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Implications of Ramsey Choice Principles in ZF

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The Ramsey Choice principle for families of n-element sets, denoted RC_n, states that every infinite set X has an infinite subset $Y \subseteq X$ with a choice function on $[Y]^n := \{z \subseteq Y : |z| = n\}$. We investigate for which positive integers m and n the implication $\mathrm{RC}_m \Rightarrow \mathrm{RC}_n$ is provable in ZF. It will turn out that beside the trivial implications $\mathrm{RC}_m \Rightarrow \mathrm{RC}_m$, under the assumption that every odd integer $n > 5$ is the sum of three primes (known as ternary Goldbach conjecture), the only non-trivial implication which is provable in ZF is $\mathrm{RC}_2 \Rightarrow \mathrm{RC}_4$.

1 Introduction

For positive integers n, the Ramsey Choice principle for families of n-element sets, denoted RC_n, is defined as follows: For every infinite set X there is an infinite subset $Y \subseteq X$ such that the set $[Y]^n := \{z \subseteq Y : |z| = n\}$ has a choice function. The Ramsey Choice principle was introduced by Montenegro [1] who showed that for $n = 2, 3, 4$, $\mathrm{RC}_n \Rightarrow \mathrm{C}_n^-$. where C_n^- is the statement that every infinite family of n-element has an infinite subfamily with a choice function. However, the question of whether or not $\mathrm{RC}_n \to \mathsf{C}_n^-$ for $n \geq 5$ is still open (for partial answers to this question see [2, 3]).

In this paper, we investigate the relation between RC_n and RC_m for positive integers n and m. First, for each positive integer m we construct a permutation models $\mathbf{MOD}_m$ in which RC_m holds, and then we show that RC_n fails in $\mathbf{MOD}_m$ for certain integers n. In particular, assuming the ternary Goldbach conjecture, which states that every odd integer $n > 5$ is the sum of three primes, and by the transfer principles of Pincus [4], we we obtain that for $m, n \geq 2$, the implication $\mathrm{RC}_m \Rightarrow \mathrm{RC}_n$ is not provable in ZF except in the case when $m = n$, or when $m = 2$ and $n = 4$.

FACT 1.1 *The implications* $\mathrm{RC}_m \Rightarrow \mathrm{RC}_m$ *(for* $m \geq 1$*) and* $\mathrm{RC}_2 \Rightarrow \mathrm{RC}_4$ *are provable in* ZF.

P r o o f. The implication $\mathrm{RC}_m \Rightarrow \mathrm{RC}_m$ is trivial. To see that $\mathrm{RC}_2 \Rightarrow \mathrm{RC}_4$ is provable in ZF, we assume RC_2. If X is an infinite set, then by RC_2 there is an infinite subset $Y \subseteq X$ such that $[Y]^2$ has a choice function f_2. Now, for any $z \in [Y]^4$, $[z]^2$ is a 6-element subset of $[Y]^2$, and by the choice function f_2 we can select an element from each 2-element subset of z. For any $z \in [Y]^4$ and each $a \in z$, let $\nu_z(a) := |\{x \in [z]^2 : f_2(x) = a\}|$, $m_z := \min\{\nu_z(a) : a \in z\}$, and $M_z := \{a \in z : \nu_z(a) = m_z\}$. Since f_2 is a choice function, we have

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$\sum_{a\in z}\nu_z(a)=6$, and since $4\nmid 6$, the function $f:[Y]^4\to Y$ defined by stipulating

$$f(z):=\begin{cases} a & \text{if } M_z=\{a\},\\ b & \text{if } z\setminus M_z=\{b\},\\ c & \text{if } |M_z|=2 \text{ and } f_2(M_z)=c,\end{cases}$$

is a choice function on $[Y]^4$, which shows that RC_4 holds. □

2 A model in which RC_m holds

In this section we construct a permutation model $\mathbf{MOD}_m$ in which RC_m holds. According to [5, p. 211 *ff.*], the model $\mathbf{MOD}_m$ is a *Shelah Model of the Second Type*.

