AN INTRODUCTION TO MICROLOCAL ANALYSIS

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Contents

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I am grateful to everybody who sends me corrections and comments via email!

1. Introduction

Microlocal analysis is a paradigm for the study of distributions and their singularities. Interesting distributions mostly arise in two ways:

(1) as solutions of partial differential equations (PDE), and
(2) as integral kernels of operators used to localize, transform, or otherwise 'test' a partial differential operator.

In these notes, we explicitly mostly focus on the first kind, and prove very general results about solutions of linear PDE. The second kind will be present throughout, starting in §4, though mostly not explicitly so.

Following a quick reminder on Schwartz functions and tempered distributions in §2, the notes can be roughly divided into two parts. The **first part** (§§3–4) introduces *pseudodifferential operators* (ps.d.o.s) on $\mathbb{R}^n$ and their basic properties. Consider for example the *Laplacian*

$$\Delta = \sum_{j=1}^{n} D_{x_j}^2, \quad D_{x_j} := \frac{1}{i}\partial_{x_j}, \tag{1.1}$$

which is a differential operator of order 2:

$$\Delta \in \mathrm{Diff}^2(\mathbb{R}^n). \tag{1.2}$$

Consider the operator $L \in \mathrm{Diff}^2(\mathbb{R}^n)$ defined by

$$L := \Delta + 1. \tag{1.3}$$

Then $L\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is invertible (see Exercise 2.1); what kind of object is its inverse L^{-1}? Morally, it should be an operator of order -2, since composing it with L gives the identity operator, which has order 0. And indeed, L^{-1} is a *pseudodifferential operator of order* -2,

$$L^{-1} \in \Psi^{-2}(\mathbb{R}^n). \tag{1.4}$$

By means of the Fourier transform and its inverse (see §2.1), we can write

$$(L^{-1}u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} (1+|\xi|^2)^{-1} u(y)\,\mathrm{d}y\,\mathrm{d}\xi \tag{1.5}$$

More generally, we shall define spaces of operators

$$\Psi^m(\mathbb{R}^n), \quad m \in \mathbb{R}, \tag{1.6}$$

acting on Schwartz functions (and much larger function spaces too, such as tempered distributions), with $\mathrm{Diff}^m(\mathbb{R}^n) \subset \Psi^m(\mathbb{R}^n)$ for $m = 0, 1, 2, \ldots$, and forming a graded algebra:

$$\Psi^m(\mathbb{R}^n) \circ \Psi^{m'}(\mathbb{R}^n) \subset \Psi^{m+m'}(\mathbb{R}^n). \tag{1.7}$$

Roughly speaking, a typical element $A \in \Psi^m(\mathbb{R}^n)$ is defined similarly to (1.5), but with $(1+|\xi|^2)^{-1}$ replaced by a more general *symbol* $a(x,\xi)$ with $|a(x,\xi)| \lesssim (1+|\xi|^2)^{m/2}$; see §3 for the definition of symbols. In §4, we will define $\Psi^m(\mathbb{R}^n)$ precisely, prove (1.7), as well as the boundedness of ps.d.o.s on a variety of useful function spaces. We will also discuss generalizations of (1.4) for *elliptic* (pseudo)differential operators. (Ellipticity is a notion concerning only the *principal symbol* of A; the latter is, roughly speaking, the leading order part of a, i.e. a modulo symbols of order $m-1$, and ellipticity is the requirement that the principal symbol be invertible.) In particular, we shall prove that on closed manifolds (compact without boundary) M, every elliptic operator $L \in \Psi^m(M)$ is *Fredholm* as a map on $\mathcal{C}^\infty(M)$, or as a map $L\colon H^s(M) \to H^{s-m}(M)$ $(s \in \mathbb{R})$; thus, we can solve the equation $Lu = f$ provided f satisfies a finite number of linear constraints, and then u is unique modulo elements of the finite-dimensional space $\ker L$.

While there are many more interesting things one can say about linear elliptic operators (index theory, Weyl's law, degenerate or non-compact problems, etc.), we will switch gears in the **second part** (§§6–8) of the notes and study *non-elliptic phenomena*. We begin in §6 by defining the *wave front set* of a distribution $u \in \mathscr{S}'(\mathbb{R}^n)$, which is a subset

$$\mathrm{WF}(u) \subset T^*\mathbb{R}^n \setminus o = \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}), \tag{1.8}$$

conic in the second factor. (Here, o is the zero section of the cotangent bundle $T^*\mathbb{R}^n$.) Its projection onto $\mathbb{R}^n$ coincides with the singular support, $\operatorname{sing\,supp} u$; roughly speaking, $\mathrm{WF}(u)$ measures *where* and *in what (co)directions* u is singular. As a basic example, see Exercise 6.2, the wave front set of the characteristic function 1_Ω of a smooth domain $\Omega \subset \mathbb{R}^n$ is given by the conormal bundle of the boundary (minus the zero section)

$$\mathrm{WF}(1_\Omega) = N^*\partial\Omega \setminus o. \tag{1.9}$$

Elliptic regularity can then be *microlocalized*: if $L \in \Psi^m(\mathbb{R}^n)$ has principal symbol ℓ, and if $u \in \mathscr{S}'(\mathbb{R}^n)$ is such that Lu is smooth, then $\mathrm{WF}(u)$ is contained in the *characteristic set* $\mathrm{Char}(L)$ of L: roughly speaking, the set of those (x, ξ) where ℓ is not elliptic. For example, the *wave operator*

$$\Box = -D_t^2 + \sum_{j=1}^n D_{x_j}^2 \tag{1.10}$$

on $\mathbb{R}^{1+n}_{t,x}$ has (principal) symbol $\ell = -\sigma^2 + |\xi|^2$, $|\xi|^2 = \sum_{j=1}^n \xi_j^2$, where we write (σ, ξ) for the momentum variables (dual under the Fourier transform) to (t, x). Thus,

$$\mathrm{Char}(\Box) = \{(t, x, \sigma, \xi) \in T^*\mathbb{R}^{1+n} \setminus o \colon \sigma^2 = |\xi|^2\}. \tag{1.11}$$

As a very concrete example, note that

$$u = H(t - x_1) \quad \Longrightarrow \quad \Box u = 0, \tag{1.12}$$

and indeed $\mathrm{WF}(u) \subset \mathrm{Char}(\Box)$ by (1.9).

The theorem on the *propagation of singularities*, proved in §8, gives a complete description of the structure of $\mathrm{WF}(u)$ for u a distributional solution of an equation $Lu = f \in \mathcal{C}^\infty$: it states that $\mathrm{WF}(u) \subset \mathrm{Char}(L)$ is invariant under the flow along the Hamiltonian vector field of the principal symbol of L. In the case of $\Box$, this flow, for time $s \in \mathbb{R}$, maps (t, x, σ, ξ) to $(t - 2s\sigma, x + 2s\xi, \sigma, \xi)$; use this to verify the theorem for (1.12)!

We shall prove this using the method of *positive commutators*, which showcases the utility of ps.d.o.s as *tools*, rather than as interesting operators in their own right as in (1.4), and exploits the link between symplectic geometry and ps.d.o.s (a form of the 'classical–quantum correspondence'). More importantly, this is a very flexible method, which allows one to control solutions of PDE also in more degenerate situations—which arise frequently in applications. We give one example concerning *radial points* in §9.

As an application which makes use of all these tools, we sketch the proof of resonance expansions for solutions of linear wave equations on a spacetime of interest in General Relativity (de Sitter space) in §10.

These notes draw material from Richard Melrose's lecture notes [Mel07], available under `www-math.mit.edu/~rbm/iml90.pdf`, the textbooks *Microlocal Analysis for Differential Operators: an Introduction* by Grigis and Sjöstrand [GS94] and *Partial Differential Equations* by Michael E. Taylor [Tay11], lecture notes by Jared Wunsch [Wun13], lecture notes by András Vasy [Vas18], as well as my own notes from lectures by Rafe Mazzeo and András Vasy at Stanford University and Ingo Witt at the University of Göttingen.

Hörmander's reference works [Hör03, Hör05, Hör07, Hör09] go significantly beyond the material developed here up until §8. The radial point estimates and applications to general relativity in §§9–10 however are not covered there; the book by Dyatlov–Zworski [DZ19] contains further material on these. We have limited references to the literature to a bare minimum (or quite possibly even less than that), in particular with regards to the earlier stages of the development of microlocal analysis which however are described in detail in Hörmander's treatise. All chapters except the last end with a list of exercises; some of these exercises are taken directly from the literature cited above.

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2. Schwartz functions and tempered distributions

Let $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$. For an open set $\Omega \subset \mathbb{R}^n$, we denote by $\mathcal{C}^k(\Omega)$ the space of k times continuously differentiable functions (with no growth restrictions), and $\mathcal{C}^\infty(\Omega) = \bigcap_{k\in\mathbb{N}} \mathcal{C}^k(\Omega)$. By $\mathcal{C}^k_b(\Omega) \subset \mathcal{C}^k(\Omega)$ we denote the space of functions which are bounded, together with their derivatives up to order k. We denote by $\mathcal{C}^k_c(\Omega)$ the space of compactly supported elements of $\mathcal{C}^k(\Omega)$. Unless otherwise noted, all functions will be complex-valued.

We use standard multiindex notation: for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we set

$$x^\alpha := \prod_{j=1}^n x_j^{\alpha_j}, \quad \partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad D_x^\alpha := D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \qquad D = \frac{1}{i}\partial. \tag{2.1}$$

When the context is clear, we shall often simply write $D^\alpha := D_x^\alpha$, and $D_j := D_{x_j}$. We also put

$$|\alpha| := \sum_{j=1}^n \alpha_j, \quad \alpha! := \prod_{j=1}^n \alpha_j!. \tag{2.2}$$

We will moreover use the *Japanese bracket*, defined for $x \in \mathbb{R}^n$ by

$$\langle x\rangle = (1 + |x|^2)^{1/2}. \tag{2.3}$$

Definition 2.1 (Schwartz space)**.** The space $\mathscr{S}(\mathbb{R}^n)$ of *Schwartz functions* consists of all $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that for all $k \in \mathbb{N}_0$,

$$\|\phi\|_k := \sup_{\substack{x\in\mathbb{R}^n \\ |\alpha|+|\beta|\leq k}} |x^\alpha D^\beta \phi(x)| < \infty. \tag{2.4}$$

Example 2.2. We have $\exp(-|x|^2) \in \mathscr{S}(\mathbb{R}^n)$. Moreover, we have a (continuous) inclusion $\mathcal{C}^\infty_c(\mathbb{R}^n) \hookrightarrow \mathscr{S}(\mathbb{R}^n)$ with dense range. Recall that there are lots of smooth functions with compact support; indeed, when $K \subset U \subset \mathbb{R}^n$ with K compact and U open and bounded, there exists $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi \equiv 1$ on K and $\phi \equiv 0$ on $\mathbb{R}^n \setminus U$.

Equipped with the countably many seminorms $\|\cdot\|_k$, $\mathscr{S}(\mathbb{R}^n)$ is a Fréchet space. Directly from the definition, we have continuous maps

$$\begin{aligned} x_j\colon \mathscr{S}(\mathbb{R}^n) &\to \mathscr{S}(\mathbb{R}^n) \qquad (\phi \mapsto x_j\phi), \\ D_j\colon \mathscr{S}(\mathbb{R}^n) &\to \mathscr{S}(\mathbb{R}^n) \qquad (\phi \mapsto D_j\phi). \end{aligned} \tag{2.5}$$

Given $a \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, pointwise multiplication by a is also continuous. Furthermore, integration is a continuous map

$$\int\colon \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}. \tag{2.6}$$

Indeed, this follows from

$$\left|\int_{\mathbb{R}^n} \phi(x)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^n} \langle x\rangle^{-n-1}\bigl(\langle x\rangle^{n+1}\phi(x)\bigr)\,\mathrm{d}x\right| \le C_n\|\phi\|_{n+1}. \tag{2.7}$$

Other useful operations are the pointwise product

$$\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \ni (\phi,\psi) \to \phi\psi \in \mathscr{S}(\mathbb{R}^n), \quad (\phi\psi)(x) = \phi(x)\psi(x), \tag{2.8}$$

and the exterior product

$$\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \ni (\phi,\psi) \to \phi\boxtimes\psi \in \mathscr{S}(\mathbb{R}^{2n}), \quad (\phi\boxtimes\psi)(x,y) = \phi(x)\psi(y). \tag{2.9}$$

Definition 2.3 (Tempered distributions)**.** The space $\mathscr{S}'(\mathbb{R}^n)$ of *tempered distributions* is the space of all continuous linear functionals $u\colon \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$, equipped with the weak topology: the seminorms are $|u|_\phi := |u(\phi)|$ for $\phi \in \mathscr{S}(\mathbb{R}^n)$. We shall usually write $\langle u,\phi\rangle := u(\phi)$.

Example 2.4. The δ-distribution is defined by $\langle \delta,\phi\rangle := \phi(0)$. We have $\delta \in \mathscr{S}'(\mathbb{R}^n)$ since $|\langle\delta,\phi\rangle| \le \|\phi\|_0$.

Combining (2.6) and (2.8), we can define a continuous map

$$\mathscr{S}(\mathbb{R}^n) \ni \phi \to T_\phi \in \mathscr{S}'(\mathbb{R}^n), \quad T_\phi(\psi) = \int_{\mathbb{R}^n} \phi(x)\psi(x)\,\mathrm{d}x. \tag{2.10}$$

Proposition 2.5 (Functions as distributions)**.** *The map $\phi \mapsto T_\phi$ is injective, and has dense range in the weak topology.*

Proof. Regarding injectivity: $T_\phi(\bar\phi) = \int_{\mathbb{R}^n} |\phi(x)|^2\,\mathrm{d}x = 0$ implies $\phi = 0$. To prove the density, it suffices to show that, given $u \in \mathscr{S}'(\mathbb{R}^n)$ and $\phi_1,\ldots,\phi_N \in \mathscr{S}(\mathbb{R}^n)$ as well as any $\epsilon > 0$, there exists $\phi \in \mathscr{S}(\mathbb{R}^n)$ such that $|\langle u,\phi_j\rangle - T_\phi(\phi_j)\rangle| < \epsilon$ for all $j = 1,\ldots,N$. Assuming, as one may, that the ϕ_j are orthonormal with respect to the $L^2(\mathbb{R}^n)$ inner product, this holds (with '$<\epsilon$' replaced by '$=0$') for $\phi = \sum_{j=1}^N \langle u,\phi_j\rangle\bar\phi_j$. A better proof, based on a mollification argument, is suggested in Exercise 2.2. ☐

Now on the one hand, we can extend the maps (2.5) by duality to $\mathscr{S}'(\mathbb{R}^n)$: indeed, for $u \in \mathscr{S}'(\mathbb{R}^n)$ and $\phi \in \mathscr{S}(\mathbb{R}^n)$, we define $x_j u, D_j u \in \mathscr{S}'(\mathbb{R}^n)$ by

$$\langle x_j u,\phi\rangle := \langle u, x_j\phi\rangle, \quad \langle D_j u,\phi\rangle := \langle u, -D_j\phi\rangle. \tag{2.11}$$

On the other hand, when $u \in \mathscr{S}(\mathbb{R}^n)$, then $T_{x_j u}(\phi) = T_u(x_j\phi)$ and $T_{D_j u}(\phi) = T_u(-D_j\phi)$, i.e. on the image of $\mathscr{S}(\mathbb{R}^n)$ inside of $\mathscr{S}'(\mathbb{R}^n)$, the definitions 2.11 agree with the usual definitions of multiplication and differentiation of Schwartz functions. The density statement of Proposition 2.5 then shows that (2.11) defines the *unique* continuous extensions of multiplication or differentiation from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$.

Similarly, by duality and starting from (2.9), pointwise multiplication by a Schwartz function extends in a unique manner to a continuous map on $\mathscr{S}'(\mathbb{R}^n)$; more generally, this is true for multiplication by a function in $\mathcal{C}_b^\infty(\mathbb{R}^n)$.

Other notions, which will be significantly refined later, are:

Definition 2.6 ((Singular) support of a distribution)**.** Let $u \in \mathscr{S}'(\mathbb{R}^n)$. Then the *support*, $\operatorname{supp} u$, is the complement of the set of $x \in \mathbb{R}^n$ such that there exists $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\chi(x) \neq 0$, such that $\chi u = 0$.

The *singular support*, $\operatorname{sing\,supp} u$, is the complement of the set of $x \in \mathbb{R}^n$ such that there exists $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\chi(x) \neq 0$, such that χu is smooth, i.e. $\chi u = T_\phi$, $\phi \in \mathscr{S}(\mathbb{R}^n)$.

Example 2.7. We have $\operatorname{supp} \delta = \operatorname{sing\,supp} \delta = \{0\}$. For $u = \delta + e^{-|x|^2} \in \mathscr{S}'(\mathbb{R}^n)$, we have $\operatorname{supp} u = \mathbb{R}^n$, but $\operatorname{sing\,supp} u = \{0\}$ still.

2.1. **Fourier transform and its inverse.** We define the Fourier transform of $\phi \in \mathscr{S}(\mathbb{R}^n)$ by

$$(\mathcal{F}\phi)(\xi) = \hat{\phi}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R}^n, \tag{2.12}$$

and the inverse Fourier transform of $\psi \in \mathscr{S}(\mathbb{R}^n)$ by

$$(\mathcal{F}^{-1}\psi)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi(\xi)\,\mathrm{d}\xi, \quad x \in \mathbb{R}^n. \tag{2.13}$$

As in (2.7), one finds $\|\mathcal{F}\phi\|_0 \leq C_n \|\phi\|_{n+1}$ and $\|\mathcal{F}^{-1}\phi\|_0 \leq C_n \|\phi\|_{n+1}$. Moreover, we have

$$\begin{aligned} \mathcal{F}(D_{x_j}\phi) &= \xi_j \mathcal{F}\phi, & \mathcal{F}(x_j\phi) &= -D_{\xi_j}\mathcal{F}\phi, \\ \mathcal{F}^{-1}(D_{\xi_j}\phi) &= -x_j\mathcal{F}^{-1}\phi, & \mathcal{F}^{-1}(\xi_j\phi) &= D_{x_j}\mathcal{F}^{-1}\phi, \end{aligned} \tag{2.14}$$

using integration by parts for the first and third statement; reading these from right to left shows that

$$\|\mathcal{F}\phi\|_k \leq C_n \|\phi\|_{k+n+1} \quad \forall\ k \in \mathbb{N}_0, \tag{2.15}$$

hence the (inverse) Fourier transform preserves the Schwartz space:

$$\mathcal{F},\ \mathcal{F}^{-1} \colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n). \tag{2.16}$$

Note then that for $u, \psi \in \mathscr{S}(\mathbb{R}^n)$,

$$\begin{aligned} \langle \mathcal{F}u, \psi \rangle &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,\mathrm{d}x\ \psi(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\psi(\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &= \langle u, \mathcal{F}\psi \rangle. \end{aligned} \tag{2.17}$$

This allows us to extend $\mathcal{F}, \mathcal{F}^{-1}$ to maps on tempered distributions,

$$\mathcal{F},\ \mathcal{F}^{-1} \colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n), \tag{2.18}$$

and the formulas (2.14) remain valid for $\phi \in \mathscr{S}'(\mathbb{R}^n)$.

Example 2.8. The Fourier transform of δ is calculated by $\langle \mathcal{F}\delta, \psi \rangle = \langle \delta, \mathcal{F}\psi \rangle = \hat{\psi}(0) = \int_{\mathbb{R}^n} \psi(x)\,\mathrm{d}x$, so $\mathcal{F}\delta = 1$.

We recall the proof that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are indeed inverses to each other.

Theorem 2.9 (Fourier transform and its inverse)**.** *We have* $\mathcal{F} \circ \mathcal{F}^{-1} = \mathcal{F}^{-1} \circ \mathcal{F} = I$ *on* $\mathscr{S}(\mathbb{R}^n)$ *and* $\mathscr{S}'(\mathbb{R}^n)$.

Proof. Let $A := \mathcal{F}^{-1}\mathcal{F} \colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$. By (2.14), we have $AD_{x_j} = \mathcal{F}^{-1}\xi_j\mathcal{F} = D_{x_j}A$ and $Ax_j = \mathcal{F}^{-1}(-D_{\xi_j})\mathcal{F} = x_jA$, i.e. A commutes with differentiation along and multiplication by coordinates. Given $\phi \in \mathscr{S}(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$, we can write

$$\phi(x) = \phi(x_0) + \sum_{j=1}^n \phi_j(x)(x_j - (x_0)_j), \quad \phi_j(x) = \int_0^1 (\partial_j\phi)(x_0 + t(x - x_0))\,\mathrm{d}t. \tag{2.19}$$

The fact that ϕ_j is in general not Schwartz is remedied by fixing a cutoff $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, identically 1 near x_0, and writing $\phi(x) = \chi(x)\phi(x) + (1 - \chi(x))\phi(x)$, so

$$\begin{split} \phi(x) = \chi(x)\phi(x_0) + \sum_{j=1}^n \tilde\phi_j(x)(x_j - (x_0)_j), \\ \tilde\phi_j(x) = \chi(x)\phi_j(x) + \frac{(1-\chi(x))\phi(x)}{|x - x_0|^2}(x_j - (x_0)_j). \end{split} \tag{2.20}$$

Since A annihilates every term in the sum, we have $(A\phi)(x_0) = \phi(x_0)(A\chi)(x_0)$; note that the constant $(A\chi)(x_0)$ here does not depend on ϕ, and not on the cutoff χ either (since the left hand side does not involve χ at all).

The same cutoff χ can be used to evaluate $A\phi$ at points x close to x_0; but

$$D_{x_j}(A\chi)(x) = A(D_{x_j}\chi)(x) = 0 \tag{2.21}$$

for $x \in \chi^{-1}(1)$. We conclude that $A = cI$ for some *constant* $c \in \mathbb{C}$. One can find c by explicitly evaluating

$$\mathcal{F}(e^{-|x|^2})(\xi) = \pi^{n/2}e^{-|\xi|^2/4}, \quad \mathcal{F}^{-1}(e^{-|\xi|^2/4})(x) = \pi^{n/2}e^{-|x|^2}, \tag{2.22}$$

so $c = 1$ indeed. The proof that $\mathcal{F}\mathcal{F}^{-1} = I$ is completely analogous. □

We also recall that $\mathcal{F}$ is an isomorphism on $L^2(\mathbb{R}^n)$; this follows from the density of $\mathscr{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ and the following fact:

Proposition 2.10 (Plancherel's theorem)**.** *For $\phi \in \mathscr{S}(\mathbb{R}^n)$, we have*

$$\|\mathcal{F}\phi\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2}\|\phi\|_{L^2(\mathbb{R}^n)}. \tag{2.23}$$

Proof. Analogously to (2.17), we have

$$\int (\mathcal{F}\phi)(\xi)\bar\psi(\xi)\,\mathrm{d}\xi = (2\pi)^n \int \phi(x)\overline{\mathcal{F}^{-1}\psi(x)}\,\mathrm{d}x, \quad \phi, \psi \in \mathscr{S}(\mathbb{R}^n). \tag{2.24}$$

Plugging in $\psi = \mathcal{F}\phi$ proves the proposition. □

2.2. **Sobolev spaces and the Schwartz representation theorem.** Using the Fourier transform, we can define operators which differentiate a 'fractional number of times':

Definition 2.11 (Fractional derivative operators on $\mathbb{R}^n$)**.** For $s \in \mathbb{R}$ (or $s \in \mathbb{C}$), we let

$$\langle D\rangle^s = (1 + |D|^2)^{s/2} \colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n), \quad \langle D\rangle^s = \mathcal{F}^{-1}\langle\xi\rangle^s\mathcal{F}. \tag{2.25}$$

This agrees for $s \in 2\mathbb{N}_0$ with the usual definition, and for $s = -2$ with the operator (1.4). What is implicitly used here is that multiplication by $(1 + |\xi|^2)^{s/2}$ is continuous on $\mathscr{S}(\mathbb{R}^n)$.

