Dual form of combinatorial problems and Laplace techniques

Lorenz Halbeisen
Department of Mathematics
Evans Hall 938
University of California at Berkeley
Berkeley, CA 94720 (USA)

E-mail: halbeis@math.berkeley.edu

Norbert Hungerbühler
Department of Mathematics
University of Alabama at Birmingham
452 Campbell Hall, 1300 University Boulevard
Birmingham, AL 35294-1170 (USA)

E-mail: buhler@uab.edu

1 Introduction

One of the central tools in enumerative combinatorics is that of generating functions. Generating functions can e.g., be used to find the asymptotic behaviour of the enumerating sequence (e.g., the Hardy-Ramanujan estimate for the partition function $P(n)$, see [3]) or even may yield an explicit formula for the solution (e.g., Rademacher's famous explicit formula for $P(n)$, see [6]).

Given a combinatorial problem, there are numerous ways to find the corresponding generating function. One possibility is to start with a recurrence relation, as, e.g., the recurrence for the Fibonacci numbers $(a_n)_{n\in\mathbb{N}_0} = (0,1,1,2,3,5,8,\ldots)$, which we write in the following form:

$$\begin{aligned} a_n &= a_{n-2} + a_{n-1} + \delta_{1,n} \qquad && \forall n \in \mathbb{Z}, \\ a_n &= 0 && \forall n < 0. \end{aligned} \tag{1}$$

($\delta_{k,n}$ denotes the Kronecker symbol.) The z-transformation method requires to multiply (1) by z^n and to sum over n. This yields an algebraic equation for the generating function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, namely

$$f(z) = z^2 f(z) + z f(z) + z,$$

which is easily solved, giving $f(z) = \frac{z}{1-z-z^2}$. The Taylor expansion of this function yields

$$f(z) = \frac{z}{1-z-z^2} = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{\sqrt{5}},$$

2000 Mathematics Subject Classification: 11B39, 05A15 i.e., we obtain the explicit Euler-Binet[1] formula for the Fibonacci numbers

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

A second way to find a generating function is to use Polya's index theorem. For example, let M be the set of all syntactic bracket figures with index n equal to the number of bracket pairs. For $n = 3$ we have the set M_3 of three bracket pairs:

$$M_3 = \{[\,][\,][\,], [[[\,]]], [[\,][\,]], [[\,]][\,], [\,][[\,]]\}.$$

By

$$\begin{array}{rcl} M & \to & M_1 \times M \times M \,\cup\, M_0 \\ [a]b & \mapsto & ([\,], a, b) \\ \varnothing & \mapsto & \varnothing \end{array}$$

we have a bijection between the sets M and $M_1 \times M \times M \cup M_0$ which is additive, that is, $\mathrm{ind}([a]b) = 1 + \mathrm{ind}(a) + \mathrm{ind}(b)$. Then, by Polya's theorem, the relation between the sets translates directly into a relation for the generating function for the numbers $c_n = \mathrm{card}(M_n)$, namely,

$$f(z) = zf^2(z) + 1.$$

Taylor expansion of the solution $f(z) = \frac{1}{2z}(1 - \sqrt{1-4z}) = \sum_{n=0}^{\infty} c_n z^n$ yields the Catalan numbers

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

A third way is to use methods from the theory of difference equations, which reach from continued fractions to Laplace transformation. As an example, we mention a recent theorem of Oberschelp (see [5]) that allows to transform a difference equation into a differential equation for the exponential generating function by a formal procedure. For example, the sequence Sloane-Plouffe sequence M1497 in [7], f_n, which counts the number of ways to build a sequence without repetition with n variables satisfies the recurrence $f_{n+1} = (n+1)f_n + 1$. Oberschelp's theorem requires the exchange

$$\tbinom{n}{k} f_{n+s-k} \quad \longleftrightarrow \quad \frac{z^k}{k!} f^{(s)},$$

i.e., to replace f_{n+1} by f', nf_n by zf', f_n by f, and 1 by e^z. This procedure yields the ordinary differential equation $(1-z)f' - f = e^z$ with the solution $f(z) = \frac{e^z}{1-z}$

[1] This formula was derived by Jacques P.M. Binet in 1843, although the result was known to Euler and to Daniel Bernoulli more than a century earlier.

determined by $f(0)=1$. Since $f(z)$ is the exponential generating function, we get in fact $f_n = n!(1+\frac{1}{1!}+\ldots+\frac{1}{n!})$.

Experience shows that the situation becomes considerably more delicate as soon as the problem requires to solve partial difference equations. In this article we want to describe methods which allow us to calculate the generating function from a recurrence relation. The idea is to link the Laplace transform directly to generating functions by interpreting the Fourier formula for the inverse Laplace transform as a residual integral. The reader who is not familiar with the Laplace or Fourier transformation might consult [1] or [8]. The idea is certainly not new; however, we would like to show that it applies also to more complicated (e.g., non-local) partial difference equations.

2 Auxiliary Results

2.1 Laplace transformation

Let $(a_n)_{n\in\mathbb{Z}}$, $a_n = 0$ for $n<0$, be a sequence of real numbers with generating function $f(z)=\sum_{n\in\mathbb{Z}} a_n z^n$. We call

$$A(z) := \sum_{n\in\mathbb{Z}} a_n \chi_{[n,n+1[}(z)$$

the associated step-function. Here, χ_I denotes the characteristic function of the set I. Then the following theorem holds.