Fix an integer $m\geq 2$ and let $\mathcal{L}_m$ be the signature containing the relation symbol Sel_m. Let T_m be the $\mathcal{L}_m$-theory containing the following axiom-schema:

For all pairwise different $x_1,\dots,x_m$, there exists a unique index $i\in\{1,\dots,m\}$ such that, whenever $\{b_1,\dots,b_m\}=\{1,\dots,m\}$,

$$\mathsf{Sel}_m(x_{b_1},\dots,x_{b_m},x_b)\iff b=i.$$

In other words, Sel_m is a selecting function which selects an element from each m-element set $\{x_1,\dots,x_m\}$. In any model of the theory T_m, the relation Sel_m is equivalent to a function Sel which selects a unique element from any m-element set.

For a model $\boldsymbol{M}$ of T_m with domain M, we will simply write $M\models\mathsf{T}_m$. Let

$$\widetilde{C}=\{M : M\in\operatorname{fin}(\omega)\wedge M\models\mathsf{T}_m\}.$$

Evidently $\widetilde{C}\neq\emptyset$. Partition $\widetilde{C}$ into maximal isomorphism classes and let C be a set of representatives. We proceed with the construction of the set of atoms for our permutation model. With the next result, taken from [5], we give an explicit construction of the Fraïssé limit of the finite models of T_m.

PROPOSITION 2.1 *Let $m\in\omega\setminus\{0\}$. There exists a model $\mathbf{F}\models\mathsf{T}_m$ with domain ω such that*

- *Given a non empty $M\in C$, $\mathbf{F}$ admits infinitely many submodels isomorphic to M.*
- *Any isomorphism between two finite submodels of $\mathbf{F}$ can be extended to an automorphism of $\mathbf{F}$.*

Proof. The construction of $\mathbf{F}$ is made by induction. Let $F_0=\emptyset$. F_0 is trivially a model of T_m and, for every element M of C with $|M|\leq 0$, F_0 contains a submodel isomorphic to M. Let F_n be a model of T_m with a finite initial segment of ω as domain and such that for every $M\in C$ with $|M|\leq n$, F_n contains a submodel isomorphic to M. Let

- $\{A_i : i\leq p\}$ be an enumeration of $[F_n]^{\leq n}$,
- $\{R_k : k\leq q\}$ be an enumeration of all the $M\in C$ such that $1\leq |M|\leq n+1$,
- $\{j_l : l\leq u\}$ be an enumeration of all the embeddings $j_l : F_n|_{A_i}\hookrightarrow R_k$, where $i\leq p$, $k\leq q$ and $|R_k|=|A_i|+1$.

For each $l\leq u$, let $a_l\in\omega$ be the least natural number such that $a_l\notin F_n\cup\{a_{l'} : l'<l\}$. The idea is to add a_l to F_n, extending $F_n|_{A_i}$ to a model $F_n|_{A_i}\cup\{a_l\}$ isomorphic to R_k, where $j_l : F_n|_{A_i}\hookrightarrow R_k$. Define $F_{n+1}:=F_n\cup\{a_l : l\leq u\}$ and make F_{n+1} into a model of T_m by choosing a way of defining the function Sel on the missing subsets. The desired model is finally given by $\mathbf{F}=\bigcup_{n\in\omega}F_n$.

We conclude by showing that every isomorphism between finite submodels can be extended to an automorphism of $\mathbf{F}$ with a back-and-forth argument. Let $i_0 : M_1 \to M_2$ be an isomorphism of T_m-models. Let a_1 be the least natural number in $\omega \setminus M_1$. Then $M_1 \cup \{a_1\}$ is contained in some F_n and by construction we can find some $a_1' \in \omega \setminus M_2$ such that $\mathbf{F}|_{M_1 \cup \{a_1\}}$ is isomorphic to $\mathbf{F}|_{M_2 \cup \{a_1'\}}$. Extend i_0 to $l_1 \colon M_1 \cup \{a_1\} \to M_2 \cup \{a_1'\}$ by imposing $l_1(a_1) = a_1'$. Let b_1' be the least integer in $\omega \setminus (M_2 \cup \{a_1'\})$ and similarly find some $b_1 \in \omega \setminus (M_1 \cup \{a_1\})$ such that we can extend l_1 to an isomorphism $i_1 \colon M_1 \cup \{a_1, b_1\} \to M_2 \cup \{a_1', b_1'\}$ which maps b_1 to b_1'. Repeating the process countably many times, the desired automorphism of $\mathbf{F}$ is given by $i = \bigcup_{n \in \omega} i_n$. □

REMARK 1 Let us fix some notations and terminology. The elements of the model $\mathbf{F}$ above constructed will be the atoms of our permutation model. Each element a corresponds to a unique embedding j. We shall call the domain of j the *ground* of a. Moreover, given two atoms a and b, we say that $a < b$ in case $a <_\omega b$ according to the natural ordering. Notice that this well ordering of the atoms will not exist in the permutation model.

