Magic sets for polynomials of degree n

Lorenz Halbeisen, Norbert Hungerbühler, Salome Schumacher*

September 17, 2020

Abstract

Let $\mathcal{P}_n$ be the family of all real, non-constant polynomials with degree at most n and let $\mathcal{Q}_n$ be the family of all complex, non-constant polynomials with degree at most n. A set $S \subseteq \mathbb{R}$ is called a set of range uniqueness (SRU) for a family $\mathcal{F} \in \{\mathcal{P}_n, \mathcal{Q}_n\}$ if for all $f, g \in \mathcal{F}$, $f[S] = g[S] \Rightarrow f = g$. And S is called a magic set if for all $f, g \in \mathcal{F}$, $f[S] \subseteq g[S] \Rightarrow f = g$. In this paper we will show that there are magic sets for $\mathcal{P}_n$ and $\mathcal{Q}_n$ of size s for every $s \geq 2n+1$. However, there are no SRUs of size at most $2n$ for $\mathcal{P}_n$ and $\mathcal{Q}_n$. Moreover we will show that SRUs and magic sets are not the same by giving examples of SRUs for $\mathcal{P}_2$ and $\mathcal{P}_3$ that are not magic.

Key words: sets of range uniqueness, polynomials, magic sets, unique range
Mathematics Subject Classification: **26C05**, 11C20

1 Introduction

Let $\mathcal{F}$ be a set of functions with a common domain X and a common range Y. A set $S \subseteq X$ is called a set of range uniqueness (SRU) for $\mathcal{F}$ if the following holds: For all $f, g \in \mathcal{F}$

$$f[S] = g[S] \Rightarrow f = g.$$

And S is called a magic set for $\mathcal{F}$ if for all $f, g \in \mathcal{F}$

$$f[S] \subseteq g[S] \Rightarrow f = g.$$

Note that every magic set is also an SRU. The existence of magic sets and SRUs has already been studied for several families of functions:

- Berarducci and Dikranjan proved in [1] that under the continuum hypothesis (CH) there exists a magic set for the family $C^n(\mathbb{R})$ of all nowhere constant, continuous functions. Halbeisen, Lischka and Schumacher showed in [6] that we can weaken the requirement by replacing CH by the assumption that the union of less than continuum many meager sets is meager, i.e. $\mathrm{add}(\mathcal{M}) = \mathfrak{c}$. However, the existence of a magic set for $C^n(\mathbb{R})$ is not provable in ZFC as Ciesielski and Shelah proved in [3].
- In [2], Burke and Ciesielski proved that SRUs always exist for the family of all Lebesgue-measurable functions on $\mathbb{R}$.
- In [4], Diamond, Pomerance and Rubel constructed SRUs for the family $C^\omega(\mathbb{C})$ of entire functions.
- In [5] the authors of this paper proved that there exist SRUs for the family $\mathcal{P}_n$ of all real, non-constant polynomials of degree at most n of size $2n+1$ but none of size $2n$.

*Partially supported by SNF grant 200021_178851.

In this paper we consider magic sets for the family $\mathcal{P}_n$ of all real, non-constant polynomials of degree at most n and for the family $\mathcal{Q}_n$ of all complex, non-constant polynomials of degree at most n. We will show that there exist no SRUs, and therefore also no magic sets, of size at most $2n$ for $\mathcal{P}_n$ and $\mathcal{Q}_n$. Then we will give examples of SRUs for $\mathcal{P}_2$ and $\mathcal{P}_3$ that are not magic. And finally we will answer one of the open questions in [5] and show that for every $s \geq 2n+1$ there is a magic set of size s for the families $\mathcal{P}_n$ and $\mathcal{Q}_n$.

2 There are no SRUs of size at most $2n$ for $\mathcal{P}_n$

In [5] we have already shown that there are no SRUs of size $2n$: For points $x_0 < x_1 < \cdots < x_{2n}$ we constructed two functions $f, g \in \mathcal{P}_n$ such that $f = 1 - g$ and

$$f(x_{2i}) = g(x_{2i-1}) \text{ and } f(x_{2i-1}) = g(x_{2i})$$

for all $1 \leq i < n$. In a similar way we can prove that there are no SRUs of size $2n-1$:

Lemma 1. *There are no SRUs of size $2n-1$.*

Proof. Let $0 < x_1 < x_2 = x_3 < x_4 < \cdots < x_{2n}$. As in [5] define

$$Y^n := \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \mid y_i \in \{x_{2i-1}, x_{2i}\} \text{ for all } 1 \leq i \leq n\}$$

and

$$A_n = A_n(x_1, x_2, \ldots, x_{2n}) = \begin{pmatrix} x_1 + x_2 & x_1^2 + x_2^2 & \ldots & x_1^n + x_2^n \\ x_3 + x_4 & x_3^2 + x_4^2 & \ldots & x_3^n + x_4^n \\ \vdots & \vdots & \ddots & \vdots \\ x_{2n-1} + x_{2n} & x_{2n-1}^2 + x_{2n}^2 & \ldots & x_{2n-1}^n + x_{2n}^n \end{pmatrix}$$

For all $y_1, y_2, \ldots, y_n \in \mathbb{R}$ let

$$V_n(y_1, y_2, \ldots, y_n) = \begin{pmatrix} y_1 & y_1^2 & \cdots & y_1^n \\ y_2 & y_2^2 & \cdots & y_2^n \\ \vdots & \vdots & \ddots & \vdots \\ y_n & y_n^2 & \cdots & y_n^n \end{pmatrix}.$$

By [5, Lemma 23] we have that

$$\begin{aligned} \det(A_n(x_1, x_2, x_3, \ldots, x_{2n})) &= \sum_{(y_1, y_2, \ldots, y_n) \in Y^n} \det(V_n(y_1, y_2, \ldots, y_n)) \\ &= \sum_{\substack{(y_1, y_2, \ldots, y_n) \in Y^n \\ y_1 \neq y_2}} \det(V_n(y_1, y_2, \ldots, y_n)) > 0, \end{aligned}$$

because $\det(V_n(y_1, y_2, \ldots, y_n)) > 0$ whenever $|\{y_1, y_2, \ldots, y_n\}| = n$. So, as in [5] we can conclude that there are functions $f, g \in \mathcal{P}_n$ with

$$f(x_{2i}) = g(x_{2i-1}) \text{ and } f(x_{2i-1}) = g(x_{2i})$$

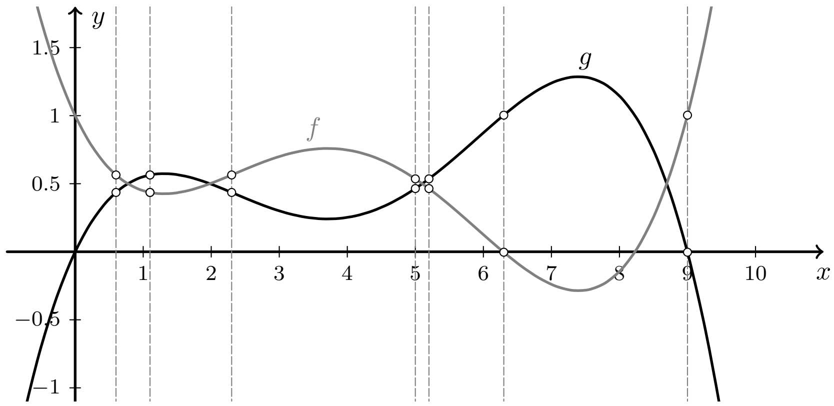
and therefore, there does not exist an SRU of size $2n-1$. □

Remark 2. *The polynomials f and g we constructed in [5] and in Lemma 1 have degree n. To see this, note that for all $1 \leq i \leq n$ we have that*

$$(f - g)(x_{2i-1}) = -(f - g)(x_{2i}).$$

By the intermediate value theorem, $(f - g)(x)$ has at least n pairwise different zeros. Since $f - g \not\equiv 0$ and since by construction $f - g$ has degree at most n, it follows that $\deg(f - g) = n$. By construction $f - g = 1 - 2g$. Therefore, $\deg(f) = \deg(g) = n$.

Example 3. Let $S := \left\{\frac{3}{5}, \frac{11}{10}, \frac{23}{10}, 5, \frac{26}{5}, \frac{63}{10}, 9\right\}$. In the following picture we can see two polynomials f and g of degree 4 with $f[S] = g[S]$ but $f \neq g$. These polynomials indicate that S is not an SRU for $\mathcal{P}_4$.



Proposition 4. *There does not exist an SRU of size less than $2n-1$.*

Proof. Let $1 \leq s < 2n-1$. Let $x_1 < x_2 < \cdots < x_s$. We want to show that $S := \{x_1, x_2, \ldots, x_s\}$ is not an SRU for $\mathcal{P}_n$.

<u>*Case 1:*</u> s is an even number.

Choose $\{x_{s+1}, x_{s+2}, \ldots, x_{2n}\} \subseteq \mathbb{R}$ with $x_s < x_{s+1} < x_{s+2} < \cdots < x_{2n}$. By [5, Lemma 23] we can find two functions $f, g \in \mathcal{P}_n$ with

$$f(x_{2i}) = g(x_{2i-1}) \text{ and } f(x_{2i-1}) = g(x_{2i})$$

for all $1 \leq i \leq n$. Therefore we have that

$$f[S] = g[S] \text{ and } f[\{x_{s+1}, x_{s+2}, \ldots, x_{2n}\}] = g[\{x_{s+1}, x_{s+2}, \ldots, x_{2n}\}].$$

So S is not an SRU for $\mathcal{P}_n$.

<u>*Case 2:*</u> s is an odd number.

