac{ap^{s-1} \cdot (ap^s - 1) \cdot \, \ldots \, \cdot (ap^s - p + 1)}{1 \cdot 2 \cdot \, \ldots \, \cdot (p-1)}\,.$$

Hence, $p^s \nmid \binom{D}{p}$ and in particular we have $D \nmid \binom{D}{p}$.

Thus, for each $\alpha \in \lambda'$ we can choose

$$\emptyset \neq D'_\alpha := \big\{d \in D_\alpha : \forall d' \in D_\alpha(\deg_\alpha(d) \leq \deg_\alpha(d'))\big\} \subsetneq D_\alpha.$$

Moreover, notice that since D_α is finite, by shrinking λ' if necessary, we can assume that for all $\alpha, \beta \in \lambda'$ we have $|D'_\alpha| = |D'_\beta|$, and we are in Case (d).

Case 2: There is a set $Z_0 \in \big[\bigcup_{\alpha\in\lambda} D_\alpha\big]^l$, a non-negative integer $j_0 < t$, and an infinite subset $\lambda' \subseteq \lambda$ such that $Z_0 \cap \bigcup_{\alpha\in\lambda'} D_\alpha = \emptyset$ and

$$\forall \alpha \in \lambda' \; \exists X, X' \in D^p_{\alpha,j_0} \; \big(h(Z_0 \cup X) \in Z_0 \wedge h(Z_0 \cup X') \in X'\big).$$

In this case, we can simultaneously refine the partition on $[D_\alpha]^p$ for each $\alpha \in \lambda'$. Moreover, since $[D_\alpha]^p$ is finite (for all $\alpha \in \lambda'$), by shrinking λ' if necessary, we can assume that for all $\alpha, \beta \in \lambda'$, the partition on $[D_\alpha]^p$ has the same type as the partition on $[D_\beta]^p$, and we are in Case (c).

Case 3: There is a set $Z_0 \in \big[\bigcup_{\alpha\in\lambda} D_\alpha\big]^l$, a non-negative integer $j_0 < t$, and an infinite subset $\lambda' \subseteq \lambda$ such that $Z_0 \cap \bigcup_{\alpha\in\lambda'} D_\alpha = \emptyset$ and

$$\forall \alpha \in \lambda' \Big(\big(\forall X \in D^p_{\alpha,j_0} \; h(Z_0 \cup X) \in Z_0\big) \wedge \exists X, X' \in D^p_{\alpha,j}\big(h(Z_0 \cup X) \neq h(Z_0 \cup X')\big)\Big).$$

In this case, we can simultaneously refine the partition on $[D_\alpha]^p$ for each $\alpha \in \lambda'$. Moreover, by shrinking λ' if necessary, we can assume that all partitions are of the same type and we are again in Case (c).

Case 4: For all $Z \in \big[\bigcup_{\alpha\in\lambda} D_\alpha\big]^l$ and for all but finitely many $\alpha \in \lambda$ we have

$$\exists j < t \; \forall X, X' \in D^p_{\alpha,j} \; \big(h(Z \cup X) = h(Z \cup X') \in Z\big). \tag{$*$}$$

Then, for each $Z \in \left[\bigcup_{\alpha\in\lambda} D_\alpha\right]^l$ let $\alpha_Z \in \lambda$ be the least element with respect to the well-ordering on λ such that $(*)$ holds for $\alpha = \alpha_Z$. Furthermore, for every $Z \in \left[\bigcup_{\alpha\in\lambda} D_\alpha\right]^l$ let $j_Z < t$ be the least integer such that $(*)$ holds for $\alpha = \alpha_Z$ and $j = j_Z$. So, for every $Z \in \left[\bigcup_{\alpha\in\lambda} D_\alpha\right]^l$ we have

$$\forall X, X' \in D^p_{\alpha_Z, j_Z}\big(h(Z \cup X) = h(Z \cup X') \wedge h(Z \cup X) \in Z\big). \tag{$\ast\ast$}$$

Finally, we define a function $h' : \left[\bigcup_{\alpha\in\lambda} D_\alpha\right]^l \to \bigcup_{\alpha\in\lambda} D_\alpha$ by stipulating

$$\begin{array}{rccc} h'\colon & \left[\bigcup_{\alpha\in\lambda} D_\alpha\right]^l & \longrightarrow & \bigcup_{\alpha\in\lambda} D_\alpha \\ & Z & \longmapsto & h(Z \cup X) \end{array}$$

where X is an arbitrary element of $D^p_{\alpha_Z, j_Z}$. Note that by $(\ast\ast)$, h' is a well-defined choice function and we are in Case (a). ⊣

Now, we are ready to prove the main result of this section.

Proposition 2.2. *Let $m, n \in \omega \setminus \{0, 1\}$ and assume that whenever we can write n in the form*

$$n = \sum_{i<k} a_i p_i,$$

where $p_0, \ldots, p_{k-1}$ are prime numbers and $a_0, \ldots, a_{k-1}$ are positive integers, then we find $b_0, \ldots, b_{k-1} \in \omega$ with

$$m = \sum_{i<k} b_i p_i.$$

Then, in ZF we have

$$\mathrm{RC}_m \Rightarrow \mathrm{WOC}_n^- .$$

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha \in \lambda\}$ be an infinite, well-ordered family of pairwise disjoint n-element sets. The goal is to construct an infinite subfamily of $\mathcal{F}$ with a choice function.

Applying RC_m to the set $X_0 := \bigcup_{\alpha\in\lambda} F_\alpha$, we obtain an infinite set $Y_0 \subseteq X_0$ such that the set $[Y_0]^m$ has a choice function. For $1 \le j \le n$, let

$$\lambda_j := \big\{\alpha \in \lambda : |F_\alpha \cap Y_0| = j\big\}.$$

Since n is finite and λ is infinite, there exists a j_0 with $1 \le j_0 \le n$ such that $\lambda_{j_0} \subseteq \lambda$ is infinite. If $j_0 = 1$ we are done since $\{F_\alpha : \alpha \in \lambda_1\} \subseteq \mathcal{F}$ has a choice function. If $1 < j_0 < n$, we apply RC_m to the set

$$X_1 := \bigcup \big\{F_\alpha \setminus Y_0 : \alpha \in \lambda_{j_0}\big\}$$

and obtain an infinite set $Y_1 \subseteq X_1$ such the set $[Y_1]^m$ has a choice function. As above, for $1 \le j \le n - j_0$, let

$$\lambda_{j_0,j} := \big\{\alpha \in \lambda_{j_0} : |F_\alpha \cap Y_1| = j\big\}.$$

Then there exists a j_1 with $1 \le j_1 \le n - j_0$ such that $\lambda_{j_0,j_1} \subseteq \lambda$ is infinite. If $j_1 = 1$, then the infinite family $\{F_\alpha : \alpha \in \lambda_{j_0,1}\} \subseteq \mathcal{F}$ has a choice function. Proceeding this way, we either find an infinite subfamily of $\mathcal{F}$ with a choice function, or for an infinite subset $\lambda_0 \subseteq \lambda$, for all $\alpha \in \lambda_0$ we can simultaneously partition the sets F_α into sets $F_{\alpha,i}$ with $i < k$ for some $k \ge 1$. Since for each

$i < k$, $|F_{\alpha,i}| \geq 2$, we have $|F_{\alpha,i}| = a_i p_i$, where p_i is prime and $a_i > 0$. Finally, for each $\alpha \in \lambda_0$, let let $D_\alpha := \{d \subseteq F_\alpha : d \text{ diagonalises } F_\alpha\}$.

Now, since $n = \sum_{i<k} a_i p_i$, by our assumption we find $b_0, \ldots, b_{k-1} \in \omega$ with $m = \sum_{i<k} b_i p_i$, and since $m \geq 2$, there is an $i_0 < k$ with $b_{i_0} \neq 0$. In particular, we have $m \geq p_{i_0}$. Let $p := p_{i_0}$ and $l := m - p$, where $l \geq 0$. Furthermore, for $t = 1$, $\{D_{\alpha,j} : j < t\} = [D_\alpha]^p$ is the trivial partition of $[D_\alpha]^p$. Thus, by RC_m, there is an infinite set $\lambda \subseteq \lambda_0$ and a choice function

$$h\colon \left[\bigcup_{\alpha\in\lambda} D_\alpha\right]^{l+p} \to \bigcup_{\alpha\in\lambda} D_\alpha\,.$$

So, we have all the requirements to apply Lemma 2.1 iteratively until — after finitely many steps — the partitions of the F_α's or of the $[D_\alpha]^p$'s contain a block with just one element, or the sets D_α are singletons: To see this, notice first that if we are in one of the cases (b), (c), or (d), or if $l = 0$, then we can either refine the partition of the F_α's or of the $[D_\alpha]^p$'s. Now, if we are in case (a) for $l > 0$, then, by the properties of

$$m = \sum_{i<k} b_i p_i$$

and since we start with $l = m - p$, $l \geq p_i$ (for some $i < k$) and we can proceed with $l' := l - p_i$.