Let A be the domain of the model $\mathbf{F}$ of the theory T_m. To build the permutation model $\mathbf{MOD}_m$, consider the normal ideal given by all the finite subsets of A and the group of permutations G defined by

$$\pi \in G \iff \forall X \in [\omega]^m, \pi(\mathsf{Sel}(X)) = \mathsf{Sel}(\pi X).$$

Theorem 2.1 *For every positive integer m, $\mathbf{MOD}_m$ is a model for RC_m.*

Proof. Let X be an infinite set with support S'. If X is well ordered, the conclusion is trivial, so let $x \in X$ be an element not supported by S' and let S be a support of x, with $S' \subseteq S$. Let $a \in S \setminus S'$. If $\mathrm{fix}_G(S \setminus \{a\}) \subseteq \mathrm{sym}_G(x)$ then $S \setminus \{a\}$ is a support of x, so by iterating the process finitely many times we can assume that there exists a permutation $\tau \in \mathrm{fix}_G(S \setminus \{a\})$ such that $\tau(x) \neq x$. Our conclusion will follow by showing that there is a bijection between an infinite set of atoms and a subset of X, namely between $I = \{\pi(a) : \pi \in \mathrm{fix}_G(S \setminus \{a\})\}$ and $\{\pi(x) : \pi \in \mathrm{fix}_G(S \setminus \{a\})\}$. First, notice that for $\pi \in \mathrm{fix}_G(S \setminus \{a\})$ the function $f \colon \pi(a) \mapsto \pi(x)$ is well defined on I. Indeed, if for some $\sigma, \pi \in \mathrm{fix}_G(S \setminus \{a\})$ we have $\sigma(x) \neq \pi(x)$, then $\pi^{-1}\sigma(x) \neq x$, which implies $\pi^{-1}\sigma(a) \neq a$ since S is a support of x. To show that f is also injective, suppose towards a contradiction that there are two permutations $\sigma, \sigma' \in \mathrm{fix}_G(S \setminus \{a\})$ such that $\sigma(x) = \sigma'(x)$ and $\sigma(a) \neq \sigma'(a)$. Then, by direct computation, the permutation $\sigma^{-1}\sigma'$ is such that $\sigma^{-1}\sigma'(a) \neq a$ and $\sigma^{-1}\sigma'(x) = x$. Let $b = \sigma^{-1}\sigma'(a)$. Now, by assumption there is a permutation $\tau \in \mathrm{fix}_G(S \setminus \{a\})$ such that $\tau(x) \neq x$. Let $y := \tau(x)$, with $c = \tau(a)$ and $d = \sigma^{-1}\sigma'(c)$. Notice that from $f(a) = f(b)$ we get $f(c) = f(d)$. Let now $e \in A$ be an atom with ground $S \cup \{c\}$ such that e behaves like b with respect to S and like d with respect to $(S \setminus \{a\}) \cup \{c\}$. This is possible by construction of the set of atoms since b and d behave in the same way with respect to $S \setminus \{a\}$. It follows that there are permutations $\pi_b \in \mathrm{fix}_G(S)$ and $\pi_d \in \mathrm{fix}_G((S \setminus \{a\}) \cup \{c\})$ with $\pi_b(b) = e$ and $\pi_d(d) = e$. Let us now consider $f(e)$. On the one hand, since $(S \setminus \{a\}) \cup \{c\}$ is a support of $y = f(d)$, we have $y = \pi_d(f(d)) = f(\pi_d(d)) = f(e)$. On the other hand, since S is a support of $x = f(b)$, we have $x = \pi_b(f(b)) = f(\pi_b(b)) = f(e)$, contradicting the fact that $x \neq y$. □

3 For which n is $\mathbf{MOD}_m$ a model for RC_n ?

The following result shows that for positive integers m, n which satisfy a certain condition, the implication $\mathrm{RC}_m \Rightarrow \mathrm{RC}_n$ is not provable in ZF. Assuming the ternary Goldbach conjecture, it will turn out that all positive integers m, n satisfy this condition, except when $m = n$, or when $m = 2$ and $n = 4$.

