CAN KIRKMAN'S SCHOOLGIRLS WALK ABREAST BY FORMING ARITHMETIC PROGRESSIONS ?

Lorenz Halbeisen

Department of Pure Mathematics, Queen's University Belfast
Belfast BT7 1NN (Northern Ireland)

Email: halbeis@qub.ac.uk

Abstract

A Kirkman triple system of order v, whose point set is consecutively numbered, is called *smooth*, if for each parallel class there is a number d, such that each triple of the parallel class forms an arithmetic progression modulo v with common difference d. It will be shown that smooth Kirkman triple systems of order v exist, if and only if v is a power of 3.

1 Smooth Kirkman triple systems

The following problem, known as Kirkman's schoolgirl problem, was introduced in 1850 by Reverend Thomas J. Kirkman (1806–1895) as "Query 6" on page 48 of the *Ladies and Gentleman's Diary* [2] (see [3] for his statement of the general problem): *Fifteen young ladies in a school walk out three abreast for seven days in succession; it is required to arrange them daily, so that no two walk twice abreast.*

The general setting of the problem is the following: One wants to find $3n+1$ arrangements of $6n+3$ girls in rows of three, such that any two girls belong to the same row in exactly one arrangement.

In modern terms of block designs, the generalized Kirkman's schoolgirl problem is equivalent to finding a resolution of some balanced incomplete block design with block size 3: For positive integers b, v, r, k, λ with $r = \frac{\lambda(v-1)}{k-1}$ and $b = \frac{\lambda v(v-1)}{k(k-1)}$, a (v, k, λ)-Balanced Incomplete Block Design **BIBD** consists of a finite set Z of v elements, called **points**, and b subsets $Z_1, Z_2, \dots, Z_b$ of Z, called **blocks**, such that the following hold:
every point occurs in exactly r blocks,
every block contains exactly k points, and
every pair of points occurs together in exactly λ blocks.

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A class of pairwise disjoint blocks such that each point occurs in one block of the class is called a **parallel class**. If the b blocks of a (v, k, λ)-BIBD can be partitioned into r parallel classes, then the (v, k, λ)-BIBD is said to be **resolvable** and the corresponding partition is called a **resolution** of the (v, k, λ)-BIBD. A resolvable $(v, 3, 1)$-BIBD, together with a resolution of it, is a **Kirkman triple system** of order v.

Kirkman's schoolgirl problem generated great interest in the late 19th century and early 20th century, and famous mathematicians like Burnside, Cayley and Sylvester made some contribution to this problem. For example, it was proved that for several infinite families of integers n, Kirkman triple systems of order $6n + 3$ exist. However, the general problem remained unsolved for more than one century until Dwijendra Ray-Chaudhuri and Richard Wilson proved in [4] that Kirkman triple systems of order v exist, if and only if $v \equiv 3 \pmod 6$. (For a bibliography of Kirkman's schoolgirl problem we refer the reader to [4], where one can also find Oscar Eckenstein's bibliography of the problem [1].)

Now we turn our attention to arithmetic progressions in cyclic groups. Let v be a positive integer and let $a, b, c \in \mathbb{Z}_v$. The triple $[a, b, c]$ forms an arithmetic progression in $\mathbb{Z}_v$, if there is an integer d satisfying $0 < d < v$ such that $d \equiv b - a \equiv c - b \pmod v$. It is easy to see that if $[a, b, c]$ forms an arithmetic progression in $\mathbb{Z}_v$ with common difference d, then $[c, b, a]$ forms an arithmetic progression with common difference $v - d$. Thus, by identifying the triples $[a, b, c]$ and $[c, b, a]$, we may always assume that $d \leq \frac{v}{2}$. In particular, if v is odd, then we may assume that $d \leq \frac{v-1}{2}$. Hence, for $v \equiv 3 \pmod 6$, there are exactly $\frac{v-1}{2}$ different types of arithmetic progression in $\mathbb{Z}_v$.

Recall that $\frac{v-1}{2}$ is equal to the number of parallel classes of a Kirkman triple system of order v, which leads to the following definition. Let $v \equiv 3 \pmod 6$. A Kirkman triple system of order v, whose point set is numbered from 0 to $v - 1$, is called **smooth**, if for each parallel class there is a positive integer d with $d \leq \frac{v-1}{2}$ so that each triple (*i.e.*, each block) of the parallel class forms an arithmetic progression in $\mathbb{Z}_v$ with common difference d.