Choose $\{x_{s+1}, x_{s+2}, \ldots, x_{2n-1}\} \subseteq \mathbb{R}$ with $x_s < x_{s+1} < x_{s+2} < \ldots x_{2n-1}$. By [5, Lemma 23] we can find two functions $f, g \in \mathcal{P}_n$ with

$$f[S] = g[S] \text{ and } f[\{x_{s+1}, x_{s+2}, \ldots, x_{2n-1}\}] = g[\{x_{s+1}, x_{s+2}, \ldots, x_{2n-1}\}].$$

So S is not an SRU for $\mathcal{P}_n$. $\square$

3 There are no SRUs of size at most $2n$ for $\mathcal{Q}_n$

We define $\mathcal{Q}_n$ to be the set of all non-constant polynomials of degree at most n with complex coefficients. Let $S := \{x_1, x_2, \ldots, x_{2n}\} \subseteq \mathbb{C}$ be a set of cardinality $2n$. Our goal is to find two polynomials $f, g \in \mathcal{Q}_n$ with $f[S] = g[S]$ but $f \neq g$. By rotating the set S around the origin of the complex plane we can assume without loss of generality that all real parts of the points in S are pairwise different. By renaming the elements in the set, we can assume that

$$\mathrm{Re}(x_1) < \mathrm{Re}(x_2) < \cdots < \mathrm{Re}(x_{2n}).$$

Define

$$Y^n := \{(y_1, y_2, \ldots, y_n) \in \mathbb{C}^n \mid y_i \in \{x_{2i-1}, x_{2i}\} \text{ for all } 1 \leq i \leq n\}$$

and let π_n be the set of all permutations of $\{1, 2, \ldots, n\}$. By translating the set S to the right in the complex plane we can also assume that for all $(y_1, y_2, \ldots, y_n) \in Y^n$, all

$M_0 \subseteq \{1,2,\ldots,n\}$ and all $M_1 \subseteq [\{1,2,\ldots,n\}]^2$ (where $[\{1,2,\ldots,n\}]^2$ is the family of all 2-element subsets of $\{1,2,\ldots,n\}$)

$$\begin{aligned}\Big|\prod_{k\in M_0}\mathrm{Im}(y_k)\prod_{\substack{1\le i<j\le n\\ \{i,j\}\in M_1}}\big(\mathrm{Im}(y_j)-\mathrm{Im}(y_i)\big)\Big| &\le \\ &\le \frac{1}{2^n 2^{\binom{n}{2}}}\prod_{k\in M_0}\mathrm{Re}(y_k)\prod_{\substack{1\le i<j\le n\\ \{i,j\}\in M_1}}\big(\mathrm{Re}(y_j)-\mathrm{Re}(y_i)\big).\end{aligned} \tag{1}$$

We will show that there are $f,g\in\mathcal{Q}_n$ with

$$f(x_{2i}) = g(x_{2i-1}) \text{ and } f(x_{2i-1}) = g(x_{2i})$$

for all $1\le i\le n$. The two polynomials will have the form

$$g(x) = \sum_{j=1}^{n} b_j x^j \text{ with } b_j\in\mathbb{C} \text{ for } j=1,2,\ldots,n$$

and

$$f(x) = 1 - g(x).$$

In order to prove that such polynomials f and g exist we have to show that the following linear equation is solvable:

$$\underbrace{\begin{pmatrix} x_1+x_2 & x_1^2+x_2^2 & \ldots & x_1^n+x_2^n \\ x_3+x_4 & x_3^2+x_4^2 & \ldots & x_3^n+x_4^n \\ \vdots & \vdots & \ddots & \vdots \\ x_{2n-1}+x_{2n} & x_{2n-1}^2+x_{2n}^2 & \ldots & x_{2n-1}^n+x_{2n}^n \end{pmatrix}}_{=:A_n}\begin{pmatrix} b_1\\ b_2\\ \vdots\\ b_n\end{pmatrix} = \begin{pmatrix}1\\1\\ \vdots\\ 1\end{pmatrix}.$$

To do this we have to show that $\det(A_n)\neq 0$ for every $n\in\mathbb{N}^*$. By [5, Lemma 23] we have that

$$\det(A_n) = \sum_{(y_1,\ldots,y_n)\in Y^n}\det(V_n(y_1,y_2,\ldots,y_n)),$$

where

$$V_n(y_1,y_2,\ldots,y_n) = \begin{pmatrix} y_1 & y_1^2 & \ldots & y_1^n\\ y_2 & y_2^2 & \ldots & y_2^n\\ \vdots & \vdots & \ddots & \vdots\\ y_n & y_n^2 & \ldots & y_n^n\end{pmatrix}.$$

Note that

$$\det(V_n(y_1,y_2,\ldots,y_n)) = \Big(\prod_{k=1}^{n} y_k\Big)\Big(\prod_{1\le i<j\le n}(y_j-y_i)\Big).$$

In particular we have that

$$\mathrm{Re}(\det(V_n(y_1,\ldots,y_n))) = \Big(\prod_{k=1}^{n}\mathrm{Re}(y_k)\Big)\Big(\prod_{1\le i<j\le n}(\mathrm{Re}(y_j)-\mathrm{Re}(y_i))\Big) + R$$

where each summand in R has the form

$$\pm\prod_{k\in M_0}\mathrm{Im}(y_k)\prod_{\substack{1\le i<j\le n\\ \{i,j\}\in M_1}}\big(\mathrm{Im}(y_j)-\mathrm{Im}(y_i)\big)\prod_{k\notin M_0}\mathrm{Re}(y_k)\prod_{\substack{1\le i<j\le n\\ \{i,j\}\notin M_1}}\big(\mathrm{Re}(y_j)-\mathrm{Re}(y_i)\big)$$

where $M_0\subseteq\{1,2,\ldots,n\}$ and $M_1\subseteq[\{1,2,\ldots,n\}]^2$ are not both empty and $M_0\cup M_1$ has even cardinality. Since R contains less than $2^n 2^{\binom{n}{2}}$ summands and by (1) we have that

$$\mathrm{Re}(\det(V_n(y_1,y_2,\ldots,y_n))) > 0$$

for all $(y_1, \dots, y_n) \in Y^n$. Therefore

$$\det(A_n(y_1, y_2, \dots, y_n)) \neq 0.$$

This implies that there are $f, g \in \mathcal{Q}_n$ with $f[S] = g[S]$ but $f \neq g$.

Note that as in Section 2 we can show that there are no SRUs for $\mathcal{Q}_n$ of size less than $2n$.

4 SRUs that are not magic for $\mathcal{P}_2$ and $\mathcal{P}_3$

Let $\mathcal{P}_n$ be the family of all real, non-constant polynomials of degree at most n. For the family $\mathcal{P}_1$ magic sets and SRUs are the same: Let $S \subseteq \mathbb{R}$ and assume that S is an SRU. If S were not magic, there were two functions $f, g \in \mathcal{P}_1$ with $f[S] \subseteq g[S]$ but $f \neq g$. But since f and g are both bijective, it follows that $f[S] = g[S]$ which then implies that $f = g$ because S is an SRU. But we assumed that $f \neq g$, which is a contradiction.

However, the following Lemmas show that magic sets and SRUs for $\mathcal{P}_2$ and $\mathcal{P}_3$ are not the same:

Lemma 5. *The set* $S := \left\{-2, -1, 2, \sqrt{8}, \sqrt{14 - \sqrt{8}}\right\}$ *is an SRU for* $\mathcal{P}_2$ *but not a magic set.*

Proof. The set S is not a magic set because for $f(x) := x^2$ and $g(x) := 2x^2 - x - 2$ we have that

$$f[S] = \left\{1, 4, 8, 14 - \sqrt{8}\right\} \subseteq \left\{1, 4, 8, 14 - \sqrt{8}, 26 - 4\sqrt{2} - \sqrt{14 - \sqrt{8}}\right\} = g[S].$$

On the other hand, we now show that $S = \{x_1, x_2, x_3, x_4, x_5\}$ is an SRU for $\mathcal{P}_2$. First of all note that $f[S] = g[S]$ with $|f[S]| \leq 2$ immediately implies $f = g = \text{const}$. Observe also that there is no polynomial $f \in \mathcal{P}_2$ with $|f[S]| = 3$. So we only have to deal with the case that $|f[S]| \geq 4$. Assume towards a contradiction that there are

$$f(x) = a_0 + a_1 x + a_2 x^2 \quad \text{and} \quad g(x) = b_0 + b_1 x + b_2 x^2$$

with $f[S] = g[S]$, $|f[S]| = |g[S]| \geq 4$ and $f \neq g$. In other words, f and g satisfy a linear equation of the form

$$\begin{pmatrix} 1 & x_1 & x_1^2 & -1 & -x_{i_1} & -x_{i_1}^2 \\ 1 & x_2 & x_2^2 & -1 & -x_{i_2} & -x_{i_2}^2 \\ 1 & x_3 & x_3^2 & -1 & -x_{i_3} & -x_{i_3}^2 \\ 1 & x_4 & x_4^2 & -1 & -x_{i_4} & -x_{i_4}^2 \\ 1 & x_5 & x_5^2 & -1 & -x_{i_5} & -x_{i_5}^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with $\{i_1, i_2, \dots, i_5\} \subseteq \{1, 2, 3, 4, 5\}$ and $|\{i_1, \dots, i_5\}| \geq 4$. By checking all cases, one finds that the only solution of such a linear equation with $f \neq g$ is

$$f(x) = 1 + \frac{1}{2}x^2 \quad \text{and} \quad g(x) = -\frac{1}{2}x + x^2.$$

But $f[S] \neq g[S]$. So S is indeed an SRU. $\square$

Lemma 6. *The set*

$$S := \left\{1, 2, 4, 10, 31, \frac{1}{2}\left(3 + \sqrt{68581}\right), \frac{1}{2}\left(3 - \sqrt{550558 + 13347\sqrt{68581}}\right)\right\}$$

is an SRU for $\mathcal{P}_3$ *but not a magic set.*

Proof. The set S is not a magic set for $\mathcal{P}_3$ because for

$$f(x) = 18(x-1)(x-2) \text{ and } g(x) := (x-1)(7x^2+120x-160)$$

we have that $f[S] \subsetneq g[S]$. Observe also that there is no polynomial $f \in \mathcal{P}_3$ with $|f[S]| = 3$. So we only have to deal with the case that $|f[S]| \geq 4$.

