A Simple Proof of Poncelet's Theorem (on the occasion of its bicentennial)

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Abstract. We present a proof of Poncelet's Theorem in the real projective plane which relies only on Pascal's Theorem.

1. INTRODUCTION In 1813, while Poncelet was in captivity as a war prisoner in the Russian city of Saratov, he discovered the following theorem.

Theorem 1.1 (Poncelet's Theorem). *Let K and C be nondegenerate conics in general position. Suppose there is an n-sided polygon inscribed in K and circumscribed about C such that none of its vertices belongs to C (in the case when K and C intersect or meet). Further suppose there is an $(n-1)$-sided polygonal chain with vertices on K such that all its sides are tangent to C and none of its vertices belongs to C. Then the side, which closes the polygonal chain, is also tangent to C.*

An immediate consequence is the following more popular version of Poncelet's Theorem.

Corollary 1.2. *Let K and C be nondegenerate conics in general position which neither meet nor intersect. Suppose there is an n-sided polygon inscribed in K and circumscribed about C. Then for any point P of K, there exists an n-sided polygon, also inscribed in K and circumscribed about C, which has P as one of its vertices.*

After his return to France, Poncelet published a proof in his book [**17**], which appeared in 1822. He derived the version displayed above from a more general statement, where the sides of the polygon are tangent to conics C_i from a pencil containing K. He first proved the statement for a pencil of circles, thus generalizing theorems of Chapple [**5**] and Euler [**9**] for triangles and of Fuss [**10**] for bicentric polygons. In order to extend his main theorem from circles to conics, Poncelet then invoked a projection theorem which states that every pair of conics with no more than two intersections can be considered as the projective image of a pair of circles. Poncelet finished by arguing that the case of more than two intersections followed by the "principle of continuity." Poncelet's treatise was a milestone in the development of projective geometry, and his theorem is widely considered the deepest and most beautiful result about conics.

Poncelet's Theorem gained immediately the attention of the mathematical community. Already in 1828, Jacobi gave in [**13**] an analytic proof for pairs of nested circles by using the addition theorem for elliptic functions. In the sequel, Cayley investigated algebraic conditions for two conics to be in Poncelet position. Using the theory of Abelian integrals, he formulated a criterion in [**4**]. Cayley actually published a series of papers dealing with Poncelet's porism. In the early 20th century, Lebesgue revisited Cayley's work and formulated the proof in the language of projective geometry and algebra, see [**15**]. He used an observation by Hart who gave in [**12**] an elegant argument for Poncelet's Theorem for triangles. In recent times, Griffiths and Harris

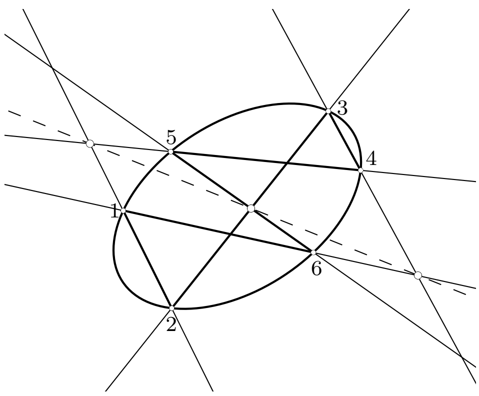
MSC: Primary 51M04, Secondary 51A20; *key words*: Poncelet's Porism, Pascal's Theorem

used Abel's Theorem and the representation of elliptic curves by means of the Weierstrass $\wp$-function to establish the equivalence of Poncelet's Theorem and the group structure on elliptic curves, see [**11**]. Poncelet's Theorem has a surprising mechanical interpretation for elliptic billiards in the language of dynamical systems: see [**8**] or [**7**] for an overview of this facet. A common approach to all four classical closing theorems (the Poncelet porism, Steiner's Theorem, the Zigzag Theorem, and Emch's Theorem) has recently been established by Protasov in [**18**]. King showed in [**14**], that Poncelet's porism is isomorphic to Tarski's plank problem (a problem about geometric set-inclusion) and to Gelfand's question (a number theoretic problem) via the construction of an invariant measure. However, according to Berger [**1**, p. 203], all known proofs of Poncelet's Theorem are rather long and recondite.

The aim of this paper is to give a simple proof of Poncelet's Theorem in the real projective plane. More precisely, we will show that Poncelet's Theorem is a purely combinatorial consequence of Pascal's Theorem. Before we give several forms of the latter, let us introduce some notation. For two points a and b, let $a - b$ denote the line through a and b, and for two lines ℓ_1 and ℓ_2, let $\ell_1 \wedge \ell_2$ denote the intersection point of these lines in the projective plane. In abuse of notation, we often write $a - b - c$ in order to emphasize that the points a, b, c are collinear. In the sequel, points are often labeled with numbers, and lines with encircled numbers like ③.

In this terminology, Pascal's Theorem and its equivalent forms read as follows.

Pascal's Theorem (*cf.* [**16**])

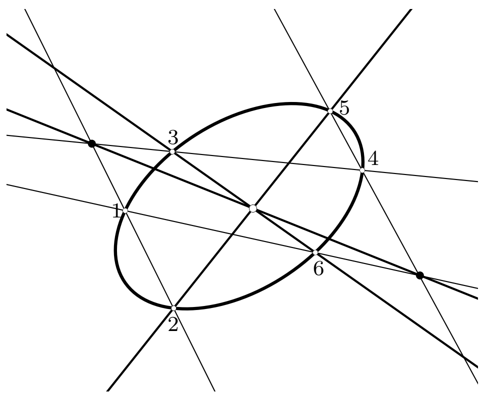


Any six points $1, \ldots, 6$ lie on a conic

if and only if

$(1-2) \wedge (4-5)$
$(2-3) \wedge (5-6)$
$(3-4) \wedge (6-1)$
are collinear.