DEFINITION 3.1 *Given $n \in \omega$, a decomposition of n is a finite sequence $(n_i)_{i \in k}$ with each $n_i \in \omega \setminus \{1\}$ so that $n = \sum_{i \in k} n_i$.*

DEFINITION 3.2 *Given two natural numbers n and m, a decomposition $(n_i)_{i \in k}$ of n is said to be beautiful for the pair (m, n) if, given any decomposition $(m_i)_{i \in k}$ of m of length k such that for all $i \in k$ we have $m_i \leq n_i$, then there is some $j \in k$ with $\gcd(m_j, n_j) = 1$.*

In what follows, when we refer to a decomposition of some n being beautiful, we mean that the decomposition is beautiful for (m, n). It will always be clear from the context to which pair (m, n) we refer.

PROPOSITION 3.3 *Let $m, n \in \omega$. If there is a decomposition of n which is beautiful, then the implication $\mathrm{RC}_m \Rightarrow \mathrm{RC}_n$ is not provable in ZF.*

REMARK 2 The condition on m and n is somewhat similar to the condition given in Theorem 2.10 of Halbeisen and Schumacher [2]. Let WOC_n^- be the statement that every infinite, well-orderable family $\mathcal{F}$ of sets of size n has an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a choice function. Then for every $m, n \in \omega \setminus \{0,1\}$, the implication $\mathrm{RC}_m \Rightarrow \mathrm{WOC}_n^-$ is provable in ZF if an only if the following condition holds: Whenever we can write n in the form

$$n = \sum_{i<k} a_i p_i,$$

where $p_0, \dots, p_{k-1}$ are prime numbers and $a_0, \dots, a_{k-1} \in \omega \setminus \{0\}$, then we find integers $b_0, \dots, b_{k-1} \in \omega$ with

$$m = \sum_{i<k} b_i p_i.$$

Proof of Propostion 3.3. We show that in $\mathbf{MOD}_m$, RC_n fails. Assume towards a contradiction that RC_n holds in $\mathbf{MOD}_m$ and let S be a support of a selection function f on the n-element subsets of an infinite subset X of the set of atoms A.

Given any finite model N of T_m extending S, we can find a submodel of $X \cup S$ isomoprhic to N. Indeed, start by noticing that, since S is a support of f and X is the domain of f, we have that X is symmetric. Then the claim follows directly from the construction in Proposition 2.1, as atoms whose ground includes the support of $X \cup S$ can belong to $X \cup S$ and can behave in arbitrarily chosen ways with respect to each other.

Our conclusion can hence follow from finding a model M of T_m which extends S with $|M \setminus S| = n$ and such that M admits an auotmorphism σ which fixes pointwise S and which does not have any other fixed point, since then $\sigma(f(M \setminus S)) \neq f(M \setminus S)$ but $\sigma(M \setminus S) = M \setminus S$. We start with the following claim:

Claim 3.1 *Given a cyclic permutation π on some set P of cardinality $|P| = q$, if a non-trivial power π^r of π fixes a proper subset P' of P, then* $\gcd(|P'|, |P|) > 1$.

To prove the claim, notice that π^r is a disjoint union of cycles of the same length $l = \frac{q}{\gcd(q,r)}$. Consider the subgroup of $\langle \pi \rangle$ given by $\langle \pi^r \rangle$. Then P' is a disjoint union of orbits of the form $\mathrm{Orb}_{<\pi^r>}(e)$ with $e \in P'$, all of them with the same cardinality s, with s being a divisor of $l = \frac{q}{\gcd(q,r)}$ and hence of q, from which we deduce the claim.

