# ON THE CARDINALITY OF SMALLEST SPANNING SETS OF RINGS

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ABSTRACT

Let $\boldsymbol{R} = (R, +, \cdot\,)$ be a ring. Then $Z \subseteq R$ is called *spanning* if the $R$-module generated by $Z$ is equal to the ring $\boldsymbol{R}$. A spanning set $Z \subseteq R$ is called *smallest* if there is no spanning set of smaller cardinality than $Z$. It will be shown that the cardinality of a smallest spanning set of a ring $\boldsymbol{R}$ is not always decidable. In particular, a ring $\boldsymbol{R} = (R, +, \cdot\,)$ will be constructed such that the cardinality of a smallest spanning set $Z \subseteq R$ depends on the underlying set theoretic model.

## 0. DEFINITIONS AND TERMINOLOGY

**Set theoretic notations.** For a set $x$ let $|x|$ denotes the **cardinality** of the set $x$, which is the least ordinal number $\alpha$ such that there exists a bijection between $x$ and $\alpha$. The least infinite ordinal number is denoted by $\omega = \{0, 1, 2, \dots\}$. Let $\mathfrak{c} := |\mathcal{P}(\omega)|$, where $\mathcal{P}(\omega)$ denotes the power-set of $\omega$. Further, let $\aleph_0 := |\omega|$ and for $n \in \omega$ let $\aleph_{n+1}$ be the least cardinal number which is strictly greater than $\aleph_n$. In particular, $\aleph_1$ is the least uncountable cardinal number. Recall that the statement $\aleph_1 = \mathfrak{c}$ is equivalent to the Continuum Hypothesis which is not decidable in ZFC, *i.e.*, in Zermelo-Fraenkel Set Theory with the Axiom of Choice.

**Spanning sets.** Let $\boldsymbol{R} = (R, +, \cdot\,)$ be a ring. We say that a set $Z \subseteq R$ is a **spanning set** of $\boldsymbol{R}$, if for all $r \in R$ there is an $n \in \omega$ and $z_0, \dots, z_{n-1} \in Z$ such that $r = \sum_{i=0}^{n-1} r_i\, z_i$ (for some $r_i \in R$). As usual, for $n = 0$ let $\sum_{i=0}^{n-1} r_i\, z_i := \mathbf{0}$, where $\mathbf{0} \in R$ denotes the neutral element with respect to addition.

In other words, a set $Z \subseteq R$ is a spanning set of $\boldsymbol{R}$ if the left $R$-module generated by $Z$ is equal to the ring $\boldsymbol{R}$.

A spanning set $Z \subseteq R$ of $\boldsymbol{R}$ is called **smallest** if there is no spanning set of $\boldsymbol{R}$ of smaller cardinality than $Z$.

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**The results.** It will be shown that the cardinality of a smallest spanning set of a ring $\boldsymbol{R}$ is not always decidable. In other words, there is a ring $\boldsymbol{R} = (R, +, \cdot)$ such that the cardinality of a smallest spanning set $Z \subseteq R$ of $\boldsymbol{R}$ depends on the underlying set theoretic model. In particular, we will construct a ring $\boldsymbol{R} = (R, +, \cdot)$ and show that for a smallest spanning set $Z \subseteq R$ there are models of $\mathsf{ZFC}$ in which we can have the following: $\aleph_1 = |Z| = \mathfrak{c}$, $\aleph_1 = |Z| < \mathfrak{c}$, $\aleph_1 < |Z| = \mathfrak{c}$, and $\aleph_1 < |Z| < \mathfrak{c}$.