Carnot's Theorem (*cf.* [**3**, no. 396])

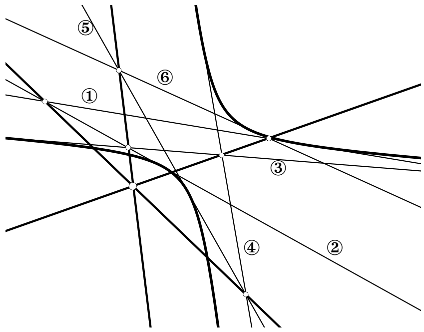


Any six points $1, \ldots, 6$ lie on a conic

if and only if

$[(1-2) \wedge (3-4)] - [(4-5) \wedge (6-1)]$
$(2-5)$
$(3-6)$
are concurrent.

Brianchon's Theorem (*cf.* [**2**])

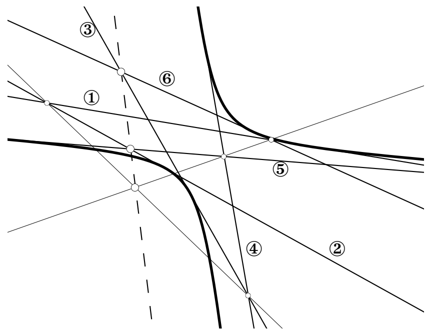


Any six lines ①, . . . , ⑥ are tangent to a conic

if and only if

(① ∧ ②) − (④ ∧ ⑤)
(② ∧ ③) − (⑤ ∧ ⑥)
(③ ∧ ④) − (⑥ ∧ ①)
are concurrent.

Carnot's Theorem*



Any six lines ①, . . . , ⑥ are tangent to a conic

if and only if

[(① ∧ ②) − (③ ∧ ④)] ∧ [(④ ∧ ⑤) − (⑥ ∧ ①)]
(② ∧ ⑤)
(③ ∧ ⑥)
are collinear.

As a matter of fact, we would like to mention that if the conic is not degenerate, then the collinear points in Pascal's Theorem are always pairwise distinct (the same applies to the concurrent lines in Brianchon's Theorem).

Since the real projective plane is self-dual, Pascal's Theorem and Brianchon's Theorem are equivalent. Moreover Carnot's Theorem and its dual Carnot's Theorem* are just reformulations of Pascal's Theorem and Brianchon's Theorem by exchanging the points 3 and 5, and the lines ③ and ⑤, respectively. Recall that if two adjacent points, say 1 and 2, coincide, then the corresponding line $1 - 2$ becomes a tangent with 1 as contact point. Similarly, if two lines, say ① and ②, coincide, then ① ∧ ② becomes the contact point of the tangent ①. As a last remark, we would like to mention that a conic is in general determined by five points, by five tangents, or by a combination like three tangents and two contact points of these tangents.

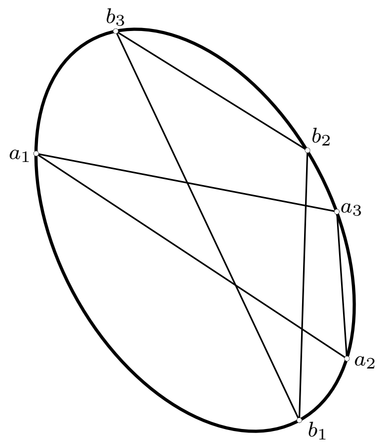
The paper is organized as follows. In Section 2, we prove Poncelet's Theorem for the special case of triangles and at the same time we develop the kind of combinatorial arguments we shall use later. Section 3 contains the crucial tool which allows to show that Poncelet's Theorem holds for an arbitrary number of edges. Finally, in Section 4, we use the developed combinatorial technics in order to prove some additional symmetry properties of Poncelet-polygons.

2. PONCELET'S THEOREM FOR TRIANGLES

In order to prove Poncelet's Theorem for triangles, we will show that if the six vertices of two triangles lie on a conic K, then the six sides of the triangles are tangents to some conic C.

The crucial point in the proof of the following theorem (as well as in the proofs of the other theorems of this paper) is to find the suitable numbering of points and edges, and to apply some form of Pascal's Theorem in order to find collinear points or concurrent lines.

Theorem 2.1. *If two triangles are inscribed in a conic and the two triangles do not have a common vertex, then the six sides of the triangles are tangent to a conic.*

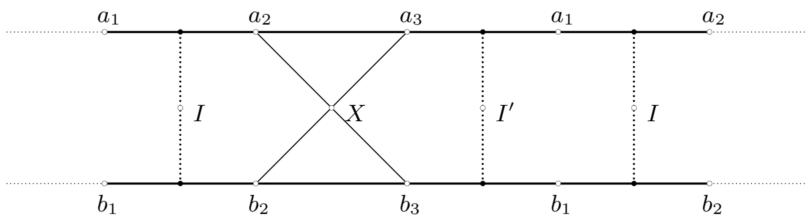


Proof. Let K be a conic in which two triangles $\triangle a_1a_2a_3$ and $\triangle b_1b_2b_3$ are inscribed where the two triangles do not have a common vertex.

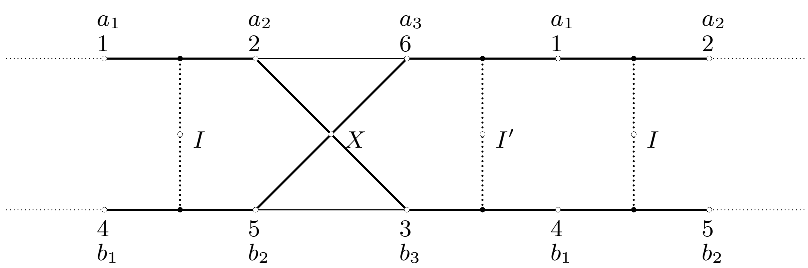
First, we introduce the following three intersection points:

$$\begin{aligned} I &:= (a_1 - a_2) \wedge (b_1 - b_2)\,, \\ X &:= (a_2 - b_3) \wedge (b_2 - a_3)\,, \\ I' &:= (a_3 - a_1) \wedge (b_3 - b_1)\,. \end{aligned}$$

In order to visualize the intersection points I, X, and I', we break up the conic K and draw it as two straight lines, one for each triangle as follows.