Definition 2.12 (Sobolev spaces on $\mathbb{R}^n$)**.** For $s \in \mathbb{R}$, the Sobolev space of order s is defined by

$$H^s(\mathbb{R}^n) := \{u \in \mathscr{S}'(\mathbb{R}^n) \colon \langle D\rangle^s u \in L^2(\mathbb{R}^n)\}. \tag{2.26}$$

With the norm

$$\|u\|_{H^s} := \|\langle D\rangle^s u\|_{L^2} = (2\pi)^{-n/2}\|\langle \xi\rangle^s \mathcal{F}u\|_{L^2}, \tag{2.27}$$

it is a Hilbert space.

Example 2.13. The δ-distribution at $0 \in \mathbb{R}^n$ satisfies $\delta \in H^s(\mathbb{R}^n)$ for all $s < -n/2$.

Since multiplication by $\langle x\rangle^r$ is continuous on $\mathscr{S}'(\mathbb{R}^n)$ for any $r \in \mathbb{R}$, we can more generally define *weighted Sobolev spaces*,

$$\langle x\rangle^r H^s(\mathbb{R}^n) := \{u \in \mathscr{S}'(\mathbb{R}^n) \colon \langle x\rangle^{-r} u \in H^s(\mathbb{R}^n)\}. \tag{2.28}$$

These are Sobolev spaces with squared norm

$$\|u\|^2_{\langle x\rangle^r H^s(\mathbb{R}^n)} := \|\langle x\rangle^{-r} u\|^2_{H^s(\mathbb{R}^n)}. \tag{2.29}$$

The second part of the following is (a version of) the *Schwartz representation theorem*:

Theorem 2.14 (Schwartz representation theorem)**.** *We have*

$$\mathscr{S}(\mathbb{R}^n) = \bigcap_{s,r\in\mathbb{R}} \langle x\rangle^r H^s(\mathbb{R}^n), \quad \mathscr{S}'(\mathbb{R}^n) = \bigcup_{s,r\in\mathbb{R}} \langle x\rangle^r H^s(\mathbb{R}^n). \tag{2.30}$$

Proof. See Exercises 2.4 and 2.7. □

It easily implies (using Sobolev embedding, Exercise 2.4) that every tempered distribution is a sum of terms of the form $x^\alpha D^\beta a$, $a \in \mathcal{C}^0_b(\mathbb{R}^n)$.

2.3. **The Schwartz kernel theorem.** The Schwartz kernel theorem is a philosophically useful fact, establishing a 1–1 correspondence between the 'most general' operators in the present context—mapping Schwartz functions to tempered distributions—and distributional integral kernels, also called *Schwartz kernels*. To state this, we note that any distribution $K \in \mathscr{S}'(\mathbb{R}^{n+m})$ induces a bounded linear operator $\mathscr{S}(\mathbb{R}^m) \to \mathscr{S}'(\mathbb{R}^n)$ by integration along the $\mathbb{R}^m$ factor, to wit

$$(O_K\phi)(\psi) := \langle K, \psi \boxtimes \phi\rangle = \int \left(\int_{\mathbb{R}^m} K(x,y)\phi(y)\,\mathrm{d}y\right)\psi(x)\,\mathrm{d}x, \quad \phi \in \mathscr{S}(\mathbb{R}^m),\ \psi \in \mathscr{S}(\mathbb{R}^n). \tag{2.31}$$

Formally, one usually writes

$$(O_K\phi)(x) = \int_{\mathbb{R}^m} K(x,y)\phi(y)\,\mathrm{d}y. \tag{2.32}$$

Theorem 2.15 (Schwartz kernel theorem: Euclidean case)**.** *The map $K \mapsto O_K$ is a bijection between $\mathscr{S}'(\mathbb{R}^{n+m})$ and the space of continuous linear operators $\mathscr{S}(\mathbb{R}^m) \to \mathscr{S}'(\mathbb{R}^n)$.*

Proof. See Exercises 2.8 and 2.9. □

Example 2.16. The Schwartz kernel of the identity operator I on $\mathscr{S}(\mathbb{R}^n)$ is given by

$$K(x,y) = \delta(x-y), \quad x,y \in \mathbb{R}^n. \tag{2.33}$$

2.4. **Differential operators.** Given $a_\alpha \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ for $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m$, we can define the m-th order differential operator

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha. \tag{2.34}$$

Since multiplication by a_α is continuous on $\mathscr{S}(\mathbb{R}^n)$, A defines a continuous linear operator on $\mathscr{S}(\mathbb{R}^n)$. By duality, A extends (uniquely) to an continuous linear operator on $\mathscr{S}'(\mathbb{R}^n)$.

Definition 2.17 (Differential operators)**.** By $\mathrm{Diff}^m(\mathbb{R}^n)$, we denote the space of all operators $A\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ of the form (2.34).

Given A as in (2.34), let us define the *full symbol* of A to be

$$\sigma(A)(x, \xi) := \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha. \tag{2.35}$$

Then, in view of (2.14), we can write

$$\begin{aligned} (Au)(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(A)(x, \xi) \hat{u}(\xi) \,\mathrm{d}\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \sigma(A)(x, \xi) u(y) \,\mathrm{d}y \right) \mathrm{d}\xi, \end{aligned} \tag{2.36}$$

which we read as an iterated integral. On the other hand, the Schwartz kernel K of A is easily verified to be

$$K(x, y) = \sum_{|\alpha| \leq m} a_\alpha(x) (D^\alpha \delta)(x - y), \tag{2.37}$$

so (formally) we have

$$K(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x, \xi) \,\mathrm{d}\xi, \tag{2.38}$$

which is indeed (rigorously) correct if one reads this as the Fourier transform of a in ξ.

Proposition 2.18 ((Pseudo)locality of differential operators)**.** *Let $A \in \mathrm{Diff}^m(\mathbb{R}^n)$. Then A is* local*, that is,*

$$\operatorname{supp} Au \subset \operatorname{supp} u, \quad u \in \mathscr{S}'(\mathbb{R}^n), \tag{2.39}$$

and A is pseudolocal*, that is,*

$$\operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u, \quad u \in \mathscr{S}'(\mathbb{R}^n). \tag{2.40}$$

The proof is straightforward. From the perspective of the Schwartz kernel K of A, (2.39) is really due to the fact that $K(x, y)$ is supported *on the diagonal* $x = y$, while (2.40) is really due to the fact that $K(x, y)$ is smooth away from $x = y$. (That is, adding to K an element of $\mathscr{S}(\mathbb{R}^{2n})$ preserves (2.40), but destroys (2.39).) Since as microlocal analysts we are interested in *singularities*, it is the property (2.40) which we care about most; and this will persist when A is a pseudodifferential operator. On the other hand, the *only* continuous linear operators $A\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ satisfying condition (2.39) are differential operators, see Exercise 2.11.

We mention three features of differential operators concerning their *principal symbol*.

Definition 2.19 (Principal symbol)**.** Given $m \in \mathbb{N}_0$ and a differential operator $A = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$, its *principal symbol* is defined as

$$\sigma^m(A)(x,\xi) := \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \tag{2.41}$$

i.e. keeping only the terms of order m.

Note that the principal symbol depends on a choice of m. For example, one may regard an operator $A \in \mathrm{Diff}^m(\mathbb{R}^n)$ as an operator of order $m+1$, and as such its principal symbol $\sigma^{m+1}(A)$ vanishes. Put differently, for $A \in \mathrm{Diff}^m(\mathbb{R}^n)$, we have $\sigma^m(A) = 0$ if and only if $A \in \mathrm{Diff}^{m-1}(\mathbb{R}^n)$.

Proposition 2.20 (Behavior of the principal symbol)**.** *Let $A \in \mathrm{Diff}^m(\mathbb{R}^n)$.*

(1) *Define the adjoint A^* of A by $\int_{\mathbb{R}^n} (A^*u)(x)\overline{v(x)}\,\mathrm{d}x = \int_{\mathbb{R}^n} u(x)\overline{(Av)(x)}\,\mathrm{d}x$, $u,v \in \mathcal{C}^\infty_\mathrm{c}(\mathbb{R}^n)$. Then $A^* \in \mathrm{Diff}^m(\mathbb{R}^n)$, and the principal symbol is*
$$\sigma^m(A^*)(x,\xi) = \overline{\sigma^m(A)(x,\xi)}. \tag{2.42}$$
(2) *Let $B \in \mathrm{Diff}^{m'}(\mathbb{R}^n)$. Then $A\circ B \in \mathrm{Diff}^{m+m'}(\mathbb{R}^n)$, and*
$$\sigma^{m+m'}(A\circ B)(x,\xi) = \sigma^m(A)(x,\xi)\sigma^{m'}(B)(x,\xi). \tag{2.43}$$
(3) *Let $\kappa\colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism which is the identity outside of a compact set. Define $A_\kappa\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ by $(A_\kappa u)(y) := (A(u\circ\kappa^{-1}))(\kappa(y))$. Then $A_\kappa \in \mathrm{Diff}^m(\mathbb{R}^n)$, and the principal symbols are related via*
$$\sigma^m(A_\kappa)(y,\eta) = \sigma^m(A)\big(\kappa(y), (\kappa'(y)^T)^{-1}\eta\big). \tag{2.44}$$

Proof. Exercise 2.13. □

Thus, the principal symbol is well-defined as a function on $T^*\mathbb{R}^n$, and it is a map—from the (non-commutative) algebra $\mathrm{Diff}(\mathbb{R}^n) = \bigcup_{m\in\mathbb{N}_0} \mathrm{Diff}^m(\mathbb{R}^n)$ into the *commutative* algebra of functions $a(x,\xi)$ which are homogeneous polynomials in ξ with coefficients in $\mathcal{C}^\infty_b(\mathbb{R}^n)$—with a number of useful properties as stated in Proposition 2.20.

2.5. **Exercises.**

Exercise 2.1 (Shifted Laplacian). Let $\Delta = \sum_{j=1}^n D_{x_j}^2$.

(1) Show that $\Delta + 1\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is an isomorphism.
(2) Find a non-trivial solution $u \in \mathcal{C}^\infty(\mathbb{R}^n)$ of $(\Delta+1)u = 0$. Why does this not contradict the first part?

Exercise 2.2 (Density of $\mathcal{C}^\infty_\mathrm{c}$ in tempered distributions). We will prove in a constructive manner that $\mathcal{C}^\infty_\mathrm{c}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ (or more precisely the image of $\mathcal{C}^\infty_\mathrm{c}(\mathbb{R}^n)$ under the map (2.10)) is dense.

(1) Let $\chi \in \mathcal{C}^\infty_\mathrm{c}(\mathbb{R}^n)$, $\chi(0) = 1$. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. Put $\phi_\epsilon(x) = \chi(\epsilon x)\phi(x)$. Show that $\phi_\epsilon \to \phi$ in $\mathscr{S}(\mathbb{R}^n)$ as $\epsilon \searrow 0$. Conclude that the space
$$\mathscr{E}'(\mathbb{R}^n) := \{u \in \mathscr{S}'(\mathbb{R}^n)\colon \operatorname{supp} u \text{ is compact}\} \tag{2.45}$$
of compactly supported distributions is dense in $\mathscr{S}'(\mathbb{R}^n)$.

(2) Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and put $\tilde\psi(x) = \psi(-x)$. For $\phi \in \mathscr{S}(\mathbb{R}^n)$, define the *convolution* of ϕ with $\tilde\psi$ by

$$(\phi * \tilde\psi)(x) := \int_{\mathbb{R}^n} \phi(x-y)\tilde\psi(y)\,\mathrm{d}y. \tag{2.46}$$

Show that $\phi * \tilde\psi \in \mathscr{S}(\mathbb{R}^n)$. Define the convolution of $u \in \mathscr{S}'(\mathbb{R}^n)$ with ψ by $\langle u * \psi, \phi\rangle := \langle u, \phi * \tilde\psi\rangle$. Check that this is the correct definition when $u \in \mathscr{S}(\mathbb{R}^n)$.

(3) Let now $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \chi(x)\,\mathrm{d}x = 1$, and set $\chi_\epsilon(x) := \epsilon^{-n}\chi(\epsilon^{-1}x)$. Show that $\chi_\epsilon * \phi \to \phi$ in $\mathscr{S}(\mathbb{R}^n)$ as $\epsilon \searrow 0$.

(4) When $u \in \mathscr{E}'(\mathbb{R}^n)$ and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, show that $u * \psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, that is, there exists $v \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ so that $u * \psi = T_v$. (*Hint.* Show that one can define $\langle u * \psi, \phi\rangle$ consistently for $\phi \in \mathscr{S}'(\mathbb{R}^n)$; define a candidate for v by using δ-distributions for ϕ. In order to show that $u * \psi = T_v$, take any $\phi \in \mathscr{S}(\mathbb{R}^n)$, apply both sides to $\chi_\epsilon * \phi$ and let $\epsilon \to 0$.)

(5) Combine the previous parts to conclude that $\mathcal{C}_c^\infty(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ is dense.

Exercise 2.3 (Fourier transform of compactly supported distributions). Let $u \in \mathscr{E}'(\mathbb{R}^n)$ (see (2.45)). Show that $\mathcal{F}u$ is an analytic function, and there exist $C, N \in \mathbb{R}$ so that $|(\mathcal{F}u)(\xi)| \le C\langle\xi\rangle^N$ for all $\xi \in \mathbb{R}^n$.

Exercise 2.4 (Sobolev embedding). Let $s > n/2$.

(1) Prove that there exists a constant $C_s < \infty$ such that for $\phi \in \mathscr{S}(\mathbb{R}^n)$, the estimate

$$\|\phi\|_{L^\infty(\mathbb{R}^n)} \le C_s\|\phi\|_{H^s(\mathbb{R}^n)}. \tag{2.47}$$

holds. (*Hint.* Pass to the Fourier transform.) Deduce that $H^s(\mathbb{R}^n) \subset \mathcal{C}_b^0(\mathbb{R}^n)$.

(2) Show more generally that $H^s(\mathbb{R}^n) \subset \mathcal{C}_b^k(\mathbb{R}^n)$ for $s > n/2 + k$.

(3) Prove the first equality in Theorem 2.14.

Exercise 2.5 (Algebra properties of Sobolev spaces). Let $n \in \mathbb{N}$.

(1) Let $u, v \in \mathscr{S}(\mathbb{R}^n)$ and recall that their convolution is defined by $(u * v)(x) = \int_{\mathbb{R}^n} u(y)v(x-y)\,\mathrm{d}y$. Show that $\mathcal{F}(u*v) = \mathcal{F}(u)\mathcal{F}(v)$. Use this to find a formula for $\mathcal{F}(uv)$.

(2) Define the function

$$a(\xi,\eta) := \frac{\langle\xi\rangle^{2s}}{\langle\eta\rangle^{2s}\langle\xi-\eta\rangle^{2s}}, \qquad \xi,\eta \in \mathbb{R}^n. \tag{2.48}$$

Show that $\sup_{\xi\in\mathbb{R}^n} \int_{\mathbb{R}^n} a(\xi,\eta)\,\mathrm{d}\eta < \infty$.

(3) Let $s > \frac{n}{2}$ and $u, v \in H^s(\mathbb{R}^n)$. Show that $uv \in H^s(\mathbb{R}^n)$, and prove an estimate $\|uv\|_{H^s(\mathbb{R}^n)} \le C\|u\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}$ for some constant C which is independent of u, v.

Exercise 2.6 (Duals of weighted Sobolev spaces). Show that $\langle x\rangle^{-r}H^{-m}(\mathbb{R}^n)$ is the L^2-dual of $\langle x\rangle^r H^m(\mathbb{R}^n)$. That is, show that the sesquilinear pairing

$$\langle -,-\rangle_{L^2} \colon \mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \ni (\phi,\psi) \mapsto \int_{\mathbb{R}^n} \phi(x)\overline{\psi(x)}\mathrm{d}x \tag{2.49}$$

extends by continuity and density to

$$\langle -,-\rangle_{L^2} \colon \langle x\rangle^r H^m(\mathbb{R}^n) \times \langle x\rangle^{-r}H^{-m}(\mathbb{R}^n) \to \mathbb{C}, \tag{2.50}$$

and that the map

$$\langle x\rangle^r H^m(\mathbb{R}^n) \to \big(\langle x\rangle^{-r} H^{-m}(\mathbb{R}^n)\big)^*, \quad \phi \mapsto \langle \phi, -\rangle_{L^2}, \tag{2.51}$$

is an antilinear isomorphism.

Exercise 2.7 (Schwartz representation theorem). Prove the second equality in Theorem 2.14 as follows.

(1) Given $u \in \mathscr{S}'(\mathbb{R}^n)$, there exist C, k such that $|u(\phi)| \leq C\|\phi\|_k$.
(2) Let $R_q = \langle x\rangle^{-q}\langle D\rangle^{-q}$. Then R_q is an isomorphism on $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$. Moreover, for sufficiently large q, we have $\|R_q\phi\|_k \leq C\|\phi\|_{L^2(\mathbb{R}^n)}$ (for some other constant C). (*Hint.* Use the previous exercise. It may be convenient to take s there and q here to be even integers.)
(3) Denoting $R_q^\dagger = \langle D\rangle^{-q}\langle x\rangle^{-q}$, deduce that $R_q^\dagger u \in L^2(\mathbb{R}^n)$, and conclude that $u \in \langle x\rangle^q H^{-q}(\mathbb{R}^n)$.

Exercise 2.8 (Schwartz kernel theorem I.). Prove the injectivity claim of Theorem 2.15. (*Hint.* Let $K \in \mathscr{S}'(\mathbb{R}^{n+m})$ be given with $O_K = 0$. Given $\phi \in \mathscr{S}(\mathbb{R}^{n+m})$, you need to show that $\langle K, \phi\rangle = 0$. You know that this is true when ϕ is a finite linear combination of exterior products $\psi_1 \boxtimes \psi_2$, $\psi_1 \in \mathscr{S}(\mathbb{R}^n)$, $\psi_2 \in \mathscr{S}(\mathbb{R}^m)$. Try to use the Fourier transform, or Fourier series, to approximate ϕ by such linear combinations. It may help to first reduce to the case that $\operatorname{supp} K$ is compact.)

Exercise 2.9 (Schwartz kernel theorem II.). Let $A\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^m)$ be continuous. Prove the surjectivity claim of Theorem 2.15 as follows.

(1) The continuity of A is equivalent to the statement that for all $\psi \in \mathscr{S}(\mathbb{R}^n)$ there exists $N > 1$ such that $|\langle A\phi, \psi\rangle| \leq N\|\phi\|_N$ for all $\phi \in \mathscr{S}(\mathbb{R}^m)$.
(2) There exist $N, M \in \mathbb{R}$ such that A extends by continuity to a bounded operator
$$A\colon \langle x\rangle^{-M} H^M(\mathbb{R}^m) \to \langle x\rangle^N H^{-N}(\mathbb{R}^n). \tag{2.52}$$
(*Hint.* An estimate from Exercise 2.7 will come in handy, in the form $\|\psi\|_k \leq C_k\|\psi\|_{\langle x\rangle^{-M}H^M(\mathbb{R}^n)}$ for given k and sufficiently large M.)
(3) The operator
$$A' := \langle D\rangle^{-N-n/2-1}\langle x\rangle^{-N} A \langle D\rangle^{-M-m/2-1}\langle x\rangle^{-M} \tag{2.53}$$
is bounded from $H^{-m/2-1}(\mathbb{R}^m)$ to $\mathcal{C}_b^0(\mathbb{R}^n)$
(4) Evaluate $A'\delta_y$ for $y \in \mathbb{R}^m$ and deduce that A' has a Schwartz kernel $K' \in \mathcal{C}_b^0(\mathbb{R}^{n+m})$.
(5) By relating the Schwartz kernels of A' and A, prove that $A = O_K$ for some $K \in \mathscr{S}'(\mathbb{R}^{n+m})$.

Exercise 2.10 (Operators with Schwartz Schwartz kernels). Let $A\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ be continuous, and denote by $K \in \mathscr{S}'(\mathbb{R}^{2n})$ its Schwartz kernel. Show that $K \in \mathscr{S}(\mathbb{R}^{2n})$ if and only if A maps $\mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ and as such moreover extends by continuity to a bounded map $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$.

Exercise 2.11 (Peetre's Theorem). Let $A\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ be a continuous linear operator, and suppose for all $u \in \mathscr{S}(\mathbb{R}^n)$, we have $\operatorname{supp} Au \subset \operatorname{supp} u$. Prove that A is a differential operator. (*Hint.* Show that the Schwartz kernel K of A has support in the diagonal $\{x = y\}$. Then show that it must be a locally finite linear combination of (differentiated) δ-distributions with smooth coefficients. To prove that A is a differential operator of *finite* order, exploit that K is a *tempered* distribution.)

Exercise 2.12 (Principal symbol via oscillatory testing). Show that the principal symbol $\sigma^m(A)$ of $A \in \mathrm{Diff}^m(\mathbb{R}^n)$ captures the 'high frequency behavior' of A in the following sense: for $x_0, \xi_0 \in \mathbb{R}^n$, we have

$$\sigma^m(A)(x_0,\xi_0) = \lim_{\lambda\to\infty} \lambda^{-m}(e^{-i\lambda\xi_0\cdot} A e^{i\lambda\xi_0\cdot})(x_0), \tag{2.54}$$

where $e^{i\xi_0\cdot}$ is the function $x \mapsto e^{i\xi_0\cdot x}$.

Exercise 2.13 (Behavior of the principal symbol). Prove Proposition 2.20.

3. Symbols

As a first step towards the definition of pseudodifferential operators, we generalize the class of symbols $a(x,\xi)$ from polynomials in ξ to more general functions:

Definition 3.1 (Symbols)**.** Let $m \in \mathbb{R}$, $n, N \in \mathbb{N}$. Then the space of *(uniform) symbols of order m*

$$S^m(\mathbb{R}^n;\mathbb{R}^N) \subset \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^N) \tag{3.1}$$

consists of all functions $a = a(x,\xi)$ which for all $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}_0^N$ satisfy the estimate

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|}. \tag{3.2}$$

for some constants $C_{\alpha\beta}$. We also write

$$S^m(\mathbb{R}^N) := S^m(\mathbb{R}^0;\mathbb{R}^N) \tag{3.3}$$

for symbols only depending on the symbolic variable ξ.

The gain of decay upon differentiation in the ξ-variables is often called *symbolic behavior* (in ξ).

Remark 3.2 (Alternative notation). Sometimes these symbol classes are denoted $S^m_\infty(\mathbb{R}^n;\mathbb{R}^N)$, the subscript '$\infty$' indicating the uniform boundedness in $\mathcal{C}^\infty$ of the 'coefficients', i.e. the x-variables. There exist many generalizations and variants of the class $S^m(\mathbb{R}^n;\mathbb{R}^N)$, such as: symbols of type ρ,δ; symbols which in addition have symbolic behavior in x (these are symbols of *scattering* (pseudo)differential operators); or symbols with joint symbolic behavior in (x,ξ) (symbols of *isotropic* operators). See [Mel07, §4] and [Hör71b, §1.1].

Equipped with the norms given by the best constants in (3.2), or more concisely

$$\|a\|_{m,k} := \sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^N} \max_{|\alpha|+|\beta|\leq k} \langle\xi\rangle^{-m+|\beta|}|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|, \tag{3.4}$$

the space $S^m(\mathbb{R}^n;\mathbb{R}^N)$ is a Fréchet space. Directly from the definition, we note that differentiations

$$\begin{aligned} D_x^\alpha\colon\ & S^m(\mathbb{R}^n;\mathbb{R}^N) \to S^m(\mathbb{R}^n;\mathbb{R}^N), \\ D_\xi^\beta\colon\ & S^m(\mathbb{R}^n;\mathbb{R}^N) \to S^{m-|\beta|}(\mathbb{R}^n;\mathbb{R}^N) \end{aligned} \tag{3.5}$$

are continuous.