Assume towards a contradiction that there are

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \quad \text{and} \quad g(x) = b_0 + b_1x + b_2x^2 + b_3x^3$$

with $f[S] = g[S]$, $|f[S]| = |g[S]| \geq 4$ and $f \neq g$. In other words, f and g satisfy a linear equation of the form

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & -1 & -x_{i_1} & -x_{i_1}^2 & -x_{i_1}^3 \\ 1 & x_2 & x_2^2 & x_2^3 & -1 & -x_{i_2} & -x_{i_2}^2 & -x_{i_2}^3 \\ 1 & x_3 & x_3^2 & x_3^3 & -1 & -x_{i_3} & -x_{i_3}^2 & -x_{i_3}^3 \\ 1 & x_4 & x_4^2 & x_4^3 & -1 & -x_{i_4} & -x_{i_4}^2 & -x_{i_4}^3 \\ 1 & x_5 & x_5^2 & x_5^3 & -1 & -x_{i_5} & -x_{i_5}^2 & -x_{i_5}^3 \\ 1 & x_6 & x_6^2 & x_6^3 & -1 & -x_{i_6} & -x_{i_6}^2 & -x_{i_6}^3 \\ 1 & x_7 & x_7^2 & x_7^3 & -1 & -x_{i_7} & -x_{i_7}^2 & -x_{i_7}^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with $\{i_1, i_2, \ldots, i_7\} \subseteq \{1,2,3,4,5,6,7\}$ and $|\{i_1, \ldots, i_7\}| \geq 4$. By checking all cases, one finds that the only solution of such a linear equation with $f \neq g$ is

$$f(x) = \frac{18}{7}x^2 - \frac{54}{7}x - \frac{124}{7} \quad \text{and} \quad g(x) = x^3 + \frac{113}{7}x^2 - 40x.$$

But $f[S] \neq g[S]$. So S is indeed an SRU. $\square$

In the above Lemma, the two polynomials showing that the set S is not magic for $\mathcal{P}_3$, are of degree 2 and 3. In the next Lemma we show that there is an SRU S and two polynomials of degree 3 showing that S is not magic.

Lemma 7. *The set*

$$S := \{1, 2, 5, 12, 23, 27, \alpha\}$$

with

$$\alpha = \frac{8}{3} - \frac{13}{3\sqrt[3]{3197764 - 9\sqrt{126243143179}}} - \frac{1}{3}\sqrt[3]{3197764 - 9\sqrt{126243143179}}$$

is an SRU for $\mathcal{P}_3$ but not a magic set.

Proof. The set S is not a magic set for $\mathcal{P}_3$ because for

$$f(x) = 21(x-1)(x-2)(x-5) \text{ and } g(x) := (x-1)(-1150x^2+17213x-13656)$$

we have that $f[S] \subsetneq g[S]$. Observe also that there is no polynomial $f \in \mathcal{P}_3$ with $|f[S]| = 3$. So we only have to deal with the case that $|f[S]| \geq 4$.

Assume towards a contradiction that there are

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \quad \text{and} \quad g(x) = b_0 + b_1x + b_2x^2 + b_3x^3$$

with $f[S] = g[S]$, $|f[S]| = |g[S]| \geq 4$ and $f \neq g$. In other words, f and g satisfy a linear equation of the form

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & -1 & -x_{i_1} & -x_{i_1}^2 & -x_{i_1}^3 \\ 1 & x_2 & x_2^2 & x_2^3 & -1 & -x_{i_2} & -x_{i_2}^2 & -x_{i_2}^3 \\ 1 & x_3 & x_3^2 & x_3^3 & -1 & -x_{i_3} & -x_{i_3}^2 & -x_{i_3}^3 \\ 1 & x_4 & x_4^2 & x_4^3 & -1 & -x_{i_4} & -x_{i_4}^2 & -x_{i_4}^3 \\ 1 & x_5 & x_5^2 & x_5^3 & -1 & -x_{i_5} & -x_{i_5}^2 & -x_{i_5}^3 \\ 1 & x_6 & x_6^2 & x_6^3 & -1 & -x_{i_6} & -x_{i_6}^2 & -x_{i_6}^3 \\ 1 & x_7 & x_7^2 & x_7^3 & -1 & -x_{i_7} & -x_{i_7}^2 & -x_{i_7}^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with $\{i_1, i_2, \dots, i_7\} \subseteq \{1,2,3,4,5,6,7\}$ and $|\{i_1, \dots, i_7\}| \geq 4$. By checking all cases, one finds that the only solution of such a linear equation with $f \neq g$ is

$$f(x) = \frac{6933}{575} - \frac{357}{1150}x + \frac{84}{575}x^2 - \frac{21}{1150}x^3 \quad \text{and} \quad g(x) = \frac{30869}{1150}x - \frac{18363}{1150}x^2 + x^3.$$

But $f[S] \neq g[S]$. So S is indeed an SRU. $\square$

5 Magic sets for $\mathcal{P}_n$

In this section we will show that for every $s \geq 2n+1$ there is a magic set of size s for the set $\mathcal{P}_n$ of all real, non-constant polynomials of degree at most n.

Remark 8. *For $n \geq 1$ the condition that $\mathcal{P}_n$ does not contain any constant polynomials is necessary for the existence of a magic set. Otherwise let $M \subseteq \mathbb{R}$ be a non-empty set, $f(x) \equiv c$ for a $c \in \mathbb{R}$ and let g be a non-constant polynomial with $g(m) = c$ for an $m \in M$. Then we have that*

$$\{c\} = f[M] \subseteq g[M]$$

but $f \neq g$.

First of all we want to give some general definitions:

Definition 9. A *directed graph* H is a pair (V, E), where V is a set (the vertices of H) and $E \subseteq V \times V$ (the edges of H). For every $v \in V$ we define

$$\begin{aligned} \mathrm{indegree}_H(v) &:= |\{v' \in V \mid (v', v) \in E\}|, \\ \mathrm{outdegree}_H(v) &:= |\{v' \in V \mid (v, v') \in E\}| \text{ and} \\ \deg_H(v) &:= \mathrm{indegree}_H(v) + \mathrm{outdegree}_H(v). \end{aligned}$$

Definition 10. Let $H = (V, E)$ be a directed graph.

- A *cycle* is a subgraph $C = (V_C, E_C)$ of H with $V_C = \{c_0, c_1, \dots, c_{m-1}\}$ and $E_C = \{(c_i, c_{(i+1) \bmod m}) \mid i \in \mathbb{N}\}$ for an $m \geq 2$.
- A *loop* is a subgraph $L = (V_L, E_L)$ of H with $V_L = \{w\}$ and $E_L = \{(w, w)\}$.
- A *solitary path* is a directed path $P = (\{v_0, v_1, \dots, v_m\}, \{(v_i, v_{i+1}) \mid i = 0, 1, \dots, m-1\})$ with $\mathrm{indegree}_H(v_0) = 0$, $\deg_H(v_m) > 2$ and $\deg_H(v_i) = 2$ for all $1 \leq i \leq m-1$.

Definition 11. Let $l \in \mathbb{N}$. Cycles and loops $C_0 = (V_{C_0}, E_{C_0}), \dots, C_l = (V_{C_l}, E_{C_l})$ are called *obviously different* if for every $0 \leq i \leq l$ there is a

$$y_i \in V_{C_i} \setminus \left(\bigcup_{j=0, j \neq i}^{l} V_{C_j} \right).$$

Definition 12. Let H be a directed graph and let H_1 and H_2 be two subgraphs of H. Then H_1 and H_2 are called *undirected edge disjoint* iff H_1 and H_2 do not share any edges even if we replace all edges in H_1 and H_2 by undirected edges.

Let $k, n \in \mathbb{N}^*$ with $k \geq 2n$ and let $\{x_0, x_1, \dots, x_k\} \subseteq \mathbb{R}$. For all $0 \leq i \leq k$ let $v_i := (x_i, x_i^2, \dots, x_i^n)$. The following family $\mathcal{H}$ will play a crucial role in the construction of magic sets of size $k+1$ for the set $\mathcal{P}_n$.

Definition 13. Let $\mathcal{H}$ be the family of all directed graphs $H = (V, E)$ with vertex set $V = \{v_0, v_1, \dots, v_k\}$ and a set E of directed edges such that for each $v \in V$ we have that

$$\mathrm{outdegree}_H(v) \geq 1.$$

We now partition the family $\mathcal{H}$ into three parts, namely the graphs of type α_n, β_n and γ_n.

Definition 14. A graph $H \in \mathcal{H}$ is of type

- γ_n iff there are more than $n-1$ solitary paths in H.
- β_n iff there are more than n obviously different loops and cycles in H and H is not of type γ_n.
- α_n iff H is neither of type γ_n nor of type β_n.

In Section 5.1, we will consider graphs of type α_n and we will show in Corollary 23, that for every graph $H = (V, E)$ of type α_n, there is a $(2n+1) \times (2n+1)$-matrix

$$M_H(x_0, x_1, \ldots, x_k) = \begin{pmatrix} 1 & v_{i_0} & -v_{j_0} \\ 1 & v_{i_1} & -v_{j_1} \\ \vdots & \vdots & \vdots \\ 1 & v_{i_{2n}} & -v_{j_{2n}} \end{pmatrix}$$

with $i_l, j_l \in \{0, 1, \ldots, k\}$ (for all $0 \le l \le 2n$) and $(v_{i_l}, v_{j_l}) \in E$ (for all $0 \le l \le 2n$), such that for all open sets $U \subseteq \mathbb{R}^{k+1}$ there is an open set $U_H \subseteq U$ with

$$\det\bigl(M_H(x_0, x_1, \ldots, x_k)\bigr) \neq 0 \tag{2}$$

for all $(x_0, x_1, \ldots, x_k) \in U_H$.