Theorem 1 *If the Laplace transform $\mathscr{L}[A]$ of the associated step-function A exists; it is related to the generating function f by*

$$\mathscr{L}[A](s) = \frac{1}{s}(1-e^{-s})f(e^{-s}).$$

Proof. Since we assume A to have at most exponential growth, we may transform term by term and get

$$\mathscr{L}[A](s) = \sum_{n=0}^{\infty} a_n \mathscr{L}[\chi_{[n,n+1[}].$$

Writing $\chi_{[n,n+1[} = H(\cdot - n) - H(\cdot-(n+1))$, where $H=\chi_{[0,\infty[}$ denotes the Heaviside function, and using that $\mathscr{L}[H](s)=\frac{1}{s}$, we obtain, by applying the basic rules for the Laplace transformation,

$$\mathscr{L}[A](s) = \sum_{n=0}^{\infty} a_n \frac{1}{s} e^{-ns}(1-e^{-s}),$$

which is what we claimed. $\square$

The following calculation provides a useful variant of Theorem 1: If $\frac{1}{z}g(e^{-z})$ is the Laplace transform of a piecewise smooth function G, we have by Fourier's formula for the inverse Laplace transformation that, for every point $x \in \mathbb{R}_+$ where G is continuous,

$$G(x) = \frac{1}{2\pi i}\,\mathrm{pv}\int_\Gamma \frac{1}{z}\,g(e^{-z})e^{xz}dz.$$

Here, Γ is the curve $\Gamma : \mathbb{R} \to \mathbb{C}, t \mapsto s+it$, with $s \in \mathbb{R}$ large enough, and "pv" denotes the principal value. If we denote $\Gamma_n : [0, 2\pi[\to \mathbb{C}, t \mapsto z := s + i(t + 2n\pi)$, we have

$$G(x) = \frac{1}{2\pi i}\,\mathrm{pv}\sum_{n\in\mathbb{Z}}\int_{\Gamma_n} \frac{1}{z}\,g(e^{-z})e^{xz}dz. \tag{2}$$

Observe that, by Fourier-series expansion, we have, for $x \notin \mathbb{Z}$,

$$\sum_{n\in\mathbb{Z}} \frac{1}{s+i(t+2n\pi)}e^{x(s+i(t+2n\pi))} = \frac{e^{\lceil x\rceil(s+it)}}{e^{s+it}-1},$$

where $\lceil\cdot\rceil$ denotes the ceiling function, i.e., $\lceil x\rceil$ is the smallest integer larger than or equal to x. Hence, by substituting $u = e^{-z}$, we obtain from (2) with $n = \lfloor x\rfloor$,

$$G(x) = \frac{1}{2\pi i}\int_\gamma \frac{g(u)}{1-u}\,\frac{du}{u^{n+1}} \tag{3}$$

where $\gamma : [0, 2\pi[\to \mathbb{C}, t \mapsto e^{-s}e^{it}$, and where $\lfloor\cdot\rfloor$ denotes the floor function, i.e., $\lfloor x\rfloor$ is the largest integer smaller than or equal to x. Thus, if g is analytic in a neighborhood of 0, we may interpret the integral in (3) as the Cauchy residue integral for the nth Taylor coefficient of the function $\frac{g(u)}{1-u}$. Thus, we have the following corollary.

Corollary 1 *Assume f and g_n are analytic functions in a neighborhood of* 0 *and a_n is given by*

$$a_n = \frac{1}{2\pi i}\,\mathrm{pv}\int_\Gamma \frac{1}{z}\,g_n(e^{-z})e^{xz}dz \tag{4}$$

for some (and hence any) $x \in]n, n+1[$ and Γ as above. If $\lim_{z\to 0}\frac{f(z)-g_n(z)}{z^n} = 0$ for all $n \in \mathbb{N}_0$, then $\frac{f(z)}{1-z}$ is the generating function of the sequence a_n.

Let us briefly mention some advantages that the use of the Laplace transformation provides: Suppose we are given a generating function $f(u)$. Only in simple cases it is possible to use direct Taylor expansion to obtain a formula for the coefficient a_n of u^n. Also, the Cauchy residue $a_n = \mathrm{Res}_{u=0}\frac{f(u)}{u^{n+1}}$ or (in case of a meromorphic function

f) $a_n = -\sum \mathrm{Res}_{u\neq 0} \frac{f(u)}{u^{n+1}}$ is often difficult to calculate. In such a situation, it may be helpful to split the residues via the Laplace transformation (as in the calculation preceding Corollary 1) in order to obtain an expansion (or at least an asymptotic formula) for the a_n. To illustrate this, let us consider the example of the generating function of the Bernoulli numbers

$$f(u) = u \cot u = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} u^n.$$

According to Theorem 1, the Laplace transform of the associated step-function G is

$$g(s) = \frac{1 - e^{-s}}{s} f(e^{-s})$$

and we may use the Fourier formula to invert g: $\mathscr{L}^{-1}[g](t) = \sum \mathrm{Res}\, g(s)e^{ts}$. The singularities of $g(s)e^{ts}$ are located at $s_{k,m} = m\pi i - \log(k\pi)$, $k \in \mathbb{N}$, $m \in \mathbb{Z}$. For $t \in \mathbb{Z}$ we have

$$\mathrm{Res}_{s_{k,m}}\, g(s)e^{ts} = \begin{cases} -\frac{1-k\pi}{s_{k,m}(k\pi)^t} & \text{if } m \text{ is even,} \\ -\frac{1+k\pi}{s_{k,m}(-k\pi)^t} & \text{if } m \text{ is odd.} \end{cases}$$