Now, given a beautiful decomposition $(n_i)_{i \in k}$ of n, we want to show that we can find a model M of T_m, which extends S with $|M \setminus S| = n$ and such that it admits an automorphism σ which fixes pointwise S and acts on $M \setminus S$ as a disjoint union of k cycles, each of length n_i for $i \in k$. This can be done as follows. Pick an m-element subset P of M for which $\mathsf{Sel}(P)$ has not been defined yet. If $P \cap S \neq \emptyset$ then let $\mathsf{Sel}(P)$ be any element in $P \cap S$. Otherwise, by our the assumptions, there is a cycle C_j of length n_j for some $j \in k$ such that $\gcd(|P \cap C_j|, |C_j|) = 1$. Define $\mathsf{Sel}(P)$ as an arbitrarily fixed element of $P \cap C_j$ and, for all permutations π in the group generated by σ, define $\mathsf{Sel}(\pi(P)) = \pi(\mathsf{Sel}(P))$. We need to argue that this is indeed well defined, i.e. that for two permutations $\pi, \pi' \in \langle \sigma \rangle$ we have that $\pi(P) = \pi'(P)$ implies $\pi(\mathsf{Sel}(P)) = \pi'(\mathsf{Sel}(P))$. Problems can arise only when $P \cap S = \emptyset$, in which case we notice that $\pi(P) = \pi'(P)$ implies $\pi(P \cap C_j) = \pi'(P \cap C_j)$, which in turn by the claim implies that $\pi^{-1} \circ \pi'$ fixes $P \cap C_j$ pointwise, from which we deduce $\pi(\mathsf{Sel}(P)) = \pi'(\mathsf{Sel}(P))$. □

Proposition 3.3 allows us to immediately deduce the following results.

Corollary 3.2 *If $m > n$, then RC_m does not imply RC_n.*

Proof. The decomposition $n = \sum_{i \in 1} n_i$ with $n_0 = n$ is clearly beautiful, so we can directly apply Proposition 3.3. □

Corollary 3.3 *If there is a prime p for which $p \mid n$ but $p \nmid m$, then RC_m does not imply RC_n.*

Proof. Given the assumption, the decomposition of n given by $n = \sum_{i \in \frac{n}{p}} n_i$, where each $n_i = p$, is beautiful, so we can apply Proposition 3.3. □

Moreover, we can show the following:

Theorem 3.4 *For any positive integers m and n, the implication $\mathrm{RC}_m \Rightarrow \mathrm{RC}_n$ is provable in* ZF *only in the case when $m = n$ or when $m = 2$ and $n = 4$.*

The proof of Theorem 3.4 is given in the following results, where in the proofs we use two well-known number-theoretical results: The first one is Bertrand's postulate, which asserts that for every positive integer $m \geq 2$ there is a prime p with $m < p < 2m$, and the second one is ternary Goldbach conjecture (assumed to be proven by Helfgott [6]), which asserts that every odd integer $n > 5$ is the sum of three primes.

PROPOSITION 3.4 *If m is prime and $n \neq m$ with $(m, n) \neq (2, 4)$, then the implication $\mathrm{RC}_m \Rightarrow \mathrm{RC}_n$ is not provable in* ZF

Proof. Given Corollary 3.3, we can assume that $n = m^k$ for some natural number $k > 1$. Let p be a prime such that $m < p < 2m$, whose existence is guaranteed by Bertrand's postulate. Then clearly $m \nmid n - p$, from which, considering that because of parity reasons $n - p \neq 1$, we get that the decomposition $n = p + (n - p)$ is beautiful. □

PROPOSITION 3.5 *If n is odd and $m \neq n$, then the implication $\mathrm{RC}_m \Rightarrow \mathrm{RC}_n$ is not provable in* ZF.

Proof. By the ternary Goldbach conjecture, let us write n as sum of three primes $n = p_0 + p_1 + p_2$. Given Proposition 3.4, we can assume that $m = p_0 + p_1$, since otherwise the decomposition $n = p_0 + p_1 + p_2$ would be beautiful.

