On Ramsey Choice and Partial Choice for infinite families of n-element sets

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Abstract

For an integer $n \geq 2$, *Ramsey Choice* RC_n is the weak choice principle "*every infinite set x has an infinite subset y such that $[y]^n$ (the set of all n-element subsets of y) has a choice function*", and C_n^- is the weak choice principle "*every infinite family of n-element sets has an infinite subfamily with a choice function*".

In 1995, Montenegro showed that for $n = 2, 3, 4$, $\mathsf{RC}_n \to \mathsf{C}_n^-$. However, the question of whether or not $\mathsf{RC}_n \to \mathsf{C}_n^-$ for $n \geq 5$ is still open. In general, for distinct $m, n \geq 2$, not even the status of "$\mathsf{RC}_n \to \mathsf{C}_m^-$" or "$\mathsf{RC}_n \to \mathsf{RC}_m$" is known.

In this paper, we provide partial answers to the above open problems and among other results, we establish the following:

1. For every integer $n \geq 2$, if RC_i is true for all integers i with $2 \leq i \leq n$, then C_i^- is true for all integers i with $2 \leq i \leq n$.
2. If $m, n \geq 2$ are any integers such that for some prime p we have $p \nmid m$ and $p \mid n$, then in ZF: $\mathsf{RC}_m \nrightarrow \mathsf{RC}_n$ and $\mathsf{RC}_m \nrightarrow \mathsf{C}_n^-$.
3. For $n = 2, 3$, $\mathsf{RC}_5 + \mathsf{C}_n^-$ implies C_5^-, and RC_5 implies neither C_2^- nor C_3^- in ZF.
4. For every integer $k \geq 2$, RC_{2k} implies "*every infinite linearly orderable family of k-element sets has a partial Kinna–Wagner selection function*" and the latter implication is not reversible in ZF (for any $k \in \omega \setminus \{0, 1\}$). In particular, RC_6 strictly implies "*every infinite linearly orderable family of 3-element sets has a partial choice function*".
5. The *Chain-AntiChain Principle* ("*every infinite partially ordered set has either an infinite chain or an infinite anti-chain*") implies neither RC_n nor C_n^- in ZF, for every integer $n \geq 2$.

Keywords Axiom of Choice, weak forms of the Axiom of Choice, Ramsey Choice, Partial Choice for infinite families of n-element sets, Ramsey's Theorem, Chain-AntiChain Principle, Fraenkel–Mostowski permutation models of $\mathsf{ZFA} + \neg\mathsf{AC}$, Pincus' Transfer Theorems.

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1 Notation and terminology

Notation 1

1. As usual, ω denotes the set of natural numbers.

2. Let $n \in \omega$ and let X be a set. Then $[X]^n$ denotes the set of n-element subsets of X. Furthermore, $[X]^{<\omega}$ denotes the set of finite subsets of X. Clearly $[X]^{<\omega} = \bigcup\{[X]^n : n \in \omega\}$.

3. ZF is Zermelo–Fraenkel set theory without the Axiom of Choice (AC).

4. ZFC is $\mathsf{ZF} + \mathsf{AC}$.

5. ZFA is ZF with the Axiom of Extensionality modified in order to allow the existence of atoms.

Next, we list the statements and notations of the weak choice principles that will be used in this paper.

Definition 1

1. *Ramsey's Theorem* (RT): For every infinite set X and for every partition of the set $[X]^2$ of two-element subsets of X into two sets A and B, there is an infinite subset Y of X such that either $[Y]^2 \subseteq A$ or $[Y]^2 \subseteq B$.

2. Let $n \in \omega \setminus \{0, 1\}$.

   Ramsey Choice RC_n: For every infinite set X there is an infinite subset $Y \subseteq X$ such that $[Y]^n$ has a choice function.

   C_n: Every family of n-element sets has a choice function.

   C_n^-: Every infinite family $\mathcal{A}$ of n-element sets has a partial choice function (i.e., $\mathcal{A}$ has an infinite subfamily $\mathcal{B}$ with a choice function).

   LOC_n^-: Every infinite linearly orderable family of n-element sets has a partial choice function.

   LOKW_n^-: Every infinite linearly orderable family $\mathcal{A}$ of n-element sets has a partial Kinna–Wagner selection function, i.e., there exists an infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ and a function f such that $\mathrm{dom}(f) = \mathcal{B}$ and for all $B \in \mathcal{B}$, $\emptyset \neq f(B) \subsetneq B$ (f is called a *Kinna–Wagner selection function* for $\mathcal{B}$).

   WOC_n^-: Every infinite well-orderable family of n-element sets has a partial choice function.

3. $\mathsf{AC}_{\mathsf{fin}}$: Every family of non-empty finite sets has a choice function.

4. $\mathsf{PAC}_{\mathsf{fin}}$: Every infinite family of non-empty finite sets has a partial choice function.

5. $\mathsf{AC}(\mathsf{LO}, \mathsf{LO})$: Every linearly orderable family of non-empty linearly orderable sets has a choice function.

6. $\mathsf{UT}(\mathsf{WO}, \mathsf{fin}, \mathsf{WO})$: The union of a well-orderable family of finite sets is well-orderable.

7. $\mathsf{DF} = \mathsf{F}$: Every Dedekind-finite set is finite (where a set X is called *Dedekind-finite* if there is no one-to-one mapping f from ω into X; otherwise, X is called *Dedekind-infinite*).

8. *Axiom of Multiple Choice* (MC): For every family $\mathcal{A}$ of non-empty sets there is a function f on $\mathcal{A}$ such that for every $x \in \mathcal{A}$, $f(x)$ is a nonempty finite subset of x (f is called a *multiple choice function* for $\mathcal{A}$).

9. *Boolean Prime Ideal Theorem* (BPI): Every Boolean algebra has a prime ideal.

10. *Ordering Principle* (OP): Every set can be linearly ordered.

11. *Chain-AntiChain Principle* (CAC): Every infinite partially ordered set has either an infinite chain or an infinite anti-chain (where for a partially ordered set $(P, \leq)$, a set $C \subseteq P$ is called a *chain* in P if $(C, \leq\restriction C)$ is a linearly ordered set, and a set $A \subseteq P$ is called an *anti-chain* in P if any two distinct elements $a, b \in A$ are incomparable, i.e., $a \not\leq b$ and $b \not\leq a$).

12. NA: There are no amorphous sets (where an infinite set X is called *amorphous* if X cannot be written as a disjoint union of two infinite sets).

2 Introduction, known and preliminary results

Ramsey Choice RC_n was introduced by Montenegro in [7], where it was asked for which n is the implication "$\mathsf{RC}_n \rightarrow \mathsf{C}_n^-$" true. In [7], it was observed that RC_n implies C_n^- for $n = 2, 3$ and it was shown that RC_4 implies C_4^-, which is a beautiful and highly non-trivial result. The status of "$\mathsf{RC}_n \rightarrow \mathsf{C}_n^-$" for $n \geq 5$ is (to the best of our knowledge) still an *open and* (in our opinion) *a quite difficult problem.* The particular question of whether RC_5 implies C_5^- is also addressed in Halbeisen [2] (see [2, Related Result 34, p. 167]).

The research in this paper is motivated by the above open questions of Montenegro's as well as the particular question of Halbeisen's. The answers to these specific questions still elude us. However, we are able to give a *partial answer* with regard to the question on the relationship between RC_5 and C_5^-. In particular, we shall prove that for $n = 2, 3$, $\mathsf{RC}_5 + \mathsf{C}_n^-$ *implies* C_5^-, and that RC_5 implies *neither* C_2^- *nor* C_3^- in ZF set theory. Furthermore, we shall provide a *plethora of new results which completely settle open problems* on the status of "RC_n implies C_m^-" for certain distinct natural numbers n, m.

We believe that the results of the current paper shed new light on this area and that they also indicate possible paths towards further study on the aforementioned open problems.

Before setting out with our main results, we shall provide some known and preliminary results in the current area of research.

Theorem 1 *The following hold:*

1. BPI *implies* OP *implies* C_n*, which in turn implies* $\mathsf{RC}_n + \mathsf{C}_n^-$*, for all* $n \in \omega \setminus \{0, 1\}$*. None of the latter implications is reversible in* ZF*.*

2. $\mathsf{DF} = \mathsf{F}$ *implies* $\mathsf{RC}_n + \mathsf{C}_n^-$ *for all* $n \in \omega \setminus \{0, 1\}$*. The statement "*$\forall n \in \omega \setminus \{0, 1\}$, $\mathsf{RC}_n + \mathsf{C}_n^-$*" does not imply* $\mathsf{DF} = \mathsf{F}$ *in* ZF*. Further, for every* $m \in \omega \setminus \{0, 1\}$*, the statement "*$\forall n \in \omega \setminus \{0, 1\}$, $\mathsf{RC}_n + \mathsf{C}_n^-$*" does not imply* C_m *in* ZF*.*

3. *([1], [6], [9])* $\mathsf{DF} = \mathsf{F}$ *implies* RT*, which in turn implies* CAC*. None of the latter implications are reversible in* ZF*.*

4. *([1], [6])* RT *implies* $\mathsf{PAC_{fin}}$*, which in turn implies* C_n^- *for all* $n \in \omega \setminus \{0, 1\}$*. None of the latter implications is reversible in* ZF*.*

5. *([1])* RT *is true in the Basic Fraenkel Model (Model* $\mathcal{N}1$ *in [4]) and it is false in the Basic Cohen Model (Model* $\mathcal{M}1$ *in [4]).*

6. *([11]) For every* $n \in \omega \setminus \{0, 1\}$*,* RC_n *and* C_n^- *are strictly weaker than* C_n *in* ZF*.*

7. *([7]) For* $n = 2, 3, 4$*,* RC_n *implies* C_n^-*.*

8. *([12]) If for some integer* $n > 1$*,* $[X]^n$ *has a choice function, then* X *is finite or not amorphous.*

9. *For all* $m, n \in \omega \setminus \{0\}$*,* C_{mn}^- *implies both* C_m^- *and* C_n^-*.*

10. *For all $m, n \in \omega \setminus \{0\}$, LOC^-_{mn} implies both LOC^-_m and LOC^-_n.*

Proof We just show 1, 2, 9, 10.

1. The first implication is well-known (see [4]). The second and third implications are straightforward. For "$\mathsf{RC}_n + \mathsf{C}^-_n \not\to \mathsf{C}_n$ in ZF", see the proof of part 2 below. For the rest of the assertions, see [4].

2. The implication is straightforward. For the second assertion, it is known that BPI is true in the basic Cohen model (Model $\mathcal{M}1$ in [4]), whereas $\mathsf{DF} = \mathsf{F}$ is false in $\mathcal{M}1$ (see [4]). It follows, by part 1 of the current theorem, that $\mathsf{RC}_n + \mathsf{C}^-_n$ is true in $\mathcal{M}1$, for all $n \in \omega \setminus \{0, 1\}$. For the third assertion, fix $m \in \omega \setminus \{0, 1\}$ and consider, for example, the ZF model $\mathcal{M}46(m, M)$ in [4]. Then $\mathsf{DF} = \mathsf{F}$ is true in $\mathcal{M}46(m, M)$, hence so is "$\forall n \in \omega \setminus \{0, 1\}$, $\mathsf{RC}_n + \mathsf{C}^-_n$", whereas C_m is false in $\mathcal{M}46(m, M)$ (see [4]).

9. Fix $n, m \in \omega \setminus \{0, 1\}$ and assume that C^-_{nm} is true. Let $\mathcal{A} = \{A_i : i \in I\}$ be a family of m-element sets (respectively, of n-element sets). Then $\mathcal{B} = \{A_i \times n : i \in I\}$ (respectively, $\mathcal{C} = \{A_i \times m : i \in I\}$) is a family of (mn)-element sets and any partial choice function for $\mathcal{B}$ (respectively, for $\mathcal{C}$) clearly yields a partial choice function for $\mathcal{A}$.

10. This can be proved similarly to 9. □

Remark 1 For use in the proofs of our forthcoming independence results, both in this section as well as in Section 5, we note here that for all $n \in \omega \setminus \{0, 1\}$, RC_n and C^-_n are *injectively boundable* (for the definition of the latter term, see [4, Note 103] or [8]). Indeed, RC_n is injectively boundable since

$$\mathsf{RC}_n \iff (\forall x)(|x|_- \leq \omega \to \text{ if } x \text{ is infinite, then there is an infinite subset } y \text{ of } x \text{ such that } [y]^n \text{ has a choice function}),$$

where $|x|_-$ denotes the *injective cardinality* of x (for the definition of injective cardinality, see [4, Note 103] or [8]), and C^-_n is injectively boundable since

$$\mathsf{C}^-_n \iff (\forall x)(|x|_- \leq \omega \to \text{ every infinite family of } n\text{-element sets whose union is } x \text{ has a partial choice function}).$$

Furthermore, we point out that $\neg\mathsf{RC}_n$ and $\neg\mathsf{C}^-_n$ are *boundable* statements, thus they are injectively boundable (see [8] for the fact that "boundable" implies "injectively boundable").