Combining residues for m and $-m$, we can easily sum the residues for fixed k over all m and obtain

$$\mathscr{L}^{-1}[g](t) = -\sum_{k=1}^{\infty} \frac{1}{(k\pi)^{2\lfloor t/2 \rfloor}}.$$

(Notice that one obtains a formula for $\sum_{m=1}^{\infty} \frac{1}{a^2+m^2}$ by expanding e^{ax} on $]-\pi,\pi[$ in a Fourier series.) Since $t \in \mathbb{Z}$ (G jumps in $\mathbb{Z}$), we finally get the zeta-function formula for the Bernoulli numbers:

$$B_{2n} = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}}.$$

A second benefit of the Laplace transformation are the various rules. For example, by the rule $\mathscr{L}[f'](s) = s\mathscr{L}[f](s) - f(0)$, we have, for $f_z(t) := t^z$, that

$$\mathscr{L}[f_z'](s) = s\mathscr{L}[f_z](s) = z\mathscr{L}[f_{z-1}](s).$$

Hence, for fixed s, the analytic function

$$h_s(z) := \mathscr{L}[f_z](s) = \int_0^{\infty} t^z e^{-st} dt$$

solves the difference equation $s\, h_s(z) = z\, h_s(z-1)$. In particular, for $s = 1$, we obtain Euler's integral representation of the Gamma-function. It is a particular feature of the Laplace-transformation method that it can be used to determine the analytic continuation of a discrete function. The Laplace transformation also yields a functional connection between the exponential generating function $e(x)$ and the ordinary generating function $f(x)$ of a sequence a_n. In fact, we have

$$\mathscr{L}[e](s) = \mathscr{L}\left[\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\right](s) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \underbrace{\mathscr{L}[x^n](s)}_{\frac{n!}{s^{n+1}}} = \tfrac{1}{s} f\left(\tfrac{1}{s}\right).$$

The translation-rule $\mathscr{L}[f(t-c)](s) = e^{-sc}\mathscr{L}[f(t)](s)$ for $c \geq 0$ allows us to transform a (linear) difference equation into an algebraic equation for the transformed function (this feature is similar to the z-transformation). In particular, it is possible to reduce a linear partial difference equation with n variables to an equation with $n-1$ variables. For an example see Section 3.4 or 3.5.

Another virtue of the Laplace transformation appears when one looks for an asymptotic expansion of a sequence or (which is a similar thing) when one treats difference equations which show oscillation and damping effects. If one is only interested in the stationary state, one can already, at the level of the transformed function, identify terms which lead to exponentially decaying terms in the solution and drop them for the rest of the calculation.

2.2 The dual of a linear difference equation

Many combinatorial problems lead to partial difference equations. As a prototype example, we investigate the two dimensional case.

Let $X \subset \mathbb{Z}^2$. For a map $p : X \to \mathbb{R}$, we consider the linear equation

$$p(z) = \sum_{\{\zeta \in X : \zeta \in \operatorname{spt} a_z\}} a_z(\zeta) p(\zeta) \tag{$*$}$$

where we assume that the cardinality of the support of a_z (spt $a_z \subset X$) is finite for all $z \in X$, i.e., that the sum in $(*)$ is always finite. A set $A \subset X$ is called *stable* if for all maps $f : A \to \mathbb{R}$ there exists a unique solution p of $(*)$ such that $p|_A = f$. A triple $(X, A, *)$ is called *triangular* if X can be written as $X = (x_i)_{i \in \mathbb{N}}$ in such a way that, for all $i \in \mathbb{N}$, there holds spt $a_{x_i} \subset A \cup \{x_1, \ldots, x_{i-1}\}$, and for all $z \in A$: spt $a_z = \{z\}$ and $a_z(z) = 1$. In particular we have that, for a triangular triple $(X, A, *)$, the set A is stable.

Now, let $(X, A, *)$ be triangular and $f : A \to \mathbb{R}$ be given. Then, for any fixed $x = x_i \in X$, the solution p of $(*)$ in x is a finite linear combination of the values of f on A, i.e.,

$$p(x) = \sum_{\zeta \in A} \alpha_x(\zeta) f(\zeta).$$

In order to determine the weights $\alpha_x(\zeta)$, we proceed as follows:

(i) Put a red mark on x.

(ii) Replace each red mark on $y \in X \setminus A$ by a blue one on y and by $a_y(\zeta)$ many red marks on ζ for all $\zeta \in \operatorname{spt} a_y$.

(iii) Iterate (ii) until no more red marks on $X \setminus A$ exist.

