Generalized Pencils of Conics derived from Cubics

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Abstract

Given a cubic K. Then for each point P there is a conic C_P associated to P. The conic C_P is called the *polar conic* of K with respect to the *pole* P. We investigate the situation when two conics C_0 and C_1 are polar conics of K with respect to some poles P_0 and P_1, respectively. First we show that for any point Q on the line P_0P_1, the polar conic C_Q of K with respect to Q belongs to the linear pencil of C_0 and C_1, and vice versa. Then we show that two given conics C_0 and C_1 can always be considered as polar conics of some cubic K with respect to some poles P_0 and P_1. Moreover, we show that P_1 is determined by P_0, but neither the cubic nor the point P_0 is determined by the conics C_0 and C_1.

1 Terminology

We will work in the real projective plane $\mathbb{RP}^2 = \mathbb{R}^3\setminus\{0\}/\sim$, where $X \sim Y \in \mathbb{R}^3\setminus\{0\}$ are equivalent, if $X = \lambda Y$ for some $\lambda \in \mathbb{R}$. Points $X = (x_1, x_2, x_3)^T \in \mathbb{R}^3\setminus\{0\}$ will be denoted by capital letters, the components with the corresponding small letter, and the equivalence class by $[X]$. However, since we mostly work with representatives, we often omit the square brackets in the notation.

Let f be a non-constant homogeneous polynomial in the variables x_1, x_2, x_3 of degree n. Then f defines a projective algebraic curve

$$C_f := \{[X] \in \mathbb{RP}^2 \mid f(X) = 0\}$$

of degree n. For a point $P \in \mathbb{RP}^2$,

$$Pf(X) := \langle P, \nabla f(X)\rangle$$

is also a homogeneous polynomial in the variables x_1, x_2, x_3. If the homogeneous polynomial f is of degree n, then C_{Pf} is an algebraic curve of degree $n-1$. The curve C_{Pf} is called the *polar curve* of C_f with respect to the *pole* P; sometimes we call it the *polar curve* of P with respect to C_f. In particular, when C_f is a cubic curve (*i.e.*, f is a homogeneous polynomial of degree 3), then C_{Pf} is a conic, which we call the *polar conic* of C_f with respect to the *pole* P, and when C_f is a conic, then C_{Pf} is a line, which we call the *polar line* of C_f with respect to the *pole* P (see, for example, Wieleitner [16]). By construction, the intersections of a curve C_f and its polar curve C_{Pf} with respect to a point P give the points of contact of the tangents from P to C_f, as well as points on C_f where $\nabla f = 0$ (see Examples 3 and 4). The geometric interpretation of poles and polar lines (or polar surface in higher dimensions) goes back to Monge, who introduced them in 1795 (see [15, §3]). The names pole and polar curve (or polar surface) were coined by Bobillier (see [1–5]) who also iterated the construction and considered higher polar curves (polar curves of polar curves). Grassmann then developed the theory of the poles using cutting methods (see [9–11], and Cremona [7, p. 61]). However, the analytical method generally used today — which we follow here — is due to Joachimsthal (see [14, p. 373]). Note that C_{Pf} is defined and can be a regular curve even if C_f is singular or reducible. We will therefore not impose any further conditions on f in the following.

A regular, symmetric matrix

$$A := \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

with eigenvalues of both signs defines a bilinear form $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, (X,Y) \mapsto \langle Y, AX \rangle$. The corresponding quadratic form $f(X) = \langle X, AX \rangle$ is homogeneous of degree 2, and it is convenient to identify the matrix A or its projective equivalence class with the conic C_f. Then, a point $[X]$ is on the polar line of C_f with respect to the pole $[Y]$ if and only if $\langle Y, AX \rangle = 0$. It follows immediately that a point $[X]$ is on the polar line of C_f with respect to $[Y]$, if and only if $[Y]$ is on the polar line of C_f with respect to $[X]$. Moreover, a line $[g]$ given by the equation $\langle g, Y \rangle = 0$ is the polar line of C_f with respect to the pole $[X] = [A^{-1}g]$.

For a conic C_0 represented by a matrix A_0, the map $\varphi_{C_0} : \mathbb{RP}^2 \to \mathbb{RP}^2$, $[X] \mapsto [A_0 X]$, which associates the pole $[X]$ to its polar line $[A_0 X]$, is called a *polarity*. Vice versa, for a conic C_1 represented by a matrix A_1, the map $\varphi_{C_1} : [Y] \mapsto [A_1^{-1} Y]$, which associates to the polar line $[Y]$ its pole $[A_1^{-1} Y]$, is also called a polarity. The composition of the two polarities $\varphi_{C_1 C_0} : [X] \mapsto [A_1^{-1} A_0 X]$ is a projective map associated to the pair C_0, C_1 of conics. More generally, a cubic f defines a polarity $\mathbb{RP}^2 \to \mathbb{RP}^5$ by associating the point $P \in \mathbb{RP}^2$ to C_{Pf} interpreted as an element of the projective space $\mathbb{RP}^5$ of conics in $\mathbb{RP}^2$. This point of view can be considered as a guiding concept in the following. It may also open the door to further research questions. For example, one may ask which projective maps from $\mathbb{RP}^2$ to $\mathbb{RP}^5$ can be realized in this way.