## 1. The Ring $\boldsymbol{R}_{\mathcal{U}}$

Let $[\omega]^\omega$ be the set of all infinite subsets of $\omega$. In the sequel, let $\mathcal{U} \subseteq [\omega]^\omega$ be an arbitrary but fixed non-principal ultrafilter over $\omega$, *i.e.*, for any $x, y \in \mathcal{U}$, $x \cap y \in \mathcal{U}$; if $x \in \mathcal{U}$ and $x \subseteq y$, then $y \in \mathcal{U}$; for all $x \in [\omega]^\omega$, either $x \in \mathcal{U}$ or $\omega \setminus x \in \mathcal{U}$. Let ${}^\omega\omega$ be the set of all functions from $\omega$ to $\omega$. On ${}^\omega\omega$ define the following equivalence relation:

$$f \overset{\mathcal{U}}{\sim} g \iff \exists x \in \mathcal{U}\, \forall n \in x \big(f(n) = g(n)\big)\,.$$

For $f \in {}^\omega\omega$ let $[f] := \big\{f' \in {}^\omega\omega : f' \overset{\mathcal{U}}{\sim} f\big\}$; and for $f, g \in {}^\omega\omega$ let

$$f <_{\mathcal{U}} g \iff \exists x \in \mathcal{U}\, \forall n \in x \big(f(n) < g(n)\big)\,,$$

and let

$$f \leq_{\mathcal{U}} g \iff [f] = [g] \text{ or } f <_{\mathcal{U}} g\,.$$

Notice that the relation '$\leq_{\mathcal{U}}$' is a total order on the equivalence classes of ${}^\omega\omega$. Indeed, for any $f, g \in {}^\omega\omega$ consider the partition of $\omega$ into the three pairwise disjoint sets $\big\{n : f(n) < g(n)\big\}$, $\big\{n : f(n) = g(n)\big\}$ and $\big\{n : f(n) > g(n)\big\}$. Since $\mathcal{U}$ is an ultrafilter, exactly one of these sets belongs to $\mathcal{U}$, and therefore $f <_{\mathcal{U}} g$ *or* $[f] = [g]$ *or* $g <_{\mathcal{U}} f$. So, for $f, g \in {}^\omega\omega$ let us define the operation '$*$' as follows:

$$f * g = \begin{cases} f & \text{if } f \leq_{\mathcal{U}} g, \\ g & \text{otherwise.} \end{cases}$$

Now, we define the ring $\boldsymbol{R}_{\mathcal{U}} = (R, +, \cdot)$ as follows: Let $\boldsymbol{S} = (S, +, \cdot)$ be any unital ring. The set $R$ consists of all elements of the form $\sum_{i=0}^{n-1} a_i\, x_{[f_i]}$, where $n \in \omega$ and for all $0 \leq i \leq n-1$: $a_i \in S$ and $f_i \in {}^\omega\omega$. Further, for $a\, x_{[f]}$, $a'\, x_{[f]}$, $b\, x_{[g]} \in R$, where $[f] \neq [g]$, let

$$\begin{aligned} a\, x_{[f]} + a'\, x_{[f]} &= (a + a')\, x_{[f]}\,, \\ a\, x_{[f]} \cdot a'\, x_{[f]} &= (a \cdot a')\, x_{[f]}\,, \\ a\, x_{[f]} + b\, x_{[g]} &= a\, x_{[f]} + b\, x_{[g]}\,, \\ a\, x_{[f]} \cdot b\, x_{[g]} &= (a \cdot b) x_{[f * g]}\,. \end{aligned}$$

Before we can start investigating the cardinality of smallest spanning sets of $\boldsymbol{R}_{\mathcal{U}}$, we have to give some facts about bounding and dominating numbers.

## 2. Bounding and dominating numbers

Let $\mathcal{F} \subseteq [\omega]^\omega$ be an arbitrary filter over $\omega$ which contains all co-finite sets. With respect to $\mathcal{F}$ define the cardinal numbers $\mathfrak{b}_{\mathcal{F}}$ and $\mathfrak{d}_{\mathcal{F}}$ as follows:

$$\begin{aligned} \mathfrak{b}_{\mathcal{F}} &= \min \big\{|F| : F \subseteq {}^\omega\omega \text{ and } \forall g \in {}^\omega\omega\, \exists f \in F (f \not<_{\mathcal{F}} g)\big\}\,, \\ \mathfrak{d}_{\mathcal{F}} &= \min \big\{|F| : F \subseteq {}^\omega\omega \text{ and } \forall g \in {}^\omega\omega\, \exists f \in F (g \leq_{\mathcal{F}} f)\big\}\,. \end{aligned}$$