We first deal with the case in which $p_0 = p_1 = p_2$ holds, for which we rename $p = p_0$. By hand we can exclude the case $p = 2$, and now we want to show that the decomposition $n = n_0 + n_1 = (3p - 2) + 2$ is beautiful. Notice that $\gcd(3p - 2, 2p - 2) \in \{1, p\}$, from which we deduce that necessarily if $m = m_0 + m_1$ is a decomposition of m with $m_0 \leq 3p - 2$ and $m_1 \leq 2$, then $m_1 = 0$. To conclude this first case, it suffices to notice that, since p is a prime grater than 2, $\gcd(3p - 2, 2p)$ necessarily equals 1.

We can now assume that it is not true that $p_0 = p_1 = p_2$. Since n is odd, $p_0 + p_1 \nmid p_2$. If $p_2 \nmid p_0 + p_1$, then the decomposition $n = n$ is actually beautiful. So, given $p_2 \mid p_0 + p_1$, without loss of generality let us assume that $p_2 < p_0$. By $p_2 \mid p_0 + p_1$ we deduce that $p_1 \neq p_2$, and we now consider the decomposition $n = n_0 + n_1 = (p_1 + p_2) + p_0$. We can't have $m_1 = p_0$ since $\gcd(p_1, p_1 + p_2) = 1$. On the other hand, we can't even have $m_1 = 0$ since $p_0 + p_1 > p_1 + p_2$, which proves that the assumptions of Proposition 3.3 are satisfied. □

PROPOSITION 3.6 *Let $m > 2$ be an even natural number and $k \in \omega$ such that $2^k + 1$ is prime. If $n = m + 2^k$, then the implication $\mathrm{RC}_m \Rightarrow \mathrm{RC}_n$ is not provable in* ZF.

Proof. We consider the decomposition $n = n_0 + n_1 = (m - 1) + (2^k + 1)$. It directly follows from the assumptions of the proposition that in order to have a decomposition $m = m_0 + m_1$ which disproves the fact that the above decomposition of n is beautiful, since $n_0 < m$, necessarily $m_1 = 2^k + 1$, from which we deduce $m_0 = m - 2^k - 1$. This immediately gives a contradiction in the case $2^k + 1 > m$, so let us assume $2^k + 1 < m$. We get again a contradiction by the fact that $\gcd(m_0, n_0) = \gcd(m - 2^k - 1, m - 1) = \gcd(2^k, m - 1) = 1$, where we used that m is even. We can hence conclude that the decomposition $n = (m - 1) + (2^k + 1)$ is indeed beautiful. □

PROPOSITION 3.7 *Let m and n be even natural numbers such that there is an odd prime p with $m < p < n$ and $n > p + 1$. Then the implication $\mathrm{RC}_m \Rightarrow \mathrm{RC}_n$ is not provable in* ZF.

Proof. If $n = p + 3$ or $n = p + 5$ the decomposition $n = p + (n - p)$ is already beautiful. Otherwise, by the ternary Goldbach conjecture, write $n - p$ as sum of three primes $n - p = p_0 + p_1 + p_2$. Consider now the decomposition $n = \sum_{i \in 4} n_i = p + p_0 + p_1 + p_2$. In order to write $m = \sum_{i \in 4} m_i$, necessarily $m_0 = 0$. If $n - p < m$ we can already conclude that $n = p + p_0 + p_1 + p_2$ is a beautiful decomposition. Otherwise, we find ourselves in the assumptions of Proposition 3.5, which again allows us to conclude that RC_m does not imply RC_n. □

The following result deals with all the remaining cases and completes the proof of Theorem 3.4.

PROPOSITION 3.8 *Let m and n be even natural numbers with $3 \leq \frac{n}{2} \leq m < n$ such that if there is a prime p with $m < p < n$, then $p = n - 1$. Then the implication $\mathrm{RC}_m \Rightarrow \mathrm{RC}_n$ is not provable in* ZF.