If n denotes the maximum of the set $\{i\colon$ there is a red mark on $x_i\}$, then, in each iteration step, n decreases at least by one due to the triangular structure. Hence, the iteration process terminates. If we denote by $\tilde{q}(\zeta)$ the number of red marks on ζ, the quantity

$$\sum_{\zeta \in X} \tilde{q}(\zeta) p(\zeta)$$

is invariant during the iteration. Hence, we obtain the result that after the iteration is completed the number of (red) marks on $\zeta \in A$, i.e., $\tilde{q}(\zeta)$, equals the weight $\alpha_x(\zeta)$.

If we denote by $q(\zeta)$ the final number of marks (blue or red) on ζ (i.e., after termination of the iteration), the iteration process described above translates into a partial difference equation for the function q:

$$q(z) = \sum_{\{\zeta \in A_x : z \in \operatorname{spt} a_\zeta\}} a_\zeta(z) q(\zeta) \tag{$**$}$$

with $q(x) = 1$ and with $A_x := \operatorname{tr} x \setminus A$, where $\operatorname{tr} x$ is the equivalence class of x with respect to the transitive hull of the relation $u \sim v :\iff u \in \operatorname{spt} a_v, v \notin A$. Notice that $(A_x, \{x\}, **)$ is triangular and finite. Let us summarize this result in a theorem.

Theorem 2 *If $(X, A, *)$ is triangular with prescribed values f on A, then the weights α_x in the solution formula $p(x) = \sum_{\zeta \in A} \alpha_x(\zeta) f(\zeta)$ can be determined by the iteration scheme (i)–(iii) or, equivalently, by solving the dual linear recursion $(**)$ with initial value $q(x) = 1$.*

Many transformation problems (for example the boustrophedon transformation in [4]) can be described as follows: Let $(X, A, *)$ be triangular; then we fix sets $A' = \{a_1, a_2, \ldots\} \subset A$ and $X' = \{b_1, b_2, \ldots\} \subset X$ and prescribe $f(a_i) = \phi_i$ and $f = 0$ on $A \setminus A'$. If we denote the solution $\psi_i = p(b_i)$, the mapping $\Psi_{X,X',A,A',*}\colon (\phi_i) \mapsto (\psi_i)$ is a linear transformation of sequences, the associated linear mapping (ALM). The problem to find its matrix (or the matrix of the inverse transformation) can often be solved by using the Laplace transformation technique for the partial difference equation for the *weights* $(**)$ even in cases where it is not possible to use directly the

Laplace transformation in the *original* partial difference equation $(*)$. We will see some examples in the following section.

Before we discuss the examples, we close this section by stating a simple path-counting lemma.

Lemma 1 *Suppose the coefficient functions a in $(*)$ satisfy the following invariance property for all $z=(n,k)$ and $z'=(n,k')$ in $X=\mathbb{Z}^2$:*

$$a_z(n+i,k+j)=a_{z'}(n+i,k'+j),\quad \forall i,j\in\mathbb{Z}. \tag{5}$$

Suppose, furthermore, that the column $\{(0,k)\colon k\in\mathbb{Z}\}$ is stable and that p denotes the solution of $(*)$ with prescribed values α_k on $(0,k)$. Then the column $\{(N,k)\colon k\in\mathbb{Z}\}$ is stable for

$$\tilde{p}(z)=\sum_{\{\zeta\in X\}}\bar{a}_z(\zeta)\tilde{p}(\zeta) \tag{$\dagger$}$$

where $\bar{a}_{u+v}(u):=a_u(u+\bar{v})$ and $(\overline{i,j}):=(i,-j)$. Finally, if we prescribe the values α_k on (N,k) for the equation $(\dagger)$, then $\tilde{p}(0,k)=p(N,k)$.

Proof of Lemma 1: We may interpret $(*)$ as a directed graph G with $a_z(\zeta)$ many edges from ζ to z. If we set $\alpha_k:=\delta_{k,k_0}$, then $p(N,k)$ is the number of paths in G from $(0,k_0)$ to (N,k). If we flip the graph horizontally by $z\mapsto\bar{z}$ and invert the orientation of the edges, we obtain a graph G'. Now, $(\dagger)$ describes G' and $\tilde{p}(0,k)$ is the number of paths in G' from (N,k_0) to $(0,k)$ which equals, by construction, the number of paths in G from $(0,k_0)$ to (N,k).

For general (α_k) the claim follows by linearity. $\square$

3 Examples and applications

3.1 The Fibonacci numbers and a variant of Faulhaber's formula

Let $X=\{(k,n):n\geq k\geq 0\}$ and $A=\big\{(k,n)\in X\colon n\in\{k,k+1\}\big\}$. Further let

$$a_{(k,n)}(i,j)=\begin{cases}\delta_{k,i}\delta_{n-1,j}+\delta_{k+1,i}\delta_{n-1,j} & \text{for } (k,n)\notin A,\\ \delta_{k,i}\delta_{n,j} & \text{otherwise,}\end{cases}$$

in the equation $(*)$. This is easily seen to be triangular. For the sets $A'=\{(k,k+1)\in A\}$ and $X'=\{(0,n)\in X\colon n\geq 0\}$, we have that the ALM $\Psi_{X,X',A,A',*}$ applied to the sequence $(1,1,\ldots)$ yields the Fibonacci sequence $\big(f(n)\big)_n$. Let us calculate the weights via $(**)$:

$$q(k,n) = q(k,n+1) + q(k-1,n+1)$$

with $q(0,l) = 1$. This is (up to renumbering) just the recursion for the binomial numbers, i.e., we get the "shallow diagonal" sum formula connecting Pascal's triangle to the Fibonacci numbers:

$$f(n+1) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{n-2k}.$$

The binomial weights always occur for this type of equation: For another example, let $p(k,n) := \sum_{i=1}^{n} i^k$. Obviously, for fixed k, p is a polynomial in n of degree $k+1$. Faulhaber's famous formula expresses this polynomial in the basis $\{1, n, n^2, n^3, \ldots\}$, and the coefficients in this basis involve the Bernoulli numbers. Here, we want to express the polynomial in the basis $\left\{\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots\right\}$. Consider again the "binomial" difference equation $f(k,n) = f(k,n-1) + f(k+1,n-1)$, this time on $X = \mathbb{N}_0^2$, with initial data $f(0,n) = p(k,n-1)$ for fixed k. The weights for the dual equation clearly are, as above, the binomial coefficients; hence, $p(k,n-1) = \sum_{i=1}^{n} \binom{n}{i} f(i,0)$, and it remains to find $f(i,0)$. Since $f(1,n) = n^k$, we use

$$\sum_{i=0}^{n} \binom{n}{i} i! S_2(k,i) = n^k, \tag{6}$$

where S_2 denotes the Stirling number of the second kind (see next section). Indeed, each term in the sum may be interpreted as the number of sequences in $\{1,\ldots,n\}^k$ with exactly i different numbers. Thus, $f(i+1,0) = i!S_2(k,i)$ and we recover the well known formula

$$p(k,n) = \sum_{i=1}^{n} \binom{n+1}{i+1} i! S_2(k,i),$$

which one also gets by summing (6).

3.2 The Stirling numbers

The Stirling numbers of the first kind $S_1(n,k)$ count the permutations of n distinct objects that can be written with exactly k disjoint cycles (cf. [2]). They can be computed recursively as follows:

$$S_1(n+1,k) \;:=\; n \cdot S_1(n,k) + S_1(n,k-1),$$

where $S_1(1,k) := \delta_{1,k}$.

Let $\tilde{S}_n(k) := S_1(n,k)$; then $\tilde{S}_n(k)$ satisfies the recurrence $\tilde{S}_{n+1}(k) = n\tilde{S}_n(k)+\tilde{S}_n(k-1)$. Let $L_n(s)$ denote the Laplace transform of the associated step-function of $\tilde{S}_n(k)$. Then we get

$$L_{n+1}(s) = nL_n(s) + e^{-s}L_n(s) = L_n(s)(n + e^{-s}),$$

with $L_1(s) = \frac{1}{s}(1 - e^{-s})$. Hence,

$$L_n(s) = \frac{1}{s}(1 - e^{-s}) \prod_{j=1}^{n-1} (j + e^{-s}).$$

Thus, by Theorem 1 we find that

$$f_n(u) = \prod_{j=1}^{n-1} (j + u)$$

is the generating function for $\big(S_1(n,k)\big)_k$.

The Stirling numbers of the second kind $S_2(n,k)$ count the number of groupings of n distinct objects into k disjoint (nonempty) groups. They can be computed recursively as follows:

$$S_2(n+1,k) \;:=\; k \cdot S_2(n,k) + S_2(n,k-1),$$

where $S_2(1,k) := \delta_{1,k}$.

Let $\tilde{S}_k(n) := S_2(n,k)$; then $\tilde{S}_k(n)$ satisfies the recurrence $\tilde{S}_k(n) = k\tilde{S}_k(n-1) + \tilde{S}_{k-1}(n-1)$. Let $L_k(s)$ denote the Laplace transform of the associated step-function of $\tilde{S}_k(n)$. Then we obtain $L_k(s) = ke^{-s}L_k(s) + e^{-s}L_{k-1}(s)$. Therefore,

$$L_k(s) = L_{k-1}(s)\frac{e^{-s}}{1 - k\,e^{-s}} = L_1(s) \prod_{j=2}^{k} \frac{e^{-s}}{1 - j\,e^{-s}}$$

with $L_1(s) = \frac{1}{s}$. Thus, by Theorem 1, we get that

$$f_k(u) = \prod_{j=1}^{k} \frac{u}{1 - j\,u}$$

is the generating function for $\big(S_2(n,k)\big)_n$.

It is well known that the matrix of the Stirling numbers of the first and second kind are inverse in the sense that

$$f(n) = \sum_{i=1}^{n} S_1(n,i)e(i)$$

if and only if

$$e(n) = \sum_{i=1}^{n} (-1)^{n-i} S_2(n,i) f(i).$$

Instead of proving this rather special formula, we now investigate more general conditions which still imply an inversion formula of the above type.