Example 3.3. Full symbols of differential operators of order m on $\mathbb{R}^n$, see (2.35), lie in $S^m(\mathbb{R}^n;\mathbb{R}^n)$. A special case of this is: given $a \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, the function $(x,\xi) \mapsto a(x)$ lies in $S^0(\mathbb{R}^n;\mathbb{R}^N)$ (for any N).

Example 3.4. Let $m \in \mathbb{R}$. Then $\langle\xi\rangle^m \in S^m(\mathbb{R}^n;\mathbb{R}^n)$. (See Exercise 3.1.)

Proposition 3.5 (Multiplication of symbols)**.** *Pointwise multiplication of symbols is a continuous bilinear map*

$$S^m(\mathbb{R}^n;\mathbb{R}^N)\times S^{m'}(\mathbb{R}^n;\mathbb{R}^N)\to S^{m+m'}(\mathbb{R}^n;\mathbb{R}^N). \tag{3.6}$$

Proof. This follows from the Leibniz rule: for $a\in S^m(\mathbb{R}^n;\mathbb{R}^N)$, $b\in S^{m'}(\mathbb{R}^n;\mathbb{R}^N)$, and $\alpha\in\mathbb{N}_0^n$, $\beta\in\mathbb{N}_0^N$, we have

$$\begin{aligned}|\partial_x^\alpha\partial_\xi^\beta(a\cdot b)| &= \Bigg|\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}(\partial_x^{\alpha'}\partial_\xi^{\beta'}a)(\partial_x^{\alpha''}\partial_\xi^{\beta''}b)\Bigg| \\ &\leq \sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} C_{\alpha'\beta'}C_{\alpha''\beta''}\langle\xi\rangle^{m+m'-|\beta'|-|\beta''|} \\ &\leq C_{\alpha\beta}\langle\xi\rangle^{m+m'-|\beta|}.\end{aligned}$$

□

We note the trivial continuous inclusion

$$m\leq m' \quad\Longrightarrow\quad S^m(\mathbb{R}^n;\mathbb{R}^N)\subseteq S^{m'}(\mathbb{R}^n;\mathbb{R}^N), \tag{3.7}$$

hence the $S^m(\mathbb{R}^n;\mathbb{R}^N)$ give a filtration of the space of all symbols $\bigcup_{m\in\mathbb{R}} S^m(\mathbb{R}^n;\mathbb{R}^N)$. In the other direction, we define the space of *residual symbols* by

$$S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N) := \bigcap_{m\in\mathbb{R}} S^m(\mathbb{R}^n;\mathbb{R}^N). \tag{3.8}$$

Equipped with the norms $\|\cdot\|_{m,k}$, $m,k\in\mathbb{N}$, this is again a Fréchet space.

Example 3.6. We have $\mathscr{S}(\mathbb{R}^n\times\mathbb{R}^N)\subset S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N)$, or more generally $\mathcal{C}_b^\infty(\mathbb{R}^n,\mathscr{S}(\mathbb{R}^N))\subset S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N)$. Moreover, given a cutoff $\chi\in\mathcal{C}_c^\infty(\mathbb{R}^N)$, its pullback along $\mathbb{R}^n\times\mathbb{R}^N\ni(x,\xi)\mapsto\xi$, i.e. $(x,\xi)\mapsto\chi(\xi)$, is a residual symbol.

While the inclusion (3.7) never has dense range for $m<m'$, there is a satisfying replacement:

Proposition 3.7 (Density properties of symbol spaces)**.** *Let $m<m'$. Then $S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N)$ is a dense subspace of $S^m(\mathbb{R}^n;\mathbb{R}^N)$ in the topology of $S^{m'}(\mathbb{R}^n;\mathbb{R}^N)$. In fact, a stronger statement is true: for any $a\in S^m(\mathbb{R}^n;\mathbb{R}^N)$ there exists a sequence $a_j\in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N)$ which is uniformly bounded in $S^m(\mathbb{R}^n;\mathbb{R}^N)$ and converges to a in the topology of $S^{m'}(\mathbb{R}^n;\mathbb{R}^N)$.*

Proof. Fix a cutoff function $\chi\in\mathcal{C}_c^\infty(\mathbb{R}^N)\subset S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N)$ (see Example 3.6) which is identically 1 in $|\xi|\leq 1$ and identically 0 when $|\xi|\geq 2$. By Proposition 3.5, we have

$$a_j(x,\xi) := a(x,\xi)\chi(\xi/j)\in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N). \tag{3.9}$$

To prove the proposition, it suffices to show, in view of Proposition 3.5, that

$$\chi_j(\xi) := \chi(\xi/j) \tag{3.10}$$

is bounded in $S^0(\mathbb{R}^N)$ and converges to 1 in the topology of $S^\epsilon(\mathbb{R}^N)$ for all $\epsilon>0$. Regarding the former, we have $|\chi_j(\xi)|\leq\|\chi\|_{0,0}$ for all j, while for $|\beta|\geq 1$ we have $\partial_\xi^\beta\chi_j(\xi)\equiv 0$ for $|\xi|\leq 1$, and

$$|\xi|^{|\beta|}\partial_\xi^\beta\chi_j(\xi)=\chi_\beta(\xi/j),\quad \chi_\beta(\xi)=|\xi|^{|\beta|}(\partial_\xi^\beta\chi)(\xi)\in\mathcal{C}_c^\infty(\mathbb{R}^N). \tag{3.11}$$

Regarding the latter, we note that $\mathrm{supp}(\chi_j - 1) \subset \{|\xi| \geq j\}$, hence

$$|\chi(\xi/j) - 1| \leq j^{-\epsilon}\langle\xi\rangle^{\epsilon}. \tag{3.12}$$

For derivatives, we note that the support observation and (3.11) give

$$|\xi|^{|\beta|-\epsilon}|\partial_\xi^\beta(\chi_j(\xi) - 1)| = |\xi|^{|\beta|-\epsilon}|\partial_\xi^\beta \chi_j(\xi)| \leq j^{-\epsilon}|\chi_\beta(\xi/j)|. \tag{3.13}$$

Thus, $\|\chi_j - 1\|_{\epsilon,k} \leq C_{k\epsilon} j^{-\epsilon} \to 0$ as $j \to \infty$, as desired. □

3.1. **Ellipticity.** We now generalize the key property of the symbol of the operator $L = \Delta + 1$ in (1.3).

Definition 3.8 (Elliptic symbols)**.** Let $m \in \mathbb{R}$. A symbol $a \in S^m(\mathbb{R}^n;\mathbb{R}^N)$ is *(uniformly) elliptic* if there exists a symbol $b \in S^{-m}(\mathbb{R}^n;\mathbb{R}^N)$ such that $ab - 1 \in S^{-1}(\mathbb{R}^n;\mathbb{R}^N)$.

Proposition 3.9 (Equivalent formulations of ellipticity)**.** *Let $m \in \mathbb{R}$, and $a \in S^m(\mathbb{R}^n;\mathbb{R}^N)$. Then the following are equivalent:*

(1) *a is elliptic.*
(2) *There exist constants $C, c > 0$ such that*

$$|\xi| \geq C \quad\Longrightarrow\quad |a(x,\xi)| \geq c|\xi|^m. \tag{3.14}$$

(3) *There exist constants $C, c > 0$ such that*

$$|a(x,\xi)| \geq c|\xi|^m - C|\xi|^{m-1}, \quad |\xi| \geq 1. \tag{3.15}$$

Proof. If a is elliptic, then in the notation of Definition 3.8, we have

$$1 - C\langle\xi\rangle^{-1} \leq |a(x,\xi)||b(x,\xi)| \leq C|a(x,\xi)|\langle\xi\rangle^{-m}, \tag{3.16}$$

for some constant $C > 0$, that is,

$$|a(x,\xi)| \geq c\langle\xi\rangle^m - \langle\xi\rangle^{m-1}. \tag{3.17}$$

This implies (3.15), since $\frac{\langle\xi\rangle}{|\xi|} \in (1, \sqrt{2}]$ for $|\xi| \geq 1$. This in turn implies (3.14) since for all $c > 0$, there exists $C > 0$ such that $|\xi|^{m-1} \leq c|\xi|^m$ for $|\xi| \geq C$ (indeed, this holds for $C = c^{-1}$).

Conversely, if (3.14) holds, choose a cutoff $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$, $\chi(\xi) = 0$ for $|\xi| \leq 2C$, $\chi(\xi) = 1$ for $|\xi| \geq 3C$, then (see Exercise 3.2)

$$b(x,\xi) := \chi(\xi)/a(x,\xi) \in S^{-m}(\mathbb{R}^n;\mathbb{R}^N), \tag{3.18}$$

and $a(x,\xi)b(x,\xi) = \chi(\xi) \in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N)$. □

Note that if $a \in S^m(\mathbb{R}^n;\mathbb{R}^N)$ is elliptic, then so is $a + a'$ for any $a' \in S^{m-1}(\mathbb{R}^n;\mathbb{R}^N)$. Thus, ellipticity is only a condition on the equivalence class

$$[a] \in S^m(\mathbb{R}^n;\mathbb{R}^N)/S^{m-1}(\mathbb{R}^n;\mathbb{R}^N). \tag{3.19}$$

For full symbols of differential operators, we can identify $[a]$ with the leading order, homogeneous of degree m, part of a. Compare with Definition 2.19 and Proposition 2.20.

3.2. **Classical symbols.** An important subclass of symbols mimics those of differential operators: they are sums of homogeneous (in ξ) functions. More precisely, we call a function $a(x,\xi)$, defined for $\xi\neq 0$, (positively) homogeneous of order $m\in\mathbb{C}$ iff

$$a(x,\lambda\xi)=\lambda^m a(x,\xi),\quad \lambda>0. \tag{3.20}$$

Definition 3.10 (Homogeneous symbols)**.** Let $m\in\mathbb{R}$.[1] Then $S^m_{\rm hom}(\mathbb{R}^n;\mathbb{R}^N\setminus\{0\})$ is the space of all functions $a(x,\xi)\in\mathcal{C}^\infty(\mathbb{R}^n\times(\mathbb{R}^N\setminus\{0\}))$, positively homogeneous of order m in ξ, such that for all $\alpha,\beta\in\mathbb{N}_0^n$

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\leq C_{\alpha\beta}|\xi|^{m-|\beta|},\quad \xi\neq 0. \tag{3.21}$$

Definition 3.11 (Classical symbols)**.** Let $m\in\mathbb{R}$, and fix a cutoff $\chi\in\mathcal{C}^\infty_{\rm c}(\mathbb{R}^N)$ which is identically 1 near 0. A symbol $a\in S^m(\mathbb{R}^n;\mathbb{R}^N)$ is called a *classical symbol of order* m if there exist functions $a_{m-j}\in S^{m-j}_{\rm hom}(\mathbb{R}^n\times(\mathbb{R}^N\setminus\{0\}))$ such that for all $J\in\mathbb{N}$, we have

$$a-\sum_{j=0}^{J-1}(1-\chi)a_{m-j}\in S^{m-J}(\mathbb{R}^n;\mathbb{R}^N). \tag{3.22}$$

The space of classical symbols of order m is denoted $S^m_{\rm cl}(\mathbb{R}^n;\mathbb{R}^N)$. Finally, we put

$$S^{-\infty}_{\rm cl}(\mathbb{R}^n;\mathbb{R}^N):=S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N). \tag{3.23}$$

Equipped with the seminorms of a_{m-j} and the remainders $a-\sum_{j=0}^{J-1}(1-\chi)a_{m-j}$ in the respective spaces, $S^m_{\rm cl}(\mathbb{R}^n;\mathbb{R}^N)$ is a Fréchet space. Proposition 3.7 fails dramatically for classical symbols; indeed (Exercise 3.4),

$$S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N)\subset S^m_{\rm cl}(\mathbb{R}^n;\mathbb{R}^N)\quad\text{is closed for any } m\in\mathbb{R}. \tag{3.24}$$

We have the following straightforward lemma (Exercise 3.5):

Lemma 3.12 (Homogeneous components of classical symbols)**.** *The homogeneous terms* a_{m-j} *in* (3.22) *are uniquely determined by* a.

For $a\in S^m_{\rm cl}(\mathbb{R}^n;\mathbb{R}^N)$ as in Definition 3.11, we can thus identify the equivalence class $[a]\in S^m(\mathbb{R}^n;\mathbb{R}^N)/S^{m-1}(\mathbb{R}^n;\mathbb{R}^N)$ with the leading order homogeneous part a_m, or even more simply with the function $\mathbb{R}^n\times\mathbb{S}^{N-1}\ni(x,\xi)\mapsto a_m(x,\xi)$, where $\mathbb{S}^{N-1}=\{\xi\in\mathbb{R}^N\colon|\xi|=1\}$ is the unit sphere. Cf. (2.41).

3.3. **Asymptotic summation.** There is a (general) 'converse' to (3.22) which is very useful when performing iterative constructions which yield lower order corrections:

Proposition 3.13 (Existence and uniqueness of asymptotic sums)**.** *Let* $a_j\in S^{m_j}(\mathbb{R}^n;\mathbb{R}^N)$, $j\geq 0$, *and suppose* $\limsup_{j\to\infty}m_j=-\infty$. *Let* $\bar m_j:=\sup_{j'\geq j}m_{j'}$, *and* $m=\bar m_0$. *Then there exists a symbol* $a\in S^m(\mathbb{R}^n;\mathbb{R}^N)$ *such that for all* $J\in\mathbb{N}$

$$a-\sum_{j=0}^{J-1}a_j\in S^{\bar m_J}(\mathbb{R}^n;\mathbb{R}^N). \tag{3.25}$$

Moreover, a *is unique modulo* $S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N)$.

[1] One can allow m to be complex without any further work, but we do not need this level of generality in these notes.

We call a 'the' asymptotic sum of the a_j, and write

$$a \sim \sum_{j=0}^{\infty} a_j. \tag{3.26}$$

Proof of Proposition 3.13. This is similar to Borel's theorem concerning the existence of a smooth function with prescribed Taylor series at 0. Uniqueness is clear, since any two asymptotic sums a, a' satisfy $a - a' \in S^{\bar m_J}(\mathbb{R}^n; \mathbb{R}^N)$, with $\bar m_J \to -\infty$, hence $a - a'$ is residual indeed.

For existence, we may partially sum finitely many of the a_j and thereby reduce to the case that $a_j \in S^{m-j}(\mathbb{R}^n; \mathbb{R}^N)$, $j \geq 0$, and $\bar m_j = m - j$. Fix a cutoff $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$, identically 0 in $|\xi| \leq 1$ and equal to 1 for $|\xi| \geq 2$. With $\epsilon_j > 0$, $\epsilon_j \to 0$, to be determined, we wish to set

$$a(x,\xi) := \sum_{j=0}^{\infty} \chi(\epsilon_j \xi) a_j(x,\xi). \tag{3.27}$$

This sum is locally finite, hence $a \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^N)$. Choosing ϵ_j more precisely, we can arrange that

$$\|\chi(\epsilon_j \cdot) a_j\|_{m-j',j'} \leq 2^{-j}, \quad j > j' \geq 0. \tag{3.28}$$

Indeed, for fixed j, j', we can choose $\epsilon_j > 0$ such that this holds since $\chi(\epsilon_j \cdot) a_j \to 0$ in $S^{m-j'}(\mathbb{R}^n; \mathbb{R}^N)$ as $\epsilon_j \to 0$, as in the proof of Proposition 3.7; but for any fixed j, (3.28) gives a *finite* number of conditions on ϵ_j, one for each $0 \leq j' < j$.

But then $\chi(\epsilon_{j'}\xi) a_{j'}(x,\xi) + \sum_{j=j'+1}^{\infty} \chi(\epsilon_j \xi) a_j(x,\xi)$ converges in $S^{m-j'}(\mathbb{R}^n; \mathbb{R}^N)$. Thus, the sequence (3.27) converges in $S^m(\mathbb{R}^n; \mathbb{R}^N)$, and we have

$$a(x,\xi) - \sum_{j=0}^{J-1} a_j(x,\xi) = \sum_{j=0}^{J-1} (1 - \chi(\epsilon_j \xi)) a_j(x,\xi) + \sum_{j=J}^{\infty} \chi(\epsilon_j \xi) a_j(x,\xi) \in S^{m-J}(\mathbb{R}^n; \mathbb{R}^N), \tag{3.29}$$

as desired. □

The space $S^m_{\mathrm{cl}}(\mathbb{R}^n; \mathbb{R}^N)$ can be characterized as the space of symbols in $S^m(\mathbb{R}^n; \mathbb{R}^N)$ which are asymptotic sums of symbols which in $|\xi| \geq 1$ are positively homogeneous of degree $m - j$, $j \in \mathbb{N}_0$.

For completeness and later use, we refine the previous result to ensure the continuous dependence of a on the sequence (a_j).

Proposition 3.14 (Continuous asymptotic summation)**.** *Denote by*

$$\ell S^m(\mathbb{R}^n; \mathbb{R}^N) := \prod_{j=0}^{\infty} S^{m-j}(\mathbb{R}^n; \mathbb{R}^N) \tag{3.30}$$

be the space of all sequences $(a_0, a_1, \ldots)$ of symbols $a_j \in S^{m-j}(\mathbb{R}^n; \mathbb{R}^N)$. Equip ℓS^m with the topology generated by the seminorms $\|(a_j)\|_J := \max_{1 \leq k \leq J} \|a_k\|_{m-k,J}$. Then there exists a continuous (nonlinear) map

$$\sum\nolimits_{\mathcal{A}} \colon \ell S^m(\mathbb{R}^n; \mathbb{R}^N) \to S^m(\mathbb{R}^n; \mathbb{R}^N) \tag{3.31}$$

with the property that $\sum_{\mathcal{A}}((a_j)_{j \in \mathbb{N}_0}) \sim \sum_{j=0}^{\infty} a_j$.

Remark 3.15 (Comparison with $\mathcal{C}^\infty(\mathbb{R}^n)$). The topology on $\ell S^m(\mathbb{R}^n;\mathbb{R}^N)$ is akin to e.g. the standard topology on $\mathcal{C}^\infty(\mathbb{R}^n)$ which is given by seminorms $\|\cdot\|_{\mathcal{C}^k(B(0,k))}$. To verify convergence of a sequence of sequences of symbols in this topology, one merely needs to check that for any fixed $J\in\mathbb{N}$, the first J terms of the sequence converge in the respective symbol spaces.

Proof of Proposition 3.14. Fix $\chi\in\mathcal{C}^\infty(\mathbb{R}^N)$, $\chi(\xi)=0$ for $|\xi|\le 1$ and $\chi(\xi)=1$ for $|\xi|\ge 2$. As in the previous proof, we shall set, for $a=(a_j)_{j\in\mathbb{N}_0}\in\ell S^m(\mathbb{R}^n;\mathbb{R}^N)$,

$$\Big(\sum\nolimits_{\mathcal{A}} a\Big)(x,\xi) := \sum_{j=0}^\infty \chi(\epsilon_j(a)\xi)a_j(x,\xi), \tag{3.32}$$

where $\epsilon_j(a)$, as in (3.28), is chosen so that for all $j\in\mathbb{N}$

$$\max_{0\le j'\le j-1}\|\chi(\epsilon_j(a)\xi)a_j(x,\xi)\|_{m-j',j'}\le 2^{-j}, \tag{3.33}$$

and we set $\epsilon_0(a)=1$. We now need to make a concrete choice of $\epsilon_j(a)$: to this effect, we note that for $|\alpha|+|\beta|\le j'\le j-1$,

$$\begin{aligned}\langle\xi\rangle^{-m+j'}\left|\partial_x^\alpha\partial_\xi^\beta\big(\chi(\epsilon_j(a)\xi)a_j(x,\xi)\big)\right| &\le C_j\langle\xi\rangle^{-m+j'}\langle\xi\rangle^{m-j}\|a_j\|_{m-j,j'}1_{|\xi|\ge\epsilon_j(a)^{-1}}\\ &\le C_j\epsilon_j(a)\|a_j\|_{m-j,j},\end{aligned} \tag{3.34}$$

where C_j only depends on χ (and j of course). Therefore (3.33) holds provided we take

$$\epsilon_j(a) := 2^{-j}\big(1+C_j\|a_j\|_{m-j,j}\big)^{-1}. \tag{3.35}$$

With this choice, $\sum_{\mathcal{A}} a$ is well-defined, and $\sum_{\mathcal{A}} a\sim\sum_{j=0}^\infty a_j$.

We now check continuity. Define

$$\chi_j(q,\xi) := \chi(2^{-j}(1+C_jq)^{-1}\xi). \tag{3.36}$$

Fix $a=(a_j)_{j\in\mathbb{N}_0}\in\ell S^m(\mathbb{R}^n;\mathbb{R}^N)$, and fix $k\in\mathbb{N}_0$, $\epsilon>0$. We need to show that there exist $\delta>0$ and $J\in\mathbb{N}$ such that

$$a'\in\ell S^m(\mathbb{R}^n;\mathbb{R}^N),\ \|a-a'\|_J\le\delta \implies \Big\|\sum\nolimits_{\mathcal{A}} a-\sum\nolimits_{\mathcal{A}} a'\Big\|_{m,k}<\epsilon, \tag{3.37}$$

which holds provided

$$\sum_{j=0}^\infty\big\|\chi_j(\|a_j\|_{m-j,j},\xi)a_j-\chi_j(\|a'_j\|_{m-j,j},\xi)a'_j\big\|_{m,k}<\epsilon. \tag{3.38}$$

The j-th summand can individually be estimated by

$$\begin{aligned}&\|\chi_j(\|a_j\|_{m-j,j},\xi)(a_j-a'_j)\|_{m,k}+\big\|\big(\chi_j(\|a_j\|_{m-j,j},\xi)-\chi_j(\|a'_j\|_{m-j,j},\xi)\big)a'_j\big\|_{m,k}\\ &\quad\le C_j\|a_j-a'_j\|_{m,k}+\big|\|a_j\|_{m-j,j}-\|a'_j\|_{m-j,j}\big|\,\|a'_j\|_{m,k}\\ &\quad\le\big(C_j+\|a_j\|_{m,k}+\|a-a'\|_{\max(j,k)}\big)\|a-a'\|_{\max(j,k)},\end{aligned} \tag{3.39}$$

which tends to zero as $a'\to a$ in $\ell S^m(\mathbb{R}^n;\mathbb{R}^N)$.

The tail of the sum (3.38) on the other hand is estimated simply using (3.33)

$$\|\chi_j(\|a_j\|_{m-j,j},\xi)a_j\|_{m,k}+\|\chi_j(\|a'_j\|_{m-j,j},\xi)a'_j\|_{m,k}\le 2^{-j}+2^{-j}=2^{-j+1} \tag{3.40}$$

provided $j>k$. Thus, we first choose $J_0\in\mathbb{N}$, $J_0>k$, such that $\sum_{j=J_0}^\infty 2^{-j+1}<\epsilon/2$, and then $\delta>0$, $J\in\mathbb{N}$ such that $a'\in\ell S^m(\mathbb{R}^n;\mathbb{R}^N)$, $\|a-a'\|_J<\delta$ implies that the j-th summand in (3.38) is bounded by $\epsilon/(2J_0)$ for $j = 0, \ldots, J_0 - 1$. This achieves (3.37) and thus finishes the proof. □

3.4. **Exercises.**

Exercise 3.1 (Symbols and classical symbols). Let $m \in \mathbb{R}$. Prove $\langle\xi\rangle^m \in S^m(\mathbb{R}^N)$. By expanding into Taylor series in $1/|\xi|$, show that indeed $\langle\xi\rangle^m \in S^m_{\mathrm{cl}}(\mathbb{R}^N)$.

Exercise 3.2 (Inverses of elliptic symbols).
1. Show that if $a \in S^m(\mathbb{R}^n; \mathbb{R}^N)$ satisfies (3.14), and $\chi \in S^0(\mathbb{R}^N)$ vanishes for $|\xi| \le 2C$, then $\chi/a \in S^{-m}(\mathbb{R}^n; \mathbb{R}^N)$.
2. If in addition a and χ are classical symbols, show that χ/a is classical as well.