Notice that if $\mathcal{F}$ is not an ultrafilter, then the relation '$<_{\mathcal{F}}$' is just a partial order on ${}^{\omega}\omega$. If $\mathcal{F}$ is the filter of all co-finite sets, then we write $\mathfrak{b}$ and $\mathfrak{d}$ instead of $\mathfrak{b}_{\mathcal{F}}$ and $\mathfrak{d}_{\mathcal{F}}$ respectively. It is easy to see that for any filter $\mathcal{F}$ which contains all co-finite sets we have $\mathfrak{b} \leq \mathfrak{b}_{\mathcal{F}} \leq \mathfrak{d}_{\mathcal{F}} \leq \mathfrak{d}$. In particular, if $\mathfrak{b} = \mathfrak{d}$, then $\mathfrak{b} = \mathfrak{b}_{\mathcal{F}}$. Further, it is also easy to see that $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$. On the other hand, by the following fact this is all one can prove in ZFC (see for example [1, Theorem 5.1]).

Fact. Each of the following statements is consistent with ZFC:

(1) $\aleph_1 = \mathfrak{b} = \mathfrak{d} = \mathfrak{c}$
(2) $\aleph_1 = \mathfrak{b} = \mathfrak{d} < \mathfrak{c}$
(3) $\aleph_1 < \mathfrak{b} = \mathfrak{d} = \mathfrak{c}$
(4) $\aleph_1 < \mathfrak{b} = \mathfrak{d} < \mathfrak{c}$

## 3. The Cardinality of Smallest Spanning Sets of $\boldsymbol{R}_{\mathcal{U}}$

Combining the previous observations we get the following.

Theorem. Let $Z \subseteq R$ be a smallest spanning set of the ring $\boldsymbol{R}_{\mathcal{U}} = (R, +, \cdot)$. Then there are models of set theory in which we can have the following:

(1) $\aleph_1 = |Z| = \mathfrak{c}$
(2) $\aleph_1 = |Z| < \mathfrak{c}$
(3) $\aleph_1 < |Z| = \mathfrak{c}$
(4) $\aleph_1 < |Z| < \mathfrak{c}$

*Proof.* Let $Z \subseteq R$ be a smallest spanning set of $\boldsymbol{R}_{\mathcal{U}}$. Then without loss of generality we may assume that $Z = \{1\, x_{[f_\iota]} : \iota < \kappa\}$, where $\kappa$ is a cardinal and 1 is the unit of $\boldsymbol{S}$. Let us first show that $|Z| = \mathfrak{d}_{\mathcal{U}}$:

Take any $g \in {}^{\omega}\omega$ and consider $x_{[g]}$. Since $Z$ is a spanning set of $\boldsymbol{R}_{\mathcal{U}}$, there must be an element $r \in R$ and some $1\, x_{[f]} \in Z$ such that $1\, x_{[g]} = r\, x_{[f]}$, which implies that $g \leq_{\mathcal{U}} f$. Hence, for every $g \in {}^{\omega}\omega$ there must be an element $1\, x_{[f]} \in Z$ with $g \leq_{\mathcal{U}} f$, and therefore, $|Z| \geq \mathfrak{d}_{\mathcal{U}}$.

Now, let $F \subseteq {}^{\omega}\omega$ be such that $|F| = \mathfrak{d}_{\mathcal{U}}$ and $\forall g \in {}^{\omega}\omega\, \exists f \in F (g \leq_{\mathcal{U}} f)$, and let $Z = \{1\, x_{[f]} : f \in F\}$. Take any non-zero $r \in R$. Then $r = \sum_{i=0}^{n-1} a_i\, x_{[g_i]}$, where $n$ is a positive integer. By the definition of $F$, there is a function $f \in F$ such that $g_i \leq_{\mathcal{U}} f$ (for all $0 \leq i \leq n-1$). Thus, $r = r \cdot 1\, x_{[f]}$, which proves that $Z$ is a spanning set of $\boldsymbol{R}_{\mathcal{U}}$.