Let now f be a homogeneous polynomial of degree $n > 2$, and let C_{Pf} be the polar curve of C_f with respect to a point P. Moreover, let C_{QPf} be the polar curve of C_{Pf} with respect to a point Q. Then we have

$$C_{QPf} = \big\{[X] \mid \langle P, Hf(X)Q\rangle = 0\big\},$$

where $Hf := \big(\frac{\partial f^2}{\partial x_i \partial x_j}\big)_{ij}$ is the Hessian of f. If C_{Qf} denotes the polar curve of C_f with respect to Q and C_{PQf} is the polar curve of C_{Qf} with respect to P, then, obviously,

$$C_{PQf} = C_{QPf}. \tag{1}$$

For two given conics C_0 and C_1, represented as matrices A_0 and A_1 as indicated above, the *linear pencil* of C_0 and C_1 is defined as the set of conics represented by the linear pencil of matrices

$$A_{\lambda,\mu} = \lambda A_0 + \mu A_1 \quad \text{where } \lambda, \mu \in \mathbb{R}, (\lambda, \mu) \neq (0, 0).$$

In the next section we will find for a fixed pair of conics C_0, C_1 points P_0, P_1 and a cubic E such that C_i is the polar conic of E with respect to P_i, and each conic in the linear pencil of C_0, C_1 is the polar conic of E with respect to a point on the line through P_0 and P_1.

2 Conics as Polar Conics of Cubics

We investigate now the situation when two conics C_{Pf} and C_{Qf} are polar conics of some cubic C_f with respect to some poles P and Q, respectively. First we show that for any point R on the line PQ, the polar conic C_{Rf} of C_f with respect to R belongs to the linear pencil of C_{Pf} and C_{Qf}, and vice versa (see Fact 1). A necessary condition for $C_0 = C_{Pf}$ and $C_1 = C_{Qf}$ is, as we have seen in (1), that the polar line of C_0 with respect to Q coincides with the polar line of C_1 with respect to P. A general solution to this problem is given in Proposition 3. Finally, we show how to construct a cubic C_f and two points P and Q, such that C_0 and C_1 are the polar conics of C_f with respect to P and Q, respectively (see Theorem 5).

Fact 1. *Let C_f be a cubic, and let P and Q be two distinct points. Furthermore, let C_{Pf} and C_{Qf} be the polar conics of C_f with respect to P and Q, respectively. Then every conic in the linear pencil of C_{Pf} and C_{Qf} is the polar conic of C_f with a pole on PQ; and vice versa, for every point R on PQ, the polar conic of C_f with respect to R is a conic in the linear pencil of C_{Pf} and C_{Qf}.*

Proof. Note that for any R on the line PQ, there exist $\lambda, \mu \in \mathbb{R}$ such that $R = \lambda P + \mu Q$. Hence, C_{Rf} is given by the equation

$$\langle R, \nabla f(X)\rangle = \lambda\langle P, \nabla f(X)\rangle + \mu\langle Q, \nabla f(X)\rangle = 0,$$

which shows that C_{Rf} belongs to the linear pencil of C_{Pf} and C_{Qf}. On the other hand, the conic in the linear pencil of C_{Pf} and C_{Qf} with this equation is the polar conic of C_f with respect to the pole $R = \lambda P + \mu Q$ on the line PQ. *q.e.d.*

So, in the case when two given conics C_0, C_1 are polar conics of a cubic C_f with respect to two points P, Q, we can interpret the linear pencil of C_0, C_1 in a new way: Namely as the polar conics of C_f with respect to points on the straight line joining P, Q. We will see in Theorem 5, that it is indeed always possible to interpret two conics C_0, C_1 as polar conics of a cubic C_f with respect to two points P, Q. Therefore, by Fact 1, we can generalize the notion of the pencil of two conics C_0, C_1 in the following way:

Definition 2. Let C_f be a cubic, let P and Q be two distinct points, and let C_{Pf} and C_{Qf} be the polar conics of C_f with respect to P and Q, respectively. Furthermore, let Γ be a curve which contains P and Q. Then the set of conics

$$\left\{C_{Rf} : R \in \Gamma\right\}$$

is the Γ-*pencil* of C_{Pf} and C_{Qf} with respect to C_f.