Exercise 3.3 (Compositions of functions with symbols).
1. Let $f \in \mathcal{C}^\infty(\mathbb{R})$. Show that if $a \in S^0(\mathbb{R}^n; \mathbb{R}^N)$, then also $f \circ a \in S^0(\mathbb{R}^n; \mathbb{R}^n)$.
2. Show that if $a \in S^0(\mathbb{R}^n; \mathbb{R}^N)$ is elliptic and positive, then there exists $b \in S^0(\mathbb{R}^n; \mathbb{R}^N)$ such that $a - b^2 \in S^{-1}(\mathbb{R}^n; \mathbb{R}^N)$.

Exercise 3.4 (Residual symbols and classical symbols). Prove (3.24).

Exercise 3.5 (Homogeneous components of classical symbols). Prove Lemma 3.12. (*Hint.* Use induction on j; the case $j = 0$ is the main content.)

Exercise 3.6 (Nonlinear character of asymptotic summation). Show that there does not exist a map $\sum_{\mathcal{A}}\colon \ell S^m(\mathbb{R}^n; \mathbb{R}^N) \to S^m(\mathbb{R}^n; \mathbb{R}^N)$ with $\sum_{\mathcal{A}}((a_j)_{j\in\mathbb{N}_0}) \sim \sum_{j=0}^\infty a_j$ which is both continuous and linear.

4. Pseudodifferential operators

For developing the theory of ps.d.o.s, it is useful to consider slightly more general symbols, in the class

$$\langle x-y\rangle^w S^m(\mathbb{R}^n_x \times \mathbb{R}^n_y; \mathbb{R}^n_\xi) = \{\langle x-y\rangle^w \tilde a\colon \tilde a \in S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)\}, \tag{4.1}$$

where $w \in \mathbb{R}$. Our immediate goal will be to make sense of the following definition.

Definition 4.1 (Quantization). Let $m, w \in \mathbb{R}$, and $a \in \langle x-y\rangle^w S^m(\mathbb{R}^n_x \times \mathbb{R}^n_y; \mathbb{R}^n_\xi)$. Then we define its *quantization* $\mathrm{Op}(a)$ by

$$(\mathrm{Op}(a)u)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,y,\xi) u(y)\,\mathrm{d}y\,\mathrm{d}\xi, \quad u \in \mathscr{S}(\mathbb{R}^n). \tag{4.2}$$

Previously, see (2.36), we only considered the special case of the *left quantization* of a *left symbol* $a \in S^m(\mathbb{R}^n_x; \mathbb{R}^n_\xi)$, independent of y:

$$(\mathrm{Op}_L(a)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,\xi) u(y)\,\mathrm{d}y\,\mathrm{d}\xi; \tag{4.3}$$

this immediately makes sense as an iterated integral for $u \in \mathscr{S}(\mathbb{R}^n)$, and should be thought of as 'differentiate first, then multiply by coefficients'. Dually, we can consider the *right quantization* of a *right symbol* $a \in S^m(\mathbb{R}^n_y; \mathbb{R}^n_\xi)$,

$$(\mathrm{Op}_R(a)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(y,\xi) u(y)\,\mathrm{d}y\,\mathrm{d}\xi, \tag{4.4}$$

which does not immediately make sense (similarly to (4.2)); this should be thought of as 'multiply by coefficients, then differentiate'. Indeed, for $a(z,\xi)=\xi^\alpha a_\alpha(z)$ with $a_\alpha\in\mathcal{C}_b^\infty(\mathbb{R}^n)$, we have

$$(\mathrm{Op}_L(a)u)(x)=a_\alpha(x)D_x^\alpha u(x),\qquad (\mathrm{Op}_R(a)u)(x)=D_x^\alpha(a_\alpha(x)u(x)). \tag{4.5}$$

The quantization map (4.2) should be read as 'multiply (y), then differentiate (ξ), then multiply (x)'. (Try this with $a(x,y,\xi)=a_1(x)\xi^\alpha a_2(y)$.) We shall see below that every operator $\mathrm{Op}(a)$ can be written as $\mathrm{Op}(a)=\mathrm{Op}_L(a_L)=\mathrm{Op}_R(a_R)$ for suitable left and right symbols a_L and a_R of the same order as a, see §4.1. (You have done most of the work for proving this for *differential* operators, i.e. in the case that a is a polynomial in ξ, in Exercise 2.13.)

Lemma 4.2 (Quantization of symbols of very negative order)**.** *Let $w\in\mathbb{R}$, $m<-n$, and let $a=\langle x-y\rangle^w\tilde a$, $\tilde a\in S^m(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$. Then the integral (4.2) is absolutely convergent and defines a continuous operator*

$$\mathrm{Op}(a)\colon \mathscr{S}(\mathbb{R}^n)\to\langle x\rangle^w\mathcal{C}_b^0(\mathbb{R}^n). \tag{4.6}$$

More precisely, for $N>n+|w|$, there exists a constant $C<\infty$ such that

$$\|\mathrm{Op}(a)u\|_{\langle x\rangle^w\mathcal{C}_b^0(\mathbb{R}^n)}\le C\|\tilde a\|_{m,0}\|u\|_N,\quad u\in\mathscr{S}(\mathbb{R}^n). \tag{4.7}$$

For the proof, we need a simple lemma:

Lemma 4.3 (Peetre's inequality)**.** *Let $w\in\mathbb{R}$. Then $\langle x+y\rangle^w\le 2^{|w|/2}\langle x\rangle^w\langle y\rangle^{|w|}$.*

Proof. By the triangle and Cauchy–Schwarz inequalities, we have

$$1+|x+y|^2\le 1+2|x|^2+2|y|^2\le 2(1+|x|^2)(1+|y|^2). \tag{4.8}$$

If $w>0$, then taking this to the power $w/2$ proves the lemma. For $w=0$, the lemma is the equality $1=1$. For $w<0$, hence $-w>0$, we obtain, analogously to (4.8),

$$\langle x\rangle^{-w}\le 2^{-w/2}\langle x+y\rangle^{-w}\langle y\rangle^{-w}, \tag{4.9}$$

which upon multiplication by $\langle x\rangle^w\langle x+y\rangle^w$ gives the desired result. □

Proof of Lemma 4.2. Since u is Schwartz, we have $|u(y)|\le C_N\|u\|_N\langle y\rangle^{-N}$ for all $N\in\mathbb{N}_0$. Therefore, the integrand in (4.2) satisfies

$$\begin{aligned}|e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)| &\le C\langle x-y\rangle^w\|\tilde a\|_{m,0}\langle\xi\rangle^m\cdot\|u\|_N\langle y\rangle^{-N}\\ &\le C\langle x\rangle^w\cdot\langle\xi\rangle^m\langle y\rangle^{|w|-N}\cdot\|\tilde a\|_{m,0}\|u\|_N.\end{aligned} \tag{4.10}$$

This is integrable in (y,ξ) provided $m<-n$ and $|w|-N<-n$, proving the lemma. □

Proposition 4.4 (Bounds on quantizations of residual symbols)**.** *Let $w\in\mathbb{R}$ and $a=\langle x-y\rangle^w\tilde a$, $\tilde a\in S^{-\infty}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$. Then the quantization $\mathrm{Op}(a)\colon\mathscr{S}(\mathbb{R}^n)\to\mathscr{S}(\mathbb{R}^n)$ is continuous. In fact, for all $k\in\mathbb{N}_0$, $m\in\mathbb{R}$, there exist $N\in\mathbb{N}$ and a constant C such that*

$$\|\mathrm{Op}(a)u\|_k\le C\|\tilde a\|_{m,N}\|u\|_N. \tag{4.11}$$

Lemma 4.5 (Differentiation of weighted symbols)**.** *Differentiations D_x^α and D_y^α are continuous maps $\langle x-y\rangle^w S^m(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)\to\langle x-y\rangle^w S^m(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$. More precisely,*

$$\|\langle x-y\rangle^{-w}D_x^\alpha a\|_{m,k}\le C\|\langle x-y\rangle^{-w}a\|_{m,k+|\alpha|}, \tag{4.12}$$

likewise for $D_y^\alpha a$.

Proof. It suffices to prove the claim for D_{x_1}. For $a(x,y,\xi)=\langle x-y\rangle^w\tilde a(x,y,\xi)$, $\tilde a\in S^m(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$, we have

$$\partial_{x_1}a=\langle x-y\rangle^w(\partial_{x_1}\tilde a)+w\langle x-y\rangle^{w-2}(x_1-y_1)\tilde a. \tag{4.13}$$

The first summand lies in $\langle x-y\rangle^w S^m(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$, and the second summand even lies in the smaller space $\langle x-y\rangle^{w-1}S^m(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$. □

Proof of Proposition 4.4. The key is that for $\xi\neq 0$, the phase $(x-y)\cdot\xi$ has no critical points in y. We exploit this by writing

$$(1-\xi\cdot D_y)e^{i(x-y)\cdot\xi}=\langle\xi\rangle^2 e^{i(x-y)\cdot\xi}, \tag{4.14}$$

so upon integrating by parts in y, one gains decay in ξ. Concretely, for $N\in\mathbb{N}$, we have

$$\begin{aligned}\mathrm{Op}(a)u(x)&=(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}((1-\xi\cdot D_y)^N e^{i(x-y)\cdot\xi})\langle\xi\rangle^{-2N}a(x,y,\xi)u(y)\,\mathrm{d}y\,\mathrm{d}\xi\\&=(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i(x-y)\cdot\xi}(1+\xi\cdot D_y)^N\big(\langle\xi\rangle^{-2N}a(x,y,\xi)u(y)\big)\,\mathrm{d}y\,\mathrm{d}\xi.\end{aligned} \tag{4.15}$$

By the Leibniz rule, we have

$$(1+\xi\cdot D_y)^N\big(\langle\xi\rangle^{-2N}a(x,y,\xi)u(y)\big)=\sum_{|\gamma|\le N}a_\gamma(x,y,\xi)\cdot D_y^\gamma u, \tag{4.16}$$

where

$$a_\gamma(x,y,\xi)=\sum_{|\delta|,|\epsilon|\le N}c_{\gamma\delta\epsilon}\langle\xi\rangle^{-2N}\xi^\delta D_y^\epsilon a(x,y,\xi) \tag{4.17}$$

for some combinatorial constants $c_{\gamma\delta\epsilon}$. By Lemma 4.5, we have $\tilde a_\gamma:=\langle x-y\rangle^{-w}a_\gamma\in S^{-\infty}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$, and setting $\tilde a:=\langle x-y\rangle^{-w}a$, we have, for any $m\in\mathbb{R}$,

$$\|\tilde a_\gamma\|_{m-N,0}\le C\|\tilde a\|_{m,N}. \tag{4.18}$$

Thus, if $N>m+n$, Lemma 4.2 applies, giving

$$\|\mathrm{Op}(a_\gamma)D^\gamma u\|_{\langle x\rangle^w\mathcal{C}^0(\mathbb{R}^n)}\le C\|\tilde a_\gamma\|_{m-N,0}\|D^\gamma u\|_M,\quad M>n+|w|, \tag{4.19}$$

and therefore

$$\|\mathrm{Op}(a)u\|_{\langle x\rangle^w\mathcal{C}^0(\mathbb{R}^n)}\le C\|\tilde a\|_{m,N}\|u\|_M,\quad M>n+N+|w|. \tag{4.20}$$

To get higher regularity and decay, let now $\alpha,\beta\in\mathbb{N}_0^n$, then

$$\begin{aligned}x^\alpha D_x^\beta\,\mathrm{Op}(a)u(x)&=(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big((D_\xi+y)^\alpha e^{i(x-y)\cdot\xi}\big)(\xi+D_x)^\beta a(x,y,\xi)u(y)\,\mathrm{d}y\,\mathrm{d}\xi\\&=(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i(x-y)\cdot\xi}(y-D_\xi)^\alpha(\xi+D_x)^\beta\big(a(x,y,\xi)u(y)\big)\,\mathrm{d}y\,\mathrm{d}\xi.\end{aligned} \tag{4.21}$$

This can be expanded using the Leibniz rule; note that powers of y are acceptable since u is Schwartz. We thus obtain

$$\|x^\alpha D_x^\beta\,\mathrm{Op}(a)u\|_{\langle x\rangle^w\mathcal{C}^0(\mathbb{R}^n)}\le C\|\tilde a\|_{m,N}\|u\|_N \tag{4.22}$$

for N sufficiently large (depending on m,n,α,β). Thus, $\mathrm{Op}(a)u\in\mathscr{S}(\mathbb{R}^n)$, finishing the proof. □

This shows that the map

$$\langle x-y\rangle^w S^{-\infty}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)\times\mathscr{S}(\mathbb{R}^n)\ni(a,u)\mapsto\mathrm{Op}(a)u\in\mathscr{S}(\mathbb{R}^n)\tag{4.23}$$

is a continuous bilinear map when putting the topology of $\langle x-y\rangle^w S^{m'}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$ on the first factor (for any $m'\in\mathbb{R}$). By Proposition 3.7, it thus extends by continuity to a continuous bilinear map

$$\langle x-y\rangle^w S^{m}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)\times\mathscr{S}(\mathbb{R}^n)\ni(a,u)\mapsto\mathrm{Op}(a)u\in\mathscr{S}(\mathbb{R}^n).\tag{4.24}$$

Identifying $\mathrm{Op}(a)$ with its Schwartz kernel, we thus get a continuous map

$$\mathrm{Op}\colon\langle x-y\rangle^w S^{m}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)\to\mathscr{S}'(\mathbb{R}^n\times\mathbb{R}^n),\tag{4.25}$$

which is given (interpreted as a limit along a sequence of residual symbols) by

$$\mathrm{Op}(a)(x,y)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{i(x-y)\cdot\xi}a(x,y,\xi)\,\mathrm{d}\xi.\tag{4.26}$$

(This is of course much weaker than (4.24).)

Remark 4.6 (Quantization via an explicit limit). Let $\chi\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ be identically 1 near 0. Given $a\in\langle x-y\rangle^w S^m(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$, (the proof of) Proposition 3.7 implies that

$$\mathrm{Op}(a)u(x)=\lim_{j\to\infty}(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i(x-y)\cdot\xi}\chi(\xi/j)a(x,y,\xi)u(y)\,\mathrm{d}y\,\mathrm{d}\xi,\tag{4.27}$$

with convergence in $\mathscr{S}(\mathbb{R}^n)$.

Definition 4.7 (Pseudodifferential operators)**.** Let $m\in\mathbb{R}$. The space of *(uniform) pseudodifferential operators of order* m,

$$\Psi^m(\mathbb{R}^n),\tag{4.28}$$

is the space of all operators of the form $\mathrm{Op}(a)\colon\mathscr{S}(\mathbb{R}^n)\to\mathscr{S}(\mathbb{R}^n)$, where $a\in\langle x-y\rangle^w S^m(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$ and $w\in\mathbb{R}$. (As we show in the next section, one can take $w=0$. See Exercise 4.1 for the case of differential operators.) We set

$$\Psi^{-\infty}(\mathbb{R}^n):=\bigcap_{m\in\mathbb{R}^n}\Psi^m(\mathbb{R}^n).\tag{4.29}$$

Note that a priori it is not clear that $\Psi^{-\infty}(\mathbb{R}^n)$ is equal to the space of quantizations of residual symbols (it is certainly contained in the latter); we show this in Proposition 4.10 below.

By duality, we can define the action of $A=\mathrm{Op}(a)\in\Psi^m(\mathbb{R}^n)$ on tempered distributions: for $u,v\in\mathscr{S}(\mathbb{R}^n)$ and $a\in\langle x-y\rangle^w S^{-\infty}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$, we have

$$\begin{aligned}\langle\mathrm{Op}(a)u,v\rangle&=(2\pi)^{-n}\iiint_{\mathbb{R}^{3n}}e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)v(x)\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}x\\&=(2\pi)^{-n}\iiint_{\mathbb{R}^{3n}}e^{i(x-y)\cdot\xi}a(y,x,-\xi)v(y)u(x)\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}x\\&=\langle u,\mathrm{Op}(a^\dagger)v\rangle,\end{aligned}\tag{4.30}$$

where we put

$$a^\dagger(x,y,\xi)=a(y,x,-\xi).\tag{4.31}$$

Since $a\mapsto a^\dagger$ is an isomorphism on $\langle x-y\rangle^w S^m(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$, the equality

$$\mathrm{Op}(a)^\dagger=\mathrm{Op}(a^\dagger),\quad\text{that is,}\quad\langle\mathrm{Op}(a)u,v\rangle=\langle u,\mathrm{Op}(a^\dagger)v\rangle\tag{4.32}$$

continues to hold for $a \in \langle x-y\rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. By the density $\mathscr{S}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$, we can thus uniquely extend, by continuity, $\mathrm{Op}(a)$ to an operator on $\mathscr{S}'(\mathbb{R}^n)$ via (4.32).

4.1. **Left/right reduction, adjoints.** In this section, we shall prove:

Theorem 4.8 (Ps.d.o.s as left/right quantizations)**.** *Let $a \in \langle x-y\rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. Then there exists a unique left symbol $a_L \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ such that*

$$\mathrm{Op}(a) = \mathrm{Op}_L(a_L), \tag{4.33}$$

and a unique right symbol $a_R \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$\mathrm{Op}(a) = \mathrm{Op}_R(a_R). \tag{4.34}$$

The symbols a_L, a_R depend continuously on a. Modulo residual symbols, they are given by asymptotic sums

$$a_L(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\bigl(\partial_\xi^\alpha D_y^\alpha a(x,y,\xi)\bigr)|_{y=x}, \tag{4.35}$$

$$a_R(y,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{(-1)^{|\alpha|}}{\alpha!}\bigl(\partial_\xi^\alpha D_x^\alpha a(x,y,\xi)\bigr)|_{x=y}. \tag{4.36}$$

(The summands are ordered by increasing $|\alpha|$.)

Definition 4.9 (Left/right reduction)**.** In the notation of Theorem 4.8, we call a_L, resp. a_R the *left*, resp. *right* reduction of the full symbol a. Writing $A = \mathrm{Op}(a)$, we write

$$a_L =: \sigma_L(A), \quad a_R =: \sigma_R(A). \tag{4.37}$$

We first consider the case '$m = -\infty$' of Theorem 4.8 and give a description of kernels of residual operators, i.e. elements of $\Psi^{-\infty}(\mathbb{R}^n)$:

Proposition 4.10 (Schwartz kernel characterization of residual operators)**.** *An operator $A\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is a residual operator if and only if its Schwartz kernel $K(x,y)$ is smooth and satisfies*

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \le C_{\alpha\beta N}\langle x-y\rangle^{-N} \quad \forall\ \alpha,\beta,N. \tag{4.38}$$

Moreover, any such A can be written as $A = \mathrm{Op}_L(a_L) = \mathrm{Op}_R(a_R)$ for unique symbols $a_L, a_R \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$.

Proof. Since $A \in \Psi^{-N}(\mathbb{R}^n)$ for all $N \in \mathbb{R}$, we can write $A = \mathrm{Op}(a_N)$ with $a_N = \langle x-y\rangle^{w_N}\tilde a_N$, $\tilde a_N \in S^{-N}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, for some $w_N \in \mathbb{R}$. Taking $N > n$, the Schwartz kernel K of A is then given by the absolutely convergent integral

$$K(x,y) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_N(x,y,\xi)\,\mathrm{d}\xi. \tag{4.39}$$

Let $M \in \mathbb{N}_0$. For $|x-y| < 1$, and α,β with $|\alpha|+|\beta| \le M$, and taking $N > n+M$, we can thus bound

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \le C_{\alpha\beta}\|\tilde a_N\|_{-N,M} \tag{4.40}$$

using the triangle inequality. This gives (4.38) in this region.

For $|x-y|>1$, we fix $N=n+1$. We use $(\frac{x-y}{|x-y|^2}\cdot D_\xi)e^{i(x-y)\cdot\xi}=e^{i(x-y)\cdot\xi}$ and repeated integration by parts to deduce that

$$\begin{aligned}|K(x,y)|&=(2\pi)^{-n}\left|\int_{\mathbb{R}^n}e^{i(x-y)\cdot\xi}\Big(-\frac{x-y}{|x-y|^2}\cdot D_\xi\Big)^M a_{n+1}(x,y,\xi)\,\mathrm{d}\xi\right|\\&\le C_M|x-y|^{-M}\langle x-y\rangle^{w_{n+1}}\int_{\mathbb{R}^n}\langle\xi\rangle^{-n-1-M}\,\mathrm{d}\xi\\&\le C'_M\langle x-y\rangle^{-M+w_{n+1}}.\end{aligned}\tag{4.41}$$

Since M is arbitrary, this proves (4.38) for $\alpha=\beta=0$. Up to k-fold derivatives in x,y are estimated in the same way, but now working with a_{n+1+k} instead of a_{n+1}.

For the converse, note that if K satisfies (4.38), we can define

$$a_L(x,\xi)=\int_{\mathbb{R}^n}e^{-iz\cdot\xi}K(x,x-z)\,\mathrm{d}z.\tag{4.42}$$

Then $A=\mathrm{Op}(a_L)$ has Schwartz kernel K by the Fourier inversion formula, and the estimates (4.38) imply $a_L\in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$. Similarly, the operator $A=\mathrm{Op}(a_R)$ has Schwartz kernel K for

$$a_R(y,\xi)=\int_{\mathbb{R}^n}e^{-iz\cdot\xi}K(y+z,y)\,\mathrm{d}z.\tag{4.43}$$

□

Remark 4.11 (Continuity of left/right reduction for residual operators). Define seminorms on the space of all $K\in\mathcal{C}^\infty(\mathbb{R}^n_x\times\mathbb{R}^n_y)$ satisfying the estimates (4.38) to be the optimal constants: $|K|_{\alpha\beta N}:=\sup_{x,y\in\mathbb{R}^n}\langle x-y\rangle^N|\partial_x^\alpha\partial_y^\beta K(x,y)|$. Then the proof of Proposition 4.10 shows that the maps $K\mapsto a_{L/R}\in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$ and $S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)\ni a\mapsto K=\mathrm{Op}_{L/R}(a)$ are continuous.

To handle the case of general orders $m\in\mathbb{R}$, we first note that integration by parts in ξ implies the equality of Schwartz kernels

$$\begin{aligned}\mathrm{Op}((y-x)^\alpha a)(x,y)&=(2\pi)^{-n}\int((-D_\xi)^\alpha e^{i(x-y)\cdot\xi})a(x,y,\xi)\,\mathrm{d}\xi\\&=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}D_\xi^\alpha a(x,y,\xi)\,\mathrm{d}\xi\\&=\mathrm{Op}(D_\xi^\alpha a)(x,y),\end{aligned}\tag{4.44}$$

first for $a\in\langle x-y\rangle^w S^{-\infty}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$, and then for symbols of order m by density and continuity. The additional off-diagonal growth of $(y-x)^\alpha a$ is the reason for working with the more general symbol class (4.1).