The above observations together with Pincus' Transfer Theorem [8, Thm. 3A3] (which states that if Φ is a conjunction of injectively boundable statements which hold in a Fraenkel–Mostowski model V_0, then there is a model $V \supset V_0$ of ZF with the same ordinals and cofinalities as V_0 in which Φ holds), show that all the independence results on RC_n and C^-_n which are obtained in this paper via Fraenkel–Mostowski models of $\mathsf{ZFA} + \neg\mathsf{AC}$ are transferable into ZF set theory.

Next, we provide some preliminary results on the connection between RC_n, C^-_n, CAC and NA.

Theorem 2 *The following hold:*

1. *For all $n \in \omega \setminus \{0, 1\}$, RC_n implies NA, and NA does not imply RC_n in ZF.*

2. *RT does not imply RC_n in ZF, for all $n \in \omega \setminus \{0, 1\}$.*

3. *For all $n \in \omega \setminus \{0, 1\}$, C^-_n does not imply RC_n in ZF.*

Proof 1. Fix $n \in \omega \setminus \{0, 1\}$ and let X be an infinite set. By RC_n, there is an infinite subset $Y \subseteq X$ such that $[Y]^n$ has a choice function. Then, by Theorem 1(8), we have that Y is not amorphous, hence neither is X.

For the second assertion, fix $n \in \omega \setminus \{0,1\}$. We consider the following permutation model which is a generalization of the Second Fraenkel Model (Model $\mathcal{N}2$ in [4]): Start with a ground model M of ZFA + AC with a set A of atoms which is a countable disjoint union $\bigcup\{A_i : i \in \omega\}$ of n-element sets. Let G be the group of all permutations of A which fix each A_i. Let Γ be the filter of subgroups of G which is generated by the subgroups $\mathrm{fix}_G(E) = \{\phi \in G : \forall e \in E(\phi(e) = e)\}$, $E \in [A]^{<\omega}$. Let $\mathcal{N}$ be the Fraenkel–Mostowski model determined by M, G, and Γ.

As in the Second Fraenkel Model, one may show that MC is true in $\mathcal{N}$ (see also [5, proof of Theorem 9.2(i), p. 135]), hence NA is true in $\mathcal{N}$. However, RC_n is false for the infinite set A of the atoms as can be easily checked via standard Fraenkel–Mostowski techniques.

Now NA is an injectively boundable statement (see [4, Note 103] or [8]) and $\neg\mathsf{RC}_n$ is boundable, hence injectively boundable, and $\mathcal{N}$ is a permutation model which satisfies the conjunction $\mathsf{NA} + \neg\mathsf{RC}_n$ of two injectively boundable statements, thus by [8, Theorem 3A3] it follows that there is a ZF model $\mathcal{M}$ such that $\mathcal{M} \models \mathsf{NA} + \neg\mathsf{RC}_n$.

2. From Theorem 1(5) we have that RT is true in the Basic Fraenkel Model (Model $\mathcal{N}1$ in [4]). On the other hand, the infinite set A of the atoms of $\mathcal{N}1$ is amorphous (see [4], [5]), hence, by part 1 of the current theorem, we have that RC_n is false in $\mathcal{N}1$ for all $n \in \omega \setminus \{0,1\}$. The independence result can be transferred to ZF via Pincus' transfer theorems, since RT is injectively boundable (see [1], [4, Note 103]) and $\neg\mathsf{RC}_n$ is boundable, thus injectively boundable.

3. This follows from the proof of part 2 of the current theorem, Theorem 1(4), and Pincus' Transfer Theorems. □

Theorem 3 *The following hold:*

1. *For all $n \in \omega \setminus \{0,1\}$, CAC and RC_n are independent of each other in ZF, and also C_n (and hence C_n^-) does not imply CAC in ZF.*
2. *CAC does not imply C_2^- in ZF.*
3. *RC_2 does not imply C_3^- in ZF. Therefore neither does RC_2 imply RC_3 in ZF.*
4. *RC_2 implies RC_4.*

Proof 1. In the proof of [9, Theorem 2.1], a Fraenkel–Mostowski model $\mathcal{N}$ is constructed, in which it is shown that CAC is true. Furthermore, in [9], it is shown that, in $\mathcal{N}$, there exist amorphous sets, and thus—by Theorem 2(1)—it follows that RC_n is false in $\mathcal{N}$ for all $n \in \omega \setminus \{0,1\}$.

To see that for all $n \in \omega \setminus \{0,1\}$, RC_n does not imply CAC, fix $n \in \omega \setminus \{0,1\}$. We consider first the permutation model $\mathcal{N}6$ in [4]: We start with a ground model M of ZFA + AC with a countably infinite set of atoms $A = \{a_n : n \in \omega\}$ such that A is a disjoint union $A = \bigcup\{P_n : n \in \omega\}$, where $P_0 = \{a_0\}$, $P_1 = \{a_1, a_2\}$, $P_2 = \{a_3, a_4, a_5\}$, ..., and in general for $n > 0$, $|P_n| = p_n$, where p_n is the nth prime. G is the group generated by $\{\pi_n : n \in \omega\}$, where if $P_n = \{a_{m+1}, a_{m+2}, \ldots, a_{m+p_n}\}$, then

$$\pi_n : a_{m+1} \mapsto a_{m+2} \mapsto \ldots \mapsto a_{m+p_n} \mapsto a_{m+1} \text{ and } \pi_n(x) = x, \text{ for all } x \in A \setminus P_n.$$

(G is the weak direct product of $\aleph_0$ cyclic groups of order p_n.) The ideal $\mathcal{I}$ of supports is the set of all finite subsets of A. $\mathcal{N}6$ is the permutation model determined by M, G and $\mathcal{I}$.

It is known that for all $n \in \omega \setminus \{0,1\}$, C_n is true in $\mathcal{N}6$ (see [4], [5, Theorem 7.11]). (Note also that the countably infinite family $\{P_n : n \in \omega\}$ has no partial choice function in $\mathcal{N}6$.) Thus, RC_n and C_n^- are true in $\mathcal{N}6$ for all $n \in \omega \setminus \{0,1\}$. We show that CAC is false in $\mathcal{N}6$. To this end, define a binary relation $\leq$ on A by requiring for all $x, y \in A$,

$$x \leq y \text{ if and only if } x = y, \text{ or } x \in P_n,\ y \in P_m, \text{ and } n < m.$$

It is easy to verify that $\leq$ is a partial order on A, which is in $\mathcal{N}6$, since it has empty support (i.e., every permutation of A in G fixes $\leq$). Clearly, the poset $(A, \leq)$ has no infinite anti-chains; the subsets of the P_n's are the only anti-chains of $(A, \leq)$. Furthermore, since the countable family $\{P_n : n \in \omega\}$ has no partial choice function, it follows that $(A, \leq)$ has no infinite chains, either. Thus, CAC is false in $\mathcal{N}6$.

Since CAC, RC_n, C_n^-, as well as their negations, are all injectively boundable (see Remark 1 and [9]), it follows by Pincus' Theorem 3A3 in [8] that all of the above ZFA independence results can be transferred to ZF.

2. In the Fraenkel–Mostowski model $\mathcal{N}$ of the proof of [9, Theorem 2.1], CAC is true, whereas there is a (amorphous) family of pairs of atoms without a (partial) choice function. Thus, C_2^- is false in $\mathcal{N}$. The result is transferable into ZF.

3. For the result, we will use the permutation model $\mathcal{N}2^*(3)$ in [4]: The set A of atoms is a countable disjoint union $\bigcup\{T_i : i \in \omega\}$, where $T_i = \{a_i, b_i, c_i\}$ for all $i \in \omega$. For each $i \in \omega$, let η_i be the three-cycle (a_i, b_i, c_i). Let G be the group of permutations π of A such that for each $i \in \omega$, $\pi \restriction T_i$ is either the identity, or η_i, or η_i^2. Let Γ be the finite support filter.

It is known that C_2 is true in $\mathcal{N}2^*(3)$ (see [4], [5, Example 7.13]), hence RC_2 is also true in $\mathcal{N}2^*(3)$. However, the family $\{T_i : i \in \omega\}$ has no partial choice function in $\mathcal{N}2^*(3)$, hence C_3^- is false in $\mathcal{N}2^*(3)$. Hence, by Theorem 1(7), it follows that RC_3 is also false in $\mathcal{N}2^*(3)$. The independence result can be transferred to ZF.

4. This can be proved as Tarski's result that C_2 implies C_4 (see [5, Example 7.12, p. 107]). $\square$

Remark 2 We would like to point out here that in Example 7.13 of [5] (that we referred to in the proof of Theorem 3(3)), Jech actually proves that C_2 is true in a permutation model $\mathscr{V}$, whose setting is the same as the one for $\mathcal{N}2^*(3)$, except for the *smaller* (than G) group $\mathscr{G}$, which is generated by the following permutations π_i of A:

$$\pi_i : a_i \mapsto b_i \mapsto c_i \mapsto a_i,$$

$$\pi_i(x) = x \text{ for all } x \in A \setminus T_i.$$

$\mathscr{G}$ is the weak direct product of $\aleph_0$ cyclic groups of order 3, and clearly $\mathscr{G} \subset G$, where G is the (unrestricted wreath product) group (of $\aleph_0$ cyclic groups of order 3) used for the construction of $\mathcal{N}2^*(3)$. However, the two models, $\mathcal{N}2^*(3)$ and $\mathscr{V}$, are *equal*, as we establish below. Similarly, for the proof of Theorem 3(1), one could argue in the permutation model $\mathcal{M}$ whose setting is the same as the one for $\mathcal{N}6$, except for the larger group G', which comprises all permutations π of A such that for each $n \in \omega$, π is a cycle on P_n; again, it is true that $\mathcal{N}6 = \mathcal{M}$. We now argue that $\mathscr{V} = \mathcal{N}2^*(3)$. (The proof that $\mathcal{N}6 = \mathcal{M}$ is identical.) We prove by $\in$-induction that for every $x \in M$ (the ground model), $\Phi(x)$ is true, where

$$\Phi(x) : \; x \in \mathscr{V} \iff x \in \mathcal{N}2^*(3).$$

Clearly $\Phi(x)$ is true, if $x = \emptyset$, or if $x \in A$. Assume that $y \in M$ and that for all $x \in y$, $\Phi(x)$ is true. We will show that $\Phi(y)$ is true. Assume that $y \in \mathscr{V}$. Then the following hold:

(1) y has a finite support $E \subset A$ relative to the group $\mathscr{G}$ (i.e., for every $\psi \in \text{fix}_{\mathscr{G}}(E)$, $\psi(y) = y$);

(2) for every $x \in y$, $x \in \mathscr{V}$ ($\mathscr{V}$ is a transitive class);

(3) for every $x \in y$, $x \in \mathcal{N}2^*(3)$ (by (2) and the induction hypothesis).

We assert that E is a support of y relative to the group G. It suffices to show that for all $\phi \in \text{fix}_G(E)$ and for all $x \in y$, $\phi(x) \in y$ (since then $\phi(y) = y$ follows from "$\phi(y) \subseteq y$ and $\phi^{-1}(y) \subseteq y$").

Let $\phi \in \text{fix}_G(E)$ and let $x \in y$. By (3), x has a finite support $E' \subset A$ relative to G. The permutation ϕ may not be in $\mathscr{G}$, but we construct a permutation $\phi' \in \text{fix}_{\mathscr{G}}(E)$ which agrees with ϕ on E' as follows: For each $a \in E'$, the set $\{\phi^n(a) : n \in \mathbb{Z}\}$ is (clearly) finite. Therefore, since E' is finite, so is $D = \bigcup_{a \in E'} \{\phi^n(a) : n \in \mathbb{Z}\}$. (Essentially, D is a finite union of T_i's.) Furthermore, D contains E' and is closed under ϕ. We define a mapping $\phi' : A \to A$ by

$$\phi'(a) = \begin{cases} \phi(a), & \text{if } a \in D; \\ a, & \text{otherwise.} \end{cases}$$

Then the following hold:

(4) $\phi' \in \mathscr{G}$;

(5) ϕ' fixes E pointwise (since ϕ fixes E pointwise); and

(6) ϕ' agrees with ϕ on E'.

By (4) and (5), $\phi' \in \mathrm{fix}_{\mathscr{G}}(E)$ so $\phi'(y) = y$. It follows that $\phi'(x) \in y$. Now, (6) gives us $\phi'(x) = \phi(x)$, and hence $\phi(x) \in y$.