Hence, by Fact 1, if Γ is the straight line joining P and Q, then the Γ-pencil coincides with the linear pencil. However, if Γ is not a straight line, then the Γ-pencil shows, depending on the curve Γ, a very rich geometry which can be quite different from that of the linear pencil. Below, two examples of Γ-pencils are given where Γ is not a straight line.

Example 1. Figure 1 shows the Γ-pencil of the two hyperbolas $3x^2 - y^2 - 2y + 3 = 0$ and $3x^2 - y^2 + 2y + 3 = 0$ with respect to the cubic

$$x^3 + 3x^2 - y^2 + 1 \;=\; 0\,,$$

where $P_0 = (0, 1)$, $P_1 = (0, -1)$, and Γ is the circle $x^2 + y^2 = 1$.

Example 2. Figure 2 shows the Γ-pencil of the two circles $x^2 + y^2 = 1$ and $x^2 - 4x + y^2 = \frac{561}{100}$ with respect to the cubic

$$\frac{461x^3}{600} + x^2 + y^2 + \frac{461xy^2}{200} - \frac{1}{3} \;=\; 0\,,$$

where $P_0 = (0, 0)$, $P_1 = (-\frac{200}{561}, 0)$, and Γ is the ellipse

$$\frac{314721\,x^2}{10000} + \frac{561\,x}{50} + \frac{314721\,y^2}{6400} \;=\; 0\,.$$

Remark 1. There is also another type of pencils of conics, called *exponential pencils* (introduced and investigated in [12]). It would be interesting to study the relation between Γ-pencils and exponential pencils.

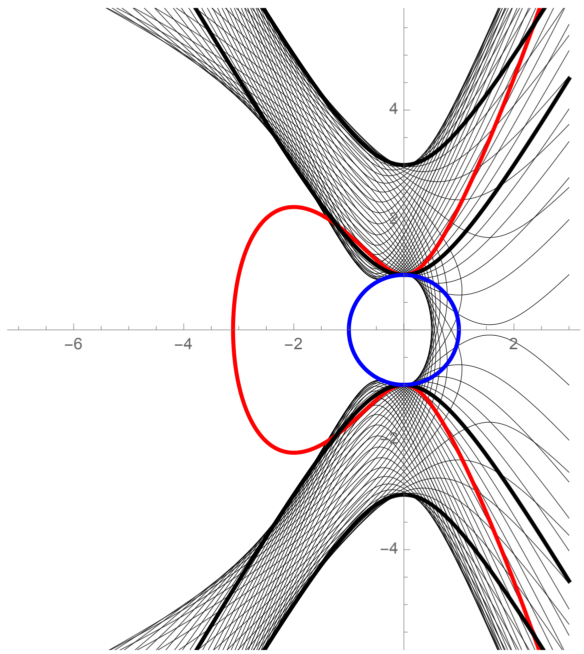

Figure 1: The Γ-pencil (thin black lines) of the black hyperbolas with respect to the red cubic curve. Γ is the blue circle joining $P_0 = (0, 1)$ and $P_1 = (0, -1)$.

The next result shows how we can find points P and Q on a given line g, such that for given conics C_0 and C_1, the polar line of P with respect to C_1 is the same as the polar line of Q with respect to C_0.

Proposition 3. *Given two conics C_0 and C_1 and a line g. Then we are in one of the following cases:*

(a) *There is exactly one pair of points P_0 and P_1 on g, such that the polar line of P_0 with respect to C_0 is the same as the polar line of P_1 with respect to C_1.*

(b) *For any $P_0 \in g$, there exists a unique P_1 on g such that the polar lines of C_0 with respect to P_0 and of C_1 with respect to P_1 coincide.*

In both cases, $P_1 = \varphi_{C_1C_0}(P_0)$ is the image of P_0 under the composition of the polarities associated to C_0 and C_1.

Proof. Let A_0 and A_1 be the matrices corresponding to the conics C_0 and C_1. Let P_0 be a point on the given line g, i.e., $\langle P_0, g\rangle = 0$. The polar line of C_0 with respect to P_0 is given by $\langle X, A_0P_0\rangle = 0$. The pole of this line with respect to C_1 is $A_1^{-1}A_0P_0$. We consider the projective map $\varphi_{C_1C_0} : P_0 \mapsto A_1^{-1}A_0P_0$ which is the composition of the two polarities induced by the conics C_0 and C_1. The image of g under $\varphi_{C_1C_0}$ is the line $\langle X, A_1A_0^{-1}g\rangle = 0$.