So, after finitely many steps — in particular after finitely many choices of sets Z_0 — we are in the situation where the partitions of the F_α's or of the $[D_\alpha]^p$'s contain a block with just one element, or the D_α's are reduced to singletons, which gives us an algorithm to select an element from each of the remaining F_α's — where in the case when $|D_\alpha| = 1$, we choose the element in $D_\alpha \cap F_{\alpha,0}$. $\dashv$

Corollary 2.3. *For every $n \in \omega$ we have that*

$$\mathrm{RC}_n \Rightarrow \mathrm{WOC}_n^-\,.$$

2.2 When is $\mathrm{RC}_m \not\Rightarrow \mathrm{WOC}_n^-$ consistent with ZF?

In this section we will show that for all $n, m \in \omega \setminus \{0,1\}$ which do not satisfy the conditions of Proposition 2.2 we get that

$$\mathrm{RC}_m \not\Rightarrow \mathrm{WOC}_n^-$$

is consistent with ZF. In a first step we will construct suitable Fraenkel-Mostowski permutation models — similar to those constructed in [2, Sec. 6] — in which we have $\mathrm{RC}_m \not\Rightarrow \mathrm{WOC}_n^-$. We will then see that both statements, RC_m and WOC_n^-, are injectively boundable. So, by [7, Theorem 3A3] the result is transferable to ZF.

Let p_0 and p_1 be two prime numbers. We start with a ground model $\mathcal{M}_{p_0,p_1}$ of $\mathsf{ZFA} + \mathsf{AC}$ with a set of atoms

$$\mathcal{A} := \bigcup\{A_i : i \in \omega\} \cup \bigcup\{B_j : j \in \omega\},$$

where for all $i, j \in \omega$ the sets A_i and B_j are called blocks. These blocks have the following properties:

- For all $i \in \omega$, $A_i = \{a_{i,k} : k < p_0\}$ and $B_i = \{b_{i,l} : l < p_1\}$ with $|A_i| = p_0$ and $|B_i| = p_1$.
- The blocks are pairwise disjoint.

For all $i, j \in \omega$ we define a permutation on $\mathcal{A}$ as follows:

- For all $i \in \omega$ and all $k < p_0$ let

$$\alpha_i(a_{i,k}) := \begin{cases} a_{i,0} & \text{if } k = p_0 - 1, \\ a_{i,k+1} & \text{if } k < p_0 - 1, \end{cases}$$

  and $\alpha_i(a) = a$ for all $a \in \mathcal{A} \setminus A_i$. Analogously for all $j \in \omega$ and all $l < p_1$ let

$$\beta_j(b_{j,l}) := \begin{cases} b_{i,0} & \text{if } l = p_1 - 1, \\ b_{j,l+1} & \text{if } l < p_1 - 1, \end{cases}$$

  and $\beta_j(b) = b$ for all $b \in \mathcal{A} \setminus B_j$.

Now we define an abelian group G of permutations of $\mathcal{A}$ by requiring

$$\phi \in G \iff \phi = \alpha \circ \beta,$$

where

$$\alpha = \prod_{i \in \omega} \alpha_i^{k_i} \text{ with } k_i < p_0 \text{ for each } i \in \omega$$

and

$$\beta = \prod_{j \in \omega} \beta_j^{l_j} \text{ with } l_j < p_1 \text{ for each } j \in \omega.$$

Let $\mathcal{F}$ be the normal filter on G generated by the subgroups

$$\text{fix}_G(E) = \{\phi \in G : \forall a \in E(\phi(a) = a)\}$$

with $E \in \text{fin}(\mathcal{A}) := \{A \subseteq \mathcal{A} : |A| \in \omega\}$. Let $\mathcal{V}_{p_0,p_1}$ be the class of all hereditarily symmetric sets.

Remark 2.4. *We can also work with k blocks of size $p_0, \dots, p_{k-1}$, where p_i is a prime number for every $i < k$. The corresponding permutation model is denoted by $\mathcal{V}_{p_0,\dots,p_{k-1}}$.*

Definition 2.5. *A set $E \in \text{fin}(\mathcal{A})$ is closed if and only if for all $i, j \in \omega$ we have that*

$$A_i \cap E \neq \emptyset \Rightarrow A_i \subseteq E \qquad \textit{and} \qquad B_j \cap E \neq \emptyset \Rightarrow B_j \subseteq E.$$

We now define a well-ordering on the set of closed sets.

Definition 2.6. *Let C_1 and C_2 be two blocks in $\{A_i : i \in \omega\} \cup \{B_j : j \in \omega\}$. We define*

$$C_1 < C_2 :\iff \begin{cases} C_1 = A_i \ \wedge \ C_2 = B_j, \ \textit{or} \\ C_1 = A_i \ \wedge \ C_2 = A_j \ \wedge \ i < j, \ \textit{or} \\ C_1 = B_i \ \wedge \ C_2 = B_j \ \wedge \ i < j. \end{cases}$$

Moreover, for distinct closed sets $E = \bigcup\{F_0, \dots F_n\} \in \text{fin}(\mathcal{A})$ and $E' = \bigcup\{F'_0, \dots, F'_m\} \in \text{fin}(\mathcal{A})$ with blocks $F_0, \dots, F_n$, $F'_0, \dots, F'_m$ let

$$E \prec E' \ :\iff \ \textit{The } <\textit{-least block in the symmetric difference } \{F_0, \dots, F_n\} \Delta \{F'_0, \dots, F'_m\} \textit{ belongs to } E.$$

Note that this defines a well-ordering on the set of closed sets and therefore on the set of all closed supports.

LEMMA 2.7. *Let $n \in \omega \setminus \{0,1\}$ and let p_0 and p_1 be two prime numbers such that*

$$n = cp_0 + dp_1 \neq 0$$

for $c, d \in \omega$. Then we have that

$$\mathcal{V}_{p_0,p_1} \models \neg\, \mathrm{WOC}_n^- .$$

Proof. Define

$$\mathcal{F} := \Big\{ A_l \cup A_{l+1} \cup \dots \cup A_{l+c-1} \cup B_{l+c} \cup \dots \cup B_{l+c+d-1} : l = k(c+d) \text{ for a } k \in \omega \Big\}.$$

Then $\mathcal{F}$ is an infinite family of pairwise disjoint n-element sets. Since the empty set is a support of $\mathcal{F}$, we have that $\mathcal{F} \in \mathcal{V}_{p_0,p_1}$. Moreover, $\mathcal{F}$ is well-orderable in $\mathcal{V}_{p_0,p_1}$. Assume towards contradiction that there is an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a choice function

$$g : \mathcal{G} \to \bigcup \mathcal{G}$$

in $\mathcal{V}_{p_0,p_1}$. Let $E_g \in \mathrm{fin}(\mathcal{A})$ be a closed support of g. Since E_g is finite, there is a $G_0 \in \mathcal{G}$ such that $G_0 \cap E_g = \emptyset$. Then there are $i, j \in \omega$ with

$$g(G_0) \in A_{l+i} \cup B_{l+c+j}.$$

Define $\gamma_0 := \alpha_{l+i} \circ \beta_{l+c+j}$. We have that

$$g(\gamma_0(G_0)) = g(G_0) \neq \gamma_0(g(G_0)).$$

So E_g is not a support of g which is a contradiction. ⊣

LEMMA 2.8. *Let $m \in \omega \setminus \{0,1\}$ and let p_0, p_1 be prime numbers such that*

$$m \neq cp_0 + dp_1$$

for all $c, d \in \omega$. Then we have:

$$\mathcal{V}_{p_0,p_1} \models \mathrm{RC}_m$$

Proof. Let $x \in \mathcal{V}_{p_0,p_1}$ be an infinite set with closed support $E_x \in \mathrm{fin}(\mathcal{A})$. If there is an $E \in \mathrm{fin}(\mathcal{A})$ such that

$$y := \big\{ z \in x : E \text{ is a support of } z \big\}$$

is an infinite set, then y can be well-ordered in $\mathcal{V}_{p_0,p_1}$ and we can define a choice function on $[y]^m$ by choosing the least element with respect to that well-ordering.

So, assume that for all $E \in \mathrm{fin}(\mathcal{A})$ there are only finitely many $z \in x$ with support E. For every closed set $E \in \mathrm{fin}(\mathcal{A})$ with $E_x \subsetneq E$ define

$$M_E := \big\{ z \in x : E \text{ is the minimal closed support of } z \text{ with } E_x \subseteq E \big\}.$$

Since E is a support of M_E, the sets M_E belong to $\mathcal{V}_{p_0,p_1}$, and by our assumption, the sets M_E are finite. Now, for each $z \in M_E$ define

$$[z] := \{\phi(z) : \phi \in \mathrm{fix}_G(E_x)\} \subseteq M_E.$$

To see that $[z] \subseteq M_E$, notice that since $E \in \mathrm{fin}(\mathcal{A})$ is closed, for all $\phi \in G$ we have $\phi(E) = E$.

We consider the following two cases:

Case 1: For infinitely many M_E there is a $z \in M_E$ with

$$[z] = M_E.$$

Let $y := \bigcup\{M_E : E_x \subsetneq E \wedge \exists z \in M_E(M_E = [z])\}$. The set y is in $\mathcal{V}_{p_0,p_1}$ because E_x is a support of y. Let $t \subseteq y$ with $|t| = m$ and let E be a smallest closed set such that $M_E \subseteq y$ and $|t \cap M_E|$ is not of the form $k_0p_0 + k_1p_1$ with $k_0, k_1 \in \omega$. To see that such a set t exists, notice that for $[z] = M_E$ and $[z'] = M_{E'}$, if $[z] \cap [z'] \neq \emptyset$, then $M_E = M_{E'}$.