Now, we number the six points $a_1, a_2, a_3, b_1, b_2, b_3$ on the conic K as shown by the following figure.

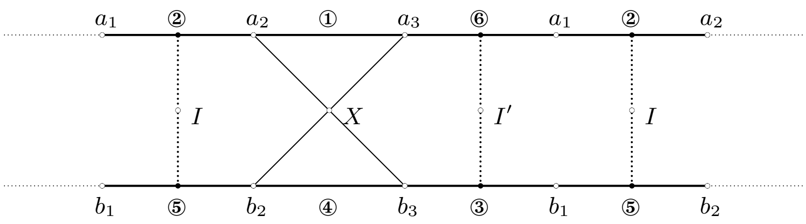


By Pascal's Theorem we get that the three intersection points

$$(1-2)\wedge(4-5)\,,\quad (2-3)\wedge(5-6)\,,\quad \text{and}\quad (3-4)\wedge(6-1)$$

are collinear, which is the same as saying that the points $I - X - I'$ are collinear.

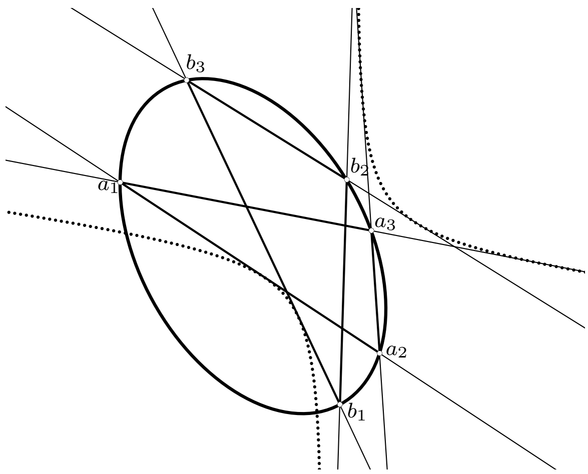
In the next step, we label the sides of the triangles as shown in the following figure.



By Carnot's Theorem* we get that the six sides ①, …, ⑥ of the two triangles are tangents to a conic if and only if

$$\begin{gathered}[(①\wedge②)-(③\wedge④)]\wedge[(④\wedge⑤)-(⑥\wedge①)]\,,\\ (②\wedge⑤)\,,\ \text{and}\\ (③\wedge⑥)\end{gathered}$$

are collinear. Now, this is the same as saying that the points $X - I - I'$ are collinear, which, as we have seen above, is equivalent to $a_1, a_2, a_3, b_1, b_2, b_3$ lying on a conic.



q.e.d.

As an immediate consequence we get Poncelet's Theorem for triangles.

Corollary 2.2 (Poncelet's Theorem for triangles). *Let K and C be nondegenerate conics. Suppose there is a triangle $\triangle a_1a_2a_3$ inscribed in K and circumscribed about C. Then for any point b_1 of K for which two tangents to C exist, there is a triangle $\triangle b_1b_2b_3$ which is also inscribed in K and circumscribed about C.*

Proof. Let K and C be two conics with a triangle $\triangle a_1a_2a_3$ inscribed in K and circumscribed about C. Let b_1 be an arbitrary point on K which is distinct from a_1, a_2, a_3, and let b_2 and b_3 be distinct points on K such that $b_1 - b_2$ and $b_1 - b_3$ are two tangents to C. By construction, we get that five sides of the triangles $\triangle a_1a_2a_3$ and $\triangle b_1b_2b_3$ are tangents to C. On the other hand, by Theorem 2.1, we know that all six sides of these triangles are tangents to some conic C'. Now, since a conic is determined by five tangents, C' and C coincide, which implies that the triangle $\triangle b_1b_2b_3$ is circumscribed about C. *q.e.d.*

As a special case of Brianchon's Theorem we get the following.

Fact 2.3. *Let C be a conic and let the triangle $\triangle a_1a_2a_3$ be circumscribed about C. Furthermore, let t_1, t_2, t_3 be the contact points of the three tangents $a_2 - a_3, a_3 - a_1, a_1 - a_2$. Then the three lines $a_1 - t_1$, $a_2 - t_2$, and $a_3 - t_3$ meet in a point.*

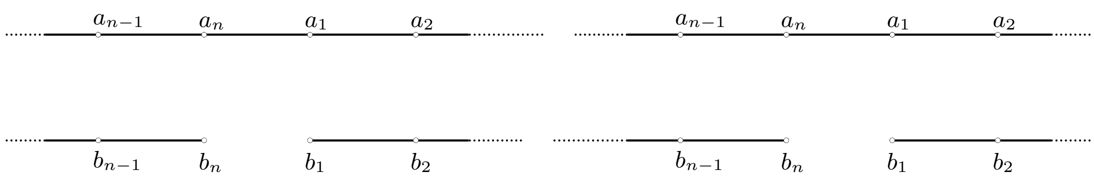
Proof. Label the three sides of the triangles as follows:

$$\textcircled{1} = a_2 - a_3 = \textcircled{2}\,, \qquad \textcircled{3} = a_3 - a_1 = \textcircled{4}\,, \qquad \textcircled{5} = a_1 - a_2 = \textcircled{6}\,.$$

Then $\textcircled{1} \wedge \textcircled{2} = t_1$, $\textcircled{3} \wedge \textcircled{4} = t_2$, $\textcircled{5} \wedge \textcircled{6} = t_3$, and by Brianchon's Theorem we get that $a_1 - t_1, a_2 - t_2, a_3 - t_3$ meet in a point. *q.e.d.*

In general, for arbitrary n-gons tangent to C the analogous statement will be false. However, if n is even and if the n-gon is at the same time inscribed in a conic K, a similar phenomenon occurs (see Theorem 4.2).