3.3 An inversion formula

We consider the following situation: Given a linear equation $(*)$ with $X = \mathbb{N}_0 \times \mathbb{Z}$, which satisfies the invariance property (5), we suppose that with $A := \{(0,k) : k \in \mathbb{Z}\}$ the triple $(X, A, *)$ is triangular. We set $A' := \{(0,k) : k \in \mathbb{N}_0\}$ and $X' := \{(n,0) : n \in \mathbb{N}_0\}$ and consider the mapping $\Psi_{X,X',A,A',*} \colon (\phi_i) \mapsto (\psi_i)$. Notice that the equation $(**)$ for the weights inherits the invariance property (5), and hence we can apply Lemma 1 to $(**)$ and obtain

$$\tilde{p}(z) = \sum_{\{\zeta \in X\}} \bar{a}_z(\zeta) \tilde{p}(\zeta), \tag{††}$$

with $\tilde{p}(n,0) = \delta_{n,0}$, where $\bar{a}_{u+v}(u) := a_{u+\bar{v}}(u)$. Then we have

$$\psi_n = \sum_{i=0}^{\infty} \tilde{p}(n,i) \phi_i. \tag{7}$$

Now we invert the previous equation: Let $Y := \mathbb{N}_0 \times \mathbb{N}_0$ and $Y' := \{(0,k) : k \in \mathbb{N}_0\}$. For any fixed $z \in X$, we can replace $(*)$ equivalently by the equation

$$p(\zeta_0) = \frac{1}{a_z(\zeta_0)} p(z) - \sum_{\{\zeta \in \operatorname{spt} a_z \setminus \{\zeta_0\}\}} \frac{a_z(\zeta)}{a_z(\zeta_0)} p(\zeta) =: \sum_{\{\zeta \in \operatorname{spt} a'_{\zeta_0}\}} a'_{\zeta_0}(\zeta) p(\zeta) \tag{$*'$}$$

for arbitrary $\zeta_0 \in \operatorname{spt} a_z$. Assume that for any $z \in X$ we can—by choosing a suitable ζ_0—replace $(*)$ by $(*')$ in such a way that

- the coefficients a'_z respect the invariance relation (5),
- the triple $(Y, Y', *')$ is triangular.

The equation for the weights for $(*')$ is

$$q(z) = \sum_{\{\zeta \in A_{(0,0)} : z \in \operatorname{spt} a'_\zeta\}} a'_\zeta(z) q(\zeta), \tag{$**'$}$$

with initial condition $q(0,0) = 1$ (because $(**')$ satisfies (5)). Then we have

$$\phi_n = \sum_{i=0}^{\infty} q(i, -n)\psi_i. \tag{8}$$

Hence, in view of (8) and (7), q and $\tilde{p}$ are inverse matrices, where q and $\tilde{p}$ satisfy certain difference equations which are related in the described manner. Notice also that, by choosing ζ_0 (see above), there is a certain freedom in the coefficients a' which can be useful sometimes.

As an example of the previous result we investigate a generalization of the Stirling numbers.

Let us define $a_{(n,k)}(i,j) := c(i)\delta_{i,n-1}\delta_{j,k} + d(i)\delta_{i,n-1}\delta_{j,k+1}$, where c and d are non-vanishing functions. Then the procedure described above yields the following proposition.

Proposition 1 *The numbers $s_1(n,k)$, $s_2(n,k)$ for $(n,k) \in \mathbb{Z} \times \mathbb{Z}$, defined by*

$$s_1(n,k) = c(n-1)s_1(n-1,k) + d(n-1)s_1(n-1,k-1)$$

and

$$s_2(n,k) = -\frac{c(n)}{d(n)} s_2(n,k-1) + \frac{1}{d(n-1)} s_2(n-1,k-1)$$

with $s_1(0,m) = s_2(m,0) = \delta_{m,0}$ are inverse in the sense that

$$\psi_n = \sum_{i=0}^{\infty} s_1(n,i)\phi_i \iff \phi_n = \sum_{i=0}^{\infty} s_2(i,n)\psi_i\,.$$

For special choices of the functions c and d, one easily gets e.g., the inversion formulas for the Stirling numbers ($c(n) = n$, $d(n) = 1$), the binomial numbers ($c(n) = 1$, $d(n) = 1$), or the numbers $Q_l(n) := \binom{n}{l} l!$ counting the number of ways to build sequences of length l with n objects without repetitions ($c(l) = -\frac{1}{l}$, $d(l) = \frac{1}{l}$)—guess what the inverse numbers are!

3.4 The partition numbers

As a further example, we consider the number $p(n,k)$ of partitions of an integer n into parts larger than or equal to k. This leads to the (non-local) partial difference equation

$$p(n,k) = p(n-k,k) + p(n,k+1), \tag{9}$$

with $p(n,k) = 0$ for $k > n > 0$ and $p(n,n) = 1$. In the above setting, the problem reads as follows: $X = \mathbb{N}^2$, $A = \{(n,k) : k \geq n\}$, $A' = \{(n,n) : n \in \mathbb{N}\}$ and $X' = \{(n,1) : n \in \mathbb{N}\}$, and for $(n,k) \in X \setminus A$ we have

$$p(n,k) = \sum_{i,j \in \mathbb{N}} (\delta_{i,n-k}\delta_{j,k} + \delta_{i,n}\delta_{j,k+1}) p(i,j) \,. \tag{10}$$