Define $t_{-1} := t \cap M_E$. Since $E \setminus E_x \neq \emptyset$ there are blocks $A_{i_0}, \dots, A_{i_{u-1}}, B_{j_u}, \dots, B_{j_{u+v-1}}$ with

$$E \setminus E_x = \bigcup\{A_{i_0}, A_{i_1} \dots, A_{i_{u-1}}, B_{j_u}, B_{j_{u+1}}, \dots, B_{j_{u+v-1}}\}.$$

Define

$$\tilde{G} := \left\{ \prod_{k \in u} \alpha_{i_k}^{\kappa_{i_k}} \circ \prod_{l \in v} \beta_{j_{u+l}}^{\lambda_{j_{u+l}}} : \forall k < u\ \forall l < v\ \left(\kappa_{i_k} < p_0 \wedge \lambda_{j_{u+l}} < p_1\right) \right\}.$$

Let $\phi = \alpha_{i_0}^{\kappa_{i_0}} \circ \dots \circ \alpha_{i_{u-1}}^{\kappa_{i_{u-1}}} \circ \beta_{j_u}^{\lambda_{j_u}} \circ \dots \circ \beta_{j_{u+v-1}}^{\lambda_{j_{u+v-1}}} \in \tilde{G}$. Define

$$\phi|_r := \kappa_{i_r} \text{ if } r < u \text{ and } \phi|_r := \lambda_{j_r} \text{ if } u \leq r < u + v.$$

The elements in $\tilde{G}$ can be ordered lexicographically. We call this well-ordering $\leq_{\tilde{G}}$. For all $s, s' < t_{-1}$ and all $r < u + v$ define

$$\mathrm{dist}_r(\langle s, s' \rangle) := \phi|_r,$$

where ϕ is the $\leq_{\tilde{G}}$-smallest element in $\tilde{G}$ with $\phi(s) = s'$.

The rest of the proof can be done as in [2, Proposition 6.6]. For the sake of completeness, we will redo it here:

Claim 1: For all $s, s', s'' < t_{-1}$ and all $r < u + v$ we have that

$$\mathrm{dist}_r(\langle s, s' \rangle) +_p \mathrm{dist}_r(\langle s', s'' \rangle) = \mathrm{dist}_r(\langle s, s'' \rangle),$$

where $p = p_0$ if $r < u$ and $p = p_1$ if $u \leq r < u + v$. Moreover, $+_p$ denotes addition modulo p.

Proof of Claim 1. Let $\phi_0, \phi_1, \phi \in \tilde{G}$ be $\leq_{\tilde{G}}$-minimal with

$$\phi_0(s) = s', \phi_1(s') = s'' \text{ and } \phi(s) = s''.$$

Assume that $\phi \neq \phi_1 \circ \phi_0$. So we have that $\phi^{-1} \circ \phi_1 \circ \phi_0 \neq \mathrm{id}$ and

$$\phi^{-1} \circ \phi_1 \circ \phi_0(s) = s.$$

Let $l < u + v$ be the largest number such that

$$\phi^{-1} \circ \phi_1 \circ \phi_0|_l \neq 0.$$

Without loss of generality we assume that $l < u$. Then let $m \in \omega$ with

$$(\phi^{-1} \circ \phi_1 \circ \phi_0)^m|_l = 1.$$

Note that $(\phi^{-1} \circ \phi_1 \circ \phi_0)^m \neq \alpha_{i_l}$ because otherwise we would have that $\alpha_{i_l}(s) = s$ which is a contradiction to the fact that E is the minimal support of s with $E_x \subseteq E$. So there is a $\varphi \in \tilde{G} \setminus \{\mathrm{id}\}$ with

$$(\phi^{-1} \circ \phi_1 \circ \phi_0)^m = \varphi \circ \alpha_{i_l} \text{ and } \varphi <_{\tilde{G}} \alpha_{i_l}.$$

Then $\varphi \circ \alpha_{i_l}(s) = s \Rightarrow \alpha_{i_l}(s) = \varphi^{-1}(s)$. Note that $\varphi^{-1} <_{\tilde{G}} \alpha_{i_l}$. We have that $\phi_0|_l \neq 0$ or $\phi_1|_l \neq 0$ or $\phi|_l \neq 0$. Without loss of generality we assume that $\phi_0|_l \neq 0$. Then

$$\phi_0 \circ \alpha_{i_l}^{-1} \circ \varphi^{-1} <_{\tilde{G}} \phi_0$$

and

$$\phi_0 \circ \alpha_{i_l}^{-1} \circ \varphi^{-1}(s) = \phi_0 \circ \alpha_{i_l}^{-1} \circ \alpha_{i_l}(s) = \phi_0(s) = s'.$$

This contradicts the minimality of ϕ_0. $\dashv_{\text{Claim 1}}$

For all $\tilde{t} \subseteq t_{-1}$, all $s < \tilde{t}$ and all $r < u + v$ define

$$\chi_r(s, \tilde{t}) := \{\mathrm{dist}_r(\langle s, s' \rangle) : s' \in \tilde{t}\}.$$

These sets have the following properties:

Claim 2: For all $\tilde{t} \subseteq t_{-1}$ and all $s, s' < \tilde{t}$ we have that

1. $1 \leq |\chi_r(s, \tilde{t})| \leq p_0$ for all $r < u$ and $1 \leq |\chi_r(s, \tilde{t})| \leq p_1$ for all $u \leq r < u + v$.
2. for all $r < u + v$ there is a $k_r \in \omega$ such that $\chi_r(s, \tilde{t}) = \chi_r(s', \tilde{t}) +_p k_r$, where $p = p_0$ if $r < u$ and $p = p_1$ if $u \leq r < u + v$.
3. $|\chi_r(s, \tilde{t})| = |\chi_r(s', \tilde{t})|$.
4. if $s \neq s'$ there is an $r < u + v$ such that $\chi_r(s, \tilde{t}) \neq \chi_r(s', \tilde{t})$.

Proof of Claim 2. 1. Note that $0 < \chi_r(s, \tilde{t})$ since $\mathrm{dist}_r(\langle s, s \rangle) = 0$.
2. Set $k_r := \phi|_r$, where ϕ is $\leq_{\tilde{G}}$-minimal with $\phi(s) = s'$ and use Claim 1.
3. This follows from 2.
4. Let $s, s' < \tilde{t}$ and let ϕ be $\leq_{\tilde{G}}$-minimal with $\phi(s) = s'$. If $\chi_r(s, \tilde{t}) = \chi_r(s', \tilde{t})$ for all $r < u + v$ it follows that $\phi|_r = k_r = 0$ for all $r < u + v$. So $\phi = \mathrm{id}$ and therefore $s = s'$. $\dashv_{\text{Claim 2}}$

We define an ordering $\preceq$ on the sets $\chi_r(s, \tilde{t})$ as follows: $\chi_r(s, \tilde{t}) \preceq \chi_r(s', \tilde{t})$ if and only if $\chi_r(s, \tilde{t}) = \chi_r(s', \tilde{t})$ or the smallest integer in the symmetric difference $\chi(s, \tilde{t}) \Delta \chi_r(s', \tilde{t})$ belongs to $\chi_r(s, \tilde{t})$.

For all non-empty sets $\tilde{t} \subseteq t_{-1}$, all $r < u+v$ and all natural numbers n define $\lambda_{r,n}(\tilde{t})$ as follows: Let $\lambda_{r,0}(\tilde{t}) := \emptyset$ and for every $n \in \omega \setminus \{0\}$ let

$$\lambda_{r,n}(\tilde{t}) := \left\{ s \in \tilde{t} \setminus \bigcup_{i=0}^{n-1} \lambda_{r,i}(\tilde{t}) : \forall s' \in \tilde{t} \setminus \bigcup_{i=0}^{n-1} \lambda_{r,i}(\tilde{t}) \left(\chi_r(s,\tilde{t}) \preceq \chi_r(s',\tilde{t})\right) \right\}.$$

Note that $\bigcup_{n\in\omega} \lambda_{r,n}(\tilde{t}) = \tilde{t}$ and only finitely many $\lambda_{r,n}(\tilde{t})$ are non-empty. Assume that t_{r-1} is defined for an $r < u+v$. Then let

$$t_r := \lambda_{r,n_0}(t_{r-1}),$$

where $n_0 \in \omega$ is the smallest natural number such that $\lambda_{r,n_0}(t_{r-1})$ is not of the form

$$cp_0 + dp_1$$

with $c, d \in \omega$. By Claim 2, t_{u+v-1} is a one-element set, i.e., there is an $s < t$ with

$$t_{u+v-1} = \{s\}.$$

So we choose s from t. This shows that RC_m holds in $\mathcal{V}_{p_0,p_1}$.

Case 2: There are infinitely many M_E such that there are $z, z' \in M_E$ with

$$[z] \cap [z'] = \emptyset.$$

Our goal is to reduce this case to Case 1. For every $E \in \mathrm{fin}(\mathcal{A})$ with $E_x \subsetneq E$ define

$$[M_E] := \{[z] : z \in M_E\}.$$

Furthermore, choose a w_0 in the ground model $\mathcal{M}_{p_0,p_1} \models \mathsf{ZFA} + \mathsf{AC}$ such that

$$w_0 \setminus \bigcup_{\substack{E \in \mathrm{fin}(\mathcal{A}) \\ E_x \subsetneq E}} [M_E] \;=\; \emptyset$$

and

for all closed sets $E \in \mathrm{fin}(\mathcal{A})$ with $E_x \subsetneq E$ and $M_E \neq \emptyset$ we have $|w_0 \cap [M_E]| = 1$.