Conversely, assume that $y \in \mathcal{N}2^*(3)$ and that y has a support E' relative to G. Then E' is a support of y relative to $\mathscr{G}$ since $\mathscr{G} \subset G$. By the induction hypothesis, every element of y is in $\mathscr{V}$, so we may conclude that $y \in \mathscr{V}$.

For the reader's complete information, we would also like to mention here that Howard [3] has shown that a formally stronger principle than C_2, namely the Principle of Consistent Choices for Pairs (see Form 141 in [4]), is true in $\mathcal{N}2^*(3)$. For a quite recent study on the set-theoretic strength of the above principle (as well as of related ones) and its connection to general topology, the reader is referred to Tachtsis [10].

3 Summary of the main results

Below, we list our main results along with their exact placement in this paper.

1. For every integer $n \geq 2$, if RC_i is true for all integers i with $2 \leq i \leq n$, then C_i^- is true for all integers i with $2 \leq i \leq n$. (Theorem 4.)

2. Let $p_0 \leq \ldots \leq p_v$ be prime numbers and let k be a positive integer. Then there exists a model $\mathcal{V}_{p_0,\ldots,p_v}$ of ZFA such that

$$\mathcal{V}_{p_0,\ldots,p_v} \models \mathsf{RC}_k \leftrightarrow \mathsf{C}_k^- \leftrightarrow \mathsf{LOC}_k^-$$

and

$$\mathcal{V}_{p_0,\ldots,p_v} \models \neg\mathsf{RC}_k \iff k \text{ is a multiple of } p_i \text{ for some } i \leq v.$$

Furthermore, for all integers $k \geq 2$ which can be written as a sum of multiples of $p_0, \ldots, p_v$,

$$\mathcal{V}_{p_0,\ldots,p_v} \models \neg\mathsf{C}_k.$$

The result is transferable into ZF. (Theorem 6.)

3. (i) If $m, n \geq 2$ are any integers such that for some prime p we have $p \nmid m$ and $p \mid n$, then in ZF: $\mathsf{RC}_m \nrightarrow \mathsf{RC}_n$ and $\mathsf{RC}_m \nrightarrow \mathsf{C}_n^-$.

(ii) There is a model $\mathcal{M}$ of ZF such that for every positive integer n, $\mathcal{M} \models \mathsf{RC}_{2n+1} \wedge \mathsf{C}_{2n+1}^- \wedge \neg\mathsf{RC}_{2n} \wedge \neg\mathsf{LOC}_{2n}^-$. Hence, for every odd integer $n \geq 3$ and for every even integer $m \geq 2$, $\mathcal{M} \models \mathsf{RC}_n \wedge \mathsf{C}_n^- \wedge \neg\mathsf{RC}_m \wedge \neg\mathsf{LOC}_m^-$.

(iii) For $k = 2, 4$, the principles RC_k and RC_3 are independent of each other in ZF.

(Corollary 1.)

4. For $n = 2, 3$, RC_5 + C_n^- implies C_5^-, and RC_5 implies neither C_2^- nor C_3^- in ZF. (Theorem 7.)

5. CAC does not imply C_n^- in ZF, for every $n \in \omega \setminus \{0, 1\}$. (Theorem 8.)

6. For every $n \in \omega \setminus \{0, 1\}$, C_n^- implies LOC_n^-, which in turn implies WOC_n^-. Furthermore, for every $n \in \omega \setminus \{0, 1\}$, none of the previous implications is reversible in ZF. (Theorem 9.)

7. (i) For every $n \in \omega \setminus \{0, 1\}$, RC_{2n} implies LOKW_n^- and the latter implication is not reversible in ZF. In particular, RC_6 strictly implies LOC_3^- in ZF.

   (ii) LOC_4^- is equivalent to LOC_2^- + LOKW_4^-. Furthermore, LOC_2^- does not imply LOKW_4^- in ZF, hence neither does it imply LOC_4^- in ZF.

   (iii) RC_6 + LOC_2^- implies LOC_6^-. Hence, RC_6 + LOC_{2n}^- implies LOC_6^- for all $n \in \omega \setminus \{0\}$.

   (iv) RC_6 + LOC_2^- implies LOC_4^-. Hence, RC_6 + LOC_{2n}^- implies LOC_4^- for all $n \in \omega \setminus \{0\}$.

   (v) RC_6 does not imply LOC_5^- in ZF, hence it does not imply C_{5n}^- in ZF, for all $n \in \omega \setminus \{0\}$.

   (vi) RC_6 implies WOC_2^-.

   (Theorem 10.)

4 Classes of Fraenkel–Mostowski models for the main results

In this section, we construct certain classes of Fraenkel–Mostowski permutation models of ZFA + $\neg\mathsf{AC}$, suitable for our independence results. The most important class which provides us with a plethora of results on the relationship between RC_k and C_l^- for certain natural numbers k and l, will be the one consisting of the models $\mathcal{V}_{n,m}$, where $n, m \in \omega$. We begin this section with the construction of this class of models and then prove some facts about them that will be the main apparatus for our independence results. We shall then provide some classes of variant models and start the investigation on RC_k and C_k^- for various natural numbers k.

The reader should recall here Remark 1 that in order to establish our forthcoming independence results in ZF, it suffices to establish them via a suitable permutation model of ZFA + $\neg\mathsf{AC}$.

The construction of permutation models is traditionally based on certain groups of permutations of atoms and normal filters of subgroups, which is the approach taken here. Another approach, which is less common, is based on automorphism groups of certain $\aleph_0$-categorical structures (see, for example, Halbeisen [2, p. 211*ff.*]). Even though the latter approach makes the construction of the models slightly shorter, we prefer the more constructive flavor of the former approach.

Fix $n, m \in \omega \setminus \{0, 1\}$. We start with a model M of ZFA + AC with a set of atoms

$$A = \bigcup\{A_q \cup B_q : q \in \mathbb{Q}\},$$

where $\mathbb{Q}$ is the set of rational numbers, such that for all q and r in $\mathbb{Q}$:

1. $A_q = \{a_{q1}, a_{q2}, \ldots, a_{qn}\}$ and $B_q = \{b_{q1}, b_{q2}, \ldots, b_{qm}\}$, so that $|A_q| = n$ and $|B_q| = m$,

2. $A_q \cap B_q = \emptyset$, and if $q \neq r$, then $(A_q \cup B_q) \cap (A_r \cup B_r) = \emptyset$.

The sets A_q and B_r (where $q, r \in \mathbb{Q}$) are called **blocks**. By 1. and 2., we have that the blocks are pairwise disjoint finite sets.

We let G be the group of all permutations η of A such that η permutes the blocks A_q and B_r *independently*; preserves the linear ordering on the q's and r's; and is a cyclic permutation when restricted to any

A_q or B_r. We make this more explicit as follows: If ψ is an order automorphism of $(\mathbb{Q}, \leq)$ (where $\leq$ is the usual dense linear order on $\mathbb{Q}$), then we let ϕ_ψ and σ_ψ be the permutations of A defined by:

$$\forall q \in \mathbb{Q}\ \forall j \in \{1, \ldots, n\}\ (\phi_\psi(a_{qj}) = a_{\psi(q)j}),\ \text{and } \phi_\psi \text{ fixes } \bigcup\{B_r : r \in \mathbb{Q}\} \text{ pointwise,}$$

and

$$\forall r \in \mathbb{Q}\ \forall k \in \{1, \ldots, m\}\ (\sigma_\psi(b_{rk}) = b_{\psi(r)k}),\ \text{and } \sigma_\psi \text{ fixes } \bigcup\{A_q : q \in \mathbb{Q}\} \text{ pointwise.}$$

Note that if ψ_1 and ψ_2 are two order automorphisms of $(\mathbb{Q}, \leq)$, then $\phi_{\psi_1}\phi_{\psi_2} = \phi_{\psi_1\psi_2}$ and $\sigma_{\psi_1}\sigma_{\psi_2} = \sigma_{\psi_1\psi_2}$. Then we require

$$\eta \in G,\ \text{if and only if, } \eta = \phi_\psi \sigma_{\psi'} \rho, \tag{1}$$

where ψ and ψ' are order automorphisms of $(\mathbb{Q}, \leq)$, ϕ_ψ and $\sigma_{\psi'}$ are respectively the (above) corresponding permutations of A, and ρ is a permutation of A with the following property:

$$\forall q \in \mathbb{Q}\ \exists j \in \{1, 2, \ldots, n\}\ \exists k \in \{1, 2, \ldots, m\}\ (\rho \restriction A_q = \tau_q^j \text{ and } \rho \restriction B_q = \sigma_q^k),$$

where for $q \in \mathbb{Q}$, τ_q is the n-cycle $a_{q1} \mapsto a_{q2} \mapsto \cdots \mapsto a_{qn} \mapsto a_{q1}$, and σ_q is the m-cycle $b_{q1} \mapsto b_{q2} \mapsto \cdots \mapsto b_{qm} \mapsto b_{q1}$. (It is clear that ρ fixes each of $\{A_q : q \in \mathbb{Q}\}$ and $\{B_q : q \in \mathbb{Q}\}$ pointwise.)

Note: When no confusion is likely to arise, we will also denote by 'τ_q' and 'σ_q' the permutations of A which, respectively, extend the above cycles τ_q and σ_q, and fix $A \setminus A_q$ and $A \setminus B_q$ pointwise. Also, for a set X, we will denote by 1_X the identity mapping on X.

Let $\mathcal{F}$ be the filter of subgroups of G which is generated by the subgroups $\text{fix}_G(E)$, where $E = \bigcup\{A_q : q \in S\} \cup \bigcup\{B_r : r \in T\}$ for finite $S, T \subseteq \mathbb{Q}$. (Note that E can be written as $\bigcup\{F_q : q \in K\}$, where $K \in [\mathbb{Q}]^{<\omega}$ and $F_q \in \{A_q, B_q, A_q \cup B_q\}$ for every $q \in K$.) Let $\mathcal{V}_{n,m}$ be the Fraenkel–Mostowski model which is determined by M, G and $\mathcal{F}$. If $x \in \mathcal{V}_{n,m}$, then there is a set $E = \bigcup\{A_q : q \in S\} \cup \bigcup\{B_r : r \in T\}$ (where $S, T \in [\mathbb{Q}]^{<\omega}$) such that for all ϕ in $\text{fix}_G(E)$, $\phi(x) = x$, that is, $\phi \in \text{Sym}_G(x) = \{\eta \in G : \eta(x) = x\}$. Such a (finite) set $E \subset A$ is called a *support* of x.

Below, we list some key facts about the model $\mathcal{V}_{n,m}$.

Fact 1 *Each of $\mathcal{A} = \{A_q : q \in \mathbb{Q}\}$ and $\mathcal{B} = \{B_q : q \in \mathbb{Q}\}$ is a linearly orderable set in $\mathcal{V}_{n,m}$.*

Proof Note that $\mathcal{A}, \mathcal{B} \in \mathcal{V}_{n,m}$ since both of these sets have empty support (i.e. every permutation of A in G fixes them (setwise)). Furthermore, since every permutation of A in G permutes the blocks (i.e. the elements of $\mathcal{A}$ and $\mathcal{B}$) preserving the ordering on $\mathbb{Q}$, it follows that the induced (by $\mathbb{Q}$) linear orders on $\mathcal{A}$ and $\mathcal{B}$ (i.e. $A_q \preceq_{\mathcal{A}} A_{q'} \Leftrightarrow q \leq q'$ and similarly for $\mathcal{B}$) are in the model (for they have empty support). □

Note: We point out that $\mathcal{C} = \{A_q \cup B_q : q \in \mathbb{Q}\} \notin \mathcal{V}_{n,m}$ (which is naturally expected since the blocks A_q and B_r are permuted independently). If not, then let $E = \bigcup\{A_q : q \in S\} \cup \bigcup\{B_r : r \in T\}$ (where $S, T \in [\mathbb{Q}]^{<\omega}$) be a support of $\mathcal{C}$. Let $q, q' \in \mathbb{Q}$ be such that $\max(S \cup T) < q < q'$, and also let ψ be an order automorphism of $(\mathbb{Q}, \leq)$ such that $\psi(r) = r$ for all $r \in S \cup T$, and $\psi(q) = q'$. Then $\phi_\psi \in \text{fix}_G(E)$, so $\phi_\psi(\mathcal{C}) = \mathcal{C}$. However, $A_q \cup B_q \in \mathcal{C}$, whereas $\phi_\psi(A_q \cup B_q) = A_{q'} \cup B_q \notin \mathcal{C} = \phi_\psi(\mathcal{C})$, a contradiction.