Proof. By Bertrand's postulate, let p be a prime with $\frac{n}{2} < p < n$. This implies by the assumption $\frac{n}{2} < p < m$ or $p = n - 1$. If we are in the latter case, apply again Bertrand's postulate to find a further prime $\frac{n}{2} - 1 < p' < n - 2$ (notice that by our assumption we have $2 \leq \frac{n}{2} - 1$). Since m is not prime we necessarily have $p' \neq m$, which together with the present assumptions makes us able to assume without loss of generality that $\frac{n}{2} < p < m$. Given that $n - m$ is even, by Proposition 3.6 we can assume $n - m > 4$, which in turn implies $n - p > 5$. Since by the ternary Goldbach conjecture we can write $n = p + p_0 + p_1 + p_2$ with $m > p_0 + p_1 + p_2$, notice that by the fact that n and m are even, we can assume that $m - p$ equals some odd prime p', since otherwise the decomposition $n = p + p_0 + p_1 + p_2$ would already be beautiful. Now, either $n = p + (n - p)$ is beautiful, or $n - p$ is a multiple of p'. We distinguish two cases, namely when $n - p$ is a power of p' and when it is not. In the second case, let p'' be a prime distinct from p' such that $p'' \mid n - p$. The decomposition of n given by $n = n_0 + \sum_{i \in \frac{n-p}{p''}} n_i = p + \sum_{i \in \frac{n-m}{p''}} p''$ is beautiful, as $n - p < m$ and hence if $m = m_0 + \sum_{i \in \frac{n-m}{p'}} m_i$ then $m_0 = p$. For the last case, without loss of generality assume that $p_0 + p_1 + p_2 = p_0^k$ for some natural number $k > 1$. If $p_0 = p_1 = p_2 = 3$, we decompose $9 = n - p$ as $5 + 2 + 2$, so we can assume $p_0^{k-1} - 2 \neq 1$. Now we get $p_2 \neq p_0$, since otherwise we would have $p_1 = p_0^k - 2p_0 = p_0(p_0^{k-1} - 2)$, which is a contradiction, and similarly we obtain $p_1 \neq p_0$. We finally assume wlog that $p_1 > p_0$, which allows us to conclude that the decomposition $n = p + p_1 + (p_0 + p_2)$ is in this case beautiful, concluding the proof. □

For the sake of completeness, we summarise the proof of our main theorem:

Proof of Theorem 3.4. Let m and n be two distinct positive integers.

$$\mathsf{ZF} \vdash \mathrm{RC}_m \Rightarrow \mathrm{RC}_n \quad \overset{\text{Cor. 3.4}}{\Longrightarrow} \quad m \leq n \quad \overset{\text{Prp. 3.8}}{\Longrightarrow} \quad n \text{ is even} \quad \overset{\text{Cor. 3.5}}{\Longrightarrow} \quad m \text{ is even}$$

Now, if m and n are both even, we have the following two cases:

$$m < \frac{n}{2} \quad \overset{\text{Prp. 3.10}}{\Longrightarrow} \quad \mathsf{ZF} \nvdash \mathrm{RC}_m \Rightarrow \mathrm{RC}_n$$

$$m \geq \frac{n}{2} \geq 3 \quad \underset{\text{Prp. 3.10}}{\overset{\text{Prp. 3.11}}{\Longrightarrow}} \quad \mathsf{ZF} \nvdash \mathrm{RC}_m \Rightarrow \mathrm{RC}_n$$

Thus, by Fact 1.1, the implication $\mathrm{RC}_m \Rightarrow \mathrm{RC}_n$ is provable in ZF if and only if $m = n$ or $m = 2$ and $n = 4$. □

REMARK 3 The proof of the implication $\mathrm{RC}_2 \Rightarrow \mathrm{RC}_4$ (Fact 1.1) is very similar to the proof of the implication $\mathrm{C}_2 \Rightarrow \mathrm{C}_4$, where C_n states that every family n-element sets has a choice function. Moreover, similar to the proof of $\mathrm{C}_2 \wedge \mathrm{C}_3 \Rightarrow \mathrm{C}_6$ one can proof the implication $\mathrm{RC}_2 \wedge \mathrm{RC}_3 \Rightarrow \mathrm{RC}_6$. So, it might be interesting to investigate which implications of the form

$$\mathrm{RC}_{m_1} \wedge \cdots \wedge \mathrm{RC}_{m_k} \Rightarrow \mathrm{RC}_n$$

are provable in ZF and compare them with the corresponding implications for C_n's. Since $\mathrm{C}_4 \Rightarrow \mathrm{C}_2$ but $\mathrm{RC}_4 \not\Rightarrow \mathrm{RC}_2$, the conditions for the RC_n's are clearly different from the conditions for the C_n's (see Halbeisen and Tachtsis [3] for some results in this direction).

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