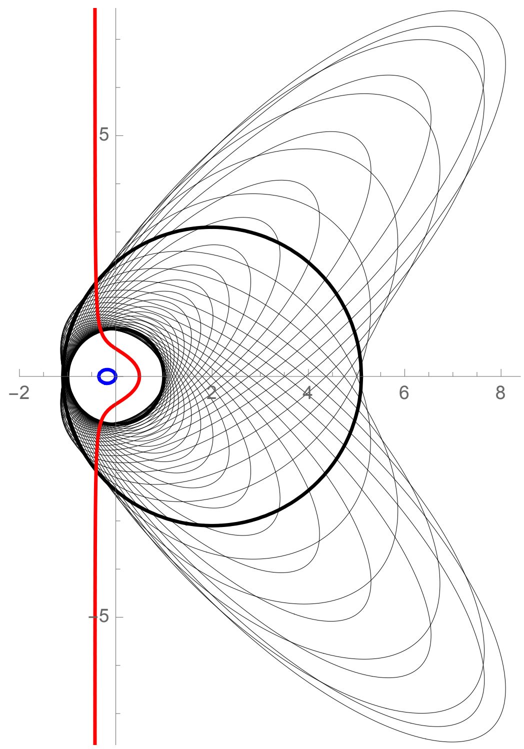

Figure 2: The Γ-pencil (thin black lines) of the black circles with respect to the red cubic curve. Γ is the blue ellips joining $P_0 = (0,0)$ and $P_1 = (-\frac{200}{561}, 0)$.

Suppose first that g is not an eigenvector of $A_1A_0^{-1}$. We want to show that points P_0 and P_1 exist on g such that $P_1 = \varphi_{C_1C_0}P_0$. Necessarily, P_1 is the intersection of g and $\varphi_{C_1C_0}(g)$, i.e., $P_1 = g \times A_1A_0^{-1}g \neq 0$, and then $P_0 = A_0^{-1}A_1P_1 = g \times A_0A_1^{-1}g$.

The second case occurs if g is an eigenvector of $A_1A_0^{-1}$, *i.e.*, if the poles of g with respect to C_0 and C_1 coincide: Then, g and $\varphi_{C_1C_0}(g)$ coincide. Hence one can choose any point P_0 on g, and $P_1 = A_1^{-1}A_0P_0$ is the corresponding point on g such that the polar lines of C_0 with respect to P_0 and of C_1 with respect P_1 agree. *q.e.d.*

Remark 2. In the previous proposition we could also fix the line g together with a projective map $\psi : \mathbb{RP}^2 \to \mathbb{RP}^2$ and ask the following question: Are there two conics C_0, C_1 and two points $P_0, P_1 \in g$ with $P_1 = \psi(P_0)$ such that the polar lines of P_i with respect to C_i coincide? This is indeed the case, since every projective map ψ can be written as the composition of two polarities (see, e.g., [8, Theorem 1.1.9]).

To motivate the main result of this section (which is Theorem 5), let us consider the following problem: Take two lines g_0, g_1 and two points P_0, P_1 in the projective plane. Is there a conic C such that g_i is the polar line of P_i with respect to C? Recall that by von Staudt's Theorem any pair of Desargues triangles are polar triangles in a certain polarity (see, e.g., [6, Section 5.7]). Hence, there must be many solutions in case of only two prescribed points and two prescribed lines. The interesting feature is, that these solutions form a pencil:

Proposition 4. *Let P_0, P_1 be two different points and g_0, g_1 two different lines in $\mathbb{RP}^2$, both points not incident with the lines. Then there is a linear pencil of real symmetric 3×3 matrices $A_0+\lambda A_1$, $\lambda\in\mathbb{R}$, such that the corresponding conics C_λ and only those, have the property that g_i is the polar line of P_i with respect to C_λ. Moreover, if P is a point on the line through P_0, P_1, then there is a line g in the linear pencil of g_0, g_1, such that for all λ, the polar line of P with respect to C_λ is g.*

Proof. By a suitable projective map we may assume without loss of generality that $P_0=(1,0,0)^T$ and $P_1=(0,0,1)^T$. Then, $g_0=(g_{01},g_{02},1)^T$ and $g_{01}\neq 0$ since P_0 is not incident with g_0 and g_1, and $g_1=(1,g_{12},g_{13})^T$ and $g_{13}\neq 0$ since P_1 is not incident with g_0 and g_1. The matrix A of a conic C with the property that g_i is the polar line of P_i with respect to C must then satisfy $AP_0=g_0$ and $AP_1=\mu g_1$ for some $\mu\neq 0$. Hence

$$A_\lambda=\begin{pmatrix} g_{01} & g_{02} & 1\\ g_{02} & 0 & g_{12}\\ 1 & g_{12} & g_{13}\end{pmatrix}+\lambda\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}.$$

$\det(A_\lambda)$ cannot vanish identically in λ since g_0 and g_1 are not the same line, therefore $\det(A_\lambda)=0$ for at most one value $\lambda=\lambda_0$. With the criterion of Hurwitz it follows that A_λ has eigenvalues of both signs for $g_{01}\lambda<g_{02}^2$ and is regular for $\lambda\neq\lambda_0$. Hence A_λ corresponds to a real, nondegenerate conic C_λ. The fact that the polar line of a point P on P_0P_1 with respect to C_λ is independent of λ follows now by a simple calculation. *q.e.d.*

It is now natural to ask whether two conics can always be considered as polar conics of a cubic with respect to two poles, and if so, to what extent the cubic and the poles are determined by the conics. The following theorem gives a complete answer to these questions.