So, if $Z \subseteq R$ is a smallest spanning set of $\boldsymbol{R}_{\mathcal{U}}$, then $|Z| = \mathfrak{d}_{\mathcal{U}}$. Now, if $\mathfrak{b} = \mathfrak{d}$, then, since $\mathfrak{b} \leq \mathfrak{d}_{\mathcal{U}} \leq \mathfrak{d}$, this implies $\mathfrak{d} = \mathfrak{d}_{\mathcal{U}}$, and as a consequence we get that if $Z \subseteq R$ is a smallest spanning set of $\boldsymbol{R}_{\mathcal{U}}$, then $|Z| = \mathfrak{d}$. Hence, by the fact mentioned before, each of the following statements is consistent with ZFC: $\aleph_1 = |Z| = \mathfrak{c}$, $\aleph_1 = |Z| < \mathfrak{c}$, $\aleph_1 < |Z| = \mathfrak{c}$, and $\aleph_1 < |Z| < \mathfrak{c}$. $\dashv$

## 4. Generating Sets of Rings

There are at least two kinds of sets generating a ring $\boldsymbol{R} = (R, +, \cdot)$ we can think of, namely subsets of $R$ generating $\boldsymbol{R}$ as a *ring* and subsets of $R$ generating $\boldsymbol{R}$ as a *left R-module*, which we called spanning sets. Let us call the former sets just **generating sets**. Further, instead of *smallest* generating or spanning sets, we can look for *minimal* generating or spanning sets (minimal in the sense that the generating / spanning set does not properly contain any other generating / spanning set). So far, we just considered smallest spanning sets of a ring. Thus, let us say a few words about the other cases.

It is quite obvious that each ring contains a smallest generating set as well as a smallest spanning set. On the other hand, for example $\boldsymbol{R}_{\mathcal{U}}$ with $\boldsymbol{S} = \mathbb{Z}$ does not have a minimal spanning set, but it has a minimal generating set, namely $\{1\,x_{[f]} : f \in {}^{\omega}\omega\}$. It is also easy to see that generating sets are also spanning sets, and therefore, the cardinality of a smallest generating set is always greater than or equal to the cardinality of a smallest spanning set. Further, by using the techniques of [2], one can show that any two infinite minimal generating sets must have the same cardinality, namely the cardinality of the ring itself. Moreover, any two infinite minimal spanning sets have the same cardinality:

PROPOSITION. Let $X, Y \subseteq R$ be two infinite minimal spanning sets of some ring $\boldsymbol{R} = (R, +, \cdot\,)$, then $|X| = |Y|$.

*Proof.* For any set $S \subseteq R$, let $\langle S \rangle$ be the linear span of $S$ and let $[S]^{<\omega}$ be the set of all finite subsets of $S$. Since $Y \subseteq R$ is a spanning set, for every $x \in X$ there exists a finite set $I_x \subseteq Y$ such that $x \in \langle I_x \rangle$. So, let us define $\varphi : X \to [Y]^{<\omega}$ such that $x \in \langle \varphi(x) \rangle$. Now, since $X$ is a minimal spanning set of $\boldsymbol{R}$, the function $\varphi$ is finite-to-one, which implies that $|X| \leq \aleph_0 \cdot \big|[Y]^{<\omega}\big|$. Further, since $Y$ is infinite, we have $\aleph_0 \cdot \big|[Y]^{<\omega}\big| = \aleph_0 \cdot |Y| = |Y|$, and therefore we get $|X| \leq |Y|$. Thus, by interchanging $X$ and $Y$, we finally have $|X| = |Y|$. $\dashv$

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