Fact 2 *(i) Neither $\mathcal{A} = \{A_q : q \in \mathbb{Q}\}$ nor $\mathcal{B} = \{B_q : q \in \mathbb{Q}\}$ has a partial Kinna–Wagner selection function in $\mathcal{V}_{n,m}$. In particular, if an integer $k \geq 2$ is a multiple of n or m, then LOC_k^- is false in $\mathcal{V}_{n,m}$. Hence, if $k \geq 2$ is a sum of multiples of n and m, then C_k is false in $\mathcal{V}_{n,m}$.*

(ii) If D is an infinite subset of A in $\mathcal{V}_{n,m}$, then there exists an infinite subset $I \subseteq \mathbb{Q}$ such that, in $\mathcal{V}_{n,m}$, $\bigcup\{A_q : q \in I\} \subseteq D$ or $\bigcup\{B_q : q \in I\} \subseteq D$.

Proof (i) By way of contradiction, assume that there exists an infinite subfamily $\mathcal{W}$ (respectively, $\mathcal{V}$) of $\mathcal{A}$ (respectively, of $\mathcal{B}$) with a Kinna–Wagner function in $\mathcal{V}_{n,m}$, f say. Let $E = \bigcup\{A_q : q \in S\} \cup \bigcup\{B_r : r \in T\}$ (where $S, T \in [\mathbb{Q}]^{<\omega}$) be a support of $\mathcal{W}$ (respectively, of $\mathcal{V}$) and f. Since $S \cup T$ is finite and $\mathcal{W}$ (respectively, $\mathcal{V}$) is infinite, there exists $q^* \in \mathbb{Q}$ such that $A_{q^*} \in \mathcal{W}$ (respectively, $B_{q^*} \in \mathcal{V}$) and $q^* \notin S \cup T$. Then $\tau_{q^*} \in \mathrm{fix}_G(E)$ (respectively, $\sigma_{q^*} \in \mathrm{fix}_G(E)$), and hence $\tau_{q^*}(f) = f$ (respectively, $\sigma_{q^*}(f) = f$). However, $\tau_{q^*}(A_{q^*}) = A_{q^*}$ (respectively, $\sigma_{q^*}(B_{q^*}) = B_{q^*}$), whereas $\tau_{q^*}(f(A_{q^*})) \neq f(A_{q^*})$ (respectively, $\sigma_{q^*}(f(B_{q^*})) \neq f(B_{q^*})$), which means that f is not supported by E. This is a contradiction.

The second assertion follows immediately from the first one and Theorem 1(10).

For the third assertion, fix an integer $k \geq 2$ such that $k = l_1 n + l_2 m$. Then

$$\mathcal{R} := \{(A_q \times l_1) \cup (B_r \times l_2) : q, r \in \mathbb{Q}\}$$

is an element of $\mathcal{V}_{n,m}$ (since it has empty support), consists of k-element sets, and from the first assertion of the current fact, we may conclude that $\mathcal{R}$ has no choice function in $\mathcal{V}_{n,m}$.

(ii) This follows immediately from part (i). □

Note: The family $\mathcal{U} = \{A_q \cup B_r : q, r \in \mathbb{Q}\}$, which is in $\mathcal{V}_{n,m}$ (having $\emptyset$ as its support) and consists of $(n+m)$-element sets, *does have* a partial choice function in $\mathcal{V}_{n,m}$. (If $\mathcal{W} = \{A_q \cup B_0 : q \in \mathbb{Q}\}$, then $\mathcal{W} \in \mathcal{V}_{n,m}$ since it has B_0 as its support, and $\mathcal{W} \subseteq \mathcal{U}$. Clearly, $f = \{(A_q \cup B_0, b_{01}) : q \in \mathbb{Q}\}$ is a choice function for $\mathcal{W}$ which is in $\mathcal{V}_{n,m}$, since it is supported by B_0.)

Fact 3 *If $x \in \mathcal{V}_{n,m}$ and E_1, E_2 are two supports of x, then $E_1 \cap E_2$ is a support of x. Hence, every $x \in \mathcal{N}_{n,m}$ has a minimum support E_x and for all $\eta, \eta' \in G$, if $\eta(E_x) \neq \eta'(E_x)$, then $\eta(x) \neq \eta'(x)$.*

Proof We may write E_1 and E_2 as $\bigcup\{F_q : q \in S_1\}$ and $\bigcup\{G_q : q \in S_2\}$, respectively, where $S_1, S_2 \in [\mathbb{Q}]^{<\omega}$ and $F_q, G_q \in \{A_q, B_q, A_q \cup B_q\}$ for every $q \in S_1 \cup S_2$. We will show that $\mathrm{fix}_G(E_1 \cap E_2) \subseteq \mathrm{Sym}_G(x)$, where $E_1 \cap E_2 = \bigcup\{F_q \cap G_q : q \in S_1 \cap S_2\}$. To this end, let $\eta \in \mathrm{fix}_G(E_1 \cap E_2)$. By the definition of G, we have $\eta = \phi_\psi \sigma_{\psi'} \rho$ (see equation (1)). Note that both ψ and ψ' must fix $S_1 \cap S_2$ pointwise and ρ must fix $E_1 \cap E_2$ pointwise. Let ρ_1 and ρ_2 be the elements of G defined by

$$\rho_1(c) = \begin{cases} \rho(c) & \text{if } c \in E_1 \setminus E_2 \\ c & \text{otherwise} \end{cases}, \qquad \rho_2(c) = \begin{cases} c & \text{if } c \in E_1 \setminus E_2 \\ \rho(c) & \text{otherwise} \end{cases}.$$

Then $\rho = \rho_1 \rho_2$, $\rho_1 \in \mathrm{fix}_G(E_2)$ and $\rho_2 \in \mathrm{fix}_G(E_1)$. Therefore, $\rho(x) = \rho_1 \rho_2(x) = x$. Now, using the same arguments as in the ordered Mostowski model (see for example [5, Proof of Lemma 4.5(a), p. 50]), ψ and ψ' can be respectively written as $\psi_1 \psi_2 \cdots \psi_m$ and $\psi'_1 \psi'_2 \cdots \psi'_k$, where for $1 \leq i \leq m$ and $1 \leq j \leq k$, each of ψ_i and ψ'_j is an order automorphism of $(\mathbb{Q}, \leq)$, which either fixes S_1 pointwise, or fixes S_2 pointwise. It follows that $\phi_{\psi_i}, \sigma_{\psi'_j} \in \mathrm{fix}_G(E_1) \cup \mathrm{fix}_G(E_2)$ for all i and j with $1 \leq i \leq m$ and $1 \leq j \leq k$. Thus $\phi_\psi(x) = \phi_{\psi_1 \psi_2 \cdots \psi_m}(x) = \phi_{\psi_1} \phi_{\psi_2} \cdots \phi_{\psi_m}(x) = x$, and similarly $\sigma_{\psi'}(x) = x$. From these two equations and the fact that $\rho(x) = x$, it follows that $\eta(x) = \phi_\psi \sigma_{\psi'} \rho(x) = x$, and so $\eta \in \mathrm{Sym}_G(x)$.

The second assertion of Fact 3 follows immediately from the first one. □

Fact 4 *Let $x \in \mathcal{V}_{n,m}$ be a non-well-orderable set. Then x has an infinite subset $y \in \mathcal{V}_{n,m}$ which has a linearly orderable partition into r-element sets, where r is a divisor of n or a divisor of m.*

Proof Assume the hypotheses on x. Let E be a support of x. Then we may write E as $\bigcup\{H_q : q \in K\}$ where $K = \{q_1, q_2, \ldots, q_\ell\} \in [\mathbb{Q}]^{<\omega} \setminus \{\emptyset\}$, $q_1 < q_2 < \ldots < q_\ell$, and $H_q \in \{A_q, B_q, A_q \cup B_q\}$ for every $q \in K$. Without loss of generality we may assume that for every $q \in K$, we have $H_q = A_q \cup B_q$. Since x is not well-orderable, there exists $z \in x$ which is not supported by E. Let $E_z = \bigcup\{F_q : q \in K'\}$, where $K' \in [\mathbb{Q}]^{<\omega}$, be a support of z. It follows that $E_z \setminus E \neq \emptyset$ and without loss of generality, we may assume that $E \subsetneq E_z$. Let such K' be of minimal size, and pick $r_0 \in K' \setminus K$. By replacing if necessary E by $\{A_r \cup B_r : r \in K' \setminus \{r_0\}\}$ (which contains the original E, and is hence a support of x), we may assume that $K' \setminus K = \{r_0\}$. We also assume that if $F_{r_0} = A_{r_0} \cup B_{r_0}$ then $E \cup A_{r_0}$ and $E \cup B_{r_0}$ are *not* supports of z. Now there are $\ell + 1$ intervals determined by $q_1, q_2, \dots, q_\ell$ in which r_0 may lie, all of which are treated similarly. Assume for instance that $q_\ell < r_0$.

There are three possibilities for the set F_{r_0}: (a) $F_{r_0} = A_{r_0}$; (b) $F_{r_0} = B_{r_0}$; (c) $F_{r_0} = A_{r_0} \cup B_{r_0}$.

Case a. $F_{r_0} = A_{r_0}$. We define

$$f = \{(\phi(z), \phi(A_{r_0})) : \phi \in \mathrm{fix}_G(E_z \setminus A_{r_0})\}.$$

Then $f \in \mathcal{V}_{n,m}$, since $E_z \setminus A_{r_0}$ is a support of f. Furthermore, f is a function with $\mathrm{dom}(f) \subseteq x$ and $\mathrm{ran}(f) = \{A_q : q > q_\ell\}$. We have $\mathrm{dom}(f) \subseteq x$ since $E \subseteq E_z \setminus A_{r_0}$, $z \in x$ and E is a support of x, and $\mathrm{ran}(f) = \{\phi(A_{r_0}) : \phi \in \mathrm{fix}_G(E_z \setminus A_{r_0})\} = \{A_q : q > q_\ell\}$, since $q_\ell < r_0$ and every element of $\mathrm{fix}_G(E_z \setminus A_{r_0})$ fixes A_{q_ℓ} and permutes the blocks A_q preserving the ordering on q's. We argue by contradiction that f is a function, so there exist $\phi, \eta \in \mathrm{fix}_G(E_z \setminus A_{r_0})$ such that $\phi(z) = \eta(z)$, but $\phi(A_{r_0}) \neq \eta(A_{r_0})$. Then $\eta^{-1}\phi(z) = z$ and $\eta^{-1}\phi(A_{r_0}) = A_q$ for some $q \in \mathbb{Q} \setminus K'$. Since E_z supports z, $\eta^{-1}\phi(E_z)$ supports $\eta^{-1}\phi(z) = z$. Thus, by Fact 3, $\eta^{-1}\phi(E_z) \cap E_z = E_z \setminus A_{r_0}$ also supports z, contradicting the minimality of K'. Thus $\phi(A_{r_0}) = \eta(A_{r_0})$, so f is a function.