Theorem 5. *Let C_0 and C_1 be any two different conics given by matrices A_0 and A_1, respectively. Then there are infinitely many pairs of points P_0, P_1, where $P_1=\varphi_{C_1C_0}(P_0)$ is the image of P_0 under the composition of the polarities associated to C_0, C_1, and there is a linear pencil of cubics given by $F_\lambda(x,y,z)=f_1(x,y,z)+\lambda f_2(x,y,z)$, $\lambda\in\mathbb{R}$, such that C_i are the polar conics of C_{F_λ} with respect to P_i.*

Proof. Given two conics C_0 and C_1. We have to find a cubic C_F and two points P_0 and P_1, such that C_0 and C_1 are the polar conics of C_F with respect to P_0 and P_1, respectively. It is convenient to consider the embedding of the affine plane $\mathbb{R}^2$ in $\mathbb{RP}^2$ given by

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}\mapsto\left[\begin{pmatrix}x_1\\x_2\\1\end{pmatrix}\right].$$

Depending on the position of C_0 and C_1 we may apply a suitable projective transformation, such that a standard situation results (see [13]):

Case A: Suppose that C_0 and C_1 have one of the following properties:

- four intersections
- no common point or two intersections
- two intersections and one first order contact
- one first order contact
- two first order contacts
- one third order contact

In these cases, we may assume that C_0 is the unit circle given by the matrix

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and that C_1 is given by

$$A_1 = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & \beta & 0 \\ \alpha & 0 & \gamma \end{pmatrix}.$$

As we have seen in the introduction, the polar line of C_0 with respect to P_1 and the polar line of C_1 with respect to P_0 must agree. Hence $[A_0\, P_1] = [A_1\, P_0]$, or equivalently $[P_1] = [A_0^{-1}\, A_1\, P_0]$. Let us first consider the case where P_0 is on the symmetry axis of C_0 and C_1, *i.e.*, $P_0 = (x_0, 0, 1)$. In this case we obtain $P_1 = (x_0 + \alpha, 0, -x_0\alpha - \gamma)$.

It is from now on a bit more convenient to write x, y, z instead of x_1, x_2, x_3. The cubic curve C_F we are looking for is given by a homogeneous polynonial F of degree 3:

$$\begin{aligned} F(x,y,z) = \;& a_1\, x^3 + a_2\, y^3 + a_3\, z^3 + a_4\, x^2 y + a_5\, x^2 z + \\ & a_6\, xy^2 + a_7\, y^2 z + a_8\, xz^2 + a_9\, yz^2 + a_{10}\, xyz. \end{aligned} \quad (2)$$

We need that $C_{P_0F} = A_0$ and $C_{P_1F} = A_1$, where $P_0F(X) = \langle P_0, \nabla F(X)\rangle$, and $P_1F(X) = \langle P_1, \nabla F(X)\rangle$. The quadratic forms of $P_0F(X)$ and $P_1F(X)$ are given by

$$\begin{pmatrix} 3a_1x_0 + a_5 & a_4x_0 + \frac{a_{10}}{2} & a_5x_0 + a_8 \\ a_4x_0 + \frac{a_{10}}{2} & a_6x_0 + a_7 & a_9 + \frac{a_{10}x_0}{2} \\ a_5x_0 + a_8 & a_9 + \frac{a_{10}x_0}{2} & 3a_3 + a_8x_0 \end{pmatrix}$$

and

$$\begin{pmatrix} 3a_1p + a_5q & a_4p + \frac{1}{2}a_{10}q & a_5p + a_8q \\ a_4p + \frac{1}{2}a_{10}q & a_6p + a_7q & a_9q + \frac{1}{2}a_{10}p \\ a_5p + a_8q & a_9q + \frac{1}{2}a_{10}p & 3a_3q + a_8p \end{pmatrix}$$

where $p := x_0 + \alpha$ and $q := -\alpha x_0 - \gamma$. The first of these two matrices has to be a multiple of A_0, the second a multiple of A_1. If we solve the resulting linear system of equations, we find: If $\alpha(1 + x_0^2) + x_0(1 + \gamma) \neq 0$ then

$$F(x, y, z) = f_1(x, y, z) + \lambda f_2(x, y, z)$$

is the linear pencil spanned by

$$\begin{aligned} f_1(x, y, z) &= (1 + x_0\alpha + \gamma)x^3 - (\alpha + x_0(1 + \gamma))z^3 + 3\alpha x^2 z + \\ &\quad + 3(\beta + \gamma + x_0\alpha)xy^2 + 3(x_0 + \alpha - x_0\beta)y^2 z - 3x_0\alpha x z^2 \\ f_2(x, y, z) &= y^3. \end{aligned}$$

Observe, that if $\alpha(1+x_0^2)+x_0(1+\gamma) = 0$ then the Hessian of F vanishes identically, and hence, the cubic curve C_F is reducible, which is precisely the case when $P_0 = P_1$.