In other words, w_0 picks exactly one element from each non-empty $[M_E]$. Note that E_x is a support of w_0. So $w_0 \in \mathcal{V}_{p_0,p_1}$. Choose

$$M'_E := M_E \cap w_0.$$

This reduces Case 2 to Case 1. ⊣

PROPOSITION 2.9. *Let $m, n \in \omega \setminus \{0,1\}$, $k \in \omega$, and let $p_0, \dots, p_{k-1}$ be prime numbers such that*

$$m \neq \sum_{i<k} c_i p_i$$

for all $c_i \in \omega$, $i < k$, and

$$n = \sum_{i<k} d_i p_i$$

for some $d_i \in \omega$, $i \in k$. Then

$$\mathrm{RC}_m \not\Rightarrow \mathrm{WOC}_n^-$$

is consistent with ZF.

Proof. Similar as in Lemma 2.7 and Lemma 2.8 we can prove that

$$\mathcal{V}_{p_0,\dots,p_{k-1}} \models \mathrm{RC}_m \wedge \neg\, \mathrm{WOC}_n^- . \tag{1}$$

In order to transfer this statement to ZF, we have to show that RC_n and WOC_n^- are injectively boundable for all $n \in \omega$. Then we can use Pincus' transfer theorem [7, Theorem 3A3]. The terms "boundable" and "injectively boundable" are defined in [7].

For a set x we define the injective cardinality

$$|x|_- := \{\alpha \in \Omega : \text{ there is an injection from } \alpha \text{ into } x\},$$

where Ω is the class of all ordinal numbers. Moreover let $\varphi(x)$ denote the following property:

if x is an infinite set, there is an infinite $y \subseteq x$ with a choice function on $[y]^n$.

Note that $\varphi(x)$ is boundable. Since $\varphi(x)$ holds when $|x|_- > \omega$, it follows that

$$\mathrm{RC}_n \iff \forall x(|x|_- \leq \omega \Rightarrow \varphi(x)).$$

So, RC_n is injectively boundable. Furthermore, we have that $\neg\,\mathrm{WOC}_n^-$ is boundable. So, (1) is transferable into ZF. ⊣

Propostion 2.2 together with Propostion 2.9 gives us the following result:

THEOREM 2.10. *Let $m, n \in \omega\backslash\{0,1\}$. Then RC_m implies WOC_n^- if an only if the following condition holds: For all prime numbers $p_0,\dots,p_{k-1}$ such that there are positive integers $a_0,\dots,a_{k-1}$ with*

$$n = \sum_{i<k} a_i p_i,$$

we can find $b_0,\dots,b_{k-1} \in \omega$ with

$$m = \sum_{i<k} b_i p_i.$$

We conclude this section by giving a few consequences. Since $\neg\,\mathrm{WOC}_n^- \Rightarrow \neg\,\mathrm{RC}_n$, Proposition 2.9 gives us:

COROLLARY 2.11. *Let $m, n \in \omega \setminus \{0,1\}$ and let $p_0,\dots,p_{k-1}$ be $k \in \omega$ prime numbers such that*

$$m \neq \sum_{i<k} c_i p_i$$

for all $c_i \in \omega$, $i < k$, and

$$n = \sum_{i<k} d_i p_i$$

for some $d_i \in \omega$, $i < k$. Then

$$\mathrm{RC}_m \not\Rightarrow \mathrm{RC}_n$$

in ZF.

Proof. This follows from $\mathrm{RC}_n \Rightarrow \mathrm{WOC}_n^-$ (Corollary 2.3) and $\mathrm{RC}_m \not\Rightarrow \mathrm{WOC}_n^-$ (Proposition 2.9). $\dashv$

COROLLARY 2.12. *Let p be a prime number, let $m \in \omega \setminus \{0\}$ and $n \in \omega \setminus \{0,1\}$. Then we have that*

$$\mathrm{RC}_{p^m} \Rightarrow \mathrm{WOC}_n^-$$

if and only if $n \mid p^m$ or $p = 2$, $m = 1$ and $n = 4$.

Proof. If n is divisible by a prime $q \neq p$ we have that

$$\mathcal{V}_q \models \mathrm{RC}_{p^m} \wedge \neg\, \mathrm{WOC}_n^- .$$

Therefore, $\mathrm{RC}_{p^m} \not\Rightarrow \mathrm{WOC}_n^-$ in ZF. So we can assume that $n = p^k$ for a $k \in \omega \setminus \{0\}$.

Case 1: $m \geq k$
Let $r \in \omega$ and let $p_0, p_1 \ldots, p_{r-1}$ be prime numbers such that there are $a_0, a_1, \ldots, a_{r-1} \in \omega$ with

$$n = p^k = \sum_{i<r} a_i p_i .$$

Then

$$p^m = p^{m-k} p^k = \sum_{i<r} p^{m-k} a_i p_i .$$

So by Proposition 2.2 we have that

$$\mathrm{RC}_{p^m} \Rightarrow \mathrm{WOC}_n^- .$$

Case 2: $m < k$
First, assume that $p \neq 2$. By Bertrand's postulate there is a prime number q_0 with

$$p^m < q_0 < 2p^m .$$

Note that $p^k - q_0 > p^k - 2p^m \geq p$ and $q_0 \neq p$. So there is a prime number $q_1 \neq p$ with

$$q_1 \mid (p^k - q_0).$$

By construction, p^k can be written as a sum of multiples of q_0 and q_1. Since $q_1 \nmid p^m$ and $p^m < q_0$, we have that

$$p^m \neq a q_0 + b q_1 .$$

for all $a, b \in \omega$. So by Proposition 2.9 we have that

$$\mathrm{RC}_{p^m} \not\Rightarrow \mathrm{WOC}_{p^k}^- .$$

Now, let $p = 2$ and $k \geq 3$. Then there is a prime number q_0 with

$$2^{k-1} - 1 < q_0 < 2^k - 2.$$

It follows that

$$2 < 2^{k-1} < q_0 < 2^k - 2.$$

So, $2^k - q_0 > 2$ and with the same argumentation as above we see that

$$\mathrm{RC}_{2^n} \not\Rightarrow \mathrm{WOC}_{2^k} .$$

Now we assume that $p = 2$ and $k = 2$ (*i.e.*, $m = 1$). This is the only remaining case. By Proposition 2.2 we have that

$$\mathrm{RC}_2 \Rightarrow \mathrm{WOC}_4^- .$$

$\dashv$

3 Results provable in ZF

In this section we shall prove four results which are provable in ZF. The first two results are about the implications $\mathrm{RC}_6 \Rightarrow \mathrm{C}_n^-$ for $n \in \{3, 9\}$, and the second two results are about the implications $\mathrm{RC}_n \Rightarrow \mathrm{LOC}_m^-$ for $m \in \{5, 7\}$.

3.1 $\mathbf{RC_6}$ implies $\mathbf{C_3^-}$

In the proof of the next result, we will closely follow the proof of $\mathrm{RC}_4 \Rightarrow \mathrm{C}_4^-$ given in [6].

Proposition 3.1. $\mathsf{ZF} \vdash \mathrm{RC}_6 \Rightarrow \mathrm{C}_3^-$, *i.e., it is provable in* ZF *that* RC_6 *implies* C_3^-.

Proof. Let $\mathcal{F}$ be an infinite family of pairwise disjoint sets of size 3. We apply RC_6 to the set $\bigcup \mathcal{F}$. This gives us an infinite subset $Y \subseteq \bigcup \mathcal{F}$ with a choice function on $[Y]^6$. For every $i \in \{1, 2, 3\}$ we define

$$\mathcal{G}_i := \{u \in \mathcal{F} \colon |u \cap Y| = i\}.$$

Without loss of generality we can assume that $\mathcal{G} := \mathcal{G}_3$ is infinite, since otherwise, we can easily define a choice function on an infinite subset of $\mathcal{F}$. So, there is a choice function

$$f : \left[\bigcup \mathcal{G}\right]^6 \to \bigcup \mathcal{G}.$$

We define a directed graph on $\mathcal{G}$ by putting a directed edge from v to u *i.e.*, $v \to u$), if and only if $f(u \cup v) \in u$. If there is direct edge from v to u we will say that the edge *points from* v *to* u. With this graph we carry out the same construction as in [6]. So, there is an infinite subset $\mathcal{H} \subseteq \mathcal{G}$ which is partitioned into finite sets $(A_n)_{n \in \omega}$ such that for every $n \in \omega$, all elements in A_n have outdegree n. Moreover, for all $n \in \omega$ we have that $|A_n|$ is odd, and for all $n < m$, the edges between A_n and A_m all point from A_m to A_n. We can assume that we are in one of the following two cases:

Case 1: There are infinitely many $n \in \omega$ with $3 \nmid |A_n|$.
In this case we follow the proof of the Claim in [6, p. 60]: Without loss of generality we can assume that $3 \nmid |A_n|$ for every $n \in \omega$. Let $n_0 \in \omega$ and $p_0 = \{x_0, x_1, x_2\} \in A_{n_0}$. For each $i \leq 2$ we define

$$\deg(x_i) := |\{q \in A_{n_0+1} : f(q \cup p_0) = x_i\}|.$$

Since $3 \nmid |A_{n_0+1}|$ we have that $3 \nmid (\deg(x_0) + \deg(x_1) + \deg(x_2))$. Therefore, we can choose one element from p_0.