Concerning graphs $H = (V, E)$ of type β_n, let $C_0 = (V_{C_0}, E_{C_0}), \ldots, C_n = (V_{C_n}, E_{C_n})$ be $n+1$ obviously different loops and cycles. Let $x_{i_0}, x_{i_1}, \ldots, x_{i_n}$ be $n+1$ vertices of H such that for each $0 \le l \le n$,

$$x_{i_l} \in V_{C_l} \setminus \left(\bigcup_{m=0, m \neq l}^{n} V_{C_m} \right).$$

We will show in Section 5.2 that for every open set $U \subseteq \mathbb{R}^{k+1}$ there is an open set $U_H \subseteq U$ such that for all $(x_0, x_1, \ldots, x_k) \in U_H$ we have

$$\det\bigl(N_H(x_0, x_1, \ldots, x_k)\bigr) \neq 0\,, \tag{3}$$

where

$$N_H(x_0, x_1, \ldots x_k) = \begin{pmatrix} |V_{C_0}| & \sum_{x \in V_{C_0}} x & \sum_{x \in V_{C_0}} x^2 & \cdots & \sum_{x \in V_{C_0}} x^n \\ |V_{C_1}| & \sum_{x \in V_{C_1}} x & \sum_{x \in V_{C_1}} x^2 & \cdots & \sum_{x \in V_{C_1}} x^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_n}| & \sum_{x \in V_{C_n}} x & \sum_{x \in V_{C_n}} x^2 & \cdots & \sum_{x \in V_{C_n}} x^n \end{pmatrix}.$$

In Section 5.3 we will show that for every graph H of type γ_n there is an $n \times n$-matrix

$$L_H(x_0, x_1, \ldots, x_k) = \begin{pmatrix} v_{j_0} - v_{i_0} \\ v_{j_1} - v_{i_1} \\ \vdots \\ v_{j_{n-1}} - v_{i_{n-1}} \end{pmatrix}$$

such that

- $j_l, i_l \in \{0, 1, \ldots, k\}$ for all $0 \le l \le n-1$;
- v_{i_l} and v_{j_l} are different but have the same successor in H and
- for all open sets $U \subseteq \mathbb{R}^{k+1}$ there is an open set $U_H \subseteq U$ such that for all $(x_0, x_1, \ldots, x_k) \in U_H$ we have that

$$\det\bigl(L_H(x_0, x_1, \ldots, x_k)\bigr) \neq 0\,. \tag{4}$$

As a consequence of (2), (3) and (4) and since $|\mathcal{H}| < \infty$, we can find a point $(m_0, m_1, \dots, m_k) \in \mathbb{R}^{k+1}$ such that for all $H \in \mathcal{H}$ of type α_n

$$\det(M_H(m_0, \dots, m_k)) \neq 0,$$

for all $H \in \mathcal{H}$ of type β_n

$$\det(N_H(m_0, \dots, m_k)) \neq 0,$$

and for all $H \in \mathcal{H}$ of type γ_n

$$\det(L_H(m_0, \dots, m_k)) \neq 0.$$

This leads to the following

Theorem 15. *The set $M := \{m_0, m_1, \dots, m_k\}$ is a magic set for $\mathcal{P}_n$.*

Proof. Assume towards a contradiction that M is not a magic set for $\mathcal{P}_n$. So, there are two non-constant polynomials

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

and

$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$

such that $f[M] \subseteq g[M]$ but $f \neq g$. Let $H = (V, E)$ with

$$V := M \text{ and } E := \{(m_i, m_j) \mid f(m_i) = g(m_j)\}.$$

Note that $H \in \mathcal{H}$. There are three cases:

<u>*Case 1:*</u> H is of type α_n.

In this case

$$M_H(m_0, m_1, \dots, m_k) = \begin{pmatrix} 1 & v_{i_0} & -v_{j_0} \\ 1 & v_{i_1} & -v_{j_1} \\ \vdots & \vdots & \vdots \\ 1 & v_{i_{2n}} & -v_{j_{2n}} \end{pmatrix}$$

has non-zero determinant. Note that for all $0 \leq l \leq n$ we have that

$$\begin{aligned} & f(m_{i_l}) = g(m_{j_l}) \iff \\ & (a_0 - b_0) + (a_1 m_{i_l} + \cdots + a_n m_{i_l}^n) - (b_1 m_{j_l} + \cdots + b_n m_{j_l}^n) = 0. \end{aligned}$$

So, f and g satisfy the following system of linear equations:

$$M_H(m_0, \dots, m_k) \cdot \begin{pmatrix} a_0 - b_0 \\ a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det(M_H(m_0, \dots, m_k)) \neq 0$, this equation has only the trivial solution. Therefore, $f = g$, which is a contradiction to our assumption that M is not a magic set.

<u>*Case 2:*</u> H is of type β_n.

In this case

$$N_H(m_0, \dots, m_k) = \begin{pmatrix} |V_{C_0}| & \sum_{x \in V_{C_0}} x & \sum_{x \in V_{C_0}} x^2 & \cdots & \sum_{x \in V_{C_0}} x^n \\ |V_{C_1}| & \sum_{x \in V_{C_1}} x & \sum_{x \in V_{C_1}} x^2 & \cdots & \sum_{x \in V_{C_1}} x^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_n}| & \sum_{x \in V_{C_n}} x & \sum_{x \in V_{C_n}} x^2 & \cdots & \sum_{x \in V_{C_n}} x^n \end{pmatrix}$$

with $n+1$ obviously different cycles $C_0 = (V_{C_0}, E_{C_0}), C_1 = (V_{C_1}, E_{C_1}), \dots, C_n = (V_{C_n}, E_{C_n})$. For all $0 \le i \le n$ we have that

$$\sum_{m \in V_{C_i}} (f-g)(m) = 0.$$

In other words, we have to solve the following system of linear equations:

$$N_H(m_0, \dots, m_k) \cdot \begin{pmatrix} a_0 - b_0 \\ a_1 - b_1 \\ \vdots \\ a_n - b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det(N_H(m_0, \dots, m_k)) \neq 0$ this equation has only the trivial solution. Therefore, $f = g$, which is again a contradiction.

<u>*Case 3:*</u> H is of type γ_n.

In this case

$$L_H(m_0, m_1, \dots, m_k) = \begin{pmatrix} v_{j_0} - v_{i_0} \\ v_{j_1} - v_{i_1} \\ \vdots \\ v_{j_{n-1}} - v_{i_{n-1}} \end{pmatrix}$$

has non-zero determinant. For all $0 \le l \le n-1$ the points m_{i_l} and m_{j_l} have the same successors in H. Therefore,

$$f(m_{j_l}) = f(m_{i_l}) \iff a_1(m_{j_l} - m_{i_l}) + a_2(m_{j_l}^2 - m_{i_l}^2) + \dots + a_n(m_{j_l}^n - m_{i_l}^n) = 0$$

for all $0 \le l \le n-1$. In other words, f satisfies the following system of linear equations:

$$L_H(m_0, m_1, \dots, m_k) \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det(L_H(m_0, m_1, \dots, m_k)) \neq 0$ this equation has only the trivial solution. Therefore, f is a constant polynomial. This is a contradiction. $\square$

5.1 Graphs and matrices of type α_n

Remark 16. *From now on we assume that there is at least one solitary path in every graph of type α_n. If a graph H of type α_n has no solitary path, it is of type 1_n (i.e., it has at most n obviously different cycles and loops) and we can find a suitable matrix as in [5].*

Definition 17. Let $G = (V, E)$ be a graph. Assume, that for each edge in E either the foot or the head is marked. The marked vertices are called *relevant*. Then $v \in V$ is called a *unique vertex* iff

$$\text{indegree}_G(v) = 0, \ \text{outdegree}_G(v) = 1$$

and v is the relevant vertex of the edge incident with v.

Definition 18. Let $n \in \mathbb{N}^*$ and let $H = (V, E)$ be a graph of type α_n with $|V| \ge 2n+1$. A *good sequence* of length $m \in \mathbb{N}$ of H is a sequence of graphs

$$(\emptyset, \emptyset) = H_0 = (V_0, E_0) \subseteq H_1 = (V_1, E_1) \subseteq \dots \subseteq H_m = (V_m, E_m) \subseteq H = (V, E)$$

such that for all $0 \le l < m$ the set $E_{l+1} \setminus E_l$ has one of the following forms:

(a) $E_{l+1} \setminus E_l = \{(v_i, v_j), (v_j, v_t)\}$ with $0 \le i, j, t \le k$, $i \neq j$ and $j \neq t$. Moreover, if v_j is contained in an edge in E_l together with a v_s, then v_s is a unique vertex of H_l. The relevant vertex of both edges (v_i, v_j) and (v_j, v_t) is v_j.

(b) $E_{l+1} \setminus E_l = \{(v_i, v_j), (v_s, v_t)\}$ with $0 \leq i, j, s, t \leq k$, $i \neq j$, $i \neq t$ and $s \neq t$. Moreover, if v_t or v_i is contained in an edge in E_l together with a v_p then v_p is a unique vertex of H_l. The relevant vertex of (v_i, v_j) is v_i and the relevant vertex of (v_s, v_t) is v_t.

(c) $E_{l+1} \setminus E_l = \{(v_i, v_i), (v_j, v_t)\}$ with $0 \leq i, j, t \leq k$ and $j \neq t$. Moreover, if v_i and v_j are contained in an edge in E_l together with a v_s, then v_s is a unique vertex of H_l. The relevant vertex of (v_i, v_i) is v_i and the relevant vertex of (v_j, v_t) is v_j.

(d) $E_{l+1} \setminus E_l = \{(v_i, v_i), (v_t, v_j)\}$ with $0 \leq i, j, t \leq k$ and $j \neq t$. Moreover, if v_i and v_j are contained in an edge in E_l together with a v_s, then v_s is a unique vertex of H_l. The relevant vertex of (v_i, v_i) is v_i and the relevant vertex of (v_t, v_j) is v_j.

(e) $E_{l+1} \setminus E_l = \{(v_i, v_j), (v_s, v_t)\}$ with $i \neq j$ and $s \neq t$. We have that $\mathrm{indegree}_H(v_i) = 0$ and for all $0 \leq q \leq l$ we have that $E_q \setminus E_{q-1}$ contains an edge with a unique vertex of H_l. Moreover we assume that if there is an edge in E_l containing v_t and a v_p we have that either v_p is a unique vertex of H_l or $(v_t, v_p) \in E_l$. The relevant vertex of (v_i, v_j) is v_i and the relevant vertex of (v_s, v_t) is v_t.

Lemma 19. *Let $n \in \mathbb{N}^*$. Every graph $H = (V_H, E_H)$ of type α_n with $|V_H| \geq 2n + 1$ has a good sequence*

$$(\emptyset, \emptyset) = H_0 = (V_0, E_0) \subseteq H_1 = (V_1, E_1) \subseteq \cdots \subseteq H_m = (V_m, E_m) \subseteq H$$

of length m with $|E_m| \geq 2n$ and an edge $z = (z_0, z_1) \notin E_m$ such that neither z_0 nor z_1 is a relevant vertex of any edge in E_m.