Now we consider a general point $P_0 = (x_0, y_0, 1)$ with $y_0 \neq 0$. In this case, we obtain $P_1 = (x_0 + \alpha, y_0\beta, -x_0\alpha - \gamma)$. The matrix of the quadratic form $P_0F(X)$ is given by

$$\begin{pmatrix} 3a_1x_0 + a_4y_0 + a_5 & a_4x_0 + a_6y_0 + \frac{1}{2}a_{10} & a_5x_0 + a_8 + \frac{1}{2}a_{10}y_0 \\ a_4x_0 + a_6y_0 + \frac{1}{2}a_{10} & 3a_2y_0 + a_6x_0 + a_7 & a_7y_0 + a_9 + \frac{1}{2}a_{10}x_0 \\ a_5x_0 + a_8 + \frac{1}{2}a_{10}y_0 & a_7y_0 + a_9 + \frac{1}{2}a_{10}x_0 & 3a_3 + a_8x_0 + a_9y_0 \end{pmatrix}$$

and the matrix for $P_1F(X)$ can now be written as

$$\begin{pmatrix} 3a_1p + a_4r + a_5q & a_4p + a_6r + \frac{1}{2}a_{10}q & a_5p + a_8q + \frac{1}{2}a_{10}r \\ a_4p + a_6r + \frac{1}{2}a_{10}q & 3a_2r + a_6p + a_7q & a_7r + a_9q + \frac{1}{2}a_{10}p \\ a_5p + a_8q + \frac{1}{2}a_{10}r & a_7r + a_9q + \frac{1}{2}a_{10}p & 3a_3q + a_8p + a_9r \end{pmatrix}$$

with $p := x_0+\alpha$, $q := -x_0\alpha-\gamma$, as above, and $r := y_0\beta$. If $\alpha(1+x_0^2)+x_0(1+\gamma) \neq 0$, we find the following solution of the resulting linear system:

$$F(x, y, z) = f_1(x, y, z) + \lambda f_2(x, y, z)$$

where

$$\begin{aligned} f_1(x, y, z) &= (1 + \alpha x_0 + \gamma)x^3 + \tfrac{1}{y_0}(\alpha(1 + x_0^2) + x_0(1 + \gamma))y^3 \\ &\qquad - (x_0(1 + \gamma) + \alpha)z^3 + 3\alpha x^2 z - 3\alpha x_0 x z^2 \\ f_2(x, y, z) &= \big(x_0y_0(\alpha x + (1 - \beta)z) + y_0(\beta + \gamma)x - (\alpha(1 + x_0^2) + (\gamma + 1)x_0)y + \alpha y_0 z\big)^3. \end{aligned}$$

Case B: Suppose that C_0 and C_1 have one second order contact and one intersection. In this case we may assume that C_0 is again the unit circle, given by the matrix A_0 above, and that C_1 is given by the matrix

$$A_1 = \begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & \nu \\ 0 & \nu & -1 \end{pmatrix}$$

with $\nu \neq 0$ (see [13]). Let $P_0 = (x_0, y_0, 1)$. Then we get this time $P_1 = A_0^{-1} A_1 P_0 = (x_0 - y_0\nu, y_0 + \nu(1-x_0), 1 - y_0\nu)$. We make the same general Ansatz for F as above in (2). Then, the quadratic forms $P_0F(X)$ and $P_1F(X)$ are

$$\begin{pmatrix} 3a_1x_0 + a_4y_0 + a_5 & a_4x_0 + a_6y_0 + \frac{a_{10}}{2} & a_5x_0 + a_8 + \frac{a_{10}}{2}y_0 \\ a_4x_0 + a_6y_0 + \frac{a_{10}}{2} & 3a_2y_0 + a_6x_0 + a_7 & a_7y_0 + a_9 + \frac{a_{10}}{2}x_0 \\ a_5x_0 + a_8 + \frac{a_{10}}{2}y_0 & a_7y_0 + a_9 + \frac{a_{10}}{2}x_0 & 3a_3 + a_8x_0 + a_9y_0 \end{pmatrix}$$