The ALM $\Psi_{X,X',A,A',(10)}$ maps the sequence $(1,1,\ldots)$ into the sequence $p(n,1) = P(n)$ of the partition numbers. The equation for the weights is given by

$$q(n,k) = q(n,k-1) + q(n+k,k)$$

with initial conditions $q(n,1) = 1$ for $n \leq N$ and $q(n,k) = 0$ for $n > N$. Then we have $P(N) = \sum_{i=1}^{N} q(i,i)$. By renumbering, this is equivalent to saying

$$\tilde{q}(n,k) = \tilde{q}(n,k-1) + \tilde{q}(n-k,k) \tag{11}$$

with $\tilde{q}(n,1) = 1$ for all n, $\tilde{q}(n,k) = 0$ for $n \leq 0$, and $P(N) = \sum_{i=1}^{N} \tilde{q}(i, N-i+1)$. Note that $\tilde{q}(n,k)$ no longer depends on N. Laplace transformation of (11) with respect to the first variable with k fixed yields

$$r_k(s) = \frac{1}{1-e^{-sk}} r_{k-1}(s)$$

with initial value $r_1(s) = \frac{1}{s}$ (since $\tilde{q}(1,k) = 1$ for $k \in \mathbb{N}$). Thus, we have

$$r_k(s) = \frac{1}{s} \prod_{j=2}^{k} \frac{1}{1-e^{-js}}$$

and, by Theorem 1, the generating function $g_k(u)$ of $\big(\tilde{r}_k(n)\big)_n$ is given by

$$g_k(u) = \prod_{j=1}^{k} \frac{1}{1-u^j} \,.$$

From this, it is easy to derive Euler's classical generating function $E(u)$ of the partition numbers $P(N)$. But, by interpreting $\tilde{q}(n,k)$ as the number of partitions of $n-1$ into k or less parts (and hence $P(n-1) = \tilde{q}(n,n-1) = \tilde{q}(n,n)$), we immediately get from the above calculation together with Corollary 1 that

$$E(u) = \prod_{j=1}^{\infty} \frac{1}{1-u^j} \,. \tag{12}$$

Also, if $f(s)$ denotes the Laplace transform of E, it follows from (12) that

$$\frac{1}{s}(1-e^{-s})\prod_{j=1}^{\infty}(1-e^{-js}) = f(s)\sum_{j=1}^{\infty}(-1)^{\lfloor\frac{j}{2}\rfloor}e^{-st_j},$$

where $t_j = 0, 1, 2, 5, 7, \ldots$ are the pentagonal numbers. Laplace inversion of the last equation yields Euler's formula $\sum_{j=1}^{\infty}(-1)^{\lfloor\frac{j}{2}\rfloor}P(n-t_j) = \delta_{n,0}$.

What about counting weighted partitions? Let $f\colon \mathbb{N}\to\mathbb{R}$ be a weight function with the meaning that we count partitions into i parts $f(i)$ many times, or—what is the same thing by considering Ferrers diagram—count partitions which largest part of size i, $f(i)$ many times. Then the calculation above gives the generating function for this problem:

$$\sum_{i=1}^{\infty}\frac{f(i)u^i}{\prod_{j=1}^{i}1-u^j}.$$

So, choosing, e.g., f as the characteristic function of the even numbers, we compute $\big(e(n)\big)_n = (0, 1, 1, 3, 3, 6, 7, 12, 14, \ldots)$.

To conclude this section let us compute the inverse of the ALM $\Psi_{X,X',A,A',(10)}$. Let us put a red mark on (L, L). In view of (10) we can replace a red mark on (n, k) (for $n \geq k > 1$) by a red mark on $(n, k-1)$, a negative red mark on $(n-k+1, k-1)$ and a blue mark on (n, k). This game terminates when all red marks are in $A\setminus A'$ (these marks are multiplied by 0) or in X' (where a mark on $(i, 1)$ is multiplied by ψ_i). Hence, $\phi_L = \sum_{n=1}^{L}\psi_n\,\omega(L, n)$, where $\omega(L, n)$ denotes the number of red marks on $(n, 1)$.

To compute $\omega(L, n)$, we consider the directed, finite graph G_L with vertices $\{(n, k) : L \geq n \geq k \geq 1\}$ and an edge from (n, k) to (n', k') if $k' = k-1$ and $n' = n$ (this edges are called v-edges) or if $k' = k$ and $n' = n-k$ (this edges are called h-edges of length k). Now let $W_L(n)$ be the number of paths through the graph G_L from the vertex (L, L) to $(n, 1)$, such that all h-edges have different length and each path is weighted by $+1$ if the number of h-edges contained in the path is even, otherwise it is weighted by -1. It is easy to see that $W_L(n) = \omega(L, n)$. To compute $W_L(n)$, let us first define the function $w(m, l, s)$, which is the number of weighted paths from (m, m) to $(m-l, 1)$, such that the maximum of the lengths of h-edges contained in the path equals s (where $s = 0$ means that the path contains no h-edge). For the function $w(m, l, s)$, we have

$$w(m,l,s) = \begin{cases} 1 & \text{if } l = s = 0,\\ 0 & \text{if } s > l \text{ or } s > \lfloor\frac{m}{2}\rfloor,\\ -\sum_{j=1}^{s} w(m-s, l-s, s-j) & \text{otherwise.}\end{cases}$$

Now, by construction, we obtain

$$W_L(n) = \sum_{s=0}^{\lfloor \frac{L}{2} \rfloor} w(L, L-n, s).$$

For example, for $L = 12$, we get $\big(W_{12}(n)\big)_n = (1, -1, -2, 0, 2, 0, 1, 0, 0, -1, -1, 1)$ and, in fact, $P(12) - P(11) - P(10) + P(7) + 2P(5) - 2P(3) - P(2) + P(1) = 77 - 56 - 42 + 15 + 2 \cdot 7 - 2 \cdot 3 - 2 + 1 = 1$.