Let $y = \mathrm{dom}(f)$ and $\mathcal{Y} = \{f^{-1}(\{A_q\}) : q > q_\ell\}$ $(= \{f^{-1}(\{\phi(A_{r_0})\}) : \phi \in \mathrm{fix}_G(E_z \setminus A_{r_0})\})$. Clearly $\mathcal{Y}$ is a partition of the infinite set y, which is linearly orderable in $\mathcal{V}_{n,m}$, since it is indexed by the linearly orderable set $\{A_q : q > q_\ell\}$ (see Fact 1). Furthermore, for any $\phi \in \mathrm{fix}_G(E_z \setminus A_{r_0})$, $f^{-1}(\{\phi(A_{r_0})\}) \subseteq \{\phi\tau_{r_0}^k(z) : k < n\}$, and hence $|f^{-1}(\{\phi(A_{r_0})\})| \leq n$. Indeed, if $\pi(z) \in f^{-1}(\{\phi(A_{r_0})\})$, then $\phi^{-1}\pi(A_{r_0}) = A_{r_0}$, so there exists $k < n$ such that $\phi^{-1}\pi$ and $\tau_{r_0}^k$ agree on A_{r_0}. Thus $(\phi^{-1}\pi)^{-1}\tau_{r_0}^k \in \mathrm{fix}_G(E_z)$, so $\phi^{-1}\pi(z) = \tau_{r_0}^k(z)$, and hence $\pi(z) = \phi\tau_{r_0}^k(z)$. Therefore,

$$\mathcal{Y} = \{U_\phi : \phi \in \mathrm{fix}_G(E_z \setminus A_{r_0})\},$$

where for $\phi \in \mathrm{fix}_G(E_z \setminus A_{r_0})$,

$$U_\phi = \{\eta z : \eta \in \mathrm{fix}_G(E_z \setminus A_{r_0}), \phi^{-1}\eta(A_{r_0}) = A_{r_0}\} \subseteq \{\phi\tau_{r_0}^k(z) : k < n\}.$$

Now fix an arbitrary ϕ in $\mathrm{fix}_G(E_z \setminus A_{r_0})$ and let $L = \{\phi^{-1}\eta : \eta \in \mathrm{fix}_G(E_z \setminus A_{r_0}), \phi^{-1}\eta(A_{r_0}) = A_{r_0}\}$. We assert that L is a subgroup of G. To see this, note firstly that $1_A \in L$, so $L \neq \emptyset$. Now let $\phi^{-1}\eta_1, \phi^{-1}\eta_2 \in L$. Then $\eta_1\phi^{-1}\eta_2 \in \mathrm{fix}_G(E_z \setminus A_{r_0})$ and $[\phi^{-1}(\eta_1\phi^{-1}\eta_2)](A_{r_0}) = \phi^{-1}\eta_1(\phi^{-1}\eta_2(A_{r_0})) = \phi^{-1}\eta_1(A_{r_0}) = A_{r_0}$, so $\phi^{-1}\eta_1\phi^{-1}\eta_2 \in L$. Also, if $\phi^{-1}\eta \in L$, then $\phi(\phi^{-1}\eta)^{-1} \in \mathrm{fix}_G(E_z \setminus A_{r_0})$ and $(\phi^{-1}\eta)^{-1}$ fixes A_{r_0} (setwise), so $\phi^{-1}(\phi(\phi^{-1}\eta)^{-1}) \in L$, and thus L is closed under inverses. Now L induces an action on A_{r_0} which is a subgroup, H say, of the cyclic group $S = \{\tau_{r_0}^k : k < n\}$ (which is isomorphic to $\mathbb{Z}_n$). Clearly $H = \{\pi \in S : \pi(z) = z\}$ (see also the above argument that $U_\phi \subseteq \{\phi\tau_{r_0}^k(z) : k < n\}$), and it is also easy to see that $|U_\phi| = (S : H)$ (the index of H in S), so $|U_\phi|$ divides n. Since ϕ was arbitrary, we conclude that all elements of $\mathcal{Y}$ have the same cardinality, which is a divisor of n.

Case b. $F_{r_0} = B_{r_0}$. This can be treated in much the same way as Case a (except that n is replaced by m and τ_{r_0} by σ_{r_0}).

Case c. $F_{r_0} = A_{r_0} \cup B_{r_0}$. Let f be given as in Case a, $f = \{(\phi(z), \phi(A_{r_0})) : \phi \in \mathrm{fix}_G(E_z \setminus A_{r_0})\}$. This is a function as before, this time using the assumption that $E_z \cup B_{r_0}$ is not a support of z. The remainder of the argument is as in Case a. □

Variants of the models $\mathcal{V}_{n,m}$

The models $\mathcal{V}_{n,m}$ can be generalized and modified as follows:

(V1) Instead of working with just two types of blocks A_q and B_r (where $q, r \in \mathbb{Q}$) of size n and m respectively, we can work with arbitrarily many types of blocks. Indeed, for a positive integer k and positive integers $m_0, \ldots, m_{k-1}$ we may define the model $\mathcal{V}_{m_0,\ldots,m_{k-1}}$ whose set of atoms A is partitioned into blocks $A_{q,0}, \ldots, A_{q,(k-1)}$, where $q \in \mathbb{Q}$ and for each $l < k$, $|A_{q,l}| = m_l$.

The group G of permutations of A and the filter $\mathcal{F}$ of subgroups of G are defined analogously.

In fact, we may also have infinitely many blocks by setting $k = \omega$; the corresponding model is denoted by $\mathcal{V}_{m\ldots}$.

We note that the corresponding Facts 1–4 also hold for the models $\mathcal{V}_{m_0,\ldots,m_{k-1}}$ and $\mathcal{V}_{m\ldots}$.

(V2) A variant model of $\mathcal{V}_{n,m}$ can be produced if we require that the blocks A_q and B_r are *not* permuted independently and that every order automorphism of $(\mathbb{Q}, \leq)$ moves for each $q \in \mathbb{Q}$ the blocks A_q and B_q *simultaneously* to the respective blocks A_r and B_r for some $r \in \mathbb{Q}$. That is, we start again with a model M of $\mathsf{ZFA} + \mathsf{AC}$ with a set of atoms $A = \bigcup\{A_q \cup B_q : q \in \mathbb{Q}\}$ where for every $q \in \mathbb{Q}$, $A_q = \{a_{q1}, a_{q2}, \ldots, a_{qn}\}$ and $B_q = \{b_{q1}, b_{q2}, \ldots, b_{qm}\}$ (so that $|A_q| = n$ and $|B_q| = m$) and $\{A_q : q \in \mathbb{Q}\} \cup \{B_q : q \in \mathbb{Q}\}$ is disjoint. The group G of permutations of A consists of all permutations η of A such that $\eta = \phi_\psi \rho$, where ψ is an order automorphism of $(\mathbb{Q}, \leq)$, $\phi_\psi(a_{qj}) = a_{\psi(q)j}$ and $\phi_\psi(b_{qk}) = b_{\psi(q)k}$ ($q \in \mathbb{Q}$, $1 \leq j \leq n$, $1 \leq k \leq m$), and ρ is a cyclic permutation when restricted to any A_q or B_r. The filter $\mathcal{F}$ of subgroups of G is generated by the subgroups $\mathrm{fix}_G(E)$, where $E = \bigcup\{A_q \cup B_q : q \in S\}$ for finite $S \subseteq \mathbb{Q}$.

We denote by $\mathcal{N}_{n,m}$ the permutation model which is determined by M, G and $\mathcal{F}$.

Facts 1, 2, 3 hold for $\mathcal{N}_{n,m}$, and a similar Fact 4 also holds true, namely if $x \in \mathcal{N}_{n,m}$ is a non-well-orderable set, then x has an infinite subset $y \in \mathcal{N}_{n,m}$ which has a linearly orderable partition into sets of the same cardinality $r \leq n \cdot m$ (the proof is much that same as the proof of Fact 4).

Furthermore, note that *in contrast* with $\mathcal{V}_{n,m}$, the family $\mathcal{C} = \{A_q \cup B_q : q \in \mathbb{Q}\}$ *is* an element of the model $\mathcal{N}_{n,m}$ and it is linearly orderable in $\mathcal{N}_{n,m}$. In addition, LOC^-_{n+m} (and hence C^-_{n+m}) is false in $\mathcal{N}_{n,m}$ for $\mathcal{C}$.

As with $\mathcal{V}_{n,m}$, the models $\mathcal{N}_{n,m}$ can be generalized and modified to models $\mathcal{N}_{m_0,\ldots,m_{k-1}}$ and $\mathcal{N}_{m\ldots}$.

We also note here that if $n \cdot m = 0$, then $\mathcal{V}_i = \mathcal{N}_i$ where $i = \max\{n, m\}$.

5 Main results

We start this section by proving that given any integer $n \geq 2$, if RC_i is true for all integers i with $2 \leq i \leq n$, then so is C^-_i for all i with $2 \leq i \leq n$.

Theorem 4 *For every integer $n \geq 2$, if RC_i is true for all integers i with $2 \leq i \leq n$, then C^-_i is true for all integers i with $2 \leq i \leq n$.*

Proof Let $2 \leq i \leq n$ and let $\mathcal{A} = \{A_j : j \in J\}$ be an infinite family of i-element sets. Let k be chosen minimal between 1 and i such that for some infinite $Y \subseteq A = \bigcup \mathcal{A}$, $\{j \in J : |Y \cap A_j| = k\}$ is infinite (this holds for i, so such k certainly exists). If $k = 1$, we already have a partial choice function for $\mathcal{A}$. Otherwise, we apply RC_k to $\bigcup\{Y \cap A_j : |Y \cap A_j| = k\}$ to find an infinite $Z \subseteq Y$ so that $[Z]^k$ has a choice function f. There is l such that $1 \leq l \leq k$ and $J_1 = \{j \in J : |Z \cap A_j| = l\}$ is infinite. By minimality of k, $k = l$. Thus f restricts to a choice function for $\{Z \cap A_j : j \in J_1\}$ and this provides a partial choice function for $\mathcal{A}$. □

Our next result provides an infinite set of pairs (m, n) of distinct positive integers m and n such that RC_m and C^-_m do not imply RC_n and LOC^-_n in ZF.

Theorem 5 *Let p be a prime number. Then for every $m \in \omega \setminus \{0,1\}$ which is not a multiple of p, and for every $r \in \omega \setminus \{0\}$,*

$$\mathcal{V}_p \models \mathsf{RC}_m \wedge \mathsf{C}_m^- \wedge \neg\mathsf{RC}_{pr} \wedge \neg\mathsf{LOC}_{pr}^-.$$

The result is transferable into ZF*.*

Proof From Fact 2(i) (of Section 4) we know that LOC_k^- is false in $\mathcal{V}_p$ for every integer k which is a multiple of p. Furthermore, using Fact 2(ii) and a similar argument to the one given for Fact 2(i), we may easily conclude that for all r in $\omega \setminus \{0\}$, RC_{pr} is false in $\mathcal{V}_p$ for the set of atoms $A = \bigcup\{A_q : q \in \mathbb{Q}\}$. Fix $r \in \omega \setminus \{0\}$ and assume, towards a contradiction, that A has an infinite subset $y \in \mathcal{V}_p$ such that $[y]^{pr}$ has a choice function, say f with support $E = \bigcup\{A_q : q \in S\}$, where $S \in [\mathbb{Q}]^{<\omega}$. By Fact 2(ii), $y = \bigcup\{A_q : q \in I\}$ for some infinite subset I of $\mathbb{Q}$ such that $\{A_q : q \in I\} \in \mathcal{V}_p$. Then there is an r-element subset W of I such that the (pr)-element set $F = \bigcup\{A_q : q \in W\}$ is disjoint from E. Assuming that $f(F) \in A_w$ for some $w \in W$, we have $\tau_w \in \mathrm{fix}_G(E)$, but $\tau_w(f) \neq f$, which is a contradiction.

We proceed now with the proofs of the rest of the assertions of the theorem.

Claim 1 RC_m is true in $\mathcal{V}_p$ for every integer m which is not a multiple of p.

Proof Let x be an infinite set in $\mathcal{V}_p$. If x is well-orderable, then RC_m is vacously true for x. So we assume that x is non-well-orderable. Let $E = \bigcup\{A_q : q \in K\}$, z, $E_z = \bigcup\{A_q : q \in K'\}$, and $r_0 \in K' \setminus K$, be as in the argument for Case a of the proof of Fact 4 of Section 4. Thus $K' = K \cup \{r_0\}$, and as there we assume that $q_\ell < r_0$, and we get the function f, and its domain $y = \{\phi(z) : \mathrm{fix}_G(E_z \setminus A_{r_0})\}$. By that proof, we know that $\mathcal{Y}$, which may be written as $\{U_\phi : \phi \in \mathrm{fix}_G(E_z \setminus A_{r_0})\}$ where $U_\phi \subseteq \{\phi(z), \phi\tau_{r_0}(z), \ldots, \phi\tau_{r_0}^{p-1}(z)\}$, is a partition of y into r-element sets for some r dividing p, which is equipped with a linear order induced from that on $\{A_q : q > q_\ell\}$ (also see Fact 1); we denote this linear order on $\mathcal{Y}$ by $\preceq$. We will show that $[y]^m$ has a choice function in $\mathcal{V}_p$. If $r = 1$, then y is linearly orderable, so this is immediate. Otherwise, as p is prime, $r = p$. Thus $U_\phi = \{\phi(z), \phi\tau_{r_0}(z), \ldots, \phi\tau_{r_0}^{p-1}(z)\}$ for all $\phi \in \mathrm{fix}_G(E_z \setminus A_{r_0})$.

Since $E_z \setminus A_{r_0}$ is a support of $(\mathcal{Y}, \preceq)$ and of y, it is also a support of $[y]^m$. Thus $[y]^m$ can be written as a disjoint union of the $\mathrm{fix}_G(E_z \setminus A_{r_0})$-orbits of its elements, i.e., $[y]^m = \bigcup\{\mathrm{Orb}_{E_z \setminus A_{r_0}}(w) : w \in [y]^m\}$, where $\mathrm{Orb}_{E_z \setminus A_{r_0}}(w) = \{\phi(w) : \phi \in \mathrm{fix}_G(E_z \setminus A_{r_0})\}$. (Note that $\{\mathrm{Orb}_{E_z \setminus A_{r_0}}(w) : w \in [y]^m\}$ is well-orderable in $\mathcal{V}_p$ since $E_z \setminus A_{r_0}$ is a support of $\mathrm{Orb}_{E_z \setminus A_{r_0}}(w)$ for all $w \in [y]^m$.)