Case 2: For all $n \in \omega$ we have that $3 \mid |A_n|$.
Let $p_0 \in \mathcal{H}$ and let $n \in \omega$ be the unique natural number with $p_0 \in A_n$. There is an $s \in \omega$ with $|A_n| = 2s + 1$. We want to find the number of elements in A_n with edges pointing to p_0. There are $\binom{|A_n|}{2}$ edges in A_n. Since the number of edges in A_n that point to an element in A_n is the same for every element of A_n, we have that the indegree of p_0 in A_n is given by

$$\mathrm{indegree}_{A_n}(p_0) = \frac{1}{|A_n|}\binom{|A_n|}{2} = \frac{1}{2}(|A_n| - 1) = s.$$

By assumption we have that $3 \mid |A_n| = 2s+1$. Therefore, $3 \nmid s$. Assume that $p_0 = \{x_0, x_1, x_2\}$. For every $i \leq 2$ we define

$$A_n^{x_i} := \{v \in A_n \colon f(v \cup p_0) = x_i\}.$$

Since $3 \nmid (|A_n^{x_0}| + |A_n^{x_1}| + |A_n^{x_2}|) = s$, we can choose an element from p_0. ⊣

3.2 $\mathbf{RC_6}$ implies $\mathbf{C_9^-}$

Lemma 3.2. *Let $\mathcal{F}$ be an infinite family of pairwise disjoint 4-element sets. If there is a choice function*

$$f : \left[\bigcup \mathcal{F}\right]^6 \to \bigcup \mathcal{F},$$

then there is a function h with $h(p \cup q) \in p \cup q$ for all $p \neq q$ in $\mathcal{F}$.

Proof. Let $p \neq q$ be elements of $\mathcal{F}$. We will show that we can choose exactly one element from $p \cup q$. There are

$$\binom{8}{6} = 28$$

6-element subsets of $p \cup q$. From each of these subsets we can choose one point with the choice function f. Let A be the set of all elements in $p \cup q$ which are chosen the most times. Note that $1 \leq |A| \leq 7$, because 8 does not divide 28.

- If $|A| = 1$ we are done.
- If $|A| = 2$, choose $f((p \cup q) \setminus A)$.
- If $|A| = 3$ and $A \subseteq p$ or $A \subseteq q$ we are done because we can choose the point in $p \setminus A$ or in $q \setminus A$. Otherwise, $|p \cap A| = 1$ or $|q \cap A| = 1$ and we are also done.
- If $|A| \in \{5, 6, 7\}$, replace A by $(p \cup q) \setminus A$. So we are in one of the cases above.
- If $|A| = 4$, the set $[(p \cup q) \setminus A]^2$ contains $\binom{4}{2} = 6$ elements. For each $B \in [(p \cup q) \setminus A]^2$ choose $f(A \cup B)$. Let C_0 and C_1 be the sets of all elements in $p \cup q$ which are chosen the most and the least often. Note that either C_0 or C_1 does not contain 4 elements. By the cases above we are done.

So there is a choice function

$$h : \{p \cup q \colon p, q \in \mathcal{F}\} \to \bigcup \mathcal{F}.$$

⊣

Lemma 3.3. *Let $\{A_n \colon n \in \omega\}$ be a countable family of pairwise disjoint non-empty finite sets of pairwise disjoint sets of size 2, and let $\mathcal{F} := \bigcup_{n \in \omega} A_n$ be the corresponding infinite family of 2-element sets. If*

$$f : \left[\bigcup \mathcal{F}\right]^6 \to \bigcup \mathcal{F}.$$

is a choice function, then there is an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a choice function.

Proof. By using a bijection between ω and an infinite subset of ω, without loss of generality we are in one of the following four cases:

Case 1: For all $n \in \omega$ we have that $2 \nmid |A_n|$.
Let $k \in \omega$. Then there are natural numbers l_0, l_1 and l_2 such that

$$|A_{3k}| = 2l_0 + 1, \ |A_{3k+1}| = 2l_1 + 1 \ \text{ and } \ |A_{3k+2}| = 2l_2 + 1.$$

For every $a \in A_{3k} \cup A_{3k+1} \cup A_{3k+2}$ define

$$\#a := |\{(a_0, a_1, a_2) \in A_{3k} \times A_{3k+1} \times A_{3k+2} \colon f(a_0 \cup a_1 \cup a_2) \in a\}|.$$

If $\#a$ is odd, we can choose an element from a, for example the element in a we choose more often than the other. Since

$$2 \nmid \prod_{i \le 2} (2l_i + 1) \ \text{ and } \sum_{a \in A_{3k} \cup A_{3k+1} \cup A_{3k+2}} \#a = \prod_{i \le 2} (2l_i + 1),$$

we have that for every $k \in \omega$ there is at least one $a \in A_{3k} \cup A_{3k+1} \cup A_{3k+2}$ such that $\#a$ is odd. So, we can find a choice function on the infinite set

$$\mathcal{G} := \{a \in \mathcal{F} \colon \#a \text{ is odd}\}.$$

Case 2: For all $n \in \omega$ we have that $|A_n| = 2$.
For every $k \in \omega$ let $A_{2k} = \{a_{2k}, b_{2k}\}$ and $B_0 := \{a_{2k}\} \cup A_{2k+1}$ and $B_1 := \{b_{2k}\} \cup A_{2k+1}$. For every $a \in A_{2k} \cup A_{2k+1}$ we define

$$\#a := \left| \left\{ i \in \{0,1\} \colon f\left(\bigcup B_i \right) \in a \right\} \right|.$$

Note that if $\#a = 1$, we can choose an element from a and we are done. So, if there are infinitely many $a \in \mathcal{F}$ such that $\#a$ is odd, we are done. Otherwise, there is an infinite subset $I \subseteq \omega$ such that for all $k \in I$ there is a unique $a_k \in A_{2k} \cup A_{2k+1}$ with $\#a_k = 2$. Then we are in the first case for the family $\{\{a_k\} \colon k \in I\}$.

Case 3: For all $n \in \omega$ we have that $|A_n| \ge 3$, $4 \nmid |A_n|$ and $2 \mid |A_n|$.
Let $n \in \omega$. Then, by the properties of $|A_n|$ we have $|A_n| = 2t$ for some odd t, and therefore we have that $\binom{|A_n|}{2}$ is odd. For every $k \in \omega$ we look at the 4-element subsets of $A_{2k} \cup A_{2k+1}$ with two elements in A_{2k} and two elements in A_{2k+1}. Note that the number of such subsets, as the product of two odd numbers, is odd. Let h be the choice function we found in Lemma 3.2. Then for every $k \in \omega$ there is at least one $a \in A_{2k} \cup A_{2k+1}$ such that

$$\#a := |\{(\{a_0, a_1\}, \{b_0, b_1\}) \in [A_{2k}]^2 \times [A_{2k+1}]^2 : h(a_0 \cup a_1 \cup b_0 \cup b_1) \in a\}|$$

is odd. So again we found a choice function on the infinite set

$$\mathcal{G} := \{a \in \mathcal{F} : \#a \text{ is odd}\}.$$

Case 4: For all $n \in \omega$ we have that $|A_n| \geq 3$ and $4 \mid |A_n|$.
Let $n \in \omega$. Then there is a $k \in \omega$ with $|A_n| = 4k$. We have that

$$2|A_n| \nmid \binom{|A_n|}{3}, \tag{2}$$

since otherwise we would have that

$$\frac{|A_n|(|A_n|-1)(|A_n|-2)}{2 \cdot |A_n| \cdot 2 \cdot 3} = \frac{2(4k^2-3k)+1}{2 \cdot 3} \in \omega,$$

but this is not the case since the numerator is odd. We define

$$\#a \coloneqq |\{\{a_0, a_1, a_2\} \in [A_n]^3 : f(a_0 \cup a_1 \cup a_2) \in a\}|$$

and for all $y \in \bigcup A_n$ let

$$\#(y) \coloneqq |\{\{a_0, a_1, a_2\} \in [A_n]^3 : f(a_0 \cup a_1 \cup a_2) = y\}|.$$

Note that by (2)

$$\left|\left\{y \in \bigcup A_n : \#(y) = \max\left\{\#(z) : z \in \bigcup A_n\right\}\right\}\right| < 2|A_n|.$$

If there is an $a = \{a_0, a_1\} \in A_n$ with

$$\#(a_0) \neq \#(a_1)$$

choose the element a_i with lower $\#(a_i)$. Otherwise we have that

$$B_n \coloneqq \{a \in A_n : \#a = \max\{\#b : b \in A_n\}\} \subsetneq A_n.$$

Repeat the procedure with $A_n \coloneqq B_n$ until either $4 \nmid |A_n|$ or there is an $a = \{a_0, a_1\} \in A_n$ with

$$\#(a_0) \neq \#(a_1).$$

Note that we have to repeat the procedure at most $|A_n|$ times. In the end we either found a choice function on an infinite subset of $\mathcal{F}$ or we reduced Case 4 to one of the other cases. ⊣

Corollary 3.4. *Let $\mathcal{F}$ be an infinite family of pairwise disjoint 4-element sets. If*

$$f : \left[\bigcup \mathcal{F}\right]^6 \to \bigcup \mathcal{F}$$

is a choice function, then there is an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a choice function on $\mathcal{G}$.