Proof of Theorem 4.8. Let $N\in\mathbb{N}$, then Taylor's formula states

$$\begin{aligned}a(x,y,\xi)&=\sum_{|\alpha|<N}\frac{1}{\alpha!}(y-x)^\alpha\big(\partial_y^\alpha a(x,y,\xi)\big)|_{y=x}+r_N(x,y,\xi),\\r_N(x,y,\xi)&=\sum_{|\alpha|=N}\frac{N}{\alpha!}(y-x)^\alpha\int_0^1(1-t)^{N-1}(\partial_y^\alpha a)(x,x+t(y-x),\xi)\,\mathrm{d}t.\end{aligned}\tag{4.45}$$

Using the identity (4.44), we have

$$\mathrm{Op}\left(a - \sum_{|\alpha|<N} \frac{1}{\alpha!}(D_\xi^\alpha \partial_y^\alpha a)|_{y=x}\right) = \mathrm{Op}(\tilde r_N) \in \Psi^{m-N}(\mathbb{R}^n), \tag{4.46}$$

where

$$\tilde r_N(x,y,\xi) = \sum_{|\alpha|=N} \frac{N}{\alpha!}\int_0^1 (1-t)^{N-1}(D_\xi^\alpha \partial_y^\alpha a)(x, x+t(y-x),\xi)\,\mathrm{d}t. \tag{4.47}$$

In view of the symbolic estimates for a, the remainder here satisfies the estimate

$$|\partial_x^\beta \partial_y^\gamma \partial_\xi^\delta \tilde r_N(x,y,\xi)| \le C_{\beta\gamma\delta N}\langle x-y\rangle^w \langle\xi\rangle^{m-N-|\delta|}, \tag{4.48}$$

hence

$$\tilde r_N \in \langle x-y\rangle^w S^{m-N}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n). \tag{4.49}$$

for all N. Note that for $|\alpha| = k$, we have $D_\xi^\alpha \partial_y^\alpha a|_{y=x} \in S^{m-k}(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$. Thus, we can let $b \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ be an asymptotic sum

$$b \sim \sum_\alpha \frac{1}{\alpha!}(D_\xi^\alpha \partial_y^\alpha a)|_{y=x}, \tag{4.50}$$

and then

$$R := \mathrm{Op}(a-b) \in \bigcap_{N\in\mathbb{N}} \Psi^{m-N}(\mathbb{R}^n) = \Psi^{-\infty}(\mathbb{R}^n). \tag{4.51}$$

By Proposition 4.10, we then have $R = \mathrm{Op}_L(r)$ for some $r \in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$. Therefore,

$$A = \mathrm{Op}_L(a_L), \quad a_L := b + r. \tag{4.52}$$

The continuous dependence of a_L on a follows by using the explicit asymptotic summation procedure of Proposition 3.14 to define b, which thus depends continuously on a, and then noting that the optimal constants for the Schwartz kernel K of R in (4.38), and thus the $S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N)$ seminorms of r (see Remark 4.11), depend continuously on a, b.

Reduction to a right symbol is proved analogously. Instead of going through the argument, one can instead use duality as in (4.30), the idea being that the adjoint of a left quantization is a right quantization (and vice versa). Namely, using (4.31), we write the adjoint of $\mathrm{Op}(a)$ as $\mathrm{Op}(a)^\dagger = \mathrm{Op}(a^\dagger) = \mathrm{Op}_L(a'_L)$ for $a'_L \in S^m(\mathbb{R}^n;\mathbb{R}^n)$, and then

$$\mathrm{Op}(a) = \mathrm{Op}(a^\dagger)^\dagger = (\mathrm{Op}_L(a'_L))^\dagger = \mathrm{Op}_R((a'_L)^\dagger) = \mathrm{Op}_R(a_R), \tag{4.53}$$

where $a_R(y,\xi) = a'_L(y,-\xi)$. The formula for left reductions gives

$$a'_L(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}((-\partial_\xi)^\alpha D_x^\alpha a)(y,x,-\xi)|_{y=x}, \tag{4.54}$$

yielding the asymptotic description (4.36) of a_R.

It remains to prove the uniqueness of a_L, a_R. Suppose that $\mathrm{Op}_L(a_L) = 0$. Then for all $\chi \in \mathscr{S}(\mathbb{R}^n)$, $\epsilon > 0$, and $\xi_0 \in \mathbb{R}^n$, we have

$$0 = \mathrm{Op}_L(a_L)(\chi(\epsilon\cdot)e^{i\xi_0\cdot}). \tag{4.55}$$

Since the Fourier transform of $x \mapsto \chi(\epsilon x)e^{i\xi_0\cdot x}$ is given by $\xi \mapsto \epsilon^{-n}\hat\chi(\epsilon^{-1}(\xi-\xi_0))$, this means

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi} a_L(x,\xi)\epsilon^{-n}\hat\chi\Big(\frac{\xi-\xi_0}{\epsilon}\Big)\,\mathrm{d}\xi = 0 \tag{4.56}$$

for all $x \in \mathbb{R}^n$. If we require $\chi(0) = 1$ and thus $\int_{\mathbb{R}^n} \hat{\chi}(\xi)\,\mathrm{d}\xi = 1$, then $\epsilon^{-n}\hat{\chi}(\frac{\xi-\xi_0}{\epsilon}) \to \delta(\xi-\xi_0)$ in $\mathscr{S}'(\mathbb{R}^n)$ as $\epsilon \searrow 0$. Upon letting $\epsilon \searrow 0$, we thus obtain $a_L(x,\xi_0) = 0$. Since $x, \xi_0 \in \mathbb{R}^n$ are arbitrary, this proves $a_L = 0$. (One may alternatively argue as follows: a left symbol a_L can be viewed as an element $a_L \in \mathcal{C}^\infty(\mathbb{R}^n_x; \mathscr{S}'(\mathbb{R}^n_\xi))$, and the Schwartz kernel of $\mathrm{Op}(a_L)$ is

$$\mathrm{Op}(a_L)(x, x-z) = (\mathcal{F}_2^{-1} a_L)(x,z). \tag{4.57}$$

Since $\mathcal{F}_2$ is an isomorphism of $\mathcal{C}^\infty(\mathbb{R}^n; \mathscr{S}'(\mathbb{R}^n))$, $\mathrm{Op}(a_L) = 0$ implies $a_L = 0$.) The proof for a_R is similar. □

Corollary 4.12 (Ps.d.o.s as left/right quantizations)**.** *Let $m \in \mathbb{R}$ or $m = -\infty$. Then $\Psi^m(\mathbb{R}^n) = \mathrm{Op}_{L/R}(S^m(\mathbb{R}^n; \mathbb{R}^n))$.*

A slight variant of (4.30) gives the first part of the following corollary; the second part is an immediate application of Theorem 4.8.

Corollary 4.13 (Adjoints of ps.d.o.s)**.** *Let $A \in \Psi^m(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} (A^* u)(x)\overline{v(x)}\,\mathrm{d}x = \int_{\mathbb{R}^n} u(x)\overline{(Av)(x)}\,\mathrm{d}x, \quad u, v \in \mathscr{S}(\mathbb{R}^n). \tag{4.58}$$

defines an operator $A^* \in \Psi^m(\mathbb{R}^n)$. If $A = \mathrm{Op}(a)$, then $A^* = \mathrm{Op}(a^*)$, $a^*(x,y,\xi) = \bar{a}(y,x,\xi)$. If $A = \mathrm{Op}_L(a_L)$, then $A^* = \mathrm{Op}_L(a_L^*)$ with

$$a_L^*(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \overline{a_L}(x,\xi) \tag{4.59}$$

4.2. **Topology on spaces of pseudodifferential operators.** Let $m \in \mathbb{R}$ or $m = -\infty$. Since $\mathrm{Op}_L \colon S^m(\mathbb{R}^n; \mathbb{R}^n) \to \Psi^m(\mathbb{R}^n)$ is an isomorphism of vector spaces, it is natural to transport the Fréchet space structure of $S^m(\mathbb{R}^n; \mathbb{R}^n)$ to $\Psi^m(\mathbb{R}^n)$ via Op_L. For instance:

Lemma 4.14 (Mollifiers)**.** *Let $\chi \in \mathcal{C}_\mathrm{c}^\infty(\mathbb{R}^n_\xi)$ be identically 1 near 0, and put $J_\epsilon = \mathrm{Op}(\chi(\epsilon\cdot))$, $\epsilon > 0$. Then $J_\epsilon \in \Psi^{-\infty}(\mathbb{R}^n)$ is uniformly bounded in $\Psi^0(\mathbb{R}^n)$ and converges to the identity operator $I = \mathrm{Op}(1)$ in the topology of $\Psi^\eta(\mathbb{R}^n)$ for any $\eta > 0$.*

Proof. This is equivalent to the main part of (the proof of) Proposition 3.7. □

It is reassuring to note that one can equally well define the topology on $\Psi^m(\mathbb{R}^n)$ using the right quantization. This is a consequence of the following result.

Proposition 4.15 (Topology on $\Psi^m(\mathbb{R}^n)$)**.** *Let $m \in \mathbb{R}$ or $m = -\infty$. Then the isomorphism of vector spaces $\mathrm{Op}_R \colon S^m(\mathbb{R}^n; \mathbb{R}^n) \to \Psi^m(\mathbb{R}^n)$ is an isomorphism of Fréchet spaces.*

Proof. Right reduction σ_R is the inverse of Op_R. By definition of the Fréchet space structure of $\Psi^m(\mathbb{R}^n)$, the proposition is thus equivalent to the continuity of $\sigma_R \circ \mathrm{Op}_L$, which is part of Theorem 4.8. □

4.3. **Composition.** Proving that composition of ps.d.o.s produces another ps.d.o. is now straightforward:

Theorem 4.16 (Composition of ps.d.o.s)**.** *Let* $A \in \Psi^m(\mathbb{R}^n)$*,* $B \in \Psi^{m'}(\mathbb{R}^n)$*. Then* $A \circ B \colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ *is a pseudodifferential operator,*

$$A \circ B \in \Psi^{m+m'}(\mathbb{R}^n), \tag{4.60}$$

and its left symbol is given as an asymptotic sum

$$\sigma_L(A \circ B) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_L(A) \cdot D_x^\alpha \sigma_L(B). \tag{4.61}$$

The bilinear map $(A, B) \mapsto A \circ B$ *is continuous.*

Note that the symbolic expansion (4.61) is local in (x, ξ): the symbols of A and B do not 'interact' at all, modulo residual terms, at distinct points in phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$.

Proof of Theorem 4.16. Write $A = \mathrm{Op}_L(a)$ and $B = \mathrm{Op}_R(b_R)$. Assume first that $A, B \in \Psi^{-\infty}(\mathbb{R}^n)$, then for $u, v \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\begin{aligned} Av(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{v}(\xi)\,\mathrm{d}\xi, \\ \widehat{Bu}(\xi) &= \int_{\mathbb{R}^n} e^{-iy\cdot\xi} b_R(y,\xi)u(y)\,\mathrm{d}y. \end{aligned} \tag{4.62}$$

Thus,

$$ABu(x) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,\xi) b_R(y,\xi) u(y)\,\mathrm{d}y\,\mathrm{d}\xi, \tag{4.63}$$

giving $A \circ B = \mathrm{Op}(c)$, $c(x, y, \xi) = a(x, \xi) b_R(y, \xi)$. (This is one of the reasons for considering such general symbols!) By density and continuity, this continues to hold for A, B as in the statement of the theorem.

To get the asymptotic expansion (4.61), let us write $a = \sigma_L(A)$, $b = \sigma_L(B)$, then[2]

$$\begin{aligned} \sigma_L(A \circ B)(x,\xi) &\sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha \big(a(x,\xi) D_y^\alpha b_R(y,\xi)|_{y=x} \big) \\ &\sim \sum_{\beta,\gamma} \frac{1}{\beta!\gamma!} \partial_\xi^\beta a(x,\xi) \cdot \partial_\xi^\gamma D_x^{\beta+\gamma} \left(\sum_\delta \frac{(-1)^{|\delta|}}{\delta!} (\partial_\xi^\delta D_x^\delta b)(x,\xi) \right) \\ &\sim \sum_\beta \frac{1}{\beta!} \partial_\xi^\beta a(x,\xi) \cdot D_x^\beta \left(\sum_\epsilon \frac{1}{\epsilon!} \partial_\xi^\epsilon D_x^\epsilon b(x,\xi) \sum_{\gamma+\delta=\epsilon} \frac{\epsilon!}{\gamma!\delta!} (-1)^{|\delta|} \right) \end{aligned} \tag{4.64}$$

and the observation that for $\epsilon = 0$, the final sum evaluates to 1, while for $\mathbb{N}_0^n \ni \epsilon \neq 0$,

$$\sum_{\gamma+\delta=\epsilon} \frac{\epsilon!}{\gamma!\delta!} (-1)^{|\delta|} = \prod_{\epsilon_j \neq 0} (1-1)^{\epsilon_j} = 0. \tag{4.65}$$

This finishes the proof. □

As a simple application, we can now prove:

[2] Since these are asymptotic sums, it suffices to consider only those terms which have symbolic order bigger than some fixed but arbitrary number; in particular, there are no convergence or rearrangement issues.

Proposition 4.17 (Pseudolocality of ps.d.o.s)**.** *Let* $A \in \Psi^m(\mathbb{R}^n)$*. Then*

$$\operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u, \quad u \in \mathscr{S}'(\mathbb{R}^n). \tag{4.66}$$

To prove this, we record:

Lemma 4.18 (Residual operators acting on tempered distributions)**.** *A residual operator* $A \in \Psi^{-\infty}(\mathbb{R}^n)$ *is continuous as a map*

$$A\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n). \tag{4.67}$$

More precisely, for any $u \in \mathscr{S}'(\mathbb{R}^n)$ *we have* $Au \in \langle x\rangle^N \mathcal{C}^\infty_{\mathrm{b}}(\mathbb{R}^n)$ *for some* N *(depending on* u*).*

Proof. Let K denote the Schwartz kernel of A; recall that it satisfies the estimates (4.38). For $u \in \mathscr{S}'(\mathbb{R}^n)$, we then have, for some $N \in \mathbb{N}$,

$$\begin{aligned} |(Au)(x)| &= |\langle K(x,\cdot), u\rangle| \leq C\|K(x,\cdot)\|_N = C \sup_{\substack{y\in\mathbb{R}^n \\ |\alpha|+|\beta|\leq N}} |y^\alpha D_y^\beta K(x,y)| \\ &\leq C \sup_{\substack{y\in\mathbb{R}^n \\ |\beta|\leq N}} |\langle y\rangle^N D_y^\beta K(x,y)| = C \sup_{\substack{y\in\mathbb{R}^n \\ |\beta|\leq N}} \langle y\rangle^N \langle x-y\rangle^{-N} |\langle x-y\rangle^N D_y^\beta K(x,y)|. \end{aligned} \tag{4.68}$$

Using Lemma 4.3, we see that $\langle y\rangle^N \langle x-y\rangle^{-N} \leq C_N \langle x\rangle^N$, hence

$$|(Au)(x)| \leq C\langle x\rangle^N. \tag{4.69}$$

Derivatives in x are estimated analogously, so $Au \in \mathcal{C}^\infty(\mathbb{R}^n)$, and in fact

$$|\partial_x^\alpha (Au)(x)| \leq C_\alpha \langle x\rangle^N. \tag{4.70}$$

Note here that the number N above only depends on u, not on K itself. □

Proof of Proposition 4.17. Suppose $x \notin \operatorname{sing\,supp} u$. There exist cutoffs $\chi, \widetilde{\chi} \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ such that

$$\chi(x) \neq 0, \quad \widetilde{\chi} \equiv 1 \text{ on } \operatorname{supp}\chi, \quad \widetilde{\chi}u \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n). \tag{4.71}$$

Then

$$\chi A u = \chi A(\widetilde{\chi}u) + \chi A(1-\widetilde{\chi})u. \tag{4.72}$$

Since A acts on $\mathscr{S}(\mathbb{R}^n)$, we have $\chi A(\widetilde{\chi}u) \in \mathscr{S}(\mathbb{R}^n)$. For the second term, note that χ and $1-\widetilde{\chi}$ have disjoint supports; hence we have

$$\sigma_L(\chi A \circ (1-\widetilde{\chi}))(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\chi(x)\partial_\xi^\alpha \sigma_L(A)(x,\xi) \cdot D_x^\alpha(1-\widetilde{\chi}(x)) = 0, \tag{4.73}$$

which implies

$$\chi A(1-\widetilde{\chi}) \in \Psi^{-\infty}(\mathbb{R}^n). \tag{4.74}$$

By Lemma 4.18, we conclude that $\chi A(1-\widetilde{\chi})u \in \mathcal{C}^\infty(\mathbb{R}^n)$, finishing the proof. □

Returning to the observation (4.74), note that if $A = \mathrm{Op}(a)$ has Schwartz kernel $K \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n)$, then the Schwartz kernel of $\chi A(1-\widetilde{\chi})$ is $\chi(x)(1-\widetilde{\chi}(y))K(x,y)$. Thus, (4.74) can equivalently be stated as:

Proposition 4.19 (Schwartz kernels of ps.d.o.s)**.** *The Schwartz kernel* K *of a pseudodifferential operator is smooth away from the diagonal* $\Delta = \{(x,x)\colon x \in \mathbb{R}^n\}$*. That is,* $\operatorname{sing\,supp} K \subset \Delta$*.*

4.4. **Principal symbols.** Similarly to Proposition 2.20, the 'leading order part' of the left or right symbol of an operator $A \in \Psi^m(\mathbb{R}^n)$ has particularly simple properties.

Definition 4.20 (Principal symbol of ps.d.o.s)**.** Let $m \in \mathbb{R}$. The *principal symbol* $\sigma^m(A)$ of a ps.d.o. $A \in \Psi^m(\mathbb{R}^n)$ is the equivalence class

$$\sigma^m(A) := [\sigma_L(A)] \in S^m(\mathbb{R}^n;\mathbb{R}^n)/S^{m-1}(\mathbb{R}^n;\mathbb{R}^n). \tag{4.75}$$

We shall often omit from the notation the passage to the equivalence class.

Directly from the definition, this gives a short exact sequence for every $m \in \mathbb{R}$:

$$0 \to \Psi^{m-1}(\mathbb{R}^n;\mathbb{R}^n) \to \Psi^m(\mathbb{R}^n;\mathbb{R}^n) \xrightarrow{\sigma^m} S^m(\mathbb{R}^n;\mathbb{R}^n)/S^{m-1}(\mathbb{R}^n;\mathbb{R}^n) \to 0. \tag{4.76}$$

The surjectivity of σ^m is clear: given a representative $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ of an equivalence class of symbols, we have $\sigma^m(\mathrm{Op}_L(a)) = [a]$.

Proposition 4.21 (Behavior of the principal symbol)**.** *The principal symbol map has the following properties:*

1. $\sigma^m(\mathrm{Op}_R(a)) = [a]$*, i.e. using the right symbol in* (4.75) *gives the same principal symbol map.*
2. *For* $A \in \Psi^m(\mathbb{R}^n)$*, we have* $\sigma^m(A^*) = \overline{\sigma^m(A)}$*.*
3. *For* $A \in \Psi^m(\mathbb{R}^n)$*,* $B \in \Psi^{m'}(\mathbb{R}^n)$*, we have* $\sigma^{m+m'}(A \circ B) = \sigma^m(A)\sigma^{m'}(B)$*.*

(The behavior under changes of variables will be discussed in §5.1.) Notice that the principal symbol map translates operator composition (a highly non-commutative operation) to the multiplication of (equivalence classes of) functions (a commutative operation), though of course at what seems to be an enormous loss of information compared to the full expansion (4.61) (which itself gives up information on the residual part of $A \circ B$). However, in most situations, the principal symbol, and sometimes a 'subprincipal' part of the full symbol, dominate the behavior of the operator, while lower order parts are irrelevant; cf. the discussion of ellipticity for symbols in §3.1.

One crucial calculation is the following. For $A \in \Psi^m(\mathbb{R}^n)$, $B \in \Psi^{m'}(\mathbb{R}^n)$, note that $\sigma^{m+m'}(A \circ B) = \sigma^m(A)\sigma^{m'}(B) = \sigma^{m+m'}(B \circ A)$, so

$$\sigma^{m+m'}([A,B]) = 0, \quad [A,B] = A \circ B - B \circ A. \tag{4.77}$$

In view of (4.76), we thus have $[A,B] \in \Psi^{m+m'-1}(\mathbb{R}^n)$, and it is natural to inquire about its principal symbol as an operator of order $m+m'-1$. It turns out that it can be computed solely in terms of the principal symbols of A and B:

Proposition 4.22 (Principal symbols of commutators)**.** *For* $A \in \Psi^m(\mathbb{R}^n)$*,* $B \in \Psi^{m'}(\mathbb{R}^n)$*, we have*

$$\sigma^{m+m'-1}(i[A,B]) = \{\sigma^m(A), \sigma^{m'}(B)\}, \tag{4.78}$$

where the Poisson bracket *of* $a, b \in \mathcal{C}^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ *is defined as*

$$\{a,b\} := \sum_{j=1}^n (\partial_{\xi_j} a)(\partial_{x_j} b) - (\partial_{x_j} a)(\partial_{\xi_j} b). \tag{4.79}$$

This will be the key connection between 'quantum mechanics' (quantizations of symbols, noncommutative algebra of operators) and 'classical mechanics' (symbols themselves, commutative algebra of functions), which will play a central role in §8.

Proof of Proposition 4.22. We leave it to the reader to verify that (4.78) is well-defined, i.e. that the image of the right hand side in the quotient space $S^{m+m'-1}/S^{m+m'-2}$ does not depend on the choice of representatives of the principal symbols of A and B.

The proof is an immediate application of (4.61). Let $a = \sigma_L(A)$, $b = \sigma_L(B)$. Working modulo $S^{m+m'-2}(\mathbb{R}^n; \mathbb{R}^n)$, we have

$$\sigma_L(A \circ B) \equiv ab + \frac{1}{i} \sum_{j=1}^{n} (\partial_{\xi_j} a)(\partial_{x_j} b), \quad \sigma_L(B \circ A) \equiv ab + \frac{1}{i} \sum_{j=1}^{n} (\partial_{\xi_j} b)(\partial_{x_j} a), \tag{4.80}$$

and (4.78) follows. □

4.5. **Classical operators.** Following Definition 3.11, we have a subclass of classical operators:

Definition 4.23 (Classical ps.d.o.s)**.** For $m \in \mathbb{R}$, we define the space of *classical pseudo-differential operators of order* m by

$$\Psi^m_{\mathrm{cl}}(\mathbb{R}^n) := \mathrm{Op}_L(S^m_{\mathrm{cl}}(\mathbb{R}^n; \mathbb{R}^n)) \subset \Psi^m(\mathbb{R}^n), \tag{4.81}$$

equipped with the structure of a Fréchet space which makes Op_L into an isomorphism. We put $\Psi^{-\infty}_{\mathrm{cl}}(\mathbb{R}^n) := \Psi^{-\infty}(\mathbb{R}^n)$.

The symbol expansions in Theorem 4.16 and Corollary 4.13 imply that compositions and adjoints of classical operators are still classical:

Proposition 4.24 (Compositions and adjoints of classical ps.d.o.s)**.** *Composition of ps.d.o.s restricts to a continuous bilinear map*

$$\Psi^m_{\mathrm{cl}}(\mathbb{R}^n) \times \Psi^{m'}_{\mathrm{cl}}(\mathbb{R}^n) \ni (A, B) \mapsto A \circ B \in \Psi^{m+m'}_{\mathrm{cl}}(\mathbb{R}^n). \tag{4.82}$$

Similarly, the map

$$\Psi^m_{\mathrm{cl}}(\mathbb{R}^n) \ni A \mapsto A^* \in \Psi^{\bar{m}}_{\mathrm{cl}}(\mathbb{R}^n) \tag{4.83}$$

is a continuous conjugate-linear map.

For a classical operator $A = \mathrm{Op}_L(a)$, with $a \in S^m_{\mathrm{cl}}(\mathbb{R}^n; \mathbb{R}^n)$, we can identify the principal symbol $\sigma^m(A)$ with the homogeneous leading order part of a, as discussed after Lemma 3.12. The corresponding short exact sequence is

$$0 \to \Psi^{m-1}_{\mathrm{cl}}(\mathbb{R}^n) \to \Psi^m_{\mathrm{cl}}(\mathbb{R}^n) \to S^m_{\mathrm{hom}}(\mathbb{R}^n; \mathbb{R}^n \setminus \{0\}) \to 0. \tag{4.84}$$

4.6. **Elliptic parametrix.** Recall Definition 3.8 and the discussion around (3.19). Then:

Definition 4.25 (Elliptic ps.d.o.s)**.** We call an operator $A \in \Psi^m(\mathbb{R}^n)$ *(uniformly) elliptic* if its principal symbol $\sigma^m(A)$ is elliptic.