Proof. Let $H = (V_H, E_H)$ be a graph of type α_n. If there is a vertex $v \in V_H$ with $\mathrm{outdegree}_H(v) \geq 2$ and $\mathrm{indegree}_H(v) = 0$ remove all but one edge containing v. The resulting graph is still of type α_n. Let $\mathcal{L}$ be the set of all isolated loops of H. To be more precise

$$\mathcal{L} := \{(\{v\}, \{(v, v)\}) \subseteq H \mid \deg_H(v) = 2\}.$$

Let $\mathcal{T} = \{S_0, S_1, \ldots, S_l\}$ (for an $l \in \mathbb{N}$) be the set of all solitary paths in H. Let $0 \leq i \leq l$. If S_i ends in a vertex v in which only solitary paths end we have that $(v, v) \in E_H$. Add this edge to S_i iff this loop has not already been added to a S_j with $j < i$. Define $Z := S_0$. Note that $|\mathcal{T}| \geq 1$ by Remark 16. Remove Z from $\mathcal{T}$. Let $\mathcal{S}$ be the set of all first edges of the remaining solitary paths in $\mathcal{T}$ that contain an odd number of edges.

Step 1: Removing isolated loops with solitary paths.

Assume that $\mathcal{S} \neq \emptyset$ and $\mathcal{L} \neq \emptyset$. Let $s = (s_0, s_1) \in \mathcal{S}$ and let $t = (t_0, t_0) \in \mathcal{L}$. Add s, t and the corresponding edges to H_0. Call the resulting graph H_1. Note that $E_1 \setminus E_0$ has the form (c) and that s contains a unique vertex. Remove t from $\mathcal{L}$ and remove s from $\mathcal{S}$. The relevant vertex of s is s_0 and the relevant vertex of t is t_0. Redo this construction until either $\mathcal{S} = \emptyset$ or $\mathcal{L} = \emptyset$.

From now on we assume that $\mathcal{L} = \emptyset$. The construction in the other case is similar. Let

$$(\emptyset, \emptyset) = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{m_0}$$

with $m_0 \in \mathbb{N}$ be the good sequence we constructed so far.

Step 2: Adding cycles.

Let $C_0 = (V_{C_0}, E_{C_0}), C_1 = (V_{C_1}, E_{C_1}), \ldots, C_{l_1} = (V_{C_{l_1}}, E_{C_{l_1}})$ be a maximal family of pairwise disjoint cycles in H. If there is a cycle $C = C_j$ for a $0 \leq j \leq l_1$ that contains a vertex to which Z points, assume that $C = C_{l_1}$. This is important because we might have to add edges of the form (e). Assume that we have already added $C_0, C_1, \ldots, C_{i-1}$ for a $0 \leq i \leq l_1$ to H_{m_1} and defined a good sequence

$$(\emptyset, \emptyset) = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{m'}$$

for an $m' \geq m_0$. Now we want to add C_i. If the solitary path Z points to a vertex in V_{C_i} mark this vertex v_0 with a cross.

<u>Case 1:</u> There is a vertex $v_0 \in V_{C_i}$ that is marked with a cross.

If $\mathcal{S} \neq \emptyset$ let $\mathcal{M}_i \subseteq \mathcal{S}$ be maximal with $0 \leq |\mathcal{M}_i| + 1 \leq |E_{C_i}|$ and such that $|\mathcal{M}_i| + |E_{C_i}|$ is even. If $\mathcal{S} = \emptyset$ let $\mathcal{M}_i = \emptyset$. Remove $\mathcal{M}_i$ from $\mathcal{S}$.

Case 1.1: $|\mathcal{M}_i| + |E_{C_i}|$ is even.

There are two subcases:

- $\mathcal{M}_i \neq \emptyset$.

  Let $e = (e_0, e_1)$ be the first edge in E_{C_i} coming after v_0 and let $s = (s_0, s_1) \in \mathcal{M}_i$. Add e, s and the corresponding vertices to $H_{m'}$. We call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ is of the form (b). Remove e and s from E_{C_i} and $\mathcal{M}_i$. The relevant vertex of e is e_1 and the relevant vertex of s is s_0. Note that $e_1 \neq v_0$ because $|\mathcal{M}_i| + 1 \leq |E_{C_i}|$. In particular v_0 is not a relevant vertex of any edge in $H_{m'+1}$.

- $\mathcal{M}_i = \emptyset$.

  There is a vertex $w \in V_{C_i} \setminus \{v_0\}$ such that both edges $e = (e_0, e_1)$ and $f = (f_0, f_1)$ containing w are still in E_{C_i}. We assume that w is the first vertex with this property coming after v_0 in C_i. Add e, f and the corresponding vertices to $H_{m'}$. We call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ is of the form (a). Remove e and f from E_{C_i}. The relevant vertex of e and of f is w. Note that v_0 is not a relevant vertex of any edge in $H_{m'+1}$.

Case 1.2: $|\mathcal{M}_i| + |E_{C_i}|$ is odd.

Note that we are only in this case when $\mathcal{M}_i = \emptyset$ and C_i is still the original cycle. Let $y = (y_0, y_1)$ be the first edge in E_{C_i} coming after v_0. By the assumption in Case 1 we have in particular that $i = l_1$. So there is no cycle C_{i+1}. If $|E_Z|$ is even, add y, the third last (or if this is not possible the first) edge $f = (f_0, f_1)$ of Z and the corresponding vertices to $E_{m'}$. We call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ has the form (b). Remove f from Z and y from E_{C_i}. The relevant vertex of y is y_1 and the relevant vertex of f is f_0. If there is no cycle C_{i+1} and $|E_Z|$ is odd, remove y from E_{C_i}.

<u>Case 2:</u> There is no vertex in V_{C_i} that is marked with a cross.

Let $\mathcal{M}_i \subseteq \mathcal{S}$ be maximal with $|\mathcal{M}_i| \leq |E_{C_i}|$. Remove $\mathcal{M}_i$ from $\mathcal{S}$.

Case 2.1: $|\mathcal{M}_i| + |E_{C_i}|$ is odd.

Note that in this case $|\mathcal{M}_i| < |E_{C_i}|$ and therefore, $\mathcal{S} = \emptyset$ (we removed $\mathcal{M}_i$ from $\mathcal{S}$). So for all $j > i$ we will have that $\mathcal{M}_j = \emptyset$.

- There is a $j > i$ such that $|E_{C_j}| = |E_{C_j}| + |\mathcal{M}_j|$ is odd.

  Let $e = (e_0, e_1) \in E_{C_i}$ be an arbitrary edge. Note that C_i is still equal to the original cycle. Otherwise we would not be in this subcase. Let $f = (f_0, f_1) \in E_{C_j}$ be an arbitrary edge. That is, if possible, ending in a vertex that is marked with a cross. Add e, f and the corresponding edges to $H_{m'}$. We call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ is of the form (b). Remove e from E_{C_i} and remove f from E_{C_j}. The relevant vertex of e is e_1 and the relevant vertex of f is f_0.

- There is no $j > i$ such that $|E_{C_j}| = |E_{C_j}| + |\mathcal{M}_j|$ is odd.

  If $|E_Z|$ is even, let $e = (e_0, e_1) \in E_{C_i}$ be an arbitrary edge and let $f = (f_0, f_1)$ be the third last (or if this is not possible the first) edge in Z. Add e, f and the corresponding vertices to $H_{m'}$. Call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ has the form (b). Remove f from Z and e from E_{C_j}.

  If $|E_Z|$ is odd let $e = (e_0, e_1) \in E_{C_i}$ be an arbitrary edge. Remove e from E_{C_i}.

Case 2.2: $|\mathcal{M}_i| + |E_{C_i}|$ is even.

There are two subcases:

- $\mathcal{M}_i \neq \emptyset$.

  If E_{C_i} does not contain all edges of the original cycle C_i let $e = (e_0, e_1)$ be the first edge in E_{C_i}. Otherwise let e be an arbitrary edge in E_{C_i}. Let $s = (s_0, s_1) \in \mathcal{M}_i$. Add e, s and the corresponding edges to $H_{m'}$. We call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ has the form (b) or (e). Remove e and s from $\mathcal{M}_i$ and E_{C_i}. The relevant variable of e is e_1 and the relevant variable of s is s_0.

- $\mathcal{M}_i = \emptyset$.

  In this case let w be the first vertex in C_i with $\deg_{C_i}(w) = 2$ (or if C_i is still the original cycle choose a $w \in V_{C_i}$ with $\deg_{C_i}(w) = 2$). Add the edges $e, f \in E_{C_i}$ that contain w to $H_{m'}$. We call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ has the form (a). Remove e and f from E_{C_i}. The relevant vertex of e and of f is w.

Assume that we have done this construction for all cycles $C_0, C_1, \ldots, C_{l_1}$. Let

$$(\emptyset, \emptyset) = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{m_1}$$

with $m_1 \geq m_0$ be the good sequence we constructed so far.

Step 3: Adding paths.

Let $P_0 = (V_{P_0}, E_{P_0})$ be a maximal path in H which is undirected edge disjoint from H_{m_1}. In addition we require that all vertices (except possibly the first or the last one) are disjoint from the vertices in H_{m_1}. If possible let P_0 be a path such that Z points to a vertex v_0 in $V_{P_0} \setminus V_{m_1}$. Let $p_0 \in \mathbb{N}$ be the number of vertices in V_{P_0} that are not in V_{m_1}.

Case 1: The solitary path Z points to a vertex $v_0 \in V_{P_0} \setminus V_{m_1}$.

If $\mathcal{S} \neq \emptyset$ let $\mathcal{N}_0 \subseteq \mathcal{S}$ be maximal with $|\mathcal{N}_0| + 1 \leq p_0$ such that $|\mathcal{N}_0| + p_0$ is even. If $\mathcal{S} = \emptyset$ let $\mathcal{N}_0 := \emptyset$. Remove $\mathcal{N}_0$ from $\mathcal{S}$.

Case 1.1: $|\mathcal{N}_0| + p_0$ is even.

There are two subcases:

- $\mathcal{N}_0 \neq \emptyset$.

  Let $e = (e_0, e_1)$ be the first edge in P_0. If it points to v_0 remove it from P_0 and from H. Otherwise let $s = (s_0, s_1) \in \mathcal{S}$. Add e, s and the corresponding vertices to H_{m_2}. Call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \setminus E_{m_1}$ is of the form (b), (c) or (d). The relevant vertex of e is e_1 and the relevant vertex of s is s_0. Remove s and e from $\mathcal{N}_0$ and from E_{P_0}.

- $\mathcal{N}_0 = \emptyset$.