Proof. Let h be the choice function we found in Lemma 3.2. We can define a complete, directed graph on $\mathcal{F}$ by putting an edge from p to q if and only if $h(p \cup q) \in q$. With this graph we can do the same construction as in [6]. So, we can find an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ such that we can choose exactly 1 or 2 elements from each $G \in \mathcal{G}$. So either we found a choice function on an infinite subset of $\mathcal{G}$ or we can find an infinite family of 2-element sets $\mathcal{H}$. Then we apply Lemma 3.3 to $\mathcal{H}$ and we are done. ⊣

LEMMA 3.5. *Let $\mathcal{F} := \{F_\lambda : \lambda \in \Lambda\}$ be an infinite family of 10-element sets. Assume that each $F_\lambda \in \mathcal{F}$ is a disjoint union of five 2-element sets $F_{\lambda,i}$, $0 \leq i \leq 4$. Moreover, assume that*

$$f : \left[\bigcup \mathcal{F}\right]^6 \to \bigcup \mathcal{F}$$

is a choice function. Then there is an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a Kinna-Wagner selection function.

Proof. For all 4-element sets $A \subseteq \bigcup \mathcal{F}$, we define the degree of A by

$$\deg(A) := |\{F_{\lambda,i} : F_\lambda \in \mathcal{F} \wedge i \leq 4 \wedge F_{\lambda,i} \cap A = \emptyset \wedge f(A \cup F_{\lambda,i}) \in F_{\lambda,i}\}|.$$

If there is an $A_0 \in [\bigcup \mathcal{F}]^4$ with infinite degree we are done, because then the set

$$\mathcal{G} := \{F_\lambda \in \mathcal{F} : \exists i \leq 4 \; (f(A_0 \cup F_{\lambda,i}) \in F_{\lambda,i})\}$$

is infinite and from every $G \in \mathcal{G}$ we can choose the set

$$\{f(A_0 \cup G_i) : i \leq 4\} \cap G \subsetneq G.$$

Thus, we can assume that each $A \in [\bigcup \mathcal{F}]^4$ has finite degree. Define $\mathcal{F}^2 = \{F_{\lambda,i} : F_\lambda \in \mathcal{F} \wedge i \leq 4\}$ and for all $F_\lambda \in \mathcal{F}$ let $F^2_\lambda := \{F_{\lambda,i} : i \leq 4\}$.

Case 1: There is an $n \in \omega$ such that for infinitely many $\lambda \in \Lambda$ there are distinct $A, B \in F^2_\lambda$ with $\deg(A \cup B) = n$.
Let $\mathcal{G} := \{F_\lambda \in \mathcal{F} : \exists A, B \in F^2_\lambda(\deg(A \cup B) = n)\}$. By assumption this is an infinite set. Choose an $(n+3)$-element set $\{X_i : i \leq n+2\} \subseteq \mathcal{F}^2$. For all $G \in \mathcal{G}$ and all $A, B \in G^2$ with $\deg(A \cup B) = n$ put an edge pointing from A to B if and only if

$$f(A \cup B \cup X_{i_0}) \in B,$$

where

$$i_0 := \min\{i \leq n+2 : f(A \cup B \cup X_i) \notin X_i\}.$$

Notice that this gives us a directed graph with at least one edge in each G^2 with $G \in \mathcal{G}$. If for infinitely many $G \in \mathcal{G}$ not all elements of G^2 have the same outdegree, we are done. So, we either have a cycle on infinitely many G^2 or we have a complete graph in which every node has outdegree 2. In the former case we can choose a point in each $A \cup B$, where $A, B \in G^2$ are neighbours. Thus, we can choose 5 elements in each $G \in \mathcal{G}$. In the latter case, we can choose 5 edges as follows: For the node $A \in G^2$, let $B, C \in G^2$ be the two successors of A in the graph. Consider the edge which connects B and C (see Figure 1). If this edge points to C, then we go to B and consider the two successors of B. Proceeding this way, we obtain a cycle on infinitely many G^2's and can again choose 5 elements from G.

Case 2: For all $n \in \omega$ there are only finitely many $\lambda \in \Lambda$ such that there are $A, B \in F^2_\lambda$ with $\deg(A \cup B) = n$.
Let $A_{-1} := \emptyset$ and for every $n \in \omega$ define

$$A_n := \{A \in \mathcal{F}^2 : \exists B \in \mathcal{F}^2(\deg(A \cup B) = n)\} \setminus A_{n-1}.$$

Note that these sets are pairwise disjoint families of 2-element sets. So we can apply Lemma 3.3 and we are done. $\dashv$

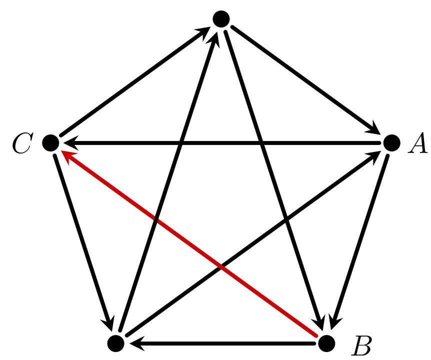

Figure 1: How to choose the edges

Now, we are ready to prove the following:

Proposition 3.6. $\mathsf{ZF} \vdash \mathrm{RC}_6 \Rightarrow \mathrm{C}_9^-$.

Proof. Let $\mathcal{F}$ be an infinite family of pairwise disjoint sets of size 9. Since RC_6 holds, there is an infinite set $Y \subseteq \bigcup \mathcal{F}$ with a choice function

$$f : [Y]^6 \to Y.$$

For all $0 \leq i \leq 9$ let

$$\mathcal{G}_i := \{F \cap Y : F \in \mathcal{F} \wedge |F \cap Y| = i\}.$$

There is a $1 \leq i \leq 9$ such that $\mathcal{G}_i$ is an infinite set.

Case 1: $\mathcal{G}_1$ or $\mathcal{G}_8$ is infinite.
In the case $\mathcal{G}_8$ is infinite, we look at the complements.

Case 2: $\mathcal{G}_3$ or $\mathcal{G}_6$ is infinite.
Use Proposition 3.1.

Case 3: $\mathcal{G}_4$ is infinite.
Use Corollary 3.4.

Case 4: $\mathcal{G}_5$ is infinite.
Apply RC_6 to the complements. Then we are either in one of the preceding cases or the complements are partitioned into two sets of size two. We look at the 10 edges between the first 5 elements and the second two elements and use Lemma 3.5.

Case 5: $\mathcal{G}_7$ is infinite.
For all $G \in \mathcal{G}_7$ let $\overline{G}$ be the complement of G in the sense that for the $F \in \mathcal{F}$ with $G \subseteq F$ we have that

$$\overline{G} := F \setminus G.$$

Note that $|\overline{G}| = 2$. Let

$$\mathcal{E} := \{\{x, y\} : \exists G \in \mathcal{G}_7(x \in G \text{ and } y \in \overline{G})\}.$$

Apply RC_6 to $\mathcal{E}$. Without loss of generality we can assume that we find a choice function

$$g : [\mathcal{E}]^6 \to \mathcal{E},$$

because otherwise we are in one of the preceding cases. So, for every $G \in \mathcal{G}_7$ there are 14 edges between G and $\overline{G}$. Hence, there are

$$\binom{14}{6} = 3 \cdot 7 \cdot 11 \cdot 13$$

6-element subsets. From each of them g chooses one element. Since $\binom{14}{6}$ is not divisible by 14, we can choose less than 14 edges and we are in one of the preceding cases.

Case 6: $\mathcal{G}_9$ is infinite.
With the choice function f we can choose an element from each 6-element subset of a $G \in \mathcal{G}_9$. There are $\binom{9}{6}$ subsets of size 6. Since $9 \nmid \binom{9}{6}$ we can reduce this case to one of the cases above.

Case 7: $\mathcal{G}_2$ is infinite.
We iteratively apply RC_6 to the complements. So, we can reduce this case to one of the cases above. $\dashv$

3.3 $\mathbf{RC_5}$ implies $\mathbf{LOC_5^-}$

We will now show that RC_5 implies LOC_5^-. The beginning of the proof will be as usual: Let $\mathcal{F}$ be an infinite, linearly orderable family of 5-element sets. We apply RC_5 to $\bigcup \mathcal{F}$. This will give us an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that each $p \in \mathcal{G}$ is partitioned into two parts. If one of these parts is of size one, we have a choice function and we are done. Otherwise, the two parts are of size 2 and 3. So if we could show that RC_5 implies LOC_2^- or LOC_3^-, the proof would be finished. However, Halbeisen's and Tachtsis' result (β) shows that this idea will not lead to success — which is the reason why we will work with the set of edges between the two parts.

Theorem 3.7. $\mathsf{ZF} \vdash \mathrm{RC}_5 \Rightarrow \mathrm{LOC}_5^-$.

Proof. Let $\mathcal{F}$ be an infinite, linearly orderable collection of pairwise disjoint sets of size 5. We fix a linear order on $\mathcal{F}$ and apply RC_5 on the set $X := \bigcup \mathcal{F}$ to find an infinite subset $Y \subseteq X$ with a choice function $f : [Y]^5 \to Y$. For every $i \leq 5$ we define

$$\mathcal{F}_i := \{p \in \mathcal{F} : |p \cap Y| = i\}.$$

The only non-trivial case is when the elements p of an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ are partitioned into a set with two elements and a set with three elements, namely $p = \{a_p, b_p, c_p\} \cup \{x_p, y_p\}$.

Now we look at the set Z of all non-directed edges between a point in $\{a_p, b_p, c_p\}$ and one in $\{x_p, y_p\}$. For every $p \in \mathcal{G}$ let p^* be the set of all edges in Z belonging to p and for each subset $\mathcal{H} \subseteq \mathcal{F}$ we define $\mathcal{H}^* := \{p^* : p \in \mathcal{H}\}$.