3.5 A path counting problem

We consider paths in a three-dimensional lattice: Starting point of the paths is a point $(x, 0, 0)$, $x \in \mathbb{N}_0$, on the x-axis. If (x, y, z) is a point on the path, then a unit step in positive y or z direction is allowed or a step of length $y + z + 1$ in negative x direction. We want to count the number $H_M(x)$ of allowed paths starting in $(x, 0, 0)$ which end in a given set $M \subset \mathbb{Z}^3$.

The dual of this problem is given by the non-local linear difference equation

$$q_{z,y}(x) = q_{z-1,y}(x) + q_{z,y-1}(x) + q_{z,y}(x - y - z - 1) \tag{13}$$

with $q_{z,y}(x) := 0$ if one of the numbers x, y, or z is negative and $q_{0,0}(0) := 1$. We already used an index notation because we want to Laplace-transform equation (13) with respect to the variable x. First, we have $Q_{0,0}(s) = \frac{1}{s}$, since $q_{0,0}(x) = 1$ for $x \geq 0$. Laplace transformation of (13) yields

$$Q_{z,y}(s) = Q_{z-1,y}(s) + Q_{z,y-1}(s) + e^{-s(y+z+1)} Q_{z,y}(s).$$

Considering s as a parameter, the solution of this difference equation in y and z is given by

$$Q_{z,y}(s) = \frac{1}{s} \binom{z+y}{z} \frac{1}{\prod_{j=2}^{z+y+1} (1 - e^{-js})}.$$

Thus, the generating function of $q_{z,y}(x)$ is

$$f_{z,y}(u) = \binom{z+y}{z} \prod_{j=1}^{z+y+1} \frac{1}{1 - u^j}.$$

Hence, using the notation of Section 3.4,

$$q_{z,y}(x) = \tilde{r}_{z+y+1}(x) \binom{z+y}{z}.$$

Finally, the solution to our path counting problem is given by the formula

$$H_M(\xi) = \sum_{(\xi-x,y,z)\in M} \tilde{r}_{z+y+1}(x)\binom{z+y}{z}.$$

For example, let us count the paths starting in $(\xi, 0, 0)$ with at most h unit steps in z direction and such that the total number of unit steps in negative x and in positive y direction equals ξ. This corresponds to the set $M = \{(x, y, z) \in \mathbb{Z}^3 : x = y,\ z \le h\}$, and the solution formula yields

$$H_M(\xi) = \sum_{z\le h, x\le\xi} \tilde{r}_{z+\xi-x+1}(x)\binom{z+\xi-x}{z}.$$

3.6 Local linear difference equations

For $X = \{(k, l) : 0 \le k \le l\}$ and $A = \{(k, l) : l \in \{k, k+1, k+2\}\}$, we consider the model equation

$$z(k, l) = a_1 z(k, l-1) + a_2 z(k+1, l-1) + a_3 z(k+2, l-1). \tag{14}$$

$(X, A, (14))$ is triangular and, for $X' = \{(0, l) : l \ge 3\}$, the equation for the weights is

$$q(k, l) = a_1 q(k, l+1) + a_2 q(k-1, l+1) + a_3 q(k-2, l+1) \tag{15}$$

with initial condition $q(k, L) = \delta_{k,0}$ for a fixed $L \ge 0$. Laplace transformation of (15) with respect to the variable k with l fixed gives $Q_l(s) = Q_{l+1}(s)(a_1 + a_2 e^{-s} + a_3 e^{-2s})$ with initial condition $Q_L(s) = \frac{1}{s}(1 - e^{-s})$. The solution is

$$Q_l(s) = \frac{1}{s}(1 - e^{-s})(a_1 + a_2 e^{-s} + a_3 e^{-2s})^{L-l},$$

and Theorem 1 gives, for the generating function of the sequence $\big(q(k, l)\big)_k$, the function $(a_1 + a_2 u + a_3 u^2)^{L-l}$. Multinomial expansion yields

$$q(k, l) = \sum_{k_2+2k_3=k} \binom{L-l}{L-l-k_2-k_3, k_2, k_3} a_1^{L-l-k_2-k_3} a_2^{k_2} a_3^{k_3}.$$

Since (15) does not stop the iteration when a mark lies on A, we have to compensate by setting $\tilde{q}(k, k+2) = q(k, k+2)$, $\tilde{q}(k, k+1) = q(k, k+1) - a_1 q(k, k+2)$, and $\tilde{q}(k, k) = q(k, k) - a_1 q(k, k+1) - a_2 q(k-1, k+1)$. Then, if α_z is given on $z \in A$ as initial data for (14), we get the solution

$$z(0, l) = \sum_{i=2}^{l} \sum_{j=0}^{2} \alpha_{(i-j,i)} \tilde{q}(i-j, i). \tag{16}$$

In particular, if $\alpha_{(k+j,k)} = x_j$ (for $j = 0, 1, 2$), $z(0, l)$ is the solution of $x_n = a_1 x_{n-1} + a_2 x_{n-2} + a_3 x_{n-k}$ with initial values x_0, x_1, x_2 and (16) is a root-free representation of the solution.

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