Solution of Riddle 12256

The Logic Coffee Circle*

Abstract

We solve the following riddle:

12256. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.* Prove

$$\int_0^1 \frac{\log(1+x)\log(1-x)}{x}\,dx = -\frac{5}{8}\zeta(3),$$

where $\zeta(3)$ is Apéry's constant $\sum_{n=1}^{\infty} 1/n^3$.

Solution

With the series expansions

$$\log(1+x) = -\sum_{j=1}^{\infty} \frac{(-1)^j x^j}{j} \quad \text{and} \quad \log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

we get that

$$\begin{aligned}
\int_0^1 \frac{\log(1+x)\log(1-x)}{x}\,dx &= \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 x^{j+k-1}\,dx = \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \sum_{k=1}^{\infty} \frac{1}{k(j+k)} \\
&= \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \sum_{k=1}^{\infty} \left(\frac{1}{kj} - \frac{1}{(k+j)j} \right) \\
&= \sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+j} \right) = \sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} \left(\sum_{k=1}^{j} \frac{1}{k} \right) \\
&= -\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} \left(\sum_{k=1}^{j} \frac{1}{k} \right) = -\frac{5}{8}\zeta(3).
\end{aligned}$$

The last equation is well-known. A proof of the equation can be found here: https://math.stackexchange.com/questions/275643/proving-an-alternating-euler-sum-sum-k-1-infty-frac-1k1-h-kk?lq=1

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