In the ground model M which satisfies AC, we let F be a choice function for the family

$$\mathcal{O} = \{\mathrm{Orb}_{E_z \setminus A_{r_0}}(w) : w \in [y]^m\}.$$

Let $Y \in \mathcal{O}$ and also let $V_{F(Y)} = \min\{R \in \mathcal{Y} : 1 \leq |R \cap F(Y)| < p\}$. Note that $V_{F(Y)}$ is definable since $F(Y) \in [y]^m$ (so $F(Y)$ is finite), $(\mathcal{Y}, \preceq)$ is linearly ordered, and m is not a multiple of p. Invoking AC again in M, for each $Y \in \mathcal{O}$ pick $a_{F(Y)} \in V_{F(Y)} \cap F(Y)$. Let

$$H = \{(\phi(F(Y)), \phi(a_{F(Y)})) : Y \in \mathcal{O}, \phi \in \mathrm{fix}_G(E_z \setminus A_{r_0})\}.$$

It is clear that H is a binary relation with domain $\bigcup \mathcal{O} = [y]^m$. Furthermore, H is a function. To see this, let $Y \in \mathcal{O}$ and also let $\phi, \psi \in \mathrm{fix}_G(E_z \setminus A_{r_0})$ such that $\phi(F(Y)) = \psi(F(Y))$ (so $\phi^{-1}\psi(F(Y)) = F(Y)$). Since for every $\eta \in \mathrm{fix}_G(E_z \setminus A_{r_0})$, $U_\eta = \{\eta\tau_{r_0}^j(z) : j < p\}$, and p is prime, and $|V_{F(Y)} \cap F(Y)| < p$, it is easy to see that $\phi^{-1}\psi \restriction V_{F(Y)}$ is necessarily the identity mapping, and thus $\phi^{-1}\psi$ fixes $V_{F(Y)} \cap F(Y)$ pointwise. Since $a_{F(Y)} \in V_{F(Y)} \cap F(Y)$, $\phi^{-1}\psi(a_{F(Y)}) = a_{F(Y)}$, and thus $\phi(a_{F(Y)}) = \psi(a_{F(Y)})$.

Finally, H is a choice function of $[y]^m$, which is in $\mathcal{V}_p$ since it is supported by $E_z \setminus A_{r_0}$. Thus RC_m is true in $\mathcal{V}_p$. □

Claim 2 C_m^- is true in $\mathcal{V}_p$ for every integer m which is not a multiple of p.

Proof Assume the result for smaller m. Let $\mathcal{Z} = \{Z_i : i \in I\}$ be a disjoint infinite family of m-element sets in $\mathcal{V}_p$, and $Z = \bigcup \mathcal{Z}$. By RC_m in $\mathcal{V}_p$, there is an infinite $y \subseteq Z$ such that $[y]^m$ has a choice function f. Then for some t with $1 \leq t \leq m$, $S_t = \{i \in I : |Z_i \cap y| = t\}$ is infinite. If $t = m$, f provides a choice function for $\{Z_i : i \in S_t\}$. Otherwise, $1 \leq t < m$ and so either t or $m - t$ is not a multiple of p. If t is not a multiple of p, then as we assumed the result for values less than m, $\{Z_i \cap y : i \in S_t\}$, and hence also $\mathcal{Z}$, has a partial choice function. Otherwise, we apply the same argument to $\{Z_i \setminus y : i \in S_t\}$. □

The above arguments complete the proof of the theorem. □

With essentially the same arguments as in the proof of Theorem 5 above (and using Facts 2(i) and 4 of Section 4), one can prove a much more general and stronger result than Theorem 5. Indeed, we have the following theorem.

Theorem 6 *Let $p_0 \leq \ldots \leq p_v$ be prime numbers and let k be a positive integer. Then*

$$\mathcal{V}_{p_0,\ldots,p_v} \models \mathsf{RC}_k \leftrightarrow \mathsf{C}_k^- \leftrightarrow \mathsf{LOC}_k^-$$

and

$$\mathcal{V}_{p_0,\ldots,p_v} \models \neg\mathsf{RC}_k \iff k \text{ is a multiple of } p_i \text{ for some } i \leq v.$$

Furthermore, for all integers $k \geq 2$ which can be written as a sum of multiples of $p_0, \ldots, p_v$,

$$\mathcal{V}_{p_0,\ldots,p_v} \models \neg\mathsf{C}_k.$$

The result is transferable into ZF.

Part (iii) of the subsequent corollary to Theorems 3 and 6 *completely settles the open problem* on the relationship between RC_k and RC_3, where $k \in \{2, 4\}$.

Corollary 1 *The following hold:*

(i) If $m, n \geq 2$ are any positive integers such that for some prime p we have $p \nmid m$ and $p \mid n$, then in ZF*: $\mathsf{RC}_m \nrightarrow \mathsf{RC}_n$ and $\mathsf{RC}_m \nrightarrow \mathsf{C}_n^-$.*

(ii) There is a model $\mathcal{M}$ of ZF *such that for every positive integer n, $\mathcal{M} \models \mathsf{RC}_{2n+1} \wedge \mathsf{C}_{2n+1}^- \wedge \neg\mathsf{RC}_{2n} \wedge \neg\mathsf{LOC}_{2n}^-$. Hence, for every odd integer $n \geq 3$ and for every even integer $m \geq 2$, $\mathcal{M} \models \mathsf{RC}_n \wedge \mathsf{C}_n^- \wedge \neg\mathsf{RC}_m \wedge \neg\mathsf{LOC}_m^-$.*

(iii) For $k = 2, 4$, the principles RC_k and RC_3 are independent of each other in ZF.

Proof (i) Use model $\mathcal{V}_p$ and the result of Theorem 5.

(ii) Use model $\mathcal{V}_2$ and the result of Theorem 5 (or Theorem 6). The result is transferable into ZF via Pincus' transfer theorems.

(iii) This follows easily from Theorem 3 (parts 3. and 4.) and part (ii) of the current corollary. □

Theorem 7 *For $n = 2, 3$, RC_5 + C_n^- implies C_5^-, and RC_5 implies neither C_2^- nor C_3^- in* ZF.

Proof Assume that RC_5 + C_2^- is true. Let $\mathcal{U} = \{U_i : i \in I\}$ be a disjoint infinite family of 5-elements sets. By way of contradiction, assume that $\mathcal{U}$ has no partial choice function. Let y be an infinite subset of $\bigcup \mathcal{U}$ such that $[y]^5$ has a choice function. Since $\mathcal{U}$ has no partial choice function, we have that the set $\{i \in I : |y \cap U_i| \geq 4 \text{ or } |y \cap U_i| = 1\}$ is finite. It follows that at least one of the sets $Y_1 = \{i \in I : |y \cap U_i| = 2\}$ and $Y_2 = \{i \in I : |y \cap U_i| = 3\}$ is infinite. If Y_1 is infinite, then, by C_2^-, the family $\mathcal{Y}_1 = \{y \cap U_i : i \in Y_1\}$, and hence $\mathcal{U}$, has a partial choice function, which is a contradiction. If Y_2 is infinite, then by C_2^- again, we have $\mathcal{Y}_2 = \{U_i \setminus (y \cap U_i) : i \in Y_2\}$ has a partial choice function, which again contradicts the assumption that $\mathcal{U}$ has no partial choice function.

The proof that RC_5 + C_3^- implies C_5^- is similar.

The third assertion (that RC_5 implies neither C_2^- nor C_3^- in ZF) follows easily from Theorem 5. □

Remark 3 By Facts 1 and 2 (of section 4), we have that the family of 5-element sets of atoms, $\mathcal{C} = \{A_q \cup B_q : q \in \mathbb{Q}\}$, is linearly orderable in the permutation model $\mathcal{N}_{3,2}$ (which was constructed in (V2) of Section 4), and has no partial choice function in $\mathcal{N}_{3,2}$.

On the other hand, it is straightforward to verify that the infinite subset $y = \bigcup\{B_q : q \in \mathbb{Q}\} \subset A$ (which is an element of $\mathcal{N}_{3,2}$ since it has empty support) is such that $[y]^5$ has a choice function in $\mathcal{N}_{3,2}$. (It is also true that the subset $u = \bigcup\{A_q : q \in \mathbb{Q}\} \subset A$ is such that $[u]^5$ has a choice function in $\mathcal{N}_{3,2}$; follow the argument for Claim 1 of the proof of Theorem 5.)

It is therefore tempting to think that $\mathcal{N}_{3,2}$ may serve as the appropriate setting in order to answer (in the negative) the question of whether RC_5 implies C_5^-. However, this is not the case. In particular, Ramsey Choice RC_5 is false in $\mathcal{N}_{3,2}$ (while, by Theorem 6, it *is* true in $\mathcal{V}_{3,2}$). To see this, let

$$x = \{\{a_{qm}, b_{qn}\} : q \in \mathbb{Q}, m \in \{1, 2, 3\}, n \in \{1, 2\}\}.$$

Then $x \in \mathcal{N}_{3,2}$ since $\emptyset$ is a support of x. We assert that x has no infinite subset y in $\mathcal{N}_{3,2}$ such that $[y]^5$ has a choice function. Assume the contrary; then we may let $y \in \mathcal{N}_{3,2}$ be an infinite subset of x such that $[y]^5$ has a choice function $f \in \mathcal{N}_{3,2}$. Let $E = \bigcup\{A_q \cup B_q : q \in S\}$, where $S \in [\mathbb{Q}]^{<\omega}$, be a support of y and f. It is easy to see that there exist distinct rational numbers q and r such that $\{q, r\} \cap S = \emptyset$ (and hence $E \cap (A_q \cup B_q \cup A_r \cup B_r) = \emptyset$) and

$$w = \{\{a_{q1}, b_{q1}\}, \{a_{q2}, b_{q1}\}, \{a_{q3}, b_{q1}\}, \{a_{r1}, b_{r1}\}, \{a_{r1}, b_{r2}\}\} \in [y]^5.$$

Clearly, $\tau_q, \sigma_r \in \mathrm{fix}_G(E)$, hence $\tau_q(f) = \sigma_r(f) = f$, and also $\tau_q(w) = \sigma_r(w) = w$. If $f(w) = \{a_{qk}, b_{q1}\}$ for some $k \in \{1, 2, 3\}$, then $\tau_q(f(w)) \neq f(w)$ so $(w, \tau_q(f(w))) \notin f$, and if $f(w) = \{a_{r1}, b_{rl}\}$ for some $l \in \{1, 2\}$, then $\sigma_r(f(w)) \neq f(w)$ so $(w, \sigma_r(f(w))) \notin f$. Since each of the above two possibilities for $f(w)$ leads to a contradiction to the fact that E is a support of f, we conclude that x has no infinite subset y in $\mathcal{N}_{3,2}$ such that $[y]^5$ has a choice function. Therefore RC_5 is false in $\mathcal{N}_{3,2}$.

Next, we prove that CAC (Chain-AntiChain Principle) does not imply C_n^- in ZF for any natural number $n \geq 2$. The permutation model that will be constructed in the proof of the subsequent result will also be useful in the proof of the forthcoming Theorem 9.

Theorem 8 *For every natural number $n \geq 2$, there is a model $\mathcal{M}$ of ZF such that $\mathcal{M} \models \mathsf{CAC} \wedge \neg\mathsf{C}_n^-$.*

Proof Fix a natural number $n \geq 2$. Since $\mathsf{CAC} \wedge \neg\mathsf{C}_n^-$ is a conjunction of injectively boundable statements, it follows—by Pincus' Theorem 3A3 in [8]—that we only need to construct a Fraenkel–Mostowski model of ZFA with the required properties. Our model will be a generalization of the permutation model constructed in the proof of Theorem 2.1 of Tachtsis [9] (where Theorem 2.1 of [9] states that CAC does not imply Ramsey's Theorem in ZFA). Since all the required arguments for the proof of the current theorem are almost identical to the ones given for the proof of Theorem 2.1 of [9], we refer the interested reader to [9] for the details.

The description of the model: We start with a model M of $\mathsf{ZFA} + \mathsf{AC}$ with a set of atoms $A = \bigcup\{A_i : i \in \omega\}$ which is a denumerable disjoint union of n-element sets $A_i = \{a_{i1}, a_{i2}, \ldots, a_{in}\}$ (where $i \in \omega$).

The group G of permutations of A is defined as follows: Firstly, for all $i \in \omega$, let τ_i be the n-cycle $a_{i1} \mapsto a_{i2} \mapsto \cdots \mapsto a_{in} \mapsto a_{i1}$. Also, for every permutation ψ of ω which moves *only finitely many* natural numbers, let ϕ_ψ be the permutation of A which is defined by $\phi_\psi(a_{ij}) = a_{\psi(i)j}$ for all $i \in \omega$ and $j \in \{1, 2, \ldots, n\}$.