The following scheme illustrates a smooth Kirkman triple system of order 27:

$d = 1:$	$d = 2:$	$d = 3:$	$d = 4:$
0, 1, 2	2, 4, 6	0, 3, 6	0, 4, 8
3, 4, 5	8, 10, 12	9, 12, 15	12, 16, 20
6, 7, 8	14, 16, 18	18, 21, 24	24, 1, 5
9, 10, 11	20, 22, 24	1, 4, 7	9, 13, 17
12, 13, 14	26, 1, 3	10, 13, 16	21, 25, 2
15, 16, 17	5, 7, 9	19, 22, 25	6, 10, 14
18, 19, 20	11, 13, 15	2, 5, 8	18, 22, 26
21, 22, 23	17, 19, 21	11, 14, 17	3, 7, 11
24, 25, 26	23, 25, 0	20, 23, 26	15, 19, 23

$d=5:$	$d=6:$	$d=7:$	$d=8:$	$d=9:$
5, 10, 15	6, 12, 18	0, 7, 14	8, 16, 24	0, 9, 18
20, 25, 3	24, 3, 9	21, 1, 8	5, 13, 21	1, 10, 19
8, 13, 18	15, 21, 0	15, 22, 2	2, 10, 18	2, 11, 20
23, 1, 6	7, 13, 19	9, 16, 23	26, 7, 15	3, 12, 21
11, 16, 21	25, 4, 10	3, 10, 17	23, 4, 12	4, 13, 22
26, 4, 9	16, 22, 1	24, 4, 11	20, 1, 9	5, 14, 23
14, 19, 24	8, 14, 20	18, 25, 5	17, 25, 6	6, 15, 24
2, 7, 12	26, 5, 11	12, 19, 26	14, 2, 3	7, 16, 25
17, 22, 0	17, 23, 2	6, 13, 20	11, 19, 0	8, 17, 26

$d=10:$	$d=11:$	$d=12:$	$d=13:$
0, 10, 20	11, 22, 6	0, 12, 24	0, 13, 26
3, 13, 23	17, 1, 12	9, 21, 6	12, 25, 11
6, 16, 26	23, 7, 18	18, 3, 15	24, 10, 23
9, 19, 2	2, 13, 24	1, 13, 25	9, 22, 8
12, 22, 5	8, 19, 3	10, 22, 7	21, 7, 20
15, 25, 8	14, 25, 9	19, 4, 16	6, 19, 5
18, 1, 11	20, 4, 15	2, 14, 26	18, 4, 17
21, 4, 14	26, 10, 21	11, 23, 8	3, 16, 2
24, 7, 17	5, 16, 0	20, 5, 17	15, 1, 14

2 When can Kirkman triple systems be smooth?

In this section it will be shown that smooth Kirkman triple systems of order v exist, if and only if v is a power of 3.

In the following we always assume that v is a positive integer with $v \equiv 3 \pmod 6$.

We consider $\mathbb{Z}_v$ as the set $\{0, 1, \ldots, v-1\}$. For an integer u, let $(u)_v$ denote the unique number $w \in Z_v$ with $w \equiv u \pmod v$. If $u \in \mathbb{Z}_v$ is an odd integer, then $\left(\frac{u}{2}\right)_v := \left(\frac{v+u}{2}\right)_v$.

By an **arithmetic triple** we always mean a *non-constant arithmetic progression modulo v of length 3 with common difference δ (for some integer δ)*, or equivalently, a *non-constant arithmetic progression of length 3 in $\mathbb{Z}_v$*. We denote such arithmetic triples by triples $[a, b, c]_\delta$, where $a \in \mathbb{Z}_v$, $b = (a+\delta)_v$ and $c = (a+2\delta)_v$. In the sequel, arithmetic triples that have all elements in common will be identified. So, for $\delta \not\equiv 0 \pmod v$, $[a, b, c]_\delta = [c, b, a]_{(v-\delta)} = [c, b, a]_{-\delta}$.

As mentioned earlier, there are $\frac{v-1}{2}$ different types of arithmetic progression in $\mathbb{Z}_v$, or equivalently, $\frac{v-1}{2}$ different common differences. So, let $D_v := \left\{d \in \mathbb{Z}_v : 0 < d \leq \frac{v-1}{2}\right\}$.

In what follows we let $K_v = \{C_d : d \in D_v\}$ be a set of parallel classes (with point set $\mathbb{Z}_v$), where for each $d \in D_v$, $C_d = \left\{t_k^d : 1 \leq k \leq \frac{v}{3}\right\}$ is such that each triple $t_k^d \in C_d$ is of the form $[a,b,c]_d$ (for some $a,b,c \in \mathbb{Z}_v$). In other words, each C_d is a set of pairwise disjoint arithmetic triples with common difference d, such that the union of C_d is $\mathbb{Z}_v$.

For nonzero $u \in \mathbb{Z}_v$, let

$$(u)_{D_v} := \begin{cases} u & \text{if } u \in D_v, \\ v-u & \text{otherwise.} \end{cases}$$

The following observations are all quite obvious.

Observation 1. If, for some distinct d and $\tilde{d}$ in D_v, we find two triples $t_k^d \in C_d$ and $t_l^{\tilde{d}} \in C_{\tilde{d}}$, respectively, so that these two triples have more than one element in common, then we must have $\tilde{d} = (2d)_{D_v}$ or $\tilde{d} = \left(\frac{d}{2}\right)_{D_v}$.

Observation 2. Let $d \in D_v \setminus \{\frac{v}{3}\}$, $\delta \in \left\{2d, \left(\frac{d}{2}\right)_v\right\}$, and put $\tilde{\delta} = (\delta)_{D_v}$. If $[a,b,c]_d \in C_d$ and $[a,b',c']_{-\delta} \in C_{\tilde{\delta}}$ are any two arithmetic triples (with common differences d and $\tilde{\delta}$, respectively) with common element a, then the elements b, b', c and c' are distinct. Hence, a is the only element that belongs to both triples.