As a first, and important, application of the symbol calculus we have developed above, we construct *parametrices* (approximate inverses—a term which, almost by nature, has no precise definition, but rather depends on the context) of uniformly elliptic operators.

Theorem 4.26 (Elliptic parametrix)**.** *Let $A \in \Psi^m(\mathbb{R}^n)$ be uniformly elliptic. Then there exists an operator $B \in \Psi^{-m}(\mathbb{R}^n)$ which is unique modulo $\Psi^{-\infty}(\mathbb{R}^n)$, such that*

$$AB - I,\ BA - I \in \Psi^{-\infty}(\mathbb{R}^n). \tag{4.85}$$

We call an operator B satisfying (4.85) a *parametrix of A*.

Lemma 4.27 (Asymptotic Neumann series)**.** *Let $R \in \Psi^{-\delta}(\mathbb{R}^n)$, $\delta > 0$. Let $R' \in \Psi^{-\delta}(\mathbb{R}^n)$ with $R' \sim \sum_{j=1}^\infty R^j$, i.e. the left symbol of R' is an asymptotic sum of the left symbols of $R^j = R \circ \cdots \circ R$ (j factors). Then $(I-R)(I+R') = I + E$ where $E \in \Psi^{-\infty}(\mathbb{R}^n)$.*

Proof. For any $N \in \mathbb{N}$, we have

$$\begin{aligned}(I-R)(I+R') - I &= (I-R)\Big(I + \sum_{j=1}^N R^j\Big) - I + (I-R)\Big(R' - \sum_{j=1}^N R^j\Big) \\ &= -R^{N+1} + (I-R)\Big(R' - \sum_{j=1}^N R^j\Big).\end{aligned} \tag{4.86}$$

The first term on the right lies in $\Psi^{-(N+1)\delta}(\mathbb{R}^n)$, as does the second term since $R' - \sum_{j=1}^N R^j \in \Psi^{-(N+1)\delta}(\mathbb{R}^n)$. ☐

Proof of Theorem 4.26. Let $b \in S^{-m}(\mathbb{R}^n;\mathbb{R}^n)$ be such that $\sigma^m(A)b - 1 \in S^{-1}(\mathbb{R}^n;\mathbb{R}^n)$. Put $B_0 = \mathrm{Op}(b) \in \Psi^{-m}(\mathbb{R}^n)$, then

$$A \circ B_0 = I - R, \quad R \in \Psi^{-1}(\mathbb{R}^n). \tag{4.87}$$

Indeed, this follows from $\sigma^0(AB_0 - I) = 0$. Choosing $R' \sim \sum_{j=1}^\infty R^j \in \Psi^{-1}(\mathbb{R}^n)$ and setting

$$B := B_0(I + R') \in \Psi^{-m}(\mathbb{R}^n), \tag{4.88}$$

we conclude using Lemma 4.27 that $AB = I + E$, as desired.

An analogous argument produces $B' \in \Psi^{-m}(\mathbb{R}^n)$ with $B'A = I + E'$, $E' \in \Psi^{-\infty}(\mathbb{R}^n)$. But then abstract 'group theory' gives

$$B = IB = (B'A - E')B = B'AB - E'B = B'(I+E) - E'B = B' + (B'E - E'B). \tag{4.89}$$

Therefore $B - B' \in \Psi^{-\infty}(\mathbb{R}^n)$. In particular, any two parametrices differ by an element of $\Psi^{-\infty}(\mathbb{R}^n)$. ☐

As a simple application, we prove:

Proposition 4.28 (Elliptic regularity: smooth case)**.** *Let $A \in \Psi^m(\mathbb{R}^n)$ be uniformly elliptic, and suppose*

$$u \in \mathscr{S}'(\mathbb{R}^n), \quad Au = f \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{4.90}$$

Then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$. More precisely, we have

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp} Au. \tag{4.91}$$

Proof. We prove (4.91). Let $B \in \Psi^{-m}(\mathbb{R}^n)$ be a parametrix of A, with $BA = I + R$, $R \in \Psi^{-\infty}(\mathbb{R}^n)$. Then by Proposition 4.17, we have

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp}(BAu + Ru) = \operatorname{sing\,supp} BAu \subset \operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u. \tag{4.92}$$

Therefore, equality must hold at each step. ☐

Example 4.29. Examples to which Proposition 4.28 applies are the Laplacian $\Delta \in \Psi^2(\mathbb{R}^n)$ and the Cauchy–Riemann operator $\bar\partial = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}) \in \Psi^1(\mathbb{R}^2)$, which is identified with $\mathbb{C}$ via $(x_1, x_2) \mapsto x_1 + ix_2$. For the latter, we deduce that if $\bar\partial u = 0$ for $u \in \mathscr{S}'(\mathbb{R}^n)$, then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$. In complex analysis we learn that in fact u is *analytic*; here we are only developing microlocal analysis in the *smooth* category, hence do not directly recover this stronger conclusion.

4.7. **Boundedness on Sobolev spaces.** In practice, one typically uses function spaces other than $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$, such as Hölder or L^p spaces. Here, we focus on function spaces related to L^2, in parts because they are the most natural for the study of non-elliptic operators in §8.

As usual, we first consider residual operators:

Proposition 4.30 (L^2-boundedness of residual operators)**.** *Let $A \in \Psi^{-\infty}(\mathbb{R}^n)$. Then A extends by continuity from*[3] *$\mathscr{S}(\mathbb{R}^n)$ to a bounded linear operator $A\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.*

This will follow from the estimates (4.38) and *Schur's lemma*:

Lemma 4.31 (Schur's lemma)**.** *Let (X,μ) and (Y,ν) be measure spaces. Suppose $K(x,y)$ is measurable on $X\times Y$ and*

$$\int_X |K(x,y)|\,\mathrm{d}\mu(x) \le C_1, \qquad \int_Y |K(x,y)|\,\mathrm{d}\nu(y) \le C_2 \tag{4.93}$$

for almost all $y\in Y$ and $x \in X$, respectively. Let

$$Tu(x) = \int_Y K(x,y)u(y)\,\mathrm{d}\nu(y). \tag{4.94}$$

Then $T\colon L^2(Y)\to L^2(X)$ is bounded. Quantitatively,

$$\|Tu\|_{L^2(X)} \le (C_1C_2)^{1/2}\|u\|_{L^2(Y)}. \tag{4.95}$$

Proof. Let $u \in L^2(Y)$ and $v\in L^2(X)$, then by Cauchy–Schwarz

$$\begin{aligned}
&\left|\int_X\int_Y K(x,y)u(y)\overline{v(x)}\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x)\right| \\
&\quad\le \left(\int_{X\times Y}|K(x,y)||u(y)|^2\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)\right)^{1/2}\left(\int_{X\times Y}|K(x,y)||v(x)|^2\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)\right)^{1/2} \\
&\quad\le C_1^{1/2}\|u\|_{L^2(Y)}\cdot C_2^{1/2}\|v\|_{L^2(X)}.
\end{aligned}$$

□

Proof of Proposition 4.30. The Schwartz kernel K of A satisfies $|K(x,y)| \le C\langle x-y\rangle^{-n-1}$, hence

$$\int_{\mathbb{R}^n}|K(x,y)|\,\mathrm{d}x \le C\int_{\mathbb{R}^n}\langle z\rangle^{-n-1}\,\mathrm{d}z < \infty, \tag{4.96}$$

and likewise $\int_{\mathbb{R}^n}|K(x,y)|\,\mathrm{d}y<\infty$. The claim then follows from Lemma 4.31. □

Using 'Hörmander's square root trick' from the proof of [Hör71b, Theorem 2.2.1], we can now prove:

Theorem 4.32 (L^2-boundedness of zeroth order ps.d.o.s)**.** *Let $A\in\Psi^0(\mathbb{R}^n)$. Then*

$$A\colon L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n) \tag{4.97}$$

is bounded. In fact, the linear map $\Psi^0(\mathbb{R}^n)\to\mathcal{L}(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))$ thus defined is continuous; here we write $\mathcal{L}(X,Y)$ for the space of bounded linear operators between normed spaces $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$.

[3] We use here that $\mathscr{S}(\mathbb{R}^n)\subset L^2(\mathbb{R}^n)$ is a dense subspace.

Proof. By Corollary 4.13 and Theorem 4.16, we have $A^*A \in \Psi^0(\mathbb{R}^n)$. With $a = \sigma^0(A)$ (that is, a is any representative of $\sigma^0(A)$), we have $\sigma^0(A^*A) = |a|^2$, which is real, non-negative, and bounded. Thus, for $C > \sup_{x,\xi\in\mathbb{R}^n} |a|^2$, the symbol $C - |a|^2 \in S^0(\mathbb{R}^n;\mathbb{R}^n)$ is elliptic and positive. By Exercise 3.3, it has an approximate square root $0 < b_0 \in S^0(\mathbb{R}^n;\mathbb{R}^n)$, so $C - |a|^2 - b_0^2 \in S^{-1}(\mathbb{R}^n;\mathbb{R}^n)$. Let $B_0 = \mathrm{Op}(b_0)$, then

$$C - A^*A = B_0^*B_0 + R_1, \quad R_1 \in \Psi^{-1}(\mathbb{R}^n). \tag{4.98}$$

Assume inductively that we have found $B_j \in \Psi^{-j}(\mathbb{R}^n)$, $j = 0, \ldots, k-1$, such that

$$R_k := C - A^*A - (B_0 + \cdots + B_{k-1})^*(B_0 + \cdots + B_{k-1}) \in \Psi^{-k}(\mathbb{R}^n). \tag{4.99}$$

This holds for $k = 1$. We try to improve the error term by finding the next correction $B_k = \mathrm{Op}(b_k) \in \Psi^{-k}(\mathbb{R}^n)$; we compute

$$\begin{aligned} R_{k+1} &= C - A^*A - (B_0 + \cdots + B_k)^*(B_0 + \cdots B_k) \\ &= R_k - \bigl(B_k^*(B_0 + \cdots + B_{k-1}) + (B_0 + \cdots + B_{k-1})^*B_k + B_k^*B_k\bigr) \in \Psi^{-k}(\mathbb{R}^n). \end{aligned} \tag{4.100}$$

Thus, the requirement $R_{k+1} \in \Psi^{-k-1}(\mathbb{R}^n)$ is equivalent to a principal symbol condition,

$$\overline{b_k}b_0 + b_0b_k = \sigma^{-k}(R_k) \quad (\text{in } S^{-k}(\mathbb{R}^n;\mathbb{R}^n)/S^{-k-1}(\mathbb{R}^n;\mathbb{R}^n)). \tag{4.101}$$

Since $R_k = R_k^*$, the principal symbol $\sigma^{-k}(R_k)$ is real; hence we can take $b_k = \frac{1}{2}\sigma^{-k}(R_k)/b_0 \in S^{-k}(\mathbb{R}^n;\mathbb{R}^n)$.

Finally, we let $B \in \Psi^0(\mathbb{R}^n)$ be the asymptotic sum

$$B \sim \sum_{k=0}^{\infty} B_k. \tag{4.102}$$

We have then arranged

$$R := C - A^*A - B^*B \in \Psi^{-\infty}(\mathbb{R}^n). \tag{4.103}$$

(Thus, we have constructed a square root, modulo residual operators, of $C - A^*A$.)

Given $u \in \mathscr{S}(\mathbb{R}^n)$, we then have

$$\begin{aligned} \|Au\|_{L^2(\mathbb{R}^n)}^2 &= \langle A^*Au, u\rangle_{L^2(\mathbb{R}^n)} \\ &= C\|u\|_{L^2(\mathbb{R}^n)}^2 - \|Bu\|_{L^2(\mathbb{R}^n)}^2 - \langle Ru, u\rangle \\ &\le C\|u\|_{L^2(\mathbb{R}^n)}^2 + \|Ru\|_{L^2(\mathbb{R}^n)}\|u\|_{L^2(\mathbb{R}^n)} \\ &\le C'\|u\|_{L^2(\mathbb{R}^n)}^2 \end{aligned} \tag{4.104}$$

by Proposition 4.30. Thus, A extends by continuity to a bounded operator on $L^2(\mathbb{R}^n)$.

The second part of the theorem can be proved constructively as follows. For a neighborhood $\mathcal{U}$ of $0 \in \Psi^0(\mathbb{R}^n)$, we can take the constant C above to be equal to 1. We can choose the operators B_k, $k \in \mathbb{N}_0$, to depend continuously (albeit nonlinearly) on A, and then also B in (4.102) can be chosen to depend continuously on A in view of Proposition 3.14. Therefore also $R \in \Psi^{-\infty}(\mathbb{R}^n)$ depends continuously on A. The estimate (4.104) reads $\|A\|_{\mathcal{L}(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))}^2 \le 1 + \|R\|_{\mathcal{L}(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))}$ for $A \in \mathcal{U}$. Now, $\|R\|_{\mathcal{L}(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))}$ is bounded by some fixed continuous seminorm of R; indeed, writing $R = \mathrm{Op}_L(r)$, one can take this seminorm to be $C'\|r\|_{-n-1,n+1}$ for some universal constant C' (see Exercise 4.3). By the continuity of $\mathcal{U} \ni A \mapsto R \in \Psi^{-\infty}(\mathbb{R}^n)$, there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ of $0 \in \Psi^0(\mathbb{R}^n)$

so that this seminorm remains uniformly bounded for $A \in \mathcal{V}$. Therefore, $\|A\|_{\mathcal{L}(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))}$ remains uniformly bounded for $A \in \mathcal{V}$. See Exercise 4.4 for an alternative argument using the Closed Graph theorem. □

Proposition 4.33 (Compactness property of certain negative order ps.d.o.s)**.** *Let $m < 0$, and let $A \in \Psi^m(\mathbb{R}^n)$. Suppose the Schwartz kernel of A has compact support in $\mathbb{R}^n \times \mathbb{R}^n$. Then $A\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact.*

Proof. Write $A = \mathrm{Op}_L(a)$ where $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$, and let $B \Subset \mathbb{R}^n$ be a bounded open ball such that the support of the Schwartz kernel of A is contained in $B \times B$. Since for all $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\operatorname{supp}\phi \cap B = \emptyset$ we have $0 = \phi A = \mathrm{Op}_L(\phi(x)a(x,\xi))$, the uniqueness part of Theorem 4.8 implies that $\phi(x)a(x,\xi) = 0$; that is, $a(x,\xi) = 0$ for all $x \notin B$. The proof of Proposition 3.7 produces a sequence $a_j \in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$, $j \in \mathbb{N}_0$, of symbols which is uniformly bounded in $S^m(\mathbb{R}^n;\mathbb{R}^n)$, converges to a in $S^0(\mathbb{R}^n;\mathbb{R}^n)$ as $j \to \infty$, and satisfies $a_j(x,\xi) = 0$ for all $x \notin B$. Now, $\mathrm{Op}(a_j)$ is compact on $L^2(\mathbb{R}^n)$, as it can be factored as

$$L^2(\mathbb{R}^n) \xrightarrow{\mathrm{Op}(a_j)} \mathcal{C}^\infty_c(B) \subset \mathcal{C}^1(\bar{B}) \hookrightarrow \mathcal{C}^0(\bar{B}) \hookrightarrow L^2(\mathbb{R}^n), \tag{4.105}$$

where the penultimate inclusion map is compact by Arzelà–Ascoli, and the final map is extension by 0. By Theorem 4.32, $\mathrm{Op}(a)$ is the limit of $\mathrm{Op}(a_j)$ in $\mathcal{L}(L^2(\mathbb{R}^n);L^2(\mathbb{R}^n))$ and therefore compact as well. □

Boundedness of ps.d.o.s on Sobolev spaces is a straightforward consequence of Theorem 4.32:

Corollary 4.34 (Boundedness on Sobolev spaces)**.** *Let $s, m \in \mathbb{R}$, and $A \in \Psi^m(\mathbb{R}^n)$. Then $A\colon H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ is bounded. The linear map $\Psi^m(\mathbb{R}^n) \to \mathcal{L}(H^s(\mathbb{R}^n), H^{s-m}(\mathbb{R}^n))$ thus defined is continuous.*

Proof. Recall the operators $\langle D\rangle^\sigma = \mathcal{F}^{-1}\langle\xi\rangle^\sigma\mathcal{F}$ for $\sigma \in \mathbb{R}$ from Definition 2.11; note that $\langle D\rangle^\sigma \in \Psi^\sigma(\mathbb{R}^n)$. Moreover, $\langle D\rangle^{-s}\colon L^2(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ and $\langle D\rangle^{s-m}\colon H^{s-m}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are isometric isomorphisms. Now

$$\langle D\rangle^{s-m} A \langle D\rangle^{-s} \in \Psi^0(\mathbb{R}^n) \tag{4.106}$$

is bounded on $L^2(\mathbb{R}^n)$ by Theorem 4.32, which is equivalent to the statement of the corollary. The second part of the corollary follows from the continuity of compositions of ps.d.o.s proved in Theorem 4.16. □

In fact, this can be generalized to *weighted* Sobolev spaces, see (2.28):

Theorem 4.35 (Boundedness between weighted Sobolev spaces)**.** *Let $s, m, r \in \mathbb{R}$, and $A \in \Psi^m(\mathbb{R}^n)$. Then $A\colon \langle x\rangle^r H^s(\mathbb{R}^n) \to \langle x\rangle^r H^{s-m}(\mathbb{R}^n)$ is bounded. The linear map $\Psi^m(\mathbb{R}^n) \to \mathcal{L}(\langle x\rangle^r H^s(\mathbb{R}^n) \to \langle x\rangle^r H^{s-m}(\mathbb{R}^n))$ thus defined is continuous.*

Proof. Since $\langle x\rangle^r\langle D\rangle^{-s}\colon L^2(\mathbb{R}^n) \to \langle x\rangle^r H^s(\mathbb{R}^n)$ and $\langle D\rangle^{s-m}\langle x\rangle^{-r}\colon \langle x\rangle^r H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are isomorphisms, we need to show that

$$A' := \langle D\rangle^{s-m}\langle x\rangle^{-r} \circ A \circ \langle x\rangle^r\langle D\rangle^{-s} \in \Psi^0(\mathbb{R}^n). \tag{4.107}$$

If $a = \sigma_L(A)$, then the full symbol $a^\sharp(x,y,\xi)$ of $A^\sharp := \langle x\rangle^{-r} \circ A \circ \langle x\rangle^r$ is given by $a^\sharp(x,y,\xi) = \langle x\rangle^{-r}\langle y\rangle^r a(x,\xi)$. By Lemma 4.3, we have

$$|a^\sharp(x,y,\xi)| \le 2^{|r|/2}\langle x-y\rangle^{|r|}|a(x,\xi)| \le C\langle x-y\rangle^{|r|}\langle\xi\rangle^{m-s}, \tag{4.108}$$

which is the first step towards showing that $a^\sharp \in \langle x-y\rangle^{|r|} S^m(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$; it remains to consider derivatives. The essence of this is contained in

$$\begin{aligned} |\partial_{y_j} a^\sharp(x,y,\xi)| &= \left|-r\langle x\rangle^r \langle y\rangle^{-r-2} y_j a(x,\xi)\langle\xi\rangle^{-s}\right| \\ &\leq C\langle x-y\rangle^{|r|}\frac{y_j}{\langle y\rangle^2}\langle\xi\rangle^m \\ &\leq C\langle x-y\rangle^{|r|}\langle\xi\rangle^m. \end{aligned} \tag{4.109}$$

We conclude that $A^\sharp \in \Psi^m(\mathbb{R}^n)$, hence $A' \in \Psi^0(\mathbb{R}^n)$, finishing the proof. □

In view of the Schwartz representation theorem, Theorem 2.14, we thus obtain another proof of Lemma 4.18. Indeed, a residual operator maps $\mathscr{S}'(\mathbb{R}^n) = \bigcup_{r,s}\langle x\rangle^r H^s(\mathbb{R}^n)$ into $\bigcup_r \langle x\rangle^r H^\infty(\mathbb{R}^n) = \bigcup_r \langle x\rangle^r \mathcal{C}^\infty_{\mathrm{b}}(\mathbb{R}^n)$ (using Sobolev embedding, Exercise 2.4).

We can sharpen and upgrade the elliptic regularity result, Proposition 4.28:

Corollary 4.36 (Elliptic regularity: weighted Sobolev case)**.** *Let $A \in \Psi^m(\mathbb{R}^n)$ be uniformly elliptic, and suppose $u \in \langle x\rangle^r H^{-N}(\mathbb{R}^n)$ for some $r, N \in \mathbb{R}$. If $Au = f \in \langle x\rangle^r H^{s-m}(\mathbb{R}^n)$, then $u \in \langle x\rangle^r H^s(\mathbb{R}^n)$.*

Proof. With $B \in \Psi^{-m}(\mathbb{R}^n)$ denoting a parametrix of A, so $I = BA + R$, $R \in \Psi^{-\infty}(\mathbb{R}^n)$, we have

$$u = BAu + Ru = Bf + Ru, \tag{4.110}$$

with $Bf \in \langle x\rangle^r H^s(\mathbb{R}^n)$ and $Ru \in \langle x\rangle^r \bigcup_{\sigma\in\mathbb{R}} H^\sigma(\mathbb{R}^n)$. □

Remark 4.37 (A priori membership in weighted space)*.* It is important that the assumption on u already has the weight factor $\langle x\rangle^r$. Indeed, the conclusion would be false in general if we merely assumed $u \in \langle x\rangle^{r'} H^{-N}(\mathbb{R}^n)$ for some $r' < r$. (Convince yourself of this. For example, take $A = \Delta$, the Laplacian on $\mathbb{R}^n$, and $u = 1$.)

4.8. **Exercises.**

Exercise 4.1 (Quantization of polynomials in ξ)*.* Let $m \in \mathbb{N}_0$, and let $a \in S^m(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$ be a polynomial in the symbolic variable ξ.

(1) Show, starting from the definition as a limit of quantizations of residual symbols, that $\mathrm{Op}(a) \in \mathrm{Diff}^m(\mathbb{R}^n)$.
(2) Prove that $\mathrm{Op}(\langle x-y\rangle^w a) \in \mathrm{Diff}^m(\mathbb{R}^n)$ (which in particular entails the boundedness of the coefficients). (*Hint.* Compute its Schwartz kernel.)

Exercise 4.2 (Singularities and decay of Schwartz kernels of ps.d.o.s)*.* Let $A \in \Psi^m(\mathbb{R}^n)$, and denote by K its Schwartz kernel.

(1) Give another, direct, proof that $K \in \mathcal{C}^\infty((\mathbb{R}^n\times\mathbb{R}^n)\setminus\Delta)$, where $\Delta = \{(x,x)\colon x\in\mathbb{R}^n\}$ is the diagonal. (*Hint.* For $\phi,\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ with $\operatorname{supp}\phi\cap\operatorname{supp}\psi = \emptyset$, rewrite the pairing $\langle A\phi,\psi\rangle$ for $A \in \Psi^{-\infty}(\mathbb{R}^n)$ using integrations by parts as in the proof of Proposition 4.4. Then use a density argument.)
(2) Prove that for every $\epsilon > 0$, $N \in \mathbb{R}$, $\alpha,\beta \in \mathbb{N}_0^n$, there exists a constant C such that

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \leq C|x-y|^{-N}, \quad |x-y| \geq \epsilon. \tag{4.111}$$

Exercise 4.3 (Bounds on very negative order ps.d.o.s)*.* Let $n \in \mathbb{N}$ and fix $m < -n$.

(1) Show that the Schwartz kernel $K = K(x, y)$ of $A = \mathrm{Op}_L(a) \in \Psi^m(\mathbb{R}^n)$ is a continuous function of $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Show moreover that for all $N \in \mathbb{N}$ there exists a constant $C \in \mathbb{R}$ so that

$$|K(x,y)| \leq C\|a\|_{m,N}\langle x-y\rangle^{-N}. \tag{4.112}$$

(2) Deduce that there exists $C \in \mathbb{R}$ so that $\|A\|_{\mathcal{L}(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))} \leq C\|a\|_{m,n+1}$.