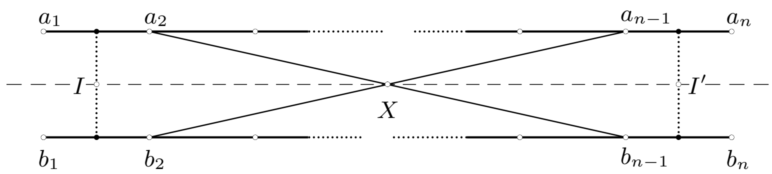
3. THE GENERAL CASE Let K and C be nondegenerate conics in general position. We assume that there is an n-sided polygon $a_1, \ldots, a_n$ which is inscribed in K such that all its n sides $a_1 - a_2, a_2 - a_3, \ldots, a_n - a_1$ are tangent to C and none of its vertices belongs to C. Let us assume that n is minimal with this property (thus, in particular, the points $a_1, \ldots, a_n$ are pairwise distinct). Further, let $b_1, \ldots, b_n$ be an $(n-1)$-sided polygonal chain on K where all $n-1$ sides $b_1 - b_2, b_2 - b_3, \ldots, b_{n-1} - b_n$ are tangent to C and none of its vertices is one of $a_1, \ldots, a_n$ or belongs to C. We do not yet know that $b_n - b_1$ is tangent to C too. If we break up the conic K and draw it as two straight lines, one for the polygon and one for the polygonal chain, we get the following situation.



In order to prove Poncelet's Theorem, we have to show that $b_n - b_1$ is also tangent to C. This will follow easily from the following result.

Lemma 3.1. *For $n \geq 4$, the three intersection points*

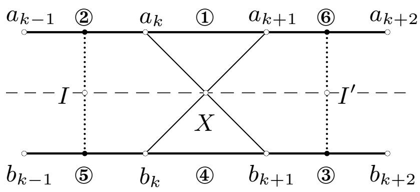
$$\begin{aligned} I \ &:= \ (a_1 - a_2) \wedge (b_1 - b_2)\,, \\ X \ &:= \ (a_2 - b_{n-1}) \wedge (b_2 - a_{n-1})\,, \text{ and} \\ I' \ &:= \ (a_{n-1} - a_n) \wedge (b_{n-1} - b_n)\,, \end{aligned}$$



are pairwise distinct and collinear, which is visualized above by the dashed line.

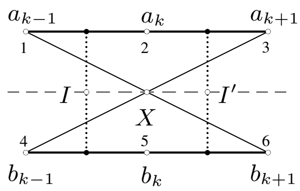
Proof. Depending on the parity of n, we have one of the following anchorings, from which we will work step by step outwards.

n even, with $k = \frac{n}{2}$:



By Carnot's Theorem* we have that $I - X - I'$ are collinear, which proves the lemma for $n = 4$.

n odd, with $k = \frac{n+1}{2}$:



By Pascal's Theorem we have that $I - X - I'$ are collinear.

For $n \geq 5$, the lemma will follow from the following two claims.

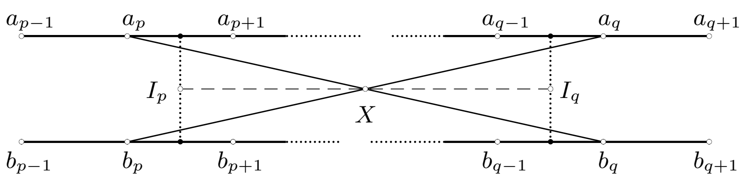
Claim 1. *Let p and q be integers with $2 \leq p < q \leq n-1$. Further, let*

$$I_{p-1} := (a_{p-1} - a_p) \wedge (b_{p-1} - b_p)\,, \qquad I_p := (a_p - a_{p+1}) \wedge (b_p - b_{p+1})\,,$$
$$I_q := (a_{q-1} - a_q) \wedge (b_{q-1} - b_q)\,, \qquad I_{q+1} := (a_q - a_{q+1}) \wedge (b_q - b_{q+1})\,,$$

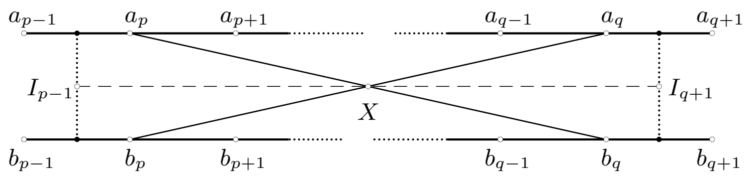
and let

$$X := (a_p - b_q) \wedge (b_p - a_q)\,.$$

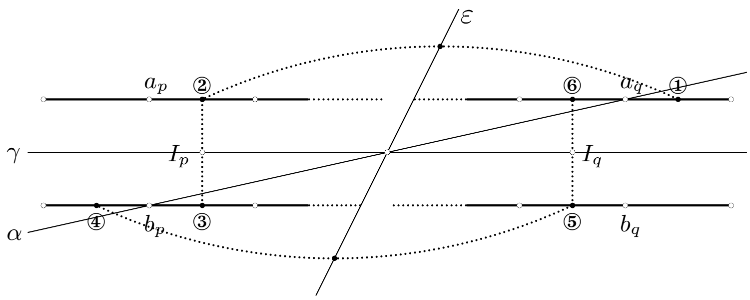
If $I_p - X - I_q$ are pairwise distinct and collinear, then $I_{p-1} - X - I_{q+1}$ are also pairwise distinct and collinear. This implication is visualized by the following figure.