Now, we require $\eta \in G$, if and only if, $\eta = \phi_\psi \rho$, where ψ is a permutation of ω which moves only finitely many natural numbers and ρ is a permutation of A for which there is a finite subset $F \subseteq \omega$ such that for every $k \in F$ we have that $\rho \restriction A_k = \tau_k^j$ for some $j < n$, and ρ fixes A_m pointwise for every $m \in \omega \setminus F$.

From the definition of the group G, it follows that if $\eta \in G$, then η moves *only finitely many* atoms, and for all $i \in \omega$ there is $k \in \omega$ such that $\eta(A_i) = A_k$.

Let Γ be the filter of subgroups of G which is generated by the subgroups $\text{fix}_G(E)$, where $E \in [A]^{<\omega}$. Let $\mathcal{N}$ be the Fraenkel–Mostowski model which is determined by M, G, and Γ.

The following hold in $\mathcal{N}$ (see [9, proof of Theorem 2.1]):

1. The set $\mathcal{A} = \{A_i : i \in \omega\}$ is amorphous and has no infinite subfamily $\mathcal{B}$ with a Kinna–Wagner selection function. Furthermore, A is also amorphous. It follows that C_n^- (as well as RC_n—see also Theorem 2(1)) is false in $\mathcal{N}$ for $\mathcal{A}$.

2. Every element $x \in \mathcal{N}$ is either well-orderable or has an infinite subset y with a partition into sets each of size at most n, indexed by a cofinite subset of $\mathcal{A}$, thus indexed by an amorphous set (note that the proof of Fact 4 of Section 4 goes through here with minor adjustments). In the second case, it follows that y is an amorphous subset of x.

3. Every linearly orderable set in $\mathcal{N}$ is well-orderable. This follows immediately from item 2.

4. The union of a well-orderable family of well-orderable sets in $\mathcal{N}$ is well-orderable.

5. $\mathsf{AC}(\mathsf{LO}, \mathsf{LO})$ is true in $\mathcal{N}$. This follows from items 3 and 4.

6. CAC is true in $\mathcal{N}$.

The above completes the (outline of the) proof of the theorem. □

Next, we completely clarify the relationship between C_n^-, LOC_n^- and WOC_n^- in ZF.

Theorem 9 *For every $n \in \omega \setminus \{0, 1\}$, C_n^- implies LOC_n^-, which in turn implies WOC_n^-. Furthermore, for every $n \in \omega \setminus \{0, 1\}$, none of the previous implications are reversible in ZF.*

Proof The implications in the statement of the theorem are straightforward.

For the second assertion of the theorem, fix $n \in \omega \setminus \{0, 1\}$. Firstly, we note that each of C_n^-, LOC_n^- and WOC_n^- is an injectively boundable statement, so in view of Pincus' Theorem 3A3 in [8], it suffices to establish our independence results using Fraenkel–Mostowski permutation models.

We shall prove something stronger than "LOC_n^- does not imply C_n^-", namely that there is a model of ZFA in which $\mathsf{AC}(\mathsf{LO}, \mathsf{LO})$ is true, whereas C_n^- is false. For our purpose, we will use the permutation model $\mathcal{N}$ of the proof of Theorem 8. From its proof we know that $\mathsf{AC}(\mathsf{LO}, \mathsf{LO})$ is true in $\mathcal{N}$, whereas C_n^- is false in $\mathcal{N}$. Thus, LOC_n^- is also true in $\mathcal{N}$.

Now, we shall also prove something stronger than "WOC_n^- does not imply LOC_n^-", namely that there is a model $\mathcal{V}$ of ZFA in which $\mathsf{UT}(\mathsf{WO}, \mathsf{fin}, \mathsf{WO})$ is true, whereas LOC_n^- is false in $\mathcal{V}$. We will use the model $\mathcal{V}_n$ of Section 4 (so that the set of atoms is $A = \bigcup\{A_q : q \in \mathbb{Q}\}$, where $A_q = \{a_{q1}, a_{q2}, \ldots, a_{qn}\}$ for every $q \in \mathbb{Q}$, and $A_q \cap A_r = \emptyset$ for distinct rationals q and r).

By Facts 1 and 2(i), we have that $\mathcal{B} = \{A_q : q \in \mathbb{Q}\}$ is a linearly orderable family of n-element sets, which admits no partial choice function in $\mathcal{V}_n$. Thus, LOC_n^- is false in $\mathcal{V}_n$.

Now we prove that $\mathsf{UT}(\mathsf{WO}, \mathsf{fin}, \mathsf{WO})$ is true in $\mathcal{V}_n$. To this end, let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$, where κ is an infinite well-ordered cardinal number, be a disjoint family of finite sets in $\mathcal{V}_n$. Let $E = \bigcup\{A_q : q \in S\}$ (where $S \in [\mathbb{Q}]^{<\omega}$) be a support of U_α for each $\alpha < \kappa$. We will show that E is a support of every element in $\bigcup \mathcal{U}$. Assuming the contrary, there exist $\alpha < \kappa$ and $u \in U_\alpha$ such that u is not supported by E. Let $E_u = \bigcup\{A_q : q \in S'\}$, where $S' \in [\mathbb{Q}]^{<\omega}$, be the minimum support of u, whose existence is guaranteed by Fact 3 of section 4. Since u is not supported by E, there exists an element $r \in S' \setminus S$ (and hence $A_r \subseteq E_u$ and $A_r \cap E = \emptyset$). Furthermore, it is not hard to verify that there is an infinite subset $P \subseteq \mathbb{Q}$ such that for all p in P, there exists an order automorphism ψ_p of $(\mathbb{Q}, \leq)$ which fixes S pointwise, and $\psi_p(r) = p$. It follows that for all p, p' in P, if $p \neq p'$ then $\psi_p(S') \neq \psi_{p'}(S')$. Thus, for all p, p' in P, if $p \neq p'$ then $\phi_{\psi_p}(E_u) \neq \phi_{\psi_{p'}}(E_u)$, and consequently, by the second part of the statement of Fact 3 of

Section 4, $\phi_{\psi_p}(u) \neq \phi_{\psi_{p'}}(u)$. Since for all p in P, ψ_p fixes S pointwise, we have $\phi_{\psi_p} \in \mathrm{fix}_G(E)$, and hence $\phi_{\psi_p}(U_\alpha) = U_\alpha$. It follows that the infinite set $\{\phi_{\psi_p}(u) : p \in P\}$ is a subset of U_α, contradicting U_α's being finite. Thus $\bigcup \mathcal{U}$ is well-orderable. $\square$

In Theorem 10 below, we elucidate the relationship of RC_6 with certain instances of LOC_n^-. The results of Theorem 10 provide *partial answers to the open problem* of whether RC_6 implies C_i^- for $i = 2, 3, 4, 6$. In part (i) of Theorem 10, we shall provide a general result, namely that for every $k \in \omega \setminus \{0, 1\}$, RC_{2k} implies LOKW_k^- and that the latter implication is not reversible in ZF (for any $k \in \omega \setminus \{0, 1\}$). The latter result gives us, in particular, that RC_6 strictly implies LOC_3^- in ZF. Our proof of (i) (of Theorem 10), uses ideas from Montenegro's ingenious proof that RC_4 implies C_4^- (see [7]). Since the subsequent theorem is centered around RC_6, we also have incorporated in the theorem's list of results, a fact which immediately follows from Theorem 5, namely that RC_6 does not imply C_{5n}^- in ZF, for all $n \in \omega \setminus \{0\}$. The latter result *completely settles* the corresponding open problems.

We would also like to point out here that Theorems 5, 6 and 10, indicate the limitations of the permutation models $\mathcal{V}_{n,m}$ and $\mathcal{N}_{n,m}$ of Section 4 with regard to the aforementioned open problems on RC_6.

Theorem 10 *The following hold:*

(i) For every $k \in \omega \setminus \{0, 1\}$, RC_{2k} implies LOKW_k^- and the latter implication is not reversible in ZF. In particular, RC_6 strictly implies LOC_3^- in ZF.

(ii) LOC_4^- is equivalent to $\mathsf{LOC}_2^- + \mathsf{LOKW}_4^-$. Furthermore, LOC_2^- does not imply LOKW_4^- in ZF, hence neither does it imply LOC_4^- in ZF.

(iii) $\mathsf{RC}_6 + \mathsf{LOC}_2^-$ implies LOC_6^-. Hence, $\mathsf{RC}_6 + \mathsf{LOC}_{2n}^-$ implies LOC_6^- for all $n \in \omega \setminus \{0\}$.

(iv) $\mathsf{RC}_6 + \mathsf{LOC}_2^-$ implies LOC_4^-. Hence, $\mathsf{RC}_6 + \mathsf{LOC}_{2n}^-$ implies LOC_4^- for all $n \in \omega \setminus \{0\}$.

(v) RC_6 does not imply LOC_5^- in ZF, hence it does not imply C_{5n}^- in ZF, for all $n \in \omega \setminus \{0\}$.

(vi) RC_6 implies WOC_2^-.

Proof (i) Fix $k \in \omega \setminus \{0, 1\}$ and assume that RC_{2k} is true. Let $\mathcal{A} = \{A_i : i \in I\}$ be a disjoint infinite family of k-element sets indexed by the set I, which is equipped with some prescribed linear order. Towards a proof by contradiction assume that $\mathcal{A}$ has no infinite subfamily with a Kinna–Wagner selection function. By RC_{2k}, let Y be an infinite subset of $A = \bigcup \mathcal{A}$ such that $[Y]^{2k}$ has a choice function, say f. Since $\mathcal{A}$ has no partial Kinna–Wagner function, we may assume, without loss of generality, that there is an infinite subset J of I such that $Y = \bigcup \{A_j : j \in J\}$. We define a binary relation R on J by requiring for all $j, j' \in J$,

$$j \; R \; j', \text{ if and only if, } f(A_j \cup A_{j'}) \in A_{j'}.$$

(Note that if $(j, j') \in R$, then $(j', j) \notin R$.)

For every $j \in J$, we let

$$K_j = \{r \in J : j \; R \; r\} = \{r \in J : f(A_j \cup A_r) \in A_r\}.$$

Since $\mathcal{A}$ has no partial choice function, it follows that for all j in J, the set K_j is finite; otherwise, if for some $j \in J$, K_j is infinite, then we let $\mathcal{B} = \{A_r : r \in K_j\}$ and we define a choice function g on $\mathcal{B}$ by requiring for all $r \in K_j$, $g(A_r) = f(A_j \cup A_r)$. Since $\mathcal{B}$ is an infinite subset of $\mathcal{A}$ with a choice function, we have arrived at a contradiction. Therefore, for all j in J, K_j is finite.

For each $n \in \omega$, let $C_n = \{j \in J : |K_j| = n\}$. It is clear that the family $\mathcal{C} = \{C_n : n \in \omega\}$ is a partition of the linearly ordered set J.

We assert that for all $n \in \omega$, C_n is finite and, in particular, $|C_n| \leq 2n + 1$. To this end, it suffices to show that for all $n \in \omega$, if U is any finite subset of C_n, then $|U| \leq 2n + 1$ (and hence C_n cannot be infinite, otherwise it would have finite subsets of arbitrarily large finite cardinality). Fix $n \in \omega$ and a finite subset $U \subseteq C_n$. For every two-element subset $\{j, j'\}$ of U, either $(j, j') \in R$ or $(j', j) \in R$, but not both. Therefore, $|R \upharpoonright U| = \binom{|U|}{2}$, where $R \upharpoonright U$ is the restriction of R on U. Since for each $j \in C_n$ we have that $|K_j| = n$, it readily follows that $\binom{|U|}{2} \leq n|U|$, which yields that $|U| \leq 2n+1$ as required.

Since J is linearly ordered and for all n in ω, $C_n \in [J]^{<\omega}$, it follows that $J = \bigcup\{C_n : n \in \omega\}$ is denumerable (i.e., countably infinite), thus, so is the (disjoint) family $\mathcal{D} = \{A_j : j \in J\}$. Let $(D_n)_{n\in\omega}$ be an enumeration of the elements of $\mathcal{D}$. Then $f \upharpoonright \mathcal{E}$, where $\mathcal{E} = \{D_{2n} \cup D_{2n+1} : n \in \omega\}$, is a choice function for the disjoint family $\mathcal{E}$ which consists of $(2k)$-element subsets of Y. Since for every $n \in \omega$, $f(D_{2n} \cup D_{2n+1})$ is an element of exactly one of the sets D_{2n} and D_{2n+1} and $\mathcal{D}$ is a denumerable subfamily of $\mathcal{A}$, it readily follows that $\mathcal{A}$ has a partial choice function. This contradicts our assumption on $\mathcal{A}$ having no partial Kinna–Wagner selection function.