Exercise 4.4 (Zeroth order ps.d.o.s as linear maps on L^2). Denote by

$$\Phi\colon \Psi^0(\mathbb{R}^n) \to \mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n)) \tag{4.113}$$

the linear map assigning to $A \in \Psi^0(\mathbb{R}^n)$ the bounded linear operator $\Phi(A)\colon u \mapsto Au$, $u \in L^2(\mathbb{R}^n)$. Use the Closed Graph theorem to show that Ψ is continuous. (*Hint.* If $A_j \to A$ in $\Psi^0(\mathbb{R}^n)$ and $\Phi(A_j) \to T$ in $\mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$, show that $T = \Phi(A)$ by evaluating both sides on Schwartz functions and using (4.24).)

Exercise 4.5 (A classical ps.d.o.). Suppose $K(x,z) \in \mathcal{C}^\infty(\mathbb{R}\times(\mathbb{R}\setminus\{0\}))$ satisfies $K(x,\lambda z) = \lambda^{-1}K(x,z)$, $\lambda > 0$, and $K(x,-z) = -K(x,z)$. Assume that $K(x,1) \in \mathcal{C}^\infty_{\mathrm{b}}(\mathbb{R}_x)$. Let $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ be identically 1 near 0. Show that the operator

$$Au(x) = \lim_{\epsilon\to 0}\int_{|x-y|\geq\epsilon} \chi(x-y)K(x,x-y)u(y)\,\mathrm{d}y, \quad u \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n), \tag{4.114}$$

is well-defined and defines an element $A \in \Psi^0_{\mathrm{cl}}(\mathbb{R})$. Compute its principal symbol.

Exercise 4.6 (Gårding's inequality). Let $A \in \Psi^{2m}(\mathbb{R}^n)$, and suppose $\operatorname{Re}\sigma^{2m}(A) \geq c\langle\xi\rangle^{2m}$ for some $c \in \mathbb{R}$. Show that for every $\epsilon > 0$ and $N \in \mathbb{R}$, there exists a constant C such that

$$\operatorname{Re}\langle Au, u\rangle_{L^2(\mathbb{R}^n)} \geq (c-\epsilon)\|u\|^2_{H^m(\mathbb{R}^n)} - C\|u\|^2_{H^{-N}(\mathbb{R}^n)}, \quad u \in \mathscr{S}(\mathbb{R}^n). \tag{4.115}$$

(*Hint.* Use the 'square root trick'.) The *sharp Gårding inequality* states that (4.115) holds for $\epsilon = 0$, but then with $-N = m-1/2$; see [Hör03, Theorem 18.1.14]. (This can be further refined to the *Fefferman–Phong* inequality, which allows $-N = m-1$.)

The following series of exercises introduces the basic properties of *scattering pseudodifferential operators* on $\mathbb{R}^n$.

Exercise 4.7 (Scattering symbols). For $m, r_1, r_2 \in \mathbb{R}$, define the space of symbols

$$S^{m,r_1,r_2}(\mathbb{R}^n_x \times \mathbb{R}^n_y; \mathbb{R}^n_\xi) \tag{4.116}$$

to consist of all $a \in \mathcal{C}^\infty(\mathbb{R}^{3n})$ such that the seminorms

$$\|a\|_{m,r_1,r_2,k} := \sup_{|\alpha_1|+|\alpha_2|+|\beta|\leq k} \langle x\rangle^{-r_1+|\alpha_1|}\langle y\rangle^{-r_2+|\alpha_2|}\langle\xi\rangle^{-m+|\beta|}|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_\xi^\beta a(x,y,\xi)| \tag{4.117}$$

are finite for all $k \in \mathbb{N}_0$.[4] Let

$$S^{-\infty,-\infty,-\infty}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n) := \bigcap_{m,r_1,r_2\in\mathbb{R}} S^{m,r_1,r_2}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n). \tag{4.118}$$

(1) Prove that $S^{-\infty,-\infty,-\infty}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n) \subset S^{m,r_1,r_2}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$ is dense in the topology of $S^{m',r_1',r_2'}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$ whenever $m < m'$, $r_1 < r_1'$, $r_2 < r_2'$.

[4] That is, such a are symbolic not only in ξ, but also in x and y.

(2) Prove the following variant of Proposition 3.13: given $a_j \in S^{m-j,r_1-j,r_2-j}(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^n)$, there exists $a \in S^{m,r_1,r_2}(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^n)$, unique modulo $S^{-\infty,-\infty,-\infty}(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^n)$, such that $a - \sum_{j=0}^{J-1} a_j \in S^{m-J,r_1-J,r_2-J}(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^n)$ for all $J \in \mathbb{N}$.

Exercise 4.8 (Scattering ps.d.o.s, I: boundedness). Let $m, r_1, r_2 \in \mathbb{R}$. Prove that

$$\mathrm{Op}(a)\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n), \quad a \in S^{m,r_1,r_2}(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^n). \tag{4.119}$$

Prove this more generally for $a \in \langle x-y\rangle^w S^{m,r_1,r_2}(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^n)$, $w \in \mathbb{R}$.

Exercise 4.9 (Scattering ps.d.o.s, II: residual operators). Let $r_2 \in \mathbb{R}$. Show that an operator A can be written as $A = \mathrm{Op}(a_N)$, $a_N \in S^{-N,-r_2,-N}(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^n)$, for all N if and only if its Schwartz kernel $K = K(x,y)$ satisfies $K \in \mathscr{S}(\mathbb{R}^n_x \times \mathbb{R}^n_y)$. Show that in this case, there exist unique $a_L, a_R \in S^{-\infty,-\infty}(\mathbb{R}^n;\mathbb{R}^n)$ such that $A = \mathrm{Op}_L(a_L) = \mathrm{Op}_R(a_R)$.

Exercise 4.10 (Scattering ps.d.o.s, III: reduction). We write

$$S^{m,r}(\mathbb{R}^n;\mathbb{R}^n) \tag{4.120}$$

for the space of $a = a(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ satisfying $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}\langle x\rangle^{r-|\alpha|}\langle\xi\rangle^{m-|\alpha|}$ for all $\alpha, \beta \in \mathbb{N}_0^n$.

Let $A = \mathrm{Op}(a)$, $a \in S^{m,r_1,r_2}(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^n)$. Prove that there exists a unique left symbol $a_L \in S^{m,r_1+r_2}(\mathbb{R}^n;\mathbb{R}^n)$ such that $A = \mathrm{Op}_L(a_L)$.

Exercise 4.11 (Scattering ps.d.o.s, IV: algebra). Define

$$\Psi_{\mathrm{sc}}^{m,r}(\mathbb{R}^n) := \mathrm{Op}(S^{m,r}(\mathbb{R}^n;\mathbb{R}^n)). \tag{4.121}$$

(1) Prove that $A \in \Psi_{\mathrm{sc}}^{m,r}(\mathbb{R}^n)$ implies $A^* \in \Psi_{\mathrm{sc}}^{m,r}(\mathbb{R}^n)$.
(2) Suppose $A \in \Psi_{\mathrm{sc}}^{m,r}(\mathbb{R}^n)$, $B \in \Psi_{\mathrm{sc}}^{m',r'}(\mathbb{R}^n)$. Prove that

$$A \circ B \in \Psi_{\mathrm{sc}}^{m+m',r+r'}(\mathbb{R}^n). \tag{4.122}$$

Exercise 4.12 (Scattering ps.d.o.s, V: principal symbol). Define the principal symbol of $A = \mathrm{Op}_L(a_L) \in \Psi_{\mathrm{sc}}^{m,r}(\mathbb{R}^n)$ by

$${}^{\mathrm{sc}}\sigma^{m,r}(A) := [a_L] \in S^{m,r}(\mathbb{R}^n;\mathbb{R}^n)/S^{m-1,r-1}(\mathbb{R}^n;\mathbb{R}^n). \tag{4.123}$$

State and prove the analogue of Proposition 4.21 for scattering ps.d.o.s.[5]

Exercise 4.13 (Scattering ps.d.o.s, VI: ellipticity). Suppose $A = \mathrm{Op}_L(a_L) \in \Psi_{\mathrm{sc}}^{m,r}(\mathbb{R}^n)$ is elliptic, that is, there exists $b \in S^{-m,-r}(\mathbb{R}^n;\mathbb{R}^n)$ such that $a_L b - 1 \in S^{-1,-1}(\mathbb{R}^n;\mathbb{R}^n)$.

(1) Prove that there exists $B \in \Psi_{\mathrm{sc}}^{-m,-r}(\mathbb{R}^n)$ such that $BA - I \in \Psi_{\mathrm{sc}}^{-\infty,-\infty}(\mathbb{R}^n;\mathbb{R}^n) = \bigcap_{m,r\in\mathbb{R}} \Psi_{\mathrm{sc}}^{m,r}(\mathbb{R}^n)$.
(2) Suppose $u \in \mathscr{S}'(\mathbb{R}^n)$, and $Au = f \in \mathscr{S}(\mathbb{R}^n)$. Prove that $u \in \mathscr{S}(\mathbb{R}^n)$. (Notice the difference to the statements of Proposition 4.28 or Corollary 4.36! For example, the Laplacian $\Delta \in \Psi^2(\mathbb{R}^n)$ is uniformly elliptic, but $\Delta u = 0$ for $u = 1$, $u = x_1 x_2$, etc. However, Δ is *not* elliptic as an element of $\Psi_{\mathrm{sc}}^{2,0}(\mathbb{R}^n)$. (Check!) What about $\Delta + 1$?)

Exercise 4.14 (Scattering ps.d.o.s, VII: boundedness on Sobolev spaces). (1) Prove that elements of $\Psi_{\mathrm{sc}}^{0,0}(\mathbb{R}^n)$ are bounded as maps on $L^2(\mathbb{R}^n)$.
(2) Show that $\Lambda_{m,r} := \langle x\rangle^r \langle D\rangle^m \in \Psi_{\mathrm{sc}}^{m,r}(\mathbb{R}^n)$ and $\Lambda'_{m,r} := \langle D\rangle^m \langle x\rangle^r \in \Psi_{\mathrm{sc}}^{m,r}(\mathbb{R}^n)$.

[5] Thus, the principal symbol is more powerful in the scattering world: it not only captures the high frequency, i.e. large ξ, behavior of an operator, but also the large x behavior.

(3) Let $A \in \Psi_{\mathrm{sc}}^{m,r}(\mathbb{R}^n)$. Show that for all $\rho,\sigma \in \mathbb{R}$, A is a bounded operator

$$A\colon \langle x\rangle^\rho H^\sigma(\mathbb{R}^n) \to \langle x\rangle^{\rho+r} H^{\sigma-m}(\mathbb{R}^n). \tag{4.124}$$

Exercise 4.15 (Scattering ps.d.o.s, VIII: elliptic scattering ps.d.o.s are Fredholm). (1) Let $m < m'$ and $r > r'$. Show that the inclusion $\langle x\rangle^{r'} H^{m'}(\mathbb{R}^n) \hookrightarrow \langle x\rangle^r H^m(\mathbb{R}^n)$ is compact.

(2) Let $A \in \Psi_{\mathrm{sc}}^{m,r}(\mathbb{R}^n)$ be elliptic (see Exercise 4.13). Show that for any $\rho,\sigma \in \mathbb{R}$, the operator

$$A\colon \langle x\rangle^\rho H^\sigma(\mathbb{R}^n) \to \langle x\rangle^{\rho+r} H^{\sigma-m}(\mathbb{R}^n) \tag{4.125}$$

is a Fredholm operator.

(3) Show that the index of A in (4.125) is independent of ρ,σ.

5. Pseudodifferential operators on manifolds

We now show how the ps.d.o. algebra on $\mathbb{R}^n$ can be transferred to smooth manifolds by using local coordinate charts. The key ingredient for showing that this is a reasonable thing to do is the invariance of the class of m-th order ps.d.o.s under changes of coordinates on $\mathbb{R}^n$.

5.1. Local coordinate invariance.

We now prove the analogue of the final part of Proposition 2.20 for ps.d.o.s.

Definition 5.1 (Ps.d.o.s with restricted Schwartz kernels)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set. Then

$$\Psi_{\mathrm{c}}^m(\Omega) := \{A \in \Psi^m(\Omega)\colon \operatorname{supp} K_A \Subset \Omega\times\Omega\}, \tag{5.1}$$

where $K_A \in \mathscr{S}'(\mathbb{R}^{2n})$ denotes the Schwartz kernel of A.

Theorem 5.2 (Ps.d.o.s under changes of variables)**.** *Suppose $\Omega,\Omega' \subset \mathbb{R}^n$ are open, and $\kappa\colon \Omega \to \Omega'$ is a diffeomorphism. Given $A \in \Psi_{\mathrm{c}}^m(\Omega')$, define $A_\kappa u := \kappa^* A(\kappa^{-1})^*(u|_\Omega)$. Then $A_\kappa \in \Psi_{\mathrm{c}}^m(\Omega)$, and the map $\Psi_{\mathrm{c}}^m(\Omega') \ni A \mapsto A_\kappa \in \Psi_{\mathrm{c}}^m(\Omega)$ is bijective. Moreover,*

$$\sigma^m(A_\kappa)(x,\xi) = \sigma^m(A)(\kappa(x), (\kappa'(x)^T)^{-1}\xi). \tag{5.2}$$

Proof. We have $A = \mathrm{Op}_L(a)$ for some $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$. Choose $\psi \in \mathcal{C}_{\mathrm{c}}^\infty(\Omega')$ such that $\psi(x)\psi(y) = 1$ on $\operatorname{supp} K_A$; thus $K_A(x,y) = \psi(x)K_A(x,y)\psi(y)$, and therefore

$$K_A = \mathrm{Op}\big(\psi(x)a(x,\xi)\psi(y)\big). \tag{5.3}$$

We localize near the diagonal: for $\epsilon > 0$ (to be determined), let $\chi_\epsilon(x,y) \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ be such that $\chi_\epsilon(x,y) = 1$ for $|x-y| < \epsilon$ and $\chi_\epsilon(x,y) = 0$ for $|x-y| > 2\epsilon$. Then

$$K_{A_\epsilon} := \mathrm{Op}(a_\epsilon), \quad a_\epsilon(x,y,\xi) = \chi_\epsilon(x,y)\psi(x)\psi(y)a(x,\xi), \tag{5.4}$$

is the Schwartz kernel of an operator $A_\epsilon \in \Psi_{\mathrm{c}}^m(\Omega')$, and

$$R_\epsilon := A - A_\epsilon \tag{5.5}$$

is a ps.d.o. with Schwartz kernel supported away from the diagonal, hence $R_\epsilon \in \Psi^{-\infty}(\mathbb{R}^n)$, and its Schwartz kernel satisfies $K_{R_\epsilon} \in \mathcal{C}_{\mathrm{c}}^\infty(\Omega'\times\Omega')$. We then have

$$\begin{aligned}(R_\epsilon)_\kappa u(x) &= \int_{\Omega'} K_{R_\epsilon}(\kappa(x), y')u(\kappa^{-1}(y'))\,\mathrm{d}y' \\ &= \int_\Omega K_{R_\epsilon}(\kappa(x),\kappa(y))|\det\kappa'(y)|u(y)\,\mathrm{d}y.\end{aligned} \tag{5.6}$$

Therefore, the Schwartz kernel of $(R_\epsilon)_\kappa$ is $K_{(R_\epsilon)_\kappa}(x,y) = K_{R_\epsilon}(\kappa(x),\kappa(y))|\det\kappa'(y)|$ for $x,y\in\Omega$, and 0 otherwise. Thus, $(R_\epsilon)_\kappa \in \Psi_{\mathrm{c}}^{-\infty}(\Omega)$.

It remains to show that $(A_\epsilon)_\kappa \in \Psi_{\mathrm{c}}^m(\Omega)$. Suppose first that $a\in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$. Note then that

$$\begin{aligned}(A_\epsilon)_\kappa u(x) &= (2\pi)^{-n}\iint e^{i(\kappa(x)-y')\xi'}a_\epsilon(\kappa(x),y',\xi')u(\kappa^{-1}(y'))\,\mathrm{d}y'\,\mathrm{d}\xi' \\ &= (2\pi)^{-n}\iint e^{i(\kappa(x)-\kappa(y))\xi'}a_\epsilon(\kappa(x),\kappa(y),\xi')|\det\kappa'(y)|u(y)\,\mathrm{d}y\,\mathrm{d}\xi',\end{aligned} \tag{5.7}$$

and thus the Schwartz kernel of $(A_\epsilon)_\kappa$ is

$$\begin{aligned}K_{(A_\epsilon)_\kappa}(x,y) &= (2\pi)^{-n}\int e^{i(\kappa(x)-\kappa(y))\xi'}b_\epsilon(x,y,\xi')\,\mathrm{d}\xi', \\ b_\epsilon(x,y,\xi') &= a_\epsilon(\kappa(x),\kappa(y),\xi')|\det\kappa'(y)| \\ &= \chi_\epsilon(\kappa(x),\kappa(y))\psi(\kappa(x))\psi(\kappa(y))a(\kappa(x),\xi)|\det\kappa'(y)|.\end{aligned} \tag{5.8}$$

We Taylor expand the exponent: denoting by κ_j the j-th component of κ, we have

$$\kappa_j(x)-\kappa_j(y) = \sum_{k=1}^n \Phi_{jk}(x,y)(x_k-y_k), \quad \Phi_{jk}(x,y) = \int_0^1 \partial_{x_k}\kappa_j(y+t(x-y)), \tag{5.9}$$

and therefore

$$(\kappa(x)-\kappa(y))\cdot\xi' = \langle\Phi(x,y)(x-y),\xi'\rangle = \langle x-y,\Phi(x-y)^T\xi'\rangle, \tag{5.10}$$

where $\Phi(x,y) = (\Phi_{jk}(x,y))_{j,k=1,\dots,n}$, and $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^n$. Note now that

$$\Phi(x,x) = \kappa'(x) \tag{5.11}$$

is invertible for $x\in\Omega$ since κ is a diffeomorphism. For

$$(x,y)\in\operatorname{supp}\chi_\epsilon(\kappa(x),\kappa(y))\psi(\kappa(x))\psi(\kappa(y)), \tag{5.12}$$

we have $(x,y)\in\kappa^{-1}(\operatorname{supp}\psi)\times\kappa^{-1}(\operatorname{supp}\psi)\Subset\Omega\times\Omega$ and $|\kappa(x)-\kappa(y)|\le 2\epsilon$. Therefore, we can choose $\epsilon>0$ such that $\Phi(x,y)$ is invertible for (x,y) in the set (5.12). In (5.8), we can then make the change of variables $\xi' = (\Phi(x,y)^T)^{-1}\xi$, so

$$\begin{aligned}(A_\epsilon)_\kappa = \mathrm{Op}(c_\epsilon), \quad c_\epsilon(x,y,\xi) &= b_\epsilon(x,y,(\Phi(x,y)^T)^{-1}\xi)|\det\Phi(x,y)|^{-1} \\ &= a_\epsilon(\kappa(x),\kappa(y),(\Phi(x,y)^T)^{-1}\xi)|\det\Phi(x,y)|^{-1}|\det\kappa'(y)|.\end{aligned} \tag{5.13}$$

We claim that $c_\epsilon\in S^{-\infty}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$; more generally, if $a\in S^m(\mathbb{R}^n;\mathbb{R}^n)$, we claim that c_ϵ defined by (5.13) satisfies $c_\epsilon\in S^m(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$ and depends continuously on a. We can drop the (smooth) Jacobian factors. We then compute

$$\begin{aligned}&\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma\big(a_\epsilon(\kappa(x),\kappa(y),(\Phi(x,y)^T)^{-1}\xi)\big) \\ &\quad= \sum_{\substack{|\alpha'|+|\alpha''|\le|\alpha| \\ |\beta'|+|\beta''|\le|\beta|}} F_{\alpha\alpha'\alpha''\gamma}^{\beta\beta'\beta''}(x,y)\xi^{\alpha''+\beta''}(\partial_x^{\alpha'}\partial_y^{\beta'}\partial_\xi^{\gamma+\alpha''+\beta''}a_\epsilon)(\kappa(x),\kappa(y),(\Phi(x,y)^T)^{-1}\xi)\end{aligned} \tag{5.14}$$

for some smooth functions $F_{\alpha\alpha'\alpha''\gamma}^{\beta\beta'\beta''}\in\mathcal{C}_{\mathrm{c}}^\infty(\Omega\times\Omega)$. Since $\Phi(x,y)^T$ and its inverse are uniformly bounded on $\operatorname{supp}a_\epsilon$, there exist $c,C>0$ such that $c|\xi|\le|(\Phi(x,y)^T)^{-1}\xi|\le C|\xi|$ on $\operatorname{supp}a_\epsilon$. Therefore, (5.14) is bounded by $\langle\xi\rangle^{m-|\gamma|}$ times a continuous seminorm on a, as desired. Finally, given $a\in S^m(\mathbb{R}^n;\mathbb{R}^n)$, we select $a^{(j)}\in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$, $j\in\mathbb{N}$, which are uniformly bounded in $S^m(\mathbb{R}^n;\mathbb{R}^n)$ and converge to a in $S^{m+1}(\mathbb{R}^n;\mathbb{R}^n)$ as $j \to \infty$. Then the corresponding symbols $c_\epsilon^{(j)}$ are uniformly bounded in $S^m(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^n)$ and converge to c_ϵ in $S^{m+1}(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^n)$. Setting $A_\epsilon^{(j)} = \mathrm{Op}(a_\epsilon^{(j)})$ with $a_\epsilon^{(j)}$ defined in terms of $a^{(j)}$ as in (5.4), we have $(A_\epsilon)_\kappa u = \lim_{j\to\infty}(A_\epsilon^{(j)})_\kappa u$ for all $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, and since $(A_\epsilon^{(j)})_\kappa u = \mathrm{Op}(c_\epsilon^{(j)})u \to \mathrm{Op}(c_\epsilon)u$ as $j \to \infty$ by (4.24), we have established (5.13) in general.

As for the principal symbol, we have $\sigma^m(A_\kappa) = \sigma^m((A_\epsilon)_\kappa) = \sigma^m(\mathrm{Op}(c_\epsilon))$, which can be read off from the (first term of the) reduction formula (4.35): using (5.11), it is given by the equivalence class in $S^m(\mathbb{R}^n;\mathbb{R}^n)/S^{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ of

$$\begin{aligned} c_\epsilon(x,x,\xi) &= b_\epsilon(x,x,(\kappa'(x)^T)^{-1}\xi)|\det\kappa'(x)|^{-1} \\ &= a_\epsilon(\kappa(x),\kappa(x),(\kappa'(x)^T)^{-1}\xi) \\ &= a(\kappa(x),(\kappa'(x)^T)^{-1}\xi). \end{aligned} \tag{5.15}$$

The proof is complete. □

5.2. **Manifolds, vector bundles, densities.** We shall only work with smooth manifolds: they are locally diffeomorphic to the unit ball $B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$. We recall the 'hands-on' definition of smooth manifolds:

Definition 5.3 (Smooth manifolds)**.** Let $n \in \mathbb{N}$. A *smooth manifold of dimension* n is a second countable, paracompact Hausdorff space M such that

(1) for each point $p \in M$, there exist an open neighborhood $U_p \ni p$ and a homeomorphism $F_p : U_p \to B(0,1) \subset \mathbb{R}^n$;
(2) for all $p,q \in M$ such that $U_p \cap U_q \neq \emptyset$, the transition map

$$F_p \circ F_q^{-1}|_{U_p\cap U_q} : F_q(U_p \cap U_q) \to F_p(U_p \cap U_q) \tag{5.16}$$

is smooth (as a map between open subsets of $\mathbb{R}^n$).