  Let $w \neq v_0$ be the first vertex in the path that is contained in exactly two edges of P_0. Let e and f be the two edges containing w. Add them and the corresponding vertices to H_{m_1} and call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \setminus E_{m_1}$ has the form (a), (c) or (d). Remove e and f from P_0. The relevant vertex of e and of f is w.

Repeat the procedure described in Case 1.1 until $|E_{P_0}| \leq 1$. Remove the remaining edge from E_{P_0}.

Case 1.2: $|\mathcal{N}_0| + p_0$ is odd.

Note that we are only in this case when $\mathcal{N}_0 = \emptyset$.

- On the right or on the left of v_0 there is an even number of edges.

  Let $w \neq v_0$ be the first vertex in the path that is contained in exactly two edges of P_0 and $w \notin \{z_0, z_1\}$ if we have already defined an edge $z = (z_0, z_1)$. Let e and f be the two edges containing w. Add e, f and the corresponding vertices to H_{m_1} and call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \setminus E_{m_1}$ has the form (a), (c) or (d). Remove e and f from E_{P_0} or from E_Z. The relevant vertex of e and of f is w.

- We are not in the first subcase and v_0 is the first vertex in $V_{P_0} \setminus V_{m_1}$.

  Let e be the first edge in P_0. Remove e from H and add back the original Z to H. This graph H is of type α_n. Redo the whole construction. Note that at one point we will never be in this case anymore.

- We are not in the first two subcases.

  If P_0 ends in a vertex of a cycle C_i that is relevant for an edge in E_{m_1} mark that last vertex of P_0 with a cross and redo the whole construction with the same cycles and paths. If necessary remove one edge s from $\mathcal{M}_i$ and add it to $\mathcal{N}_0$. So we can now assume that the last vertex in P_0 is not relevant for any edge in E_{m_1}. There are two cases we have to look at:

  – If $|\mathcal{N}_0| = 1$, let $e = (e_0, e_1)$ be the first edge in P_0 (note that $e_1 \neq v_0$) and let $s = (s_0, s_1) \in \mathcal{N}_0$. Add e, s and the corresponding vertices to H_{m_1} and remove them from P_0 and from $\mathcal{N}_0$. Call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \setminus E_{m_1}$ is of the form (b). The relevant vertex of e is e_1 and the relevant vertex of s is s_0.

  – If $\mathcal{N}_0 = \emptyset$, let $e = (e_0, e_1)$ be the first edge in P_0. Note that by assumption $e_1 \neq v_0$. Let $f = (f_0, f_1)$ be the third last (or if this is not possible the first) edge in Z. Add e, f and the corresponding vertices to H_{m_1}. Call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \setminus E_{m_1}$ is of the form (b), (c) or (d). Remove e form P_0 and f from Z.

    If now $|E_Z| = 0$ let $z = (z_0, z_1)$ be the first edge coming after v_0 in P_0. In particular we have that $z_0 = v_0$. Note that neither z_0 nor z_1 is a relevant vertex of an edge we added to H_0 so far. Moreover, it will never be a relevant vertex of any edge we will add in the future.

Repeat the procedure described in Case 1.2 until $|E_{P_0}| \leq 1$. Remove the remaining edge from P_0.

<u>*Case 2:*</u> The solitary path Z does not point to a vertex in P_0.

Let $\mathcal{N}_0 \subseteq \mathcal{S}$ be maximal with $|\mathcal{N}_0| \leq p_0$. Remove $\mathcal{N}_0$ from $\mathcal{S}$.

Case 2.1: $\mathcal{N}_0 \neq \emptyset$.

Let $e = (e_0, e_1)$ be the first edge in P_0 and let $f = (f_0, f_1) \in \mathcal{N}_0$. Add e, f and the corresponding vertices to H_{m_1}. Call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \setminus E_{m_1}$ is of the form (b). Remove e and f from $\mathcal{N}_0$ and from E_{P_0}. The relevant vertex of e is e_1 and the relevant vertex of f is f_0

Case 2.2: $\mathcal{N}_0 = \emptyset$.

Let w be the first vertex in P_0 that is contained in exactly two edges $e, f \in E_{P_0}$. Add e, f and the corresponding vertices to H_{m_1}. Call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \setminus E_{m_1}$ is of the form (a). Remove e and f from E_{P_0}. The relevant vertex of e and of f is w.

Repeat this procedure until $|E_{P_0}| \leq 1$. Remove the remaining edges from P_0.

Do the same procedure for all paths in H. Let

$$(\emptyset, \emptyset) = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{m_2}$$

with $m_2 \geq m_1$ be the good sequence we constructed so far.

Step 4: Adding the rest of the solitary paths.

Add Z to $\mathcal{T}$. And if $|E_Z| \geq 2$ is odd, add the first edge of Z to $\mathcal{S}$. Define

$$\mathcal{T}_2 := \{S \in \mathcal{T} \mid |E_S| \geq 2\} = \{T_0, T_1, \ldots, T_{l_3}\}$$

for an $l_3 \in \mathbb{N}$. Assume that $Z = T_{l_3}$ if $|E_Z| \geq 2$. Note that if Z ends in a vertex v in which only solitary paths end, Z contains the loop (v, v).

Let $F = \emptyset$. Assume that we have already added $T_0, T_1, \ldots, T_{i-1}$ for a $0 \leq i \leq l_2$ to H_{m_2} and we defined a good sequence

$$(\emptyset, \emptyset) = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{m'}$$

with a $m' \geq m_2$. Now we want to add $T_i = (V_{T_i}, E_{T_i})$.

Case 1: $|E_{T_i}| > 2$ is even and $\mathcal{S} \neq \emptyset$.

Let $s = (s_0, s_1)$ be the third last edge in E_{T_i} and let $t = (t_0, t_1) \in \mathcal{S}$. Add s, t and the corresponding vertices to $H_{m'}$. Call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ is of the form (b). Remove t from $\mathcal{S}$ and s from E_{T_i}. If t is contained in a T_j, $j > i$, remove t from E_{T_j}. The relevant vertex of s is s_1 and the relevant vertex of t is t_0.

Case 2: $|E_{T_i}| > 2$ is even and $\mathcal{S} = \emptyset$.

Let w be the first vertex in T_i with $\deg_{T_i}(w) = 2$. Let e and f be two edges containing w. Add e, f and the corresponding vertices to $H_{m'}$. Call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ is of the form (a) or (d). Remove e and f from E_{T_i}. The relevant vertex of e and of f is w.

Case 3: $|E_{T_i}| > 2$ is odd and $\mathcal{S} \setminus E_{T_i} \neq \emptyset$.

Let $e = (e_0, e_1)$ be the third last edge in E_{T_i} and let $f = (f_0, f_1) \in \mathcal{S} \setminus \{e\}$. Add e, f and the corresponding vertices to $H_{m'}$. The resulting graph is called $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ is of the form (b). The relevant vertex of e is e_1 and the relevant vertex of f is f_1. Remove e from E_{T_i} and f from $\mathcal{S}$. Remove the first edge of E_{T_i} from $\mathcal{S}$.

Case 4: $|E_{T_i}| > 2$ is odd and $\mathcal{S} \setminus E_{T_i} = \emptyset$.

Let $z = (z_0, z_1)$ be the first edge in E_{T_i}. Remove z from E_{T_i} and from $\mathcal{S}$. Note that neither z_0 nor z_1 will ever be a relevant vertex of an edge we add to H_0.

Case 5: $|E_{T_i}| = 2$.

There are two subcases:

- $T_i = Z$ and we haven't defined an edge z yet.

  Let $z = (z_0, z_1)$ be the last edge in E_{T_i}. Remove both edges from E_{T_i}. Note that neither z_0 nor z_1 are relevant vertices of any edge in $E_{m'}$.

- We are not in the first subcase and E_{T_i} does not contain a loop.

  Add the two edges in E_{T_i} to the set F and remove them from E_{T_i}.

- We are not in the first subcase and E_{T_i} does contain a loop.

  Do the same as in Case 2.

Repeat the procedure with all solitary paths. Let

$$(\emptyset, \emptyset) = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{m_3}$$

with $m_3 \geq m_2$ be the good sequence we constructed so far.

Step 5: Adding the set F.

Let $F = \{\{e_0, f_0\}, \{e_1, f_1\}, \ldots, \{e_{l_4}, f_{l_4}\}\}$ with a $l_4 \in \mathbb{N}$. The pairs of edges are enumerated in the order we added them to F. Now add e_0, f_0 and the corresponding vertices to H_{m_3}. Call the resulting graph H_{m_3+1}. Note that $E_{m_3+1} \setminus E_{m_3}$ has the form (a). The relevant vertex of e_0 and of f_0 is the vertex they share. Repeat the procedure with $\{f_1, e_1\}$, $\{f_2, e_2\}$ and so on. $\square$

Example 20. In this example we will construct a good sequence for the following graph H of type α_n:

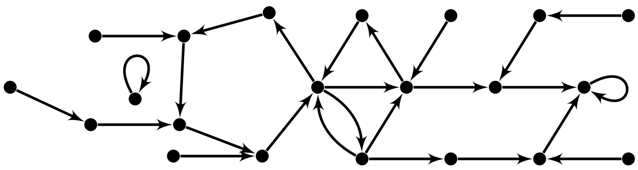


Figure 1: Graph $H = (V, E)$.

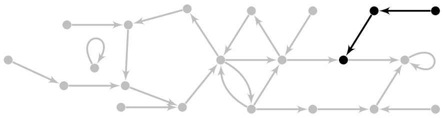


Figure 2: solitary path Z.

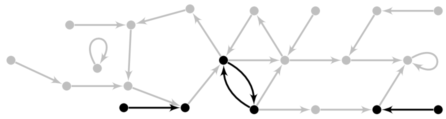


Figure 3: $\mathcal{M}_0$ and cycle C_0.

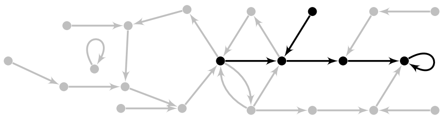


Figure 4: $\mathcal{N}_0$ and path P_0.

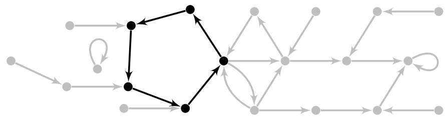


Figure 5: Path P_1.