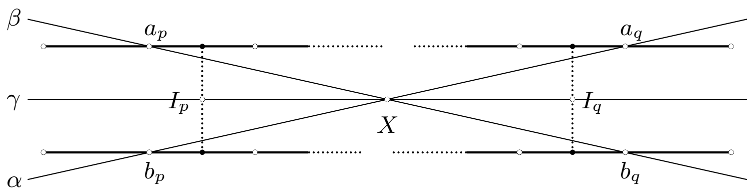
$$\Downarrow$$



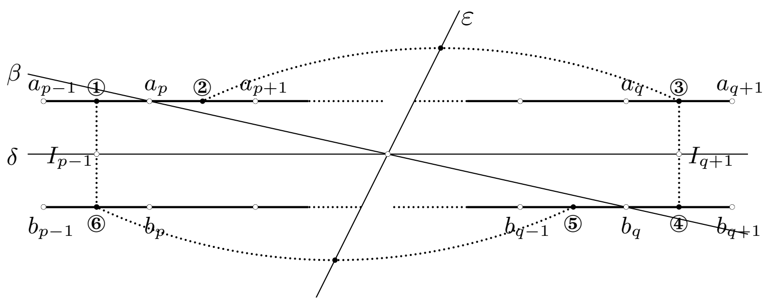
Proof of Claim 1.



(a) By Brianchon's Theorem, the lines α, γ, ε are pairwise distinct and concurrent.



(b) The lines α, β, γ meet, by assumption, in X, and they are pairwise distinct. By (a), we have that α and γ are distinct, and by symmetry also β and γ are distinct. Since a straight line meets a nondegenerate conic in at most two points, α and β are also distinct.



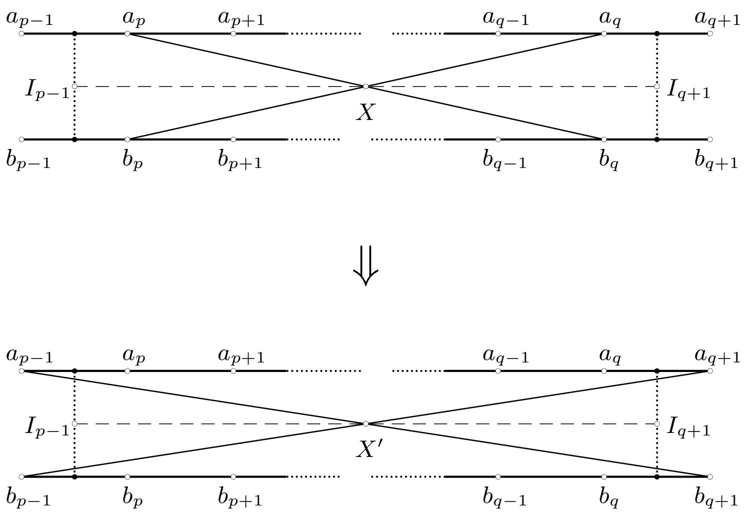
(c) By Brianchon's Theorem, the lines β, ε, δ are pairwise distinct and concurrent.

By (a) & (b) we get that α, β, and ε meet in X, and by (c) we get that also α, β, and δ meet in X, which implies that $I_{p-1} - X - I_{q+1}$ are collinear and pairwise distinct. If $I_{p-1} = I_{q+1}$, then the four lines $a_{p-1} - a_p, b_{p-1} - b_p, a_q - a_{q+1}, b_q - b_{q+1}$, which are all tangent to C, would be concurrent. But then these four lines are not pairwise distinct, and since the eight points $a_{p-1}, a_p, a_q, a_{q+1}, b_{p-1}, b_p, b_q, b_{q+1}$ are pairwise distinct (recall that $1 \leq p-1 < q+1 \leq n$), this contradicts our assumption that the conic K is nondegenerate. By similar arguments it follows that both I_{p-1} and I_{q+1} are distinct from X. *q.e.d.*

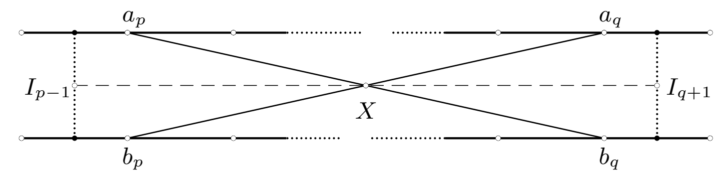
Claim 2. *Let I_{p-1}, I_{q+1}, and X be as above, and let*

$$X' := (a_{p-1} - b_{q+1}) \wedge (b_{p-1} - a_{q+1}) .$$

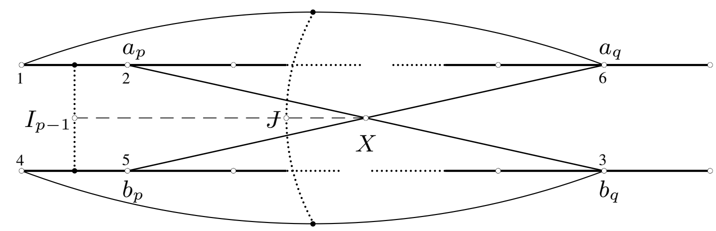
If $I_{p-1} - X - I_{q+1}$ are pairwise distinct and collinear, then $I_{p-1} - X' - I_{q+1}$ are pairwise distinct and collinear too. This implication is visualized by the following figure.



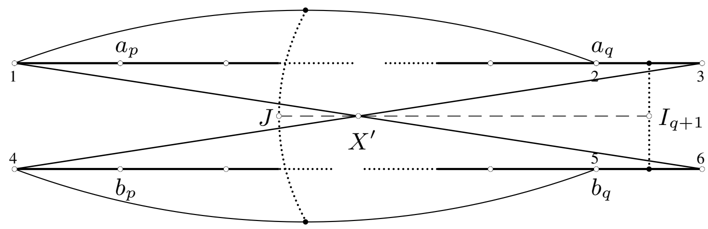
Proof of Claim 2.