Claim 1: Assume that there is an infinite subset $\mathcal{H} \subseteq \mathcal{G}$ such that we can choose between 1 and 5 elements from each $p^* \in \mathcal{H}^*$. Then there is a choice function

$$h : \mathcal{H} \to \bigcup \mathcal{H}.$$

Proof of Claim 1. Let $p \in \mathcal{H}$ and assume that we can choose $k \in \{1, 2, 3, 4, 5\}$ elements from p^*. We look at p as a graph with k edges. If $2 \nmid k$, x_p and y_p do not have the same degree and we can choose the element with lower degree. Otherwise we have that $3 \nmid k$ and we can choose an element from $\{a_p, b_p, c_p\}$. $\dashv_{\text{Claim 1}}$

Now we apply RC_5 on the set Z. Then there is an infinite subset $Q \subseteq Z$ with a choice function $g : [Q]^5 \to Q$. By Claim 1 we can without loss of generality assume that $p^* \subseteq Q$ for every p in some infinite $\mathcal{H} \subseteq \mathcal{G}$.

We can partition each $p^* \in \mathcal{H}^*$ as follows into two sets γ_0^p and γ_1^p of size three:

$$\gamma_0^p := \{\{a_p, x_p\}, \{b_p, x_p\}, \{c_p, x_p\}\} \text{ and } \gamma_1^p := \{\{a_p, y_p\}, \{b_p, y_p\}, \{c_p, y_p\}\}.$$

Analogously we can partition p^* into three sets $\beta_0, \beta_1, \beta_2$ of size two as follows:

$$\beta_0^p := \{\{a_p, x_p\}, \{a_p, y_p\}\},\ \beta_1^p := \{\{b_p, x_p\}, \{b_p, y_p\}\} \text{ and } \beta_2^p := \{\{c_p, x_p\}, \{c_p, y_p\}\}.$$

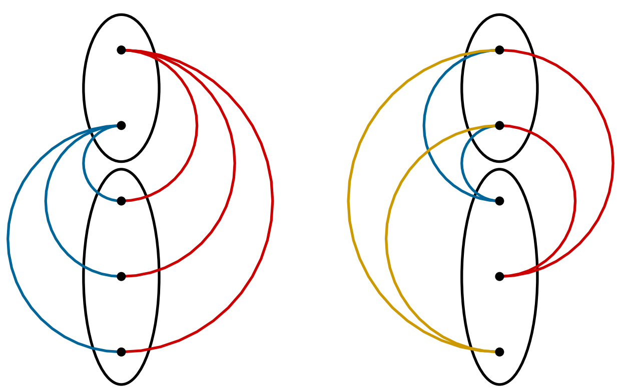

Figure 2: The partitions of a p^* into γ_0^p, γ_1^p on the left and into β_0^p, β_1^p and β_2^p on the right.

Let

$$\mathcal{H}_3^* := \{\gamma_i^p : i \leq 1 \wedge p \in \mathcal{H}\}$$

be the sets of size three appearing in the partition of a $p^* \in \mathcal{H}^*$ and let

$$\mathcal{H}_2^* := \{\beta_i^p : i \leq 2 \wedge p \in \mathcal{H}\}$$

be the family of sets of size two which appear in the partition of a $p^* \in \mathcal{H}^*$. If there is a $\gamma \in \mathcal{H}_3^*$ such that for infinitely many $\beta \in \mathcal{H}_2^*$

$$g(\gamma \cup \beta) \in \beta, \tag{3}$$

we are done by Claim 1. Otherwise, for every $\gamma \in \mathcal{H}_3^*$ there are only finitely many $\beta \in \mathcal{H}_2^*$ with (3) and we define

$$\deg(\gamma) \coloneqq |\{\beta \in \mathcal{H}_2^* : g(\gamma \cup \beta) \in \beta\}| \in \omega.$$

We are in one of the following two cases:

Case 1: There is an $n \in \omega$ such that $\deg(\gamma) = n$ for infinitely many $\gamma \in \mathcal{H}_3^*$.
Let $\mathcal{I}_3^* \coloneqq \{\gamma \in \mathcal{H}_3^* : \deg(\gamma) = n\}$. Choose an $(n+4)$-element set $\{\beta_i : i \leq n+3\} \subseteq \mathcal{H}_2^*$. For every $\gamma \in \mathcal{I}_3^*$ we define

$$j(\gamma) \coloneqq \min\{i \leq n+3 : g(\gamma \cup \beta_i) \in \gamma\}.$$

So from every $\gamma \in \mathcal{I}_3^*$ we choose the element

$$g(\gamma \cup \beta_{j(\gamma)}) \in \gamma$$

and we are done by Claim 1.

Case 2: For each $n \in \omega$ there are only finitely many $\gamma \in \mathcal{H}_3$ with $\deg(\gamma) = n$.
For every $n \in \omega$ we define

$$A_n \coloneqq \{\gamma \in \mathcal{H}_3^* : \deg(\gamma) = n\} \text{ and } B_n \coloneqq \{\beta \in \mathcal{H}_2^* : \exists \gamma \in A_n \exists p^* \in \mathcal{H}^*(\gamma \subseteq p^* \wedge \beta \subseteq p^*)\}.$$

If there are infinitely many $p \in \mathcal{H}$ such that $\gamma_0^p \in A_n$ and $\gamma_1^p \in A_m$ with $n \neq m$ we are done by Claim 1 since we can choose three edges from each of these infinitely many p's. So we can assume that for every $p \in \mathcal{H}$ both, γ_0^p and γ_1^p, have the same degree and we define

$$C_n \coloneqq \{p \in \mathcal{H} : \{\gamma_0^p, \gamma_1^p\} \subseteq A_n\}$$

for every $n \in \omega$. Moreover, let

$$\operatorname{out}(\beta) \coloneqq \left\{\gamma \in \bigcup_{m>n} A_m : g(\beta \cup \gamma) \in \gamma\right\}.$$

for every $n \in \omega$ and every $\beta \in B_n$. If there is a $\beta \in \bigcup_{n\in\omega} B_n$ with $|\operatorname{out}(\beta)| = \infty$ we are done by Claim 1. So assume that $|\operatorname{out}(\beta)| \in \omega$ for all $\beta \in \bigcup_{n\in\omega} B_n$.

Claim 2: We can find an infinite subset $\mathcal{K} \subseteq \mathcal{H}$ with a partition $\mathcal{K} = \bigcup_{n\in\omega} K_n$ where each K_n is finite and non-empty. Moreover, we can assume that for all natural numbers $n > m$, all $p \in K_n$, all $q \in K_m$ and all $j \leq 2$

$$g(\gamma_0^p \cup \beta_j^q) = g(\gamma_1^p \cup \beta_j^q) \in \beta_j^q.$$

Proof of Claim 2. For every $n \in \omega$ we define R_n to be the set of all $p \in \bigcup_{k>n} C_k$ such that there are a $q \in C_n$, an $i \in \{0,1\}$ and a $j \in \{0,1,2\}$ with

$$g(\gamma_i^p \cup \beta_j^q) \in \gamma_i^p.$$

Since $|\mathrm{out}(\beta)|$ is finite for all $\beta \in \bigcup_{n\in\omega} B_n$, the set R_n is finite. $J_n := C_n \setminus \bigcup_{k<n} R_n$. Define S_n to be the set of all $p \in \bigcup_{k>n} J_k$ such that there are a $q \in J_n$, and a $j \in \{0,1,2\}$ with

$$g(\gamma_0^p \cup \beta_j^q) \neq g(\gamma_1^p \cup \beta_j^q).$$

First of all assume that there is an $n_0 \in \omega$ such that S_{n_0} is infinite. Since J_{n_0} is finite, we can then find a $q_0 \in J_{n_0}$ and a $j_0 \in \{0,1,2\}$ such that for infinitely many $p \in S_{n_0}$

$$\beta_{j_0}^{q_0} \ni g(\gamma_0^p \cup \beta_{j_0}^{q_0}) \neq g(\gamma_1^p \cup \beta_{j_0}^{q_0}) \in \beta_{j_0}^{q_0}$$

and we can choose the set of edges γ_0^p or γ_1^p depending on the choice in $\beta_{j_0}^{p_0}$. With Claim 1 we are done. Therefore, we can assume that each S_n is finite. In this case we define $K_n := J_n \setminus \bigcup_{k<n} S_n$ for all $n \in \omega$. Infinitely many sets K_n are non-empty. By renumbering the sets K_n we can assume that each K_n is non-empty. $\dashv_{\text{Claim 2}}$

With the same construction we did in the proof of Claim 2 we can find an infinite subset $\mathcal{I} \subseteq \mathcal{K}$ with a partition $\mathcal{I} = \bigcup_{n\in\omega} I_n$, where each I_n is finite and non-empty. Moreover, we can assume that for all natural numbers $n > m$, all $p \in I_n$, all $q \in I_m$ and all $j \leq 2$

$$g(\gamma_0^p \cup \beta_j^q) = g(\gamma_1^p \cup \beta_j^q) \in \beta_j^q.$$

Note: Up to now we nowhere used the assumption that our infinite family $\mathcal{F}$ of sets of size five is linearly ordered. In the last step we will need this assumption.