and

$$\begin{pmatrix} 3a_1p + a_4r + a_5q & a_4p + a_6r + \frac{1}{2}a_{10}q & a_5p + a_8q + \frac{1}{2}a_{10}r \\ a_4p + a_6r + \frac{1}{2}a_{10}q & 3a_2r + a_6p + a_7q & a_7r + a_9q + \frac{1}{2}a_{10}p \\ a_5p + a_8q + \frac{1}{2}a_{10}r & a_7r + a_9q + \frac{1}{2}a_{10}p & 3a_3q + a_8p + a_9r \end{pmatrix}$$

where $p = x_0 - y_0\nu$, $q = 1 - y_0\nu$, and $r = y_0 + \nu(1-x_0)$. The first of these two matrices has to be a multiple of A_0, the second a multiple of A_1. Solving the linear system of equations yields the following:

If $y_0 = 0$ and $x_0 \neq 1$, then the cubic function F is the linear pencil $F(x,y,z) = f_1(x,y,z) + \lambda f_2(x,y,z)$, spanned by

$$\begin{aligned} f_1(x,y,z) &= (x_0-1)x^2z - y^2z + xy^2 - (x_0-1)x_0xz^2 + \tfrac{1}{3}(x_0-1)^2(x_0+1)z^3 \\ f_2(x,y,z) &= (x - x_0z)^3. \end{aligned}$$

If $y_0 \neq 0$ and $x_0 = 1$, then $F(x,y,z) = f_1(x,y,z) + \lambda f_2(x,y,z)$, with

$$\begin{aligned} f_1(x,y,z) &= y(y_0x^2 - xy + \tfrac{1}{3}y_0y^2 + yz - y_0z^2) \\ f_2(x,y,z) &= (x-z)^3. \end{aligned}$$

If $y_0 \neq 0$ and $x_0 = 1 + y_0^2$, then $F(x,y,z) = f_1(x,y,z) + \lambda f_2(x,y,z)$, with

$$\begin{aligned} f_1(x,y,z) &= (x-z)(y_0x^2 - 3xy + 3y_0y^2 + 3yz + y_0xz - 2y_0z^2) \\ f_2(x,y,z) &= (y - y_0z)^3. \end{aligned}$$

Finally, if $y_0 \neq 0$ and $x_0 \neq 1$ and $x_0 \neq 1 + y_0^2$, then $F(x,y,z) = f_1(x,y,z) + \lambda f_2(x,y,z)$, with

$$\begin{aligned} f_1(x,y,z) &= u(1 + x_0(y_0^2 - 1))x^3 + v^2y_0^3y^3 + \\ &\quad + (v(3x_0^2 - x_0 - v)y_0^2 - v^2x_0^3 - x_0y_0^4)z^3 \\ &\quad - 3uvy_0x^2y - 3x_0u^2x^2z - 3v^3y_0^2y^2z + 3x_0u(y_0^2 - vx_0)z^2x \\ &\quad + 3vy_0(x_0^2v - y_0^2(v + x_0))z^2y + 6uvx_0y_0xyz \\ f_2(x,y,z) &= (ux - y_0vy - wz)^3 \end{aligned}$$

where $u = 1 - x_0 + y_0^2$, $v = x_0 - 1$, and $w = x_0(1-x_0) + y_0^2$. *q.e.d.*

It is remarkable that in *Case B*, the pencil of cubics does not depend on ν.

Observe, that the situation in Proposition 4 and in Theorem 5 is somewhat different in that the point P_1 cannot be chosen independently of P_0 in Theorem 5. However, we have the following common feature:

Proposition 6. *For each point P on the line through P_0 and P_1 in Theorem 5, the polar conic of P with respect to the pencil C_{F_λ} does not depend on λ.*

Proof. For P_0, P_1 we have that the polar conic $\langle P_i, \nabla F_\lambda(X)\rangle = 0$ is independent of λ. This equation written out in full is

$$\langle P_i, \nabla f_1(X)\rangle + \lambda\langle P_i, \nabla f_2(X)\rangle = 0.$$

Direct inspection of all cases in the proof of Theorem 5 shows that $\langle P_i, \nabla f_2(X)\rangle$ vanishes identically in X, and the claim follows. *q.e.d.*

Remark 3. In order to obtain a cubic with respect to two given conics and a pole, we had to solve an over-constrained system of linear equations. Thus, it is somewhat surprising that this system is not just solvable, but has infinitely many solutions, and that the solutions lead to a linear pencil of cubics with only “few” singular or reducible cubics (see also Examples 3 and 4).

We conclude this paper by providing two linear pencils of cubics which belong to two given conics C_0 and C_1 and a point P_0 (see Theorem 5).