(a) By assumption, the points $I_{p-1} - X - I_{q+1}$ are pairwise distinct and collinear.



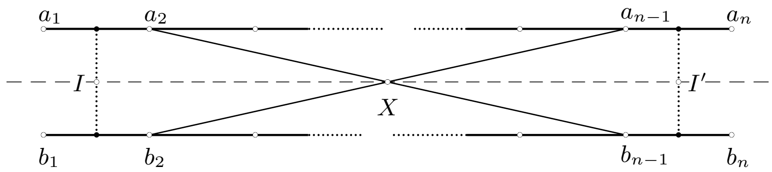
(b) By Pascal's Theorem, the points $I_{p-1} - X - J$ are pairwise distinct and collinear.



(c) By Pascal's Theorem, the points $X' - J - I_{q+1}$ are pairwise distinct and collinear.

By (a) & (b) we get that $I_{p-1} - J - I_{q+1}$ are collinear, and by (c) we get that X' lies on $J - I_{q+1}$. Hence, $I_{p-1} - X' - I_{q+1}$ are collinear. By (a), (c) and a symmetric version of (c), the three points I_{p-1}, X', I_{q+1} are pairwise distinct. *q.e.d.*

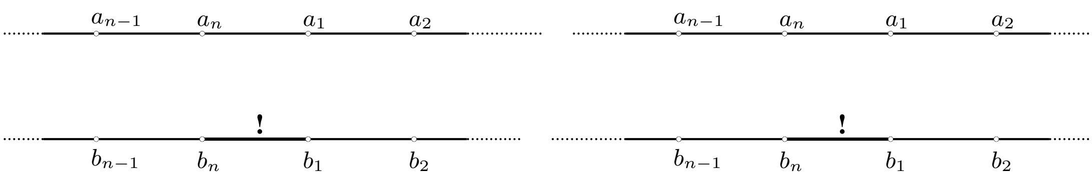
By an iterative application of Claim 1 & 2, we finally get the situation



in which $I - X - I'$ are pairwise distinct and collinear. *q.e.d.*

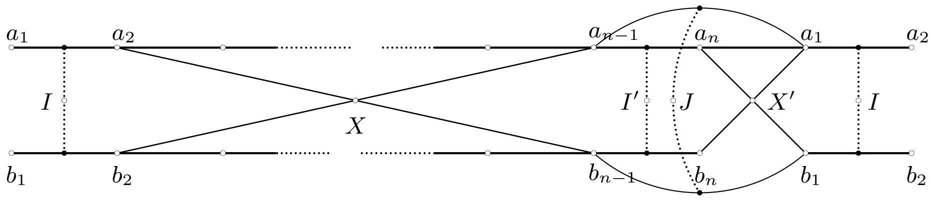
With similar arguments as in the proof of Poncelet's Theorem for triangles (Corollary 2.2), we can now prove the general case of Poncelet's Theorem (Theorem 1.1):

Proof of Poncelet's Theorem. Let K and C be nondegenerate conics in general position. We assume that there is an n-sided polygon $a_1, \ldots, a_n$ which is inscribed in K such that all its n sides $a_1 - a_2, a_2 - a_3, \ldots, a_n - a_1$ are tangent to C and none of its vertices belongs to C. Let us assume that n is minimal with this property. Further we assume that there is an $(n-1)$-sided polygonal chain $b_1, \ldots, b_n$ whose $n-1$ sides are tangent to C and none of its vertices is one of $a_1, \ldots, a_n$ or belongs to C. We have to show that $b_n - b_1$ is tangent to C.

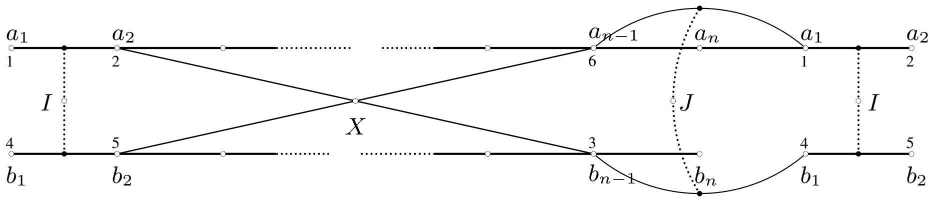


By Lemma 3.1 we know that $I - X - I'$ are pairwise distinct and collinear, where $I = (a_1 - a_2) \wedge (b_1 - b_2)$, $I' = (a_{n-1} - a_n) \wedge (b_{n-1} - b_n)$, and $X = (a_2 - b_{n-1}) \wedge (b_2 - a_{n-1})$. In order to show that $b_n - b_1$ is tangent to C, we have to introduce two more intersection points:

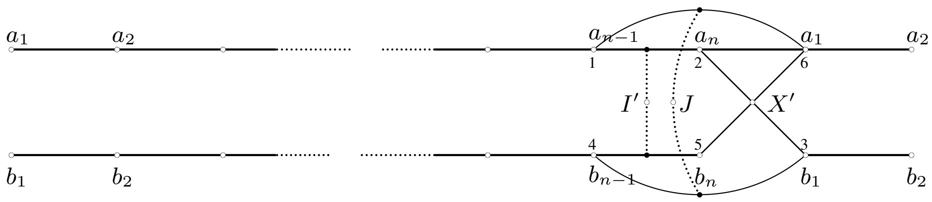
$$J := (a_{n-1} - a_1) \wedge (b_{n-1} - b_1),$$
$$X' := (a_n - b_1) \wedge (b_n - a_1).$$



We now apply Pascal's Theorem twice as illustrated below.

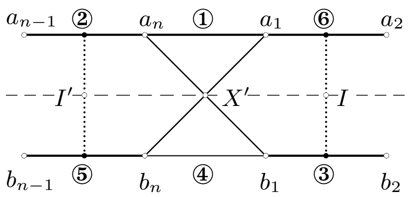


(a) By Pascal's Theorem, the points $I - X - J$ are pairwise distinct and collinear.