On the other hand, we have:

Observation 3. Let $d \in D_v \setminus \{\frac{v}{3}\}$, $\delta \in \left\{2d, \left(\frac{d}{2}\right)_v\right\}$, and put $\tilde{\delta} = (\delta)_{D_v}$. If $[a,b,c]_d \in C_d$ and $[c',a,b']_{-\delta} \in C_{\tilde{\delta}}$ are any two arithmetic triples (with common differences d and $\tilde{\delta}$, respectively) with common element a, then there are triples in C_d and $C_{\tilde{\delta}}$ that have more than one element in common.

The following defines the sets that generate the parallel classes. For each $a \in \mathbb{Z}_v$ and each nonzero $u \in \mathbb{Z}_v$, let

$$\Delta_u^a := \left\{ \left[(a+3iu)_v, (a+3iu+u)_v, (a+3iu+2u)_v\right]_u : 0 \leq i < \frac{v}{3} \right\}.$$

Observation 4. If $K_v = \{C_d : d \in D_v\}$ is a Kirkman triple system, then for any $d \in D_v$ there is a set $A \subseteq \mathbb{Z}_v$ such that $C_d = \bigcup\{\Delta_d^a : a \in A\}$.

By combining the foregoing observations, we get a characterization of the Kirkman triple systems that are smooth.

Lemma. A Kirkman triple system $K_v = \{C_d : d \in D_v\}$ of order v is smooth, if and only if for each $d \in D_v$ we have: If $\delta \in \left\{2d, \left(\frac{d}{2}\right)_v\right\}$ and $\Delta_d^a \subseteq C_d$ (for some $a \in Z_v$), then $\Delta_{-\delta}^a \subseteq C_{\tilde{\delta}}$, where $\tilde{\delta} = (\delta)_{D_v}$.

Proof: ($\Rightarrow$) This follows from Observation 4 and Observation 3.

($\Leftarrow$) This is a consequence of Observation 1 and Observation 2. $\dashv$

Now we are ready to prove the main result.

Theorem. Smooth Kirkman triple systems of order v exist, if and only if v is a power of 3.

Proof: ($\Rightarrow$) If there is a Kirkman triple system $K_v = \{C_d : d \in D_v\}$ of order v, then $v \equiv 3 \pmod 6$ (see [4]), and hence, if v is not a power of 3, then $v = q \cdot 3^k$, where $k \geq 1$, $3 \nmid q$ and $q > 3$. Since $q > 3$, we have $3^k \in D_v$. Let us consider C_{3^k}. By Observation 4, $C_{3^k} = \bigcup\{\Delta^a_{3^k} : a \in A\}$ (for some $A \subseteq \mathbb{Z}_v$). Now, since $3 \nmid q$, for any $a \in \mathbb{Z}_v$, the triples in $\Delta^a_{3^k}$ cannot be pairwise disjoint. Accordingly, C_{3^k} cannot be a set of pairwise disjoint arithmetic triples. Thus, the Kirkman triple system K_v is not smooth.

($\Leftarrow$) Take $v = 3^k$ (for some $k \geq 1$). For $u \in \mathbb{Z}_v$ write u in the form $u = \sum_{i=0}^{k-1} b_i\, 3^i$, where the b_i's belong to $\{0,1,2\}$ ($0 \leq i < k$), and let $\tau(u) := \langle b_0, \ldots, b_{k-1}\rangle$. In addition, let $\zeta\big(\tau(u)\big)$ be the first positive integer in the sequence $\tau(u)$; in other words, $\zeta\big(\tau(u)\big) = b_\iota$, where $b_\iota \neq 0$ but $b_i = 0$ whenever $i < \iota$. For $u \in \mathbb{Z}_v$, let $\sigma(u)$ be defined as follows.

$$\sigma(u) := \begin{cases} +1 & \text{if } \zeta\big(\tau(u)\big) = 1, \\ -1 & \text{if } \zeta\big(\tau(u)\big) = 2. \end{cases}$$

Notice that for $d \in D_v \setminus \{\frac{v}{3}\}$ and $\delta \in \big\{2d, \big(\frac{d}{2}\big)_v\big\}$ we have $\sigma(d) = -\sigma(\delta)$. To keep the notation short, write $u_\sigma := \sigma(u) \cdot u$ for $u \in \mathbb{Z}_v$. Finally, define $K_v = \{C_d : d \in D_v\}$, in which

$$C_d = \bigcup \big\{\Delta^a_{d_\sigma} : 0 \leq a < \gcd(d,v)\big\}$$

for each $d \in D_v$, where $\gcd(d,v)$ denotes the greatest common divisor of d and v. Bearing the lemma in mind, a moment's reflection should be sufficient to reveal that K_v is indeed a smooth Kirkman triple system of order v. This completes the proof of the theorem and the paper as well. $\dashv$

References

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