For each $n \in \omega$, let $p_n \in I_n$ be the smallest element in I_n with respect to the linear order on $\mathcal{F}$. Note that such a smallest element exists since each I_n is finite and non-empty. We define

$$\begin{aligned} h^* : \{p_n^* : n \in \omega\} &\to \left[\bigcup_{n\in\omega} p_n^*\right]^3 \\ p_n^* &\mapsto \left\{ g\left(\gamma_0^{p_{n+1}} \cup \beta_j^{p_n}\right) : j \leq 2 \right\}. \end{aligned}$$

By Claim 1 we are done. $\dashv$

3.4 $\mathbf{RC_7}$ implies $\mathbf{LOC_7^-}$

Before we prove our last result, we shall prove three lemmata.

LEMMA 3.8. *Let $\mathcal{F}$ be a linearly orderable family of pairwise disjoint 6-element sets. Assume that we can partition each $p \in \mathcal{F}$ in a unique way into three 2-element sets β_0^p, β_1^p and β_2^p and in a unique way into two 3-element sets γ_0^p, γ_1^p. Further assume that there is a choice function*

$$f : \left[\bigcup \mathcal{F}\right]^7 \to \bigcup \mathcal{F}.$$

Then there is an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a Kinna-Wagner selection function.

Proof. We define

$$\mathcal{F}_3 := \{\gamma_i^p : i \in \{0,1\} \wedge p \in \mathcal{F}\}$$

and

$$\mathcal{F}_4 := \{\beta_i^p \cup \beta_j^p : \{i,j\} \in [3]^2 \wedge p \in \mathcal{F}\}.$$

For every $\gamma \in \mathcal{F}_3$ let

$$\deg(\gamma) := \left|\{\delta \in \mathcal{F}_4 : \delta \cap \gamma = \emptyset \wedge f(\delta \cup \gamma) \in \delta\}\right|.$$

If there is a $\gamma \in \mathcal{F}_3$ with $\deg(\gamma) = \infty$, then we are done because we can choose between one and three elements from infinitely many $p \in \mathcal{F}$. The rest of the proof is similar to the proof of Theorem 3.7. ⊣

LEMMA 3.9. *Let $\mathcal{F}$ be a linearly orderable family of pairwise disjoint 12-element sets. Assume that we can partition each $p \in \mathcal{F}$ in a unique way into three 4-element sets δ_0, δ_1 and δ_2 and in a unique way into four 3-element sets $\gamma_0, \gamma_1, \gamma_2$ and γ_3. Further assume that there is a choice function*

$$f : \left[\bigcup \mathcal{F}\right]^7 \to \bigcup \mathcal{F}.$$

Then there is an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a Kinna-Wagner selection function.

Proof. The proof is similar to the proof of Theorem 3.7. ⊣

LEMMA 3.10. *Let $\mathcal{F}$ be a linearly orderable family of pairwise disjoint 10-element sets. Assume that we can partition each $p \in \mathcal{F}$ in a unique way into two 5-element sets ϵ_0 and ϵ_1 and in a unique way into five 2-element sets β_i, $i \le 4$. Further assume that there is a choice function*

$$f : \left[\bigcup \mathcal{F}\right]^7 \to \bigcup \mathcal{F}.$$

Then there is an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a Kinna-Wagner selection function.

Proof. The proof is similar to the proof of Theorem 3.7. ⊣

PROPOSITION 3.11. $\mathsf{ZF} \vdash \mathrm{RC}_7 \Rightarrow \mathrm{LOC}_7^-$.

Proof. Let $\mathcal{F}$ be a linearly orderable, infinite family of sets of size 7. We apply RC_7 on the set $X := \bigcup \mathcal{F}$ to find an infinite subset $Y \subseteq X$ with a choice function $f : [Y]^7 \to Y$. For every $i \le 7$ we define

$$\mathcal{F}_i := \{p \in \mathcal{F} : |p \cap Y| = i\}.$$

Note that we can without loss of generality assume that $\mathcal{F}_2$ or $\mathcal{F}_3$ has infinite cardinality.

Case 1: $\mathcal{F}_3$ has infinite cardinality.
For every $p \in \mathcal{F}_3$ let

$$p^* := \{\{a,x\} \in [p]^2 : a \in p \cap Y \wedge x \in p \setminus Y\}$$

and apply RC_7 on the set $X^* := \bigcup\{p^* : p \in \mathcal{F}_3\}$. We get an infinite subset $Y^* \subseteq X^*$ with a choice function $g : [Y^*]^7 \to Y^*$. For every $1 \le i \le 12$ define

$$\mathcal{F}_i^* := \{p^* : p \in \mathcal{F}_3 \wedge |p^* \cap Y^*| = i\}.$$

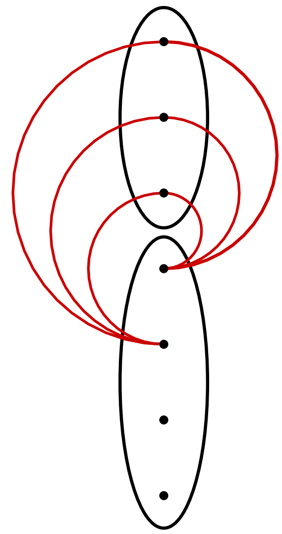

Figure 3: Case $i = 6$

There is an i with $1 \leq i \leq 12$ such that $|\mathcal{F}_i^*| = \infty$. If $i \notin \{6, 12\}$ we can choose an element from each p with $p^* \in \mathcal{F}_i^*$ and therefore we are done. If $i = 6$, the only case in which we cannot choose an element from all p with $p^* \in \mathcal{F}_6^*$ is the one illustrated in Figure 3:

But in this case we are done by Lemma 3.8. And if $i = 12$ we are done by Lemma 3.9.

Case 2: $\mathcal{F}_2$ has infinite cardinality.
For every $1 \leq i \leq 10$ we define $\mathcal{F}_i^*$ as in Case 1. The only i for which we cannot choose one element from each p with $p^* \in \mathcal{F}_i^*$ or for which we cannot choose three elements from each p with $p^* \in \mathcal{F}_i^*$ in order to reduce it to Case 1, is $i = 10$. But in this case we are done by Lemma 3.10. $\dashv$

4 Open Questions

1. By [6] we have that $\mathrm{RC}_n \Rightarrow \mathrm{C}_n^-$ in ZF for every $n \in \{2, 3, 4\}$. Does this implication hold for any other $n \in \omega \setminus \{0, 1\}$?

2. By [6], Proposition 3.11 and Theorem 3.7 we have that $\mathrm{RC}_n \Rightarrow \mathrm{LOC}_n^-$ in ZF for any $n \in \{2, 3, 4, 5, 7\}$. Does this implication hold for any other $n \in \omega \setminus \{0, 1\}$?

3. For every $n \in \omega \setminus \{0, 1\}$ the following weak choice principle was introduced in [8]:

   $\mathrm{nC}^-_{<\aleph_0}$*: For every infinite family $\mathcal{F}$ of finite sets with cardinality at least n there is an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a selection function $f\colon \mathcal{G} \to [\bigcup \mathcal{G}]^n$ such that $f(G) \in [G]^n$ for all $G \in \mathcal{G}$.*

   Moreover, as in [1] we can define a restricted version of $\mathrm{nC}^-_{<\aleph_0}$ as follows:

   $\mathrm{nRC}_{\mathrm{fin}}$*: Given any infinite set x, there is an infinite subset $y \subseteq x$ and a selection function f that chooses an n-element subset from every $z \subseteq y$ containing at least n elements.*

The relationship of RC_n and $\mathrm{nRC}_{\mathrm{fin}}$ to $\mathrm{kC}^-_{<\aleph_0}$ and C^-_j has already been studied in [3]. However, the following question is still open: For every $n \in \{2,3,4,6\}$ we have that $\mathrm{nRC}_{\mathrm{fin}} \Rightarrow \mathrm{nC}^-_{\aleph_0}$ in ZF. Does this implication hold for any other $n \in \omega \setminus \{0,1\}$?

Acknowledgement: We would like to thank the referee for her or his careful reading and the numerous comments and corrections that helped to improve the quality of this article.

References

[1] Omar De La Cruz and Carlos A. Di Prisco: Weak choice principles. Proceedings of the American Mathematical Society 126 (3) (1998), pp. 867-876.

[2] Lorenz Halbeisen, Norbert Hungerbühler, Nir Lazarovich, Waltraud Lederle, Marc Lischka and Salome Schumacher: Forms of choice in ring theory. Results in Mathematics 74(1) (2019) Article:14.

[3] Lorenz Halbeisen, Riccardo Plati and Salome Schumacher: A new weak choice principle. arXiv: 2101.07840

[4] Lorenz Halbeisen and Eleftherios Tachtsis: On Ramsey choice and partial choice for infinite families of n-element sets. Archive for Mathematical Logik 59(5) (2020), pp. 583-606.

[5] Lorenz J. Halbeisen: Combinatorial Set Theory - With a Gentle Introduction to Forcing. Springer-Verlag London, 2012.

[6] Carlos H. Montenegro: Weak versions of the axiom of choice for families of finite sets. In: ***Models, Algebras, and Proofs***, *Selected Papers of the X Latin American Symposium on Mathematical Logic,* Bogotà, Colombia, June 24-29, 1995, X. Caicedo, C. Montenegro (eds.), Lecture Notes in Pure and Applied Mathematics, vol. 203, pp. 57-60. Dekker, New York (1999)

[7] David Pincus: Zermelo-Fraenkel consistency results by Fraenkel-Mostowski methods. The Journal of Symbolic Logic 37 (1972), pp. 721-743.

[8] Salome Schumacher: The relation between two diminished choice principles. The Journal of Symbolic Logic 86(1) (2021), pp. 415-432.