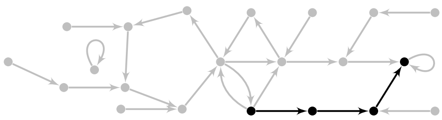


Figure 6: Path P_2.

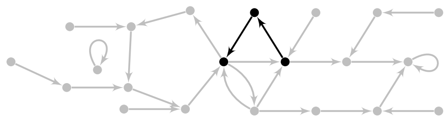


Figure 7: Path P_3.

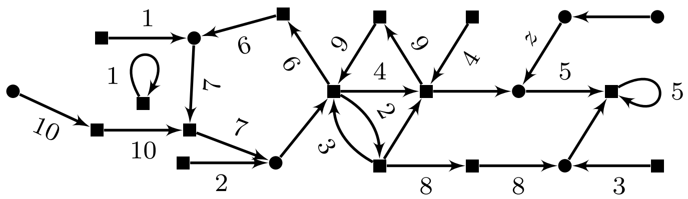


Figure 8: Graph $H = (V, E)$. The squared vertices are relevant vertices of an edge. The numbers show the order in which the edges are added.

(end example)

Let $k \geq n$, and for all $0 \leq i, j \leq k$ and all $0 \leq s \leq n$ define

$$v_i\_ - v_j := (x_i, x_i^2, \ldots, x_i^s, -x_j, -x_j^2, \ldots, -x_j^s)$$

and

$$1\_v_i\_ - v_j := (1, x_i, x_i^2, \ldots, x_i^s, -x_j, -x_j^2, \ldots, -x_j^s).$$

For every graph $H = (V, E)$ of type α_n choose a good sequence

$$(\emptyset, \emptyset) = H_0 = (V_0, E_0) \subseteq H_1 = (V_1, E_1) \subseteq \cdots \subseteq H_n = (V_n, E_n)$$

with $|E_n| = 2n$ and an additional edge $z = (z_0, z_1)$ such that neither z_0 nor z_1 is a relevant vertex for any edge in E_n. For every graph H of type α_n and all $0 \leq l \leq n$ let $M_{H_l}(x_0, \ldots, x_k)$

be a square matrix with pairwise different rows $v_{i^-}-v_j$ where $(v_i, v_j) \in E_{H_l}$. For all $0 \leq l \leq n$ we define

$$\mathcal{C}_l := \left\{ M_{H_l}(x_0, \ldots, x_k) \mid H \text{ is a graph of type } \alpha_n \right\}.$$

Furthermore, we define M_H to be the square matrix with $2n+1$ pairwise different rows $1\_v_{i^-} - v_j$ where $(v_i, v_j) \in E_n$ or $(v_i, v_j) = z$.

Definition 21. Let $R_0 := \emptyset$ and $p_0(x_0, \ldots, x_k) := 1$. For every $1 \leq l \leq n$ let R_l be the set of all relevant vertices of the edges in $E_l \setminus E_{l-1}$. We define

$$p_l(x_0, x_1, \ldots, x_k) = \Big(\prod_{v_i \in R_l} x_i^l \Big) p_{l-1}(x_0, x_1, \ldots, x_k).$$

The polynomial p_l is called the *relevant polynomial* of $M_{H_l}(x_0, x_1, \ldots, x_k)$.

Lemma 22. *Let H be a graph of type α_n, let $1 \leq l \leq n$ and let $M_{H_l} \in \mathcal{C}_l$. Then we have that*

$$\det(M_{H_l}) = \overline{p_l} + q_l,$$

where $\overline{p_l}$ is plus or minus the relevant polynomial of H_l and q_l is a polynomial that contains no term of the form $\pm p_l$.

Proof. We prove the Lemma by induction on l. For $l = 1$ it is clear. So assume that $2 \leq l \leq n$. By the induction hypothesis we have that

$$\det(M_{H_{l-1}}) = \overline{p_{l-1}} + q_{l-1}$$

with the properties described in the Lemma. There are five cases:

<u>Case 1:</u> $E_l \setminus E_{l-1}$ has the form (a).

There are two rows

$$Z_0 = v_{i^-} - v_j$$
$$Z_1 = v_{j^-} - v_t$$

in M_{H_l} such that v_j and $-v_j$ are only contained in these two rows and in rows that also contain a unique vertex of H_l. We first do a Laplace expansion of M_{H_l} along Z_0. So we have that

$$\det(M_{H_l}) = \epsilon_0 x_j^l \det(\overline{M_{H_l}}) + \gamma,$$

where γ is a polynomial, $\epsilon_0 \in \{-1, 1\}$ and $\overline{M_{H_l}}$ is the matrix we obtain from M_{H_l} when we delete the row Z_0 and the $2l$-th column. Now we do a Laplace expansion along the remainders of the row Z_1. We get

$$\det(\overline{M_{H_l}}) = \epsilon_1 x_j^l \det(M_{H_{l-1}}) + \delta = \epsilon_1 x_j^l (\overline{p_{l-1}} + q_{l-1}) + \delta,$$

where δ is a polynomial and $\epsilon_1 \in \{-1, 1\}$. So we have that

$$\det(M_{H_l}) = \epsilon_0 \epsilon_1 x_j^{2l} (\overline{p_{l-1}} + q_{l-1}) + \epsilon_0 x_j^l \delta + \gamma.$$

Define

$$\overline{p_l} := \epsilon_0 \epsilon_1 x_j^{2l} \overline{p_{l-1}} \text{ and } q_l := \epsilon_0 \epsilon_1 x_j^{2l} q_{l-1} + \epsilon_0 x_j^l \delta + \gamma.$$

It remains to prove that q_l does not contain a term of the form $\pm p_l$. First we show that γ does not contain a term of the form $\pm p_l$. If γ does not contain a term containing x_j^{2l} we are done. So there are terms in γ containing x_j^{2l}. But then not the whole x_j^{2l} comes from the rows Z_0, Z_1. Since outside of Z_0 and Z_1 the vertex v_j is only contained in rows together with unique vertices of H_{l-1}, there is a unique variable (i.e. the variable belonging to a unique vertex) which is not contained in the term with x_j^{2l} in it. So there are no terms in γ of the form $\pm p_l$.

Similarly we can show that there are no terms in $\epsilon_0 x_j \delta$ of the form $\pm p_l$. By the properties of q_{l-1} also $\epsilon_0\epsilon_1 x_j^{2l} q_{l-1}$ does not contain a term of the form $\pm p_l$. So q_l has the desired properties.

<u>*Case 2:*</u> $E_l \setminus E_{l-1}$ has the form (b).

There are two rows

$$Z_0 = v_{i^-} - v_j$$
$$Z_1 = v_{s^-} - v_t$$

in M_{H_l} such that $v_i, -v_i, v_t$ and $-v_t$ are only contained in these two rows and in rows together with a unique vertex of H_{l-1}. After doing two Laplace expansions we see that

$$\det(M_{H_l}) = \epsilon_0\epsilon_1 x_i^l x_t^l(\overline{p_{l-1}} + q_{l-1}) + \epsilon_0 x_i^l \delta + \gamma.$$

Define

$$\overline{p_l} := \epsilon_0\epsilon_1 x_i^l x_t^l \overline{p_{l-1}} \text{ and } q_l := \epsilon_0\epsilon_1 x_i^l x_t^l q_{l-1} + \epsilon_0 x_i^l \delta + \gamma.$$

If γ does not contain a term containing $x_i^l x_t^l$ we are done. Otherwise not the whole $x_i^l x_t^l$ comes from the rows Z_0 and Z_1. Since outside of Z_0 and Z_1 the vertices v_i and v_j are only contained in rows together with unique vertices of H_{l-1}, there is a unique variable (i.e. the variable belonging to a unique vertex) which is not contained in the term with $x_i^l x_t^l$ in it. So there are no terms in γ of the form $\pm p_l$. Similarly we can show that $\epsilon_0 x_i^l \delta$ does not contain terms of the form $\pm p_l$. By the properties of q_{l-1} the polynomial $\epsilon_0\epsilon_1 x_i^l x_t^l q_{l-1}$ does not contain a term of the form $\pm p_l$.

<u>*Case 3:*</u> $E_l \setminus E_{l-1}$ has the form (c).

This case is similar to Case 2.

<u>*Case 4:*</u> $E_l \setminus E_{l-1}$ has the form (d).

This case is similar to Case 2.

<u>*Case 5:*</u> $E_l \setminus E_{l-1}$ has the form (e).

There are two rows

$$Z_0 = v_{i^-} - v_j$$
$$Z_1 = v_{s^-} - v_t$$

in M_{H_l} such that $\text{indegree}_H(v_i) = 0$ and such that v_t is only contained in rows together with a unique variable or on the left side. Moreover, for all $0 \leq l' < l$ we have that one of the edges in $E_{l'} \setminus E_{l'-1}$ contains a unique vertex of H_{l-1}. After doing two Laplace expansions we see that

$$\det(M_{H_l}) = \epsilon_0\epsilon_1 x_i^l x_t^l(\overline{p_{l-1}} + q_{l-1}) + \epsilon_0 x_i^l \delta + \gamma.$$

Define

$$\overline{p_l} := \epsilon_0\epsilon_1 x_i^l x_t^l \overline{p_{l-1}} \text{ and } q_l := \epsilon_0\epsilon_1 x_i^l x_t^l q_{l-1} + \epsilon_0 x_i^l \delta + \gamma.$$

Note that there is no term in γ that contains x_i^l because Z_0 is the only row in M_{H_l} containing x_i. So γ does not contain a term of the form $\pm p_l$.

