# Solution of Riddle 12237

The Logic Coffee Circle*

**Abstract**

We solve the following riddle:

**12237**. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Let $x_0 = 1$ and $x_{n+1} = x_n + \lfloor x_n^{3/10} \rfloor$ for $n \geq 0$. What are the first 40 decimal digits of $x_n$ when $n = 10^{100}$?

## 1 Solution

We begin by calculating the first few elements of the sequence:

$$x_0 = 1,\ x_1 = 2, \dots, x_{10} = 11,\ x_{11} = 13,\ x_{12} = 15, \dots, x_{23} = 37,\ x_{24} = 39,\ x_{25} = 42,\ x_{26} = 45,\ \dots$$

We have that for all $m \in \mathbb{N}_{>0}$

$$\lfloor x^{\frac{3}{10}} \rfloor = m \quad \Longleftrightarrow \quad m^{\frac{10}{3}} \leq x < (m+1)^{\frac{10}{3}}. \tag{1}$$

For positive integers $m$ define

$$s_m := |\{n \in \mathbb{N}_{>0} \mid x_n = x_{n-1} + m\}|.$$

Note that for $n = \sum_{m=1}^{l} s_m$ we have $x_n = x_{n-1} + l$ and $x_{n+1} = x_n + (l+1)$. Define $g : \mathbb{N} \to \mathbb{R}$ by

$$g(m) := \frac{(m+1)^{\frac{10}{3}} - m^{\frac{10}{3}}}{m}.$$

Note that by (1) we have that

$$g(m) - 1 \leq s_m \leq g(m) + 1.$$

Now we want to find a suitable approximation of $g(m)$. For this, we first consider the following power series:

$$(1+y)^{\frac{10}{3}} - 1 = \sum_{k=1}^{\infty} \binom{\frac{10}{3}}{k} \cdot y^k$$

With this series and $y := \frac{1}{m}$ we obtain

$$\begin{aligned} g(m) :&= \frac{(m+1)^{\frac{10}{3}} - m^{\frac{10}{3}}}{m} = \frac{1}{y^{\frac{7}{3}}} \left( (1+y)^{\frac{10}{3}} - 1 \right) \\ &= \underbrace{\frac{10}{3} m^{\frac{4}{3}} + \frac{35}{9} m^{\frac{1}{3}} + \frac{140}{81} \frac{1}{m^{\frac{2}{3}}}}_{=:g_1(m)} + \underbrace{\frac{35}{243} \frac{1}{m^{\frac{5}{3}}} - \frac{14}{729} \frac{1}{m^{\frac{8}{3}}} \pm \dots}_{=:R(m)} \end{aligned}$$

By definition of $R(m)$, in particular since $R(m)$ is an alternating series, for each $m \in \mathbb{N}$ we have $0 < R(m) < \frac{35}{243} \frac{1}{m^{\frac{5}{3}}}$. Moreover, we have

$$0 < \sum_{m=1}^{\infty} R(m) < \frac{35}{243} \sum_{m=1}^{\infty} \frac{1}{m^{\frac{5}{3}}} = \frac{35}{243} \cdot \zeta\left(\tfrac{5}{3}\right) < \frac{35}{243} \cdot \frac{17}{8} < \frac{1}{3}.$$

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Therefore, we have that

$$g_1(m) - 1 \le s_m \le g_1(m) + 2.$$

Now, we want to compute $l_0 \in \mathbb{N}$ with

$$x_{10^{100}} = x_{10^{100}-1} + l_0.$$

In other words, we have to find the smallest $l_0 \in \mathbb{N}$ such that $\sum_{m=1}^{l_0} s_m \ge 10^{100}$. For this, we first notice that

$$\begin{aligned} \sum_{m=1}^{l} g_1(m) &= \frac{10}{3}\sum_{m=1}^{l} m^{\frac{4}{3}} + \frac{35}{9}\sum_{m=1}^{l} m^{\frac{1}{3}} + \frac{140}{81}\sum_{m=1}^{l} \frac{1}{m^{\frac{2}{3}}} \\ &= \frac{5}{81}\left(54H_l^{(-\frac{4}{3})} + 63H_l^{(-\frac{1}{3})} + 28H_l^{(\frac{2}{3})}\right), \end{aligned}$$

where $H_l^{(r)}$ is the $l$-th harmonic number of order $r$, defined by $H_l^{(r)} := \sum_{m=1}^{l} \frac{1}{m^r}$. Now, with the inequality

$$10^{100} \le \sum_{m=1}^{l_0} s_m \le \sum_{m=1}^{l_0} (g_1(m) + 2) = 2l_0 + \sum_{m=1}^{l_0} g_1(m)$$

and the help of Mathematica we find after a short search

$$l_0 \ge 6\,176\,697\,077\,775\,135\,894\,745\,105\,594\,539\,832\,252\,961\,972,$$

and with the inequality

$$\sum_{m=1}^{l_0} s_m \ge \sum_{m=1}^{l_0} (g_1(m) - 1) = -l_0 + \sum_{m=1}^{l_0} g_1(m) \ge 10^{100}$$

we find the same value for $l_0$. Hence,

$$l_0 = 6\,176\,697\,077\,775\,135\,894\,745\,105\,594\,539\,832\,252\,961\,972.$$

Since $g(m) - 1 \le s_m \le g(m) + 1$, we have

$$x_{10^{100}} \le \sum_{m=1}^{l_0} m\, s_m \le \sum_{m=1}^{l_0} m\, g(m) + \sum_{m=1}^{l_0} m = (l_0+1)^{\frac{10}{3}} - 1 + \frac{l_0(l_0+1)}{2} =: A,$$

and analogously

$$x_{10^{100}} \ge \sum_{m=1}^{l_0-1} m\, s_m \ge \sum_{m=1}^{l_0-1} m\, g(m) - \sum_{m=1}^{l_0-1} m = l_0^{\frac{10}{3}} - 1 - \frac{l_0(l_0-1)}{2} =: B.$$

With Mathematica we see that the first 40 digits of $A$ and $B$ are the same, so the first 40 digits of $x_{10^{100}}$ are

$$43236\,87954\,44259\,51263\,21573\,91617\,78825\,77073\,.$$