(b) By Pascal's Theorem, the points $I' - J - X'$ are pairwise distinct and collinear. Since, by Lemma 3.1, $I - X - I'$ are pairwise distinct and collinear, by (a) we get that $I - X - J - I'$ are collinear, and by (b) we finally get that $I - X' - I'$ are collinear.

For the last step, we apply Carnot's Theorem*.



Since $I - X' - I'$ are collinear, by Carnot's Theorem* we get that the six lines ①, ..., ⑥ are tangent to some conic C'. Now, since a conic is determined by five tangents, and the five lines ①,②,③,⑤,⑥ are tangent to C, C' and C coincide. This implies that ④ is tangent to C, which is what we had to show. *q.e.d.*

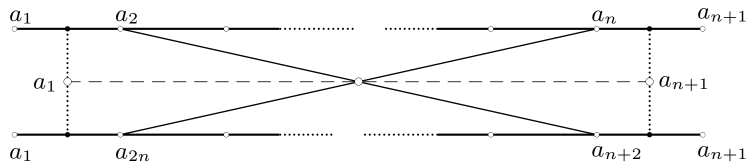
4. SYMMETRIES IN PONCELET-POLYGONS In this section we present some symmetries in $2n$-sided polygons which are inscribed in some conic K and circumscribed about another conic C. To keep the terminology short, we shall call such a polygon a $2n$*-Poncelet-polygon* with respect to $K \,\&\, C$.

Theorem 4.1. *Let K and C be nondegenerate conics in general position which neither meet nor intersect and let $a_1, \dots, a_{2n}$ be the vertices of a $2n$-Poncelet-polygon with respect to $K \,\&\, C$. Further let $t_1, \dots, t_{2n}$ be the contact points of the tangents $a_1 - a_2, \dots, a_{2n} - a_1$.*

(a) All the n diagonals $a_1 - a_{n+1}, a_2 - a_{n+2}, \dots, a_n - a_{2n}$ meet in a point H_0.

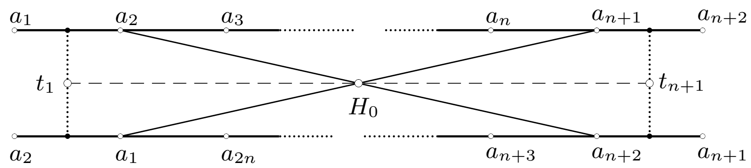
(b) All the n lines $t_1 - t_{n+1}, t_2 - t_{n+2}, \dots, t_n - t_{2n}$ meet in the same point H_0.

Proof. (a) By the proof of Lemma 3.1, we get that the three points a_1, a_{n+1}, and $(a_2 - a_{n+2}) \wedge (a_n - a_{2n})$ are collinear.



This is the same as saying that the three diagonals $a_1 - a_{n+1}$, $a_2 - a_{n+2}$, and $a_n - a_{2n}$ meet in a point, say H_0. Now, by cyclic permutation we get that all n diagonals meet in H_0.

(b) By the proof of Lemma 3.1, we get that the three points $t_1 - H_0 - t_{n+1}$ are collinear.

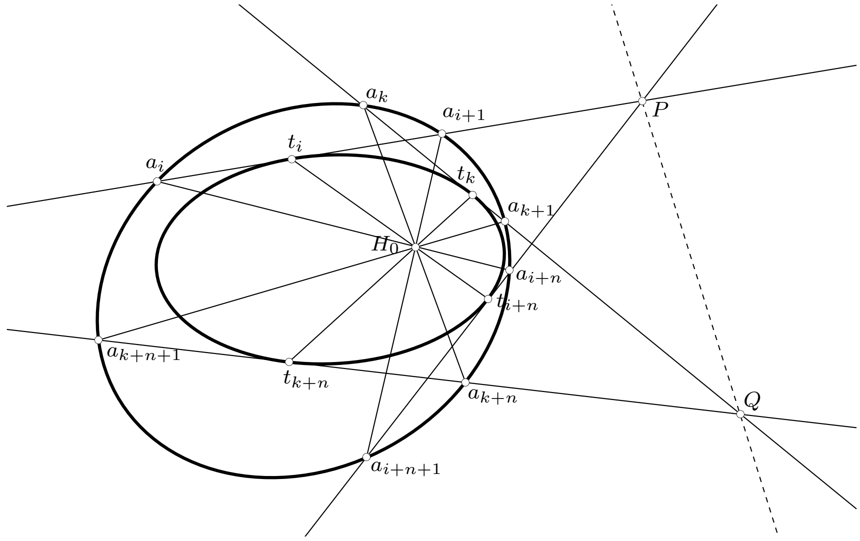


Thus, by cyclic permutation we get that all n lines $t_1 - t_{n+1}, t_2 - t_{n+2}, \dots, t_n - t_{2n}$ pass through H_0, which implies that all n lines meet in H_0. *q.e.d.*

In the last result, we show that the point H_0 is independent of the particular $2n$-Poncelet-polygon (compare with Poncelet's results no. 570 & 571 in [**17**]).