Assume towards a contradiction that there is a term in δ containing x_t^l. But then x_t^l contains an $x_t^{l'}$ with $0 < l' < l$ maximal from an other row than Z_1. If this $x_t^{l'}$ comes from a row that also contains a unique variable, then the term containing x_t^l does not contain this unique variable. So this is not possible. Therefore, the $x_t^{l'}$ comes from a row of the form

$$v_{t^-} - v_p$$

for a $p \in \{0, 1, \ldots, k\} \setminus \{t\}$. But then the term does not contain the the unique variable in p_{l-1} that has power l'. This is a contradiction. So $\epsilon_0 x_i^l \delta$ does not contain a term of the form $\pm p_l$. By the properties of q_{l-1} the polynomial $\epsilon_0\epsilon_1 x_i^l x_t^l q_{l-1}$ does not contain a term of the form $\pm p_l$. $\square$

Corollary 23. *Let H be a graph of type α_n. For every open set $U \subseteq \mathbb{R}^{k+1}$ there is an open subset $U_H \subseteq U$ such that for all $(x_0, x_1, \dots, x_k) \in U_H$*

$$\det(M_H(x_0, x_1, \dots, x_k)) \neq 0.$$

Proof. It suffices to prove that

$$\det(M_H(x_0, x_1, \dots, x_k)) \not\equiv 0.$$

By Lemma 22 we have that

$$\det(M_{H_n}) = \overline{p_n} + q_n,$$

where $\overline{p_n}$ is plus or minus the relevant polynomial of H_n and q_n is a polynomial that contains no term of the form $\pm p_n$. Let $z = (v_i, v_j)$ be the edge in E_H that does not contain a relevant vertex of any edge in E_n. Do a Laplace expansion of M_H along the row

$$1\_v_{i}\_ - v_j.$$

We have that

$$\det(M_H(x_0, \dots, x_k)) = \det(M_n) + \gamma = \overline{p_n} + q_n + \gamma,$$

where γ is a polynomial in which each term either contains x_i or x_j. Since $\overline{p_n}$ does not contain terms with x_i or x_j in it, we have that

$$\det(M_H(x_0, \dots, x_k)) \not\equiv 0.$$

This finishes the proof. $\square$

5.2 Graphs of type β_n

Let $H = (V, E) \in \mathcal{H}$ be a graph of type β_n. So H contains at least $n + 1$ obviously different loops and cycles $C_0 = (V_{C_0}, E_{C_0}), C_1 = (V_{C_1}, E_{C_1}), \dots, C_n = (V_{C_n}, E_{C_n})$. Without loss of generality we can assume that for all $0 \leq i \leq n$ we have that

$$x_i \in V_{C_i} \setminus \left(\bigcup_{j=0, j \neq i}^{n} V_{C_j} \right).$$

Let

$$N_H(x_0, x_1, \dots, x_k) = \begin{pmatrix} |V_{C_0}| & \sum_{x \in V_{C_0}} x & \sum_{x \in V_{C_0}} x^2 & \cdots & \sum_{x \in V_{C_0}} x^n \\ |V_{C_1}| & \sum_{x \in V_{C_1}} x & \sum_{x \in V_{C_1}} x^2 & \cdots & \sum_{x \in V_{C_1}} x^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_n}| & \sum_{x \in V_{C_n}} x & \sum_{x \in V_{C_n}} x^2 & \cdots & \sum_{x \in V_{C_n}} x^n \end{pmatrix}$$

Then we have that

$$\det(N_H(x_0,x_1,\ldots,x_n,0,\ldots,0)) = \det\begin{pmatrix} |V_{C_0}| & x_0 & x_0^2 & \ldots & x_0^n \\ |V_{C_1}| & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_n}| & x_n & x_n^2 & \ldots & x_n^n \end{pmatrix}$$

$$= \sum_{l=0}^{n}(-1)^{l+2}|V_{C_l}| \det\begin{pmatrix} x_0 & x_0^2 & \ldots & x_0^n \\ x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_{l-1} & x_{l-1}^2 & \ldots & x_{l-1}^n \\ x_{l+1} & x_{l+1}^2 & \ldots & x_{l+1}^n \\ x_{l+2} & x_{l+2}^2 & \ldots & x_{l+2}^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_n^2 & \ldots & x_n^n \end{pmatrix}$$

$$= \sum_{l=0}^{n}(-1)^{l}|V_{C_l}| \prod_{\substack{0\le i<j\le n \\ i,j\neq l}}(x_j - x_i) \not\equiv 0.$$

Therefore, $\det(N_H(x_0,\ldots,x_k)) \not\equiv 0$. So, for every open set $U \subseteq \mathbb{R}^{k+1}$ there is an open set $U_H \subseteq U$ such that for all $(x_0,\ldots,x_k) \in U_H$

$$\det(N_H(x_0,\ldots,x_k)) \neq 0.$$

5.3 Graphs of type γ_n

Let $H = (V,E) \in \mathcal{H}$ be a graph of type γ_n. Let $V_0 \subseteq V$ be a maximal subset such that the direct successors of the vertices in V_0 are pairwise different. Since H contains at least n solitary paths there is a set $W_0 \subseteq V \setminus V_0$ which contains at least n points. We define the matrix $L_H(x_0,x_1,\ldots,x_k)$ belonging to H as follows:

$$L_H(x_0,\ldots,x_k) = \begin{pmatrix} v_{j_0} - v_{i_0} \\ v_{j_1} - v_{i_1} \\ \vdots \\ v_{j_{n-1}} - v_{i_{n-1}} \end{pmatrix},$$

where for all $0 \le l \le n-1$ the vertices $v_{i_l} \in V_0$ and $v_{j_l} \in W_0$ have the same successor in H and the vertices v_{j_l}, $0 \le l \le n-1$, are pairwise different.

Lemma 24. *Let $H = (V,E) \in \mathcal{H}$ be a graph of type γ_n and let*

$$L_H(x_0,\ldots,x_k)$$

be a matrix belonging to H. Then we have that $\det(L_H(x_0,x_1,\ldots,x_k)) \not\equiv 0$.

Proof. Let V_0 and W_0 be as above. Without loss of generality we assume that $x_{j_l} = x_l$ for all $0 \le l \le n-1$. Since $V_0 \cap W_0 \neq \emptyset$ we have that $V_0 \subseteq \{x_{n+1},\ldots,x_k\}$. Let $x_{n+1} = x_{n+2} = \cdots = x_k = 0$. Then we have that

$$L_H(x_0,x_1,\ldots,x_n,0,\ldots,0) = \begin{pmatrix} x_0 & x_0^2 & \ldots & x_0^n \\ x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_{n-1}^2 & \ldots & x_{n-1}^n \end{pmatrix}.$$

This is a Vandermonde matrix. Its determinant is not constantly equal to zero. Therefore, $\det(L_H(x_0,x_1,\ldots,x_k)) \not\equiv 0$. $\square$

6 Magic sets for $\mathcal{Q}_n$

To construct a magic set for $\mathcal{Q}_n$ we could redo the construction from Section 5. However, this is not necessary:

Fact 25. *Let $M \subseteq \mathbb{R}$ be a magic set for $\mathcal{P}_n$ and let $f \in \mathcal{P}_n$. Then we have that $|f[M]| \geq n+1$.*

Proof. Let $M = \{m_1, m_2, \dots, m_k\} \subseteq \mathbb{R}$ be a magic set for $\mathcal{P}_n$ and assume towards a contradiction that there is an $f \in \mathcal{P}_n$ with $|f[M]| \leq n$. Note that $k \geq 2n+1$ by Section 2. So, there is a non-constant polynomial $g \in \mathcal{P}_n$ with $g \neq f$ and $g[\{m_1, \dots, m_n\}] = f[M]$. Therefore, $f[M] \subseteq g[M]$ but $f \neq g$ which contradicts the assumption that M is a magic set for $\mathcal{P}_n$. $\square$

Lemma 26. *Every magic set for $\mathcal{P}_n$ is also a magic set for $\mathcal{Q}_n$.*

Proof. Let $M \subseteq \mathbb{R}$ be a magic set for $\mathcal{P}_n$ and let $f, g \in \mathcal{Q}_n$ with $f[M] \subseteq g[M]$. Let

$$f(x) = f_0(x) + i f_1(x) \quad \text{and} \quad g(x) = g_0(x) + i g_1(x)$$

where f_0, f_1, g_0 and g_1 are polynomials of degree at most n with real coefficients. By our assumption we have that

$$f_0[M] \subseteq g_0[M] \quad \text{and} \quad f_1[M] \subseteq g_1[M],$$

because $f[M] \subseteq g[M]$ and M contains only real numbers. Note that f_0 or f_1 is not constant. Without loss of generality we assume that f_1 is not constant. Since $f_1[M] \subseteq g_1[M]$, g_1 is also not constant. So, we have that $f_1 = g_1$ because M is a magic set for $\mathcal{P}_n$. If f_0 is also not constant, it follows that $f_0 = g_0$ and therefore $f = g$. So, assume that f_0 is constantly equal to $c \in \mathbb{R}$. By Fact 25 there are $m_1, m_2, \dots, m_{n+1} \in M$ such that $f_1(m_1), f_1(m_2), \dots, f_1(m_{n+1})$ are pairwise different. Since $f[M] \subseteq g[M]$ there are pairwise different $m_{i_1}, m_{i_2}, \dots, m_{i_{n+1}} \in M$ such that for $1 \leq k \leq n+1$ we have

$$c + i f_1(m_k) = g_0(m_{i_k}) + i g_1(m_{i_k}) \;\Rightarrow\; f_1(m_k) = g_1(m_{i_k}) \wedge c = g_0(m_{i_k}).$$

So, $g_0(x) - c$ is a polynomial of degree at most n that has at least $n+1$ zeros. This shows that g_0 is constantly equal to c. Therefore we have $f_0 = g_0$ which implies $f = g$. $\square$

Acknowledgment

We would like to thank the referee for his or her careful reading of our manuscript and the comment on the proof of Lemma 26.

References

[1] Alessandro Berarducci and Dikran Dikranjan. Uniformly approachable functions and spaces. In *Proceedings of the Eleventh International Conference of Topology (Trieste, 1993)*, volume 25, pages 23–55 (1994), 1993.

[2] Maxim R. Burke and Krzysztof Ciesielski. Sets on which measurable functions are determined by their range. *Canad. J. Math.*, 49(6):1089–1116, 1997.

[3] Krzysztof Ciesielski and Saharon Shelah. A model with no magic sets. *J. Symbolic Logic*, 64(4):1467–1490, 1999.

[4] Harold G. Diamond, Carl Pomerance, and Lee Rubel. Sets on which an entire function is determined by its range. *Math. Z.*, 176(3):383–398, 1981.

[5] Lorenz Halbeisen, Norbert Hungerbühler, and Salome Schumacher. Sets and multisets of range uniqueness for polynomials. *Linear Algebra Appl.*, 589:39–61, 2020.

[6] Lorenz Halbeisen, Marc Lischka, and Salome Schumacher. Magic sets. *Real Anal. Exchange*, 43(1):187–204, 2018.