For the second assertion of (i), we may use the Fraenkel–Mostowski model $\mathcal{N}$ which was constructed in the proof of Theorem 8 (with n any natural number greater than or equal to 2). From the proof of Theorem 8, we have that $\mathsf{AC}(\mathsf{LO}, \mathsf{LO})$ is true in $\mathcal{N}$ from which LOKW_k^- follows, whereas RC_k is false for any integer $k \geq 2$, since there are amorphous sets in $\mathcal{N}$.

The third assertion is, in view of the above, straightforward.

(ii) The implication "$\mathsf{LOC}_4^- \to \mathsf{LOKW}_4^-$" is evident and the implication "$\mathsf{LOC}_4^- \to \mathsf{LOC}_2^-$" follows from Theorem 1(10).

Conversely, assume that LOC_2^- + LOKW_4^- is true and let $\mathcal{A} = \{A_i : i \in I\}$ be a linearly orderable, disjoint, infinite family of 4-element sets. Let, by LOKW_4^-, $\mathcal{B} = \{A_j : j \in J\}$ be an infinite subfamily of $\mathcal{A}$ with a Kinna–Wagner function, say f. Then there exists an infinite subset $J' \subseteq J$ such that either for all j in J', $|f(A_j)| = 3$, or for all j in J', $|f(A_j)| = 2$, or for all j in J', $|f(A_j)| = 1$. In the first case, we let $g = \{(A_j, \bigcup(A_j \setminus f(A_j))) : j \in J'\}$; then g is a partial choice function for $\mathcal{A}$. In the second case, we apply LOC_2^- to the family $\{f(A_j) : j \in J'\}$, thus obtaining a partial choice function for $\mathcal{A}$. The third case is evident.

For the second assertion of (ii), we only need to establish the independence result using a suitable Fraenkel–Mostowski permutation model, since via Pincus' Transfer Theorems, the result can be transferred into ZF (see also Remark 1 of Section 2). To this end, first let κ be any infinite well-ordered cardinal number. We start with a model M of ZFA + AC with a κ-sized set A of atoms which is a disjoint union $A = \bigcup\{A_\alpha : \alpha < \kappa\}$, where for $\alpha < \kappa$, $A_\alpha = \{a_{\alpha,1}, a_{\alpha,2}, a_{\alpha,3}, a_{\alpha,4}\}$ so that $|A_\alpha| = 4$, for all $\alpha < \kappa$.

For each $\alpha < \kappa$, let G_α be the alternating group on A_α and let G be the weak direct product of the G_α's. Hence, a permutation η of A is an element of G if and only if for every $\alpha < \kappa$, $\eta \upharpoonright A_\alpha \in G_\alpha$, and $\eta \upharpoonright A_\alpha = 1_{A_\alpha}$ for all but finitely many ordinals $\alpha < \kappa$ (and thus every element $\eta \in G$ moves *only finitely many* atoms).

Let Γ be the filter of subgroups of G which is generated by the subgroups $\mathrm{fix}_G(E)$ of G, where $E \in [A]^{<\omega}$. Let $\mathcal{M}$ be the permutation model which is determined by M, G and Γ.

We first show that LOKW_4^- is false in $\mathcal{M}$ for the well-ordered family $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ of $\mathcal{M}$ (the enumeration $\alpha \mapsto A_\alpha$, $\alpha < \kappa$, has empty support). Assume the contrary and let $\mathcal{B}$ be an infinite subfamily of $\mathcal{A}$ having a Kinna–Wagner selection function $f \in \mathcal{M}$ with support some finite set $E \subset A$. Then there exists an ordinal $\alpha_0 < \kappa$ such that $A_{\alpha_0} \in \mathcal{B}$ and $A_{\alpha_0} \cap E = \emptyset$. There are three possibilities for $f(A_{\alpha_0})$:

(a) $|f(A_{\alpha_0})| = 1$. Let t be the unique element of $f(A_{\alpha_0})$ and let $A_{\alpha_0} \setminus f(A_{\alpha_0}) = \{x, y, z\}$. Let η be the permutation of A in G which is defined by $\eta \upharpoonright A_{\alpha_0} = (t, x)(y, z)$ and $\eta \upharpoonright A \setminus A_{\alpha_0} = 1_{A \setminus A_{\alpha_0}}$. Clearly, $\eta \in \mathrm{fix}_G(E)$, hence $\eta(f) = f$. However, $\eta(A_{\alpha_0}) = A_{\alpha_0}$, but $\eta(f(A_{\alpha_0})) \neq f(A_{\alpha_0})$, which contradicts the fact that f is supported by E.

Similarly to (a), the remaining two possibilities, namely (b) $|f(A_{\alpha_0})| = 2$ and (c) $|f(A_{\alpha_0})| = 3$ lead to a contradiction, so we leave the verification of the details as an easy exercise for the interested reader.

Next, we assert that every linearly orderable set in $\mathcal{M}$ is well-orderable (in $\mathcal{M}$). Indeed, let $(x, \leq)$ be a linearly ordered set in $\mathcal{M}$ with support $E \in [A]^{<\omega}$. By way of contradiction, assume that x is not well-orderable in $\mathcal{M}$, and hence there exists an element $z \in x$ which is not supported by E. This means that there is an element $\phi \in \text{fix}_G(E)$ such that $\phi(z) \neq z$. Since $\leq$ is a linear order on x (and $\phi(z) \in \phi(x) = x$), we must have that either $\phi(z) < z$ or $z < \phi(z)$. Without loss of generality, we assume that $\phi(z) < z$ (the case where $z < \phi(z)$ can be treated as in the argument below). Since every element of G moves only finitely many atoms, it follows that every permutation of A in G has finite order, thus there is $m \in \omega$ such that $\phi^m = 1_A$. Then we have $z = \phi^m(z) < \phi^{m-1}(z) < \ldots < \phi^2(z) < \phi(z) < z$, hence $z < z$, which is impossible. Therefore, E supports every element of x, and hence x can be well-ordered in $\mathcal{M}$.

Our final step in the proof is to show that LOC_2^- is true in $\mathcal{M}$. To this end, let $\mathcal{U} = \{U_i : i \in I\}$ be a disjoint infinite family of pairs in $\mathcal{M}$, which is indexed by the linearly orderable set I. In view of the result of the previous paragraph, we have that I is well-orderable, so we may assume that $I = \lambda$ for some infinite well-ordered cardinal number λ. Let $E \in [A]^{<\omega}$ be a support of U_i, for all $i < \lambda$. Without loss of generality, we assume that $E = A_{\alpha_1} \cup A_{\alpha_2} \cup \cdots \cup A_{\alpha_m}$, where for $i = 1, 2, \ldots, m$, $\alpha_i < \kappa$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_m$. We will show that $\bigcup \mathcal{U}$ is well-orderable by proving that E supports every element of $\bigcup \mathcal{U}$. Assume the contrary; then there exist $i \in \lambda$, $u \in U_i$, and $\eta \in \text{fix}_G(E)$ such that $\eta(u) \neq u$. Let E_u be a support of u and, without loss of generality, assume that $E_u = E \cup A_\mu$, where $\mu \in \kappa \setminus \{\alpha_j : 1 \leq j \leq m\}$, and that $\eta \restriction A \setminus A_\mu = 1_{A \setminus A_\mu}$.

Let $\mathcal{G} = \prod^w_{\alpha<\kappa} \mathcal{G}_\alpha$, i.e., $\mathcal{G}$ is the weak direct product of the groups $\mathcal{G}_\alpha$, where for $\alpha < \kappa$, $\mathcal{G}_\alpha = G_\mu$ if $\alpha = \mu$ and $\mathcal{G}_\alpha = \{1_{A_\alpha}\}$ if $\alpha \in \kappa \setminus \{\mu\}$. Clearly, $\mathcal{G}$ is a subgroup of G which is isomorphic to the alternating group G_μ on A_μ. Let $\mathcal{H} = \{\rho \in \mathcal{G} : \rho(u) = u\}$. Then $\mathcal{H}$ is a subgroup of $\mathcal{G}$, which is proper, for $\eta \in \mathcal{G} \setminus \mathcal{H}$. Since $|U_i| = 2$ and $\eta \in \mathcal{G} \setminus \mathcal{H}$, we conclude that the index $(\mathcal{G} : \mathcal{H})$ of $\mathcal{H}$ in $\mathcal{G}$ is 2. As $|\mathcal{G}| = |G_\mu| = 12$, we have $|\mathcal{H}| = 6$. This contradicts the well-known group-theoretic fact that the alternating group on 4 letters has no subgroups of order 6. Therefore, E supports every element of $\bigcup \mathcal{U}$, and so $\bigcup \mathcal{U}$ is well-orderable. Thus $\mathcal{U}$ has a choice function in $\mathcal{M}$, and consequently LOC_2^- is true in $\mathcal{M}$.

(iii) Assume RC_6 + LOC_2^-. Let $\mathcal{A} = \{A_i : i \in I\}$ be a linearly orderable, disjoint, infinite family of 6-element sets. By RC_6, let y be an infinite subset of $A = \bigcup \mathcal{A}$ such that $[y]^6$ has a choice function. If the set $Z_1 = \{i \in I : |y \cap A_i| \geq 5 \text{ or } |y \cap A_i| = 1\}$ is infinite, then we easily conclude that $\mathcal{A}$ has a partial choice function. If $Z_2 = \{i \in I : |y \cap A_i| = 4\}$ is infinite, then by LOC_2^-, the family $\mathcal{B} = \{A_i \setminus (y \cap A_i) : i \in Z_2\}$ has a partial choice function, hence $\mathcal{A}$ has a partial choice function too. If $Z_3 = \{i \in I : |y \cap A_i| = 3\}$ is infinite, then the conclusion follows from part (i) of the current theorem, and if $Z_4 = \{i \in I : |y \cap A_i| = 2\}$ is infinite, then the conclusion follows from LOC_2^- again.

(iv) Assume RC_6 + LOC_2^-. Let $\mathcal{A} = \{A_i : i \in I\}$ be a linearly orderable, disjoint, infinite family of 4-element sets. Then $\mathcal{B} = \{[A_i]^2 : i \in I\}$ is a linearly orderable disjoint family of 6-element sets. By our assumption and part (iii) of the current theorem, we have that LOC_6^- is true, and thus there exists an infinite subfamily $\mathcal{C} = \{[A_j]^2 : j \in J\}$ of $\mathcal{B}$, where $J \subseteq I$ is infinite, with a choice function, say f. Since for every $j \in J$, $f([A_j]^2)$ is a 2-element subset of A_j, we apply LOC_2^- to the linearly orderable infinite family $\mathcal{D} = \{f([A_j]^2) : j \in J\}$ in order to obtain a partial choice function g for $\mathcal{D}$. Using g, we immediately obtain a partial choice function for $\mathcal{A}$.

(v) This follows from Theorem 5; in particular, the permutation model $\mathcal{V}_5$ satisfies RC_6 + $\neg\mathsf{C}_5^-$, and the result is transferable into ZF.

(vi) Assume RC_6. Let $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$, where κ is an infinite well-ordered cardinal number, be a family of 2-element sets. By RC_6, let y be an infinite subset of $A = \bigcup \mathcal{A}$ such that $[y]^6$ has a choice function, say f. Without loss of generality, assume that the set $\{\alpha : \alpha < \kappa \text{ and } |y \cap A_\alpha| = 1\}$ is finite. Therefore, since y is infinite and κ is an infinite well-ordered cardinal, it follows that there exists a strictly increasing sequence $(\alpha_n)_{n\in\omega}$ of ordinals in κ such that $A_{\alpha_n} \subset y$, for all $n < \omega$. Then $f \restriction \mathcal{B}$, where $\mathcal{B} = \{A_{\alpha_{3n}} \cup A_{\alpha_{3n+1}} \cup A_{\alpha_{3n+2}} : n < \omega\}$, is a choice function for the disjoint family $\mathcal{B}$ which consists of 6-element sets. Clearly, this yields that $\mathcal{A}$ has a partial choice function. $\square$

6 Open questions

1. Does RC_5 imply either of LOC_5^- and C_5^-?
2. Does RC_4 imply RC_2? (Recall that RC_2 implies RC_4 (see Theorem 3(4)). Also, note that RC_4 implies C_2^-, since (from [7]) RC_4 implies C_4^-, which in turn implies C_2^-.)
3. Is there a model of ZF which satisfies $\mathsf{RC}_6 + \neg\mathsf{C}_i^-$, where $i \in \{2, 3, 4, 6\}$? Same question for RC_6 and LOC_i^-, where $i \in \{2, 4, 6\}$.

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