Theorem 4.2. *Let K and C be nondegenerate conics in general position which neither meet nor intersect and let $a_1, \dots, a_{2n}$ and $b_1, \dots, b_{2n}$ be the vertices of two $2n$-Poncelet-polygons with respect to $K \,\&\, C$. Further let $t_1, \dots, t_{2n}$ and $t'_1, \dots, t'_{2n}$ be the contact points of the Poncelet-polygons. Then all $4n$ lines $a_1 - a_{n+1}, \dots$, $t_1 - t_{n+1}, \dots$, $b_1 - b_{n+1}, \dots$, $t'_1 - t'_{n+1}, \dots$ meet in a point H_0. Moreover, opposite sides of the Poncelet-polygons meet on a fixed line h, where h is the polar of H_0, both with respect to C and K.*

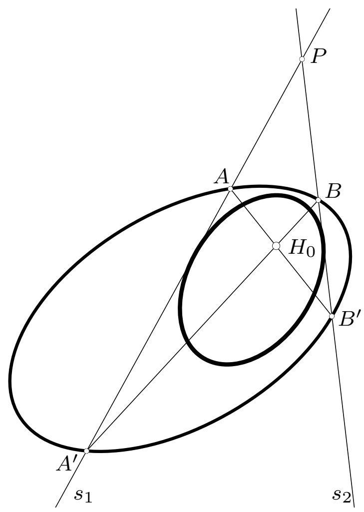
Proof. By Theorem 4.1 we know that the $2n$ lines $a_1 - a_{n+1}, \dots$, $t_1 - t_{n+1}, \dots$ meet in a point H_0. First, we show that the polar h of the pole H_0 with respect to C is the same as the polar h' of H_0 with respect to K, and then we show that the point H_0 is independent of the choice of the $2n$-Poncelet-polygon.



First notice that in the figure above, H_0 is on the polar p of P with respect to the conic C and that H_0 is also on the polar p' of P with respect to the conic K (see for example Coxeter and Greitzer [**6**, Theorem 6.51]). Thus, P lies on the polar h of H_0 with respect to C, as well as on the polar h' of H_0 with respect ot K. Since the same applies to the point Q, the polars h and h' coincide, which shows that the pole H_0 has the same polar with respect to both conics.

The fact that H_0 is independent of the choice of the $2n$-Poncelet-polygon is just a consequence of the following.

Claim. *Let H_0 be as above and let h be the polar of H_0 (with respect to K or C). Choose an arbitrary point P on h. Let s_1 & s_2 be the two tangents from P to C and let A & A' and B & B' be the intersection points of s_1 and s_2 with K.*



Then $H_0 = (A - B') \wedge (B - A')$.

Proof of Claim. By a projective transformation, we may assume that h is the line at infinity. Then, the pole H_0 becomes the common center of both conics and the claim follows by symmetry. *q.e.d.*

Now, let $a_1, \dots, a_{2n}$ and $b_1, \dots, b_{2n}$ be the vertices of two $2n$-Poncelet-polygons with respect to K & C. Furthermore, let $H_0 = (a_1 - a_{n+1}) \wedge (a_2 - a_{n+2})$ and $H_0' = (b_1 - b_{n+1}) \wedge (b_2 - b_{n+2})$, and let h and h' be their respective polars. Choose any point P which lies on both h and h', and draw the two tangents from P to C which intersect K in the points A, A', B, B'. If the conics K and C do not meet (what we assume), then these points are pairwise distinct and by the Claim we get $H_0 = (A - B') \wedge (B - A') = H_0'$. *q.e.d.*

Notice, that for $n = 3$, H_0 is the Brianchon point with respect to C of the Poncelet-hexagon, and h its Pascal line with respect to K. So, for $n > 3$, the point H_0 is the *generalized Brianchon point* with respect to C of the $2n$-Poncelet-polygon, and h its *generalized Pascal line* with respect to K.

REFERENCES

1. M. Berger, *Geometry II*, Springer-Verlag, Berlin, 1987.
2. C. J. Brianchon, Sur les surfaces courbes du second degré, *Journal de l'École Polytechnique*, Cah. XIII, (1806).
3. L. Carnot, *Géométrie de Position*, Crapelet, Paris, 1803.
4. A. Cayley, Developments on the porism of the in-and-circumscribed polygon, *Philosophical magazine* **7** (1854) 339–345.
5. W. Chapple, An essay on the properties of triangles inscribed in and circumscribed about two given circles, *Miscellanea Curiosa Mathematica* (1746) 117–124.
6. H. S. M. Coxeter and S. Greitzer, *Geometry Revisited*, The Mathematical Association of America, Washington D.C., 1967.
7. V. Dragović and M. Radnović, *Poncelet Porisms and Beyond: Integrable Billiards, Hyperelliptic Jacobians and Pencils of Quadrics*, Frontiers in Mathematics, Springer, 2011.
8. ———, Bicentennial of the great Poncelet theorem (1813–2013): current advances, *Bull. Amer. Math. Soc. (N.S.)* **51(3)** (2014) 373–445.
9. L. Euler, Solutio facilis problematum quorundam geometricorum difficillimorum, *Novi Commentarii academiae scientiarum imperialis Petropolitanae* (1767) 103–123.
10. N. Fuss, De polygonis symmetrice irregularibus circulo simul inscriptis et circumscriptis, *Nova Acta Petropol* (1798) 166–189.
11. P. Griffiths and J. Harris, On Cayley's explicit solution to Poncelet's porism, *L'Enseignement Mathématique. Revue Internationale. IIe Série* **24** (1978) 31–40.
12. A. S. Hart, On the porism of the in-and-circumscribed triangle, *The Quarterly journal of pure and applied mathematics* **2** (1858) 143.
13. C. G. J. Jacobi, Ueber die Anwendung der elliptischen Transcendenten auf ein bekanntes Problem der Elementargeometrie: "Die Relation zwischen der Distanz der Mittelpuncte und den Radien zweier Kreise zu finden, von denen der eine einem unregelmäßigen Polygon eingeschrieben, der andere demselben umgeschrieben ist", *J. Reine Angew. Math.* **3** (1828) 376–389.
14. J. L. King, Three problems in search of a measure, *The American Mathematical Monthly* **101** (1994) 609–628.
15. H. Lebesgue, *Les coniques*, Gauthier-Villars, Paris, 1942.
16. B. Pascal, *Essay pour les Coniques*, (1640).
17. J.-V. Poncelet, *Traité des propriétés projectives des figures*, Bachelier, Paris, 1822.
18. V. Y. Protasov, Generalized closing theorems, *Elemente der Mathematik* **66** (2011) 98–117.

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