Example 3. Figure 3 shows the linear pencil of cubics which belong to the conics

$$C_0 : x^2 + y^2 = 1 \qquad \text{and} \qquad C_1 : x^2 + 4x + 5y^2 + 2 = 0$$

and the points $P_0 = (0,0), P_1 = (-1,0)$: Tangents to the red cubics in points of C_0 meet in P_0, and tangents to the red cubics in points of C_1 meet in P_1. At the intersections of C_0 and C_1 the gradient of the corresponding cubic vanishes. This examples belongs to *Case A* in the proof of Theorem 5 since C_0 and C_1 have two intersections.

Example 4. Figure 4 shows the linear pencil of cubics which belong to the conics

$$C_0 : x^2 + y^2 = 1 \qquad \text{and} \qquad C_1 : x^2 - 4xy + 4y + y^2 = 1$$

and the points $P_0 = (1,-2), P_1 = (1,-\frac{2}{5})$: Tangents to the red cubics in points of C_0 meet in P_0, and tangents to the red cubics in points of C_1 meet in P_1. At the intersections of C_0 and C_1 the gradient of the corresponding cubic vanishes. This examples belongs to *Case B* in the proof of Theorem 5 since C_0 and C_1 have one second order contact and one intersection.

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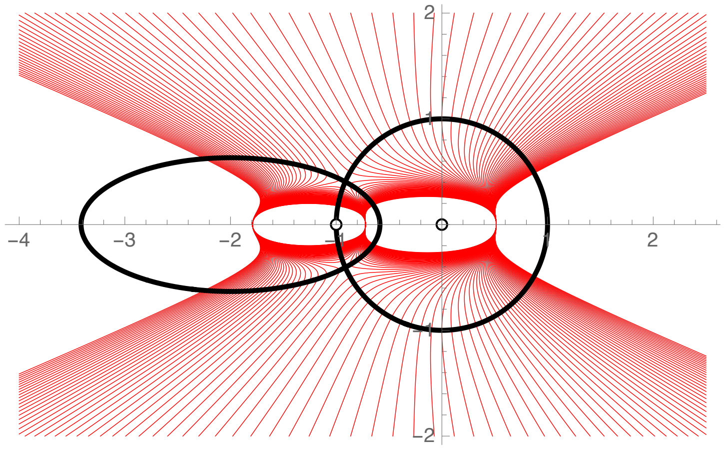

Figure 3: The linear pencil of cubics (thin red lines) which belongs to the two conics (thick black lines), and the points $P_0 = (0, 0), P_1 = (-1, 0)$ (small black circles). In this example, all members of the linear pencil given by Theorem 5 are irreducible cubic curves, and only two curves of the pencil have a singular point, namely a double point at the intersections of C_0 and C_1.

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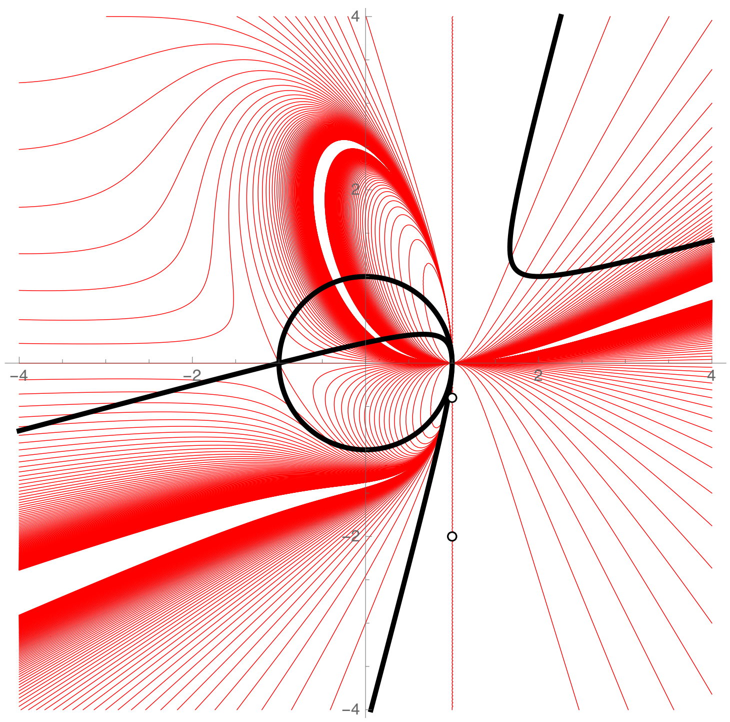

Figure 4: The linear pencil of cubics (thin red lines) which belongs to the two conics (thick black lines), and the points $P_0 = (1, -2), P_1 = (1, -\frac{2}{5})$ (small black circles). In this example, all cubics in the pencil have a singular point in $(1, 0)$. One cubic of the pencil is reducible and decomposes into the line and an ellipse trough the two intersections of C_0 and C_1.

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