Definition 5.4 (Smooth structure)**.** Let M be a smooth manifold. A *atlas* on M is a collection $\{(U_\alpha, F_\alpha)\}$ of pairs (U_α, F_α), with $U_\alpha \neq \emptyset$ and $M = \bigcup_\alpha U_\alpha$, such that $F_\alpha : U_\alpha \to \mathbb{R}^n$ is a diffeomorphism onto an open subset $F_\alpha(U_\alpha)$ of $\mathbb{R}^n$. A *maximal atlas*, or *smooth structure*, is an atlas with the property that any other atlas is contained in it. An element of the[6] maximal atlas is called a *(local coordinate) chart*.

The 'hands-on' definition of vector bundles is the following.

Definition 5.5 (Vector bundles)**.** Let M be a smooth n-dimensional manifold. A *real vector bundle of rank k over M* is a triple (π, E, M) with the following properties:

(1) E is a smooth $(n+k)$-dimensional manifold, and $\pi : E \to M$ is smooth;
(2) for each $p \in M$ there exist an open neighborhood $U_p \subset M$, $p \in U_p$, and a diffeomorphism $\tau_p : \pi^{-1}(U_p) \to U_p \times \mathbb{R}^k$ such that $\pi(\tau_p^{-1}(q,v)) = q$ is the projection onto the first factor;
(3) for all $p,q \in M$ such that $U_p \cap U_q \neq \emptyset$, the transition map

$$\tau_{pq} := \tau_p \circ \tau_q^{-1}|_{(U_p\cap U_q)\times\mathbb{R}^k} : (U_p \cap U_q) \times \mathbb{R}^k \to (U_p \cap U_q) \times \mathbb{R}^k \tag{5.17}$$

takes the form $\tau_{pq}(r,v) = (r, \Phi_{pq}(r)v)$, where $\Phi_{pq} : U_p \cap U_q \to GL(k)$ is smooth.

[6] A maximal atlas always exists and is unique.

The *fiber of E over* $p \in M$ is denoted $E_p := \pi^{-1}(p)$; it is a k-dimensional real vector space. The *zero section* of E is the submanifold of E given locally in $\pi^{-1}(U_p)$ by $\tau_p^{-1}(U_p \times \{0\})$. A *smooth section of* E is a smooth map $s\colon M \to E$ such that $\pi \circ s = \mathrm{Id}_M$. The space of smooth sections is denoted $\mathcal{C}^\infty(M; E)$.

Another useful notion for later is the following.

Definition 5.6 (Pullback of vector bundles)**.** Let M, N be smooth manifolds (not necessarily of the same dimension), and let $f\colon M \to N$ be smooth. If $\pi\colon E \to N$ is a vector bundle, then the *pullback of E by f* is the vector bundle

$$\tilde\pi\colon f^*E \to M \tag{5.18}$$

given by $f^*E = M\,{}_f\!\times_\pi E = \{(p, e) \in M \times E\colon f(p) = \pi(e)\}$, with projection map $\tilde\pi(p, e) = p$, and with linear structure on $(f^*E)_p = E_{f(p)}$ equal to that on $E_{f(p)}$.

To specify a real rank k vector bundle uniquely (up to vector bundle isomorphisms), it suffices to have the following data and conditions:

(1) a cover $\{U_\alpha\}$ of M by open non-empty subsets;
(2) for all α, β with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ a map $\tau_{\alpha\beta}\colon U_{\alpha\beta} \times \mathbb{R}^k \to U_{\alpha\beta} \times \mathbb{R}^k$ of the form $\tau_{\alpha\beta}(p, v) = (p, \Phi_{\alpha\beta}(p)v)$ with $\Phi_{\alpha\beta}\colon U_{\alpha\beta} \to GL(k)$ smooth;
(3) $\tau_{\alpha\alpha}(p, v) = (p, v)$ for all $p \in U_\alpha$, $v \in \mathbb{R}^k$;
(4) the *cocycle condition* holds: for α, β, γ with $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, we have $\tau_{\gamma\beta} \circ \tau_{\beta\alpha} = \tau_{\gamma\alpha}$ on $U_{\alpha\beta\gamma} \times \mathbb{R}^k$.

Indeed, one can then set

$$E := \left(\bigsqcup_\alpha U_\alpha \times \mathbb{R}^k \right) / \sim, \tag{5.19}$$

where we define the equivalence relation $\sim$ by

$$U_\alpha \times \mathbb{R}^k \ni (p, v) \sim (q, w) \in U_\beta \times \mathbb{R}^k \iff p = q,\ \tau_{\alpha\beta}(q, w) = (p, v). \tag{5.20}$$

(The cocycle condition guarantees that this is transitive, while reflexivity follows from the cocycle condition together with $\tau_{\alpha\alpha} = \mathrm{Id}$.) We denote the equivalence classes by $[U_\alpha, (p, v)] \in E$. Note that the same point p can lie in two (or more) open sets U_α and U_β, but typically $[U_\alpha, (p, v)] \neq [U_\beta, (p, v)]$ for $v \in \mathbb{R}^k$. The projection map $\pi\colon E \to M$ is simply given by $\pi([U_\alpha, (p, v)]) = p$. As local trivializations, we can take

$$\tau_\alpha\colon \{[U_\alpha, (p, v)]\colon p \in U_\alpha\} \mapsto (p, v) \in U_\alpha \times \mathbb{R}^k. \tag{5.21}$$

Example 5.7. Taking a cover of an n-dimensional manifold M by coordinate charts $F_i\colon U_i \to \mathbb{R}^n$, we take $\tau_{ij}(p, v) = (p, (F_i \circ F_j^{-1})'|_{F_j(p)} v)$. The resulting vector bundle is the *tangent bundle of* M, denoted TM. Note that a chart $F\colon U \to \mathbb{R}^n$ induces a trivialization of $T_U M = \pi^{-1}(U)$ via $U \times \mathbb{R}^n \ni (p, v) \mapsto [U, (p, v)] \in T_U M$. A tangent vector $V = [U, (p, v)] \in T_p M$ has several interpretations.

(1) V is a directional derivative on M at p. That is, it gives a map

$$\mathcal{C}^\infty(M) \ni f \mapsto V(f) := \frac{\mathrm{d}}{\mathrm{d}s} \Big(\big((F^{-1})^* f|_U \big)(F(p) + sv) \Big) \Big|_{s=0}. \tag{5.22}$$

(See also Exercise 5.1.)

(2) V is the tangent vector of a smooth curve on M. This comes from the following construction: consider the set of all smooth curves $\gamma\colon I \to M$ with $\gamma(0) = p$, where $I \subset \mathbb{R}$ is an open interval containing 0. An equivalence relation on this set is defined as follows: $\gamma_1 \sim \gamma_2$ if and only if, in any local coordinate system $F\colon U \to \mathbb{R}^n$ with $p \in U$, we have $\frac{\mathrm{d}}{\mathrm{d}s}F(\gamma_1(s))|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}F(\gamma_2(s))|_{s=0}$. (Check that this condition is independent of the choice of F.) Then T_pM is the set of equivalence classes of curves. The tangent vector V is identified with the equivalence class of the curve $s \mapsto F^{-1}(F(p) + sv)$.

Functorial operations on vector spaces give corresponding operations on vector bundles. For instance, given a linear map $A\colon V \to W$ between two vector spaces, the adjoint is $A^T\colon W^* \to V^*$; if A is invertible, then this gives a map $(A^T)^{-1}\colon V^* \to W^*$. In the notation of Example 5.7, we thus take

$$\tau_{ij}(p, v) = \bigl(p, ((\kappa'_{ij}|_{F_j(p)})^T)^{-1}v\bigr), \quad \kappa_{ij} := F_i \circ F_j^{-1}. \tag{5.23}$$

The resulting vector bundle is the *cotangent bundle*, denoted $\pi\colon T^*M \to M$. Note that formula (5.23) appears in (5.2) (except in the latter we also use/change local coordinates on the base M via κ_{ij}). We recall then that given a smooth function $f \in \mathcal{C}^\infty(M)$, we can define its exterior derivative $\mathrm{d}f \in \mathcal{C}^\infty(M; T^*M)$ as follows: if $F_i\colon U_i \to F_i(U_i) \subset \mathbb{R}^n$ is a coordinate chart, we define

$$\tau_i(\mathrm{d}f(p)) := \Bigl(p, \bigl(\partial_{x_j}((F_i^{-1})^*f)\bigr)_{j=1,\dots,n}\Bigr) \in U_i \times \mathbb{R}^n, \quad p \in U_i. \tag{5.24}$$

That this indeed gives a well-defined section of T^*M follows from the change of variables formula.

As in Example 5.7, a local coordinate system $F\colon U \to \mathbb{R}^n$ induces a trivialization of $T^*U = \pi^{-1}(U)$. The relationship between T^*M and TM is as follows: for any $p \in M$, we have an isomorphism of vector spaces

$$T_p^*M \cong \mathrm{Hom}(T_pM, \mathbb{R}) \tag{5.25}$$

which is defined independently of any choices. This isomorphism can be described in local trivializations of TM and T^*M induced by the same local coordinate system F: if $\omega = [U, (p, \xi)] \in T_p^*M$ and $V = [U, (p, v)] \in T_pM$ where $\xi, v \in \mathbb{R}^n$, we set $\omega(V) = \langle \xi, v\rangle$ (Euclidean inner product). See Exercise 5.2.

A natural choice for f in local coordinates near p is $f = x_k$ (i.e. $f = F_i^*x_k$), in which case (5.24) defines the differential $\mathrm{d}x_k$ with $\tau_i(\mathrm{d}x_k) = (p, (0, \dots, 1, \dots, 0))$ (with the 1 in the k-th slot). The exterior derivative $\mathrm{d}f$ is then usually written as $\mathrm{d}f = \sum_{j=1}^n (\partial_{x_j} f)\mathrm{d}x^j$, dropping the coordinate and trivialization maps from the notation.

Example 5.8. Let $E \to M$ and $F \to M$ denote two vector bundles.

(1) The fiberwise direct sum of vector spaces produces the vector bundle $E \oplus F \to M$, with fibers $(E \oplus F)_p = E_p \oplus F_p$.
(2) Likewise, taking the fiberwise tensor product gives $E \otimes F \to M$, with fibers $(E \otimes F)_p = E_p \otimes F_p$.
(3) The vector bundle $\mathrm{Hom}(E, F) \to M$ has fibers $\mathrm{Hom}(E_p, F_p)$. We have $\mathrm{Hom}(E, F) = E \otimes F^*$.
(4) Let $q \in \mathbb{N}$. The fiberwise q-th exterior power of E produces the vector bundle $\Lambda^q E \to M$. In the special case $E = T^*M$, one often writes $\Lambda^q M := \Lambda^q T^*M$.

We discuss another important vector bundle, closely related to the top exterior power $\Lambda^n T^*M$ of the cotangent bundle of an n-dimensional manifold M:

Definition 5.9 (Density bundles)**.** Let $\alpha \in \mathbb{R}$. In the notation of Example 5.7, the *α-density bundle on M* is the vector bundle

$$\Omega^\alpha M \to M \tag{5.26}$$

with transition functions $\tau_{ij}(p,v) = (p, |\det \kappa'_{ij}|_{F_j(p)}|^{-\alpha} v)$, $\kappa_{ij} = F_i \circ F_j^{-1}$. We also write

$$\Omega M := \Omega^1 M. \tag{5.27}$$

Since the transition functions τ_{ij} act in the second argument (v) via multiplication by a positive number, there is a well-defined notion of *positive α-densities*, which are those densities which in the trivialization of $\Omega^\alpha M$ over a coordinate chart on M are given by positive smooth functions.

Remark 5.10 (Functorial perspective on density bundles). $\Omega^\alpha M \to M$ arises functorially from the following operation on vector spaces, applied to TM: given a real n-dimensional vector space V, we define

$$\Omega^\alpha V := \{\omega\colon \Lambda^n V \to \mathbb{R}\colon \omega(\mu v) = |\mu|^\alpha \omega(v),\ v \in \Lambda^n V,\ \mu \in \mathbb{R}\}. \tag{5.28}$$

To see the relationship, note first that $\Lambda^n V$ is 1-dimensional. Then, given another n-dimensional vector space W and a map $\kappa\colon V \to W$, let us fix bases $e_1, \ldots, e_n$ of V and $f_1, \ldots, f_n$ of W. Consider, as a warm-up, the top exterior powers: $e_1 \wedge \cdots \wedge e_n$ and $f_1 \wedge \cdots \wedge f_n$ are bases of $\Lambda^n V$ and $\Lambda^n W$, and the map $\Lambda^n\kappa\colon \Lambda^n V \to \Lambda^n W$ is given by $e_1 \wedge \cdots \wedge e_n \mapsto \kappa(e_1) \wedge \cdots \kappa(e_n) = (\det \kappa) f_1 \wedge \cdots \wedge f_n$, where $\det \kappa$ is the determinant of the matrix of κ in these bases: that is, in the stated basis, $\Lambda^n\kappa$ is simply multiplication by $\det \kappa$. Similarly, $\Omega^\alpha V$ and $\Omega^\alpha W$ are 1-dimensional, with basis elements $\omega_V\colon \mu e_1 \wedge \cdots \wedge e_n \mapsto |\mu|^\alpha$ and $\omega_W\colon \mu f_1 \wedge \cdots \wedge f_n \mapsto |\mu|^\alpha$. Now, the map

$$\Omega^\alpha \kappa\colon \Omega^\alpha V \to \Omega^\alpha W \tag{5.29}$$

is given by

$$\Omega^\alpha\kappa(\omega)(f_1 \wedge \cdots \wedge f_n) = \omega(\kappa^{-1}(f_1) \wedge \cdots \wedge \kappa^{-1}(f_n)), \tag{5.30}$$

hence $\Omega^\alpha\kappa(\omega_V) = |\det \kappa|^{-\alpha}\omega_W$.

The proof of the following simple lemmas is left to the reader as a simple exercise.

Lemma 5.11 (Properties of density bundles)**.** *Let $\alpha, \beta \in \mathbb{R}$. Then*

(1) $(\Omega^\alpha)^* M = \Omega^{-\alpha} M$,
(2) $\Omega^\alpha M \otimes \Omega^\beta M = \Omega^{\alpha+\beta} M$,
(3) $\Omega^0 M = M \times \mathbb{R}$.

Proof. See Exercise 5.6. □

If $x \in \mathbb{R}^n$ denotes local coordinates on a manifold M, then a typical α-density is

$$|\mathrm{d}x|^\alpha\colon \mu\partial_{x_1} \wedge \cdots \wedge \partial_{x_n} \mapsto |\mu|^\alpha. \tag{5.31}$$

Similarly to differential forms, α-densities can be pulled back by smooth maps:

Lemma 5.12 (Pullback of densities)**.** *Let $f\colon M \to N$ be a smooth map between smooth manifolds of the same dimension; assume that f has no critical points. In local coordinates x, y on M, N, and $u(y) = u_0(y)|\mathrm{d}y|^\alpha$, define $(f^*u)(x) = u_0(f(x))|\det f'(x)|^\alpha|\mathrm{d}x|^\alpha$. Then f^* is a well-defined map*

$$f^*\colon \mathcal{C}^\infty(N; \Omega^\alpha N) \to \mathcal{C}^\infty(M; \Omega^\alpha M). \tag{5.32}$$

Remark 5.13 (Invariant formulation). If $V \in T_pM$, define its pushforward along a smooth map $f\colon M \to N$ to be the tangent vector $f_*V \in T_{f(p)}N$ so that, as a directional derivative, $(f_*V)(g) = V(f^*g)$ for $g \in \mathcal{C}^\infty(N)$. Then in the notation of Lemma 5.12, the pullback of an α-density u is given by $(f^*u)|_p(V_1 \wedge \cdots \wedge V_n) = u|_{f(p)}(f_*V_1 \wedge \cdots \wedge f_*V_n)$ where $n = \dim M$.

See Exercise 5.12.

For us, 1-densities are the most useful: sections of ΩM can be invariantly integrated. On $\mathbb{R}^n$, we write for $u \in \mathcal{C}^\infty_\mathrm{c}(\mathbb{R}^n; \Omega\mathbb{R}^n)$, $u = u_0(x)|\mathrm{d}x|$:

$$\int_{\mathbb{R}^n} u := \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x \tag{5.33}$$

Let $\{\phi_i\}$ be a partition of unity on M subordinate to a cover by coordinate systems $F_i\colon U_i \to F_i(U_i) \subset \mathbb{R}^n$ with $\overline{U_i}$ compact. Define the map

$$\int_M \colon \mathcal{C}^\infty_\mathrm{c}(M; \Omega M) \to \mathbb{R}, \quad u \mapsto \sum_i \int_{\mathbb{R}^n} (F_i^{-1})^*(\phi_i u). \tag{5.34}$$

(Note here that $(F_i^{-1})^*(\phi_i u) \in \mathcal{C}^\infty_\mathrm{c}(\mathbb{R}^n; \Omega\mathbb{R}^n)$!)

Proposition 5.14 (Integration of 1-densities)**.** *The map* (5.34) *is independent of the choice of local coordinates and the partition of unity.*

Proof. First, suppose $u \in \mathcal{C}^\infty_\mathrm{c}(M; \Omega M)$ is supported in the intersection of two coordinate charts, with local coordinates $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ and transition function $x = \kappa(y)$, then

$$u(x) = u_0(x)|\mathrm{d}x| = u_1(y)|\mathrm{d}y|. \tag{5.35}$$

But at $x = \kappa(y)$, we have $(\Omega^1\kappa)|\mathrm{d}y| = |\det \kappa'(y)|^{-1}|\mathrm{d}x|$, so $|\mathrm{d}x| = |\det \kappa'(y)||\mathrm{d}y|$. Therefore, $u_1(y) = u_0(\kappa(y))|\det \kappa'(y)|$, and thus

$$\int_{\mathbb{R}^n} u_1(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} u_0(\kappa(y))|\det \kappa'(y)|\,\mathrm{d}y = \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x. \tag{5.36}$$

The proposition follows easily from this: if $\{\psi_j\}$ is another partition of unity subordinate to a cover by coordinate systems $G_j\colon V_j \to G_j(U_j) \subset \mathbb{R}^n$, then $\int_M u = \sum_j \int_M \psi_j u$, and

$$\begin{aligned} \int_M u &= \sum_j \int_M \psi_j u \\ &= \sum_{i,j} \int_{\mathbb{R}^n} (F_i^{-1})^*(\phi_i \psi_j u) \\ &= \sum_{i,j} \int_{\mathbb{R}^n} (G_j \circ F_i^{-1})^*\big((G_j^{-1})^*(\psi_j \phi_i u)\big) \\ &= \sum_{i,j} \int_{\mathbb{R}^n} (G_j^{-1})^*(\psi_j \phi_i u) \\ &= \sum_j \int_{\mathbb{R}^n} (G_j^{-1})^* \psi_j u. \end{aligned} \tag{5.37}$$

This finishes the proof. ☐

In analogy with the case of $\mathbb{R}^n$, this leads us to the following definition:

Definition 5.15 (Distributions on manifolds)**.** The space $\mathscr{D}'(M)$ consists of all continuous linear maps $\mathcal{C}^\infty_c(M; \Omega M) \to \mathbb{C}$. More generally, if $E \to M$ is a vector bundle, then $\mathscr{D}'(M; E)$ consists of all continuous linear functionals $\mathcal{C}^\infty_c(M; E^* \otimes \Omega M) \to \mathbb{C}$. The space $\mathscr{E}'(M; E)$ consists of all continuous linear functionals $\mathcal{C}^\infty(M; E^* \otimes \Omega M) \to \mathbb{C}$.

Thus, $\mathcal{C}^\infty(M; E) \hookrightarrow \mathscr{D}'(M; E)$ via the pairing

$$\mathcal{C}^\infty_c(M; E^* \otimes \Omega M) \times \mathcal{C}^\infty(M; E) \ni (u, \phi) \mapsto \int_M \langle u(p), \phi(p) \rangle, \tag{5.38}$$

where $\langle \cdot, \cdot \rangle\colon E^* \times E \to \mathbb{R}$ is the dual pairing; note that $\langle u, \phi \rangle \in \mathcal{C}^\infty(M; \Omega M)$ can indeed be invariantly integrated by Proposition 5.14.

The support and singular support of a distribution are defined analogously to the local ($\mathbb{R}^n$) case, see Definition 2.6. The space $\mathscr{E}'(M; E) \subset \mathscr{D}'(M; E)$ is, as in the local theory (on $\mathbb{R}^n$), the space of distributions with compact support. (Without further structure, there is no natural analogue of the space of Schwartz functions or tempered distributions on a general smooth manifold.)

Example 5.16. Let $p \in M$, then $\delta_p \in \mathscr{E}'(M; \Omega M)$ is the distribution defined by mapping $\phi \in \mathcal{C}^\infty(M)$ to $\phi(p)$.

In order to state the Schwartz kernel theorem in this context, we define the projections

$$\begin{aligned} \pi_L\colon M^2 \to M, &\quad (p, q) \mapsto p, \\ \pi_R\colon M^2 \to M, &\quad (p, q) \mapsto q. \end{aligned} \tag{5.39}$$

Then:

Theorem 5.17 (Schwartz kernel theorem: manifold case)**.** *Let M be a smooth n-dimensional manifold, and let $E, F \to M$ be two vector bundles. Then there is a one-to-one correspondence between continuous linear operators $A\colon \mathcal{C}^\infty_c(M; E) \to \mathscr{D}'(M; F)$ and distributional*

Schwartz kernels $K \in \mathscr{D}'(M \times M; \pi_L^* F \otimes \pi_R^*(E^* \otimes \Omega M))$. *This correspondence is given by assigning to* K *the operator* $O_K \colon \mathcal{C}_c^\infty(M; E) \to \mathscr{D}'(M; F)$, *defined as*

$$(O_K \phi)(\psi) = \langle K, \pi_L^* \psi \otimes \pi_R^* \phi \rangle, \quad \phi \in \mathcal{C}_c^\infty(M; E),\ \psi \in \mathcal{C}_c^\infty(M; F^* \otimes \Omega M). \tag{5.40}$$

5.3. **Differential operators on manifolds.** Let M be a smooth n-dimensional manifold. Before we talk about ps.d.o.s on M, let us think about *differential* operators.

Definition 5.18 (Vector fields on manifolds)**.** The space of smooth vector fields on M is $\mathcal{V}(M) := \mathcal{C}^\infty(M; TM)$.

An element $V \in \mathcal{V}(M)$ can be regarded as a differential operator by assigning

$$\mathcal{C}^\infty(M) \ni f \mapsto Vf \in \mathcal{C}^\infty(M), \quad (Vf)(p) = \mathrm{d}f(p)(V(p)) \tag{5.41}$$

Definition 5.19 (Differential operators on manifolds)**.** (1) We define $\mathrm{Diff}^0(M) = \mathcal{C}^\infty(M)$.
(2) We define $\mathrm{Diff}^1(M)$ as the space of all operators $A \colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ of the form $Au = Vu + fu$ with $V \in \mathcal{V}(M)$, $f \in \mathcal{C}^\infty(M)$.
(3) Let $m \in \mathbb{N}_0$. Then $\mathrm{Diff}^m(M)$ is the space of all operators $A \colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ which are of the form

$$Au = \sum_{k=1}^{K} A_{k1} \cdots A_{kN_k} u, \quad A_{kj} \in \mathrm{Diff}^1(M),\ K \in \mathbb{N},\ N_k \le m. \tag{5.42}$$

(Check that this agrees with the standard local coordinate definition.) Of course, differential operators also map $\mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(M)$, $\mathscr{D}'(M) \to \mathscr{D}'(M)$, $\mathscr{E}'(M) \to \mathscr{E}'(M)$. What are the Schwartz kernels of differential operators? The Schwartz kernel K_I of the identity operator $I \in \mathrm{Diff}^0(M)$ should be

$$K_I(x, y) = \delta(x - y). \tag{5.43}$$

We aim to make sense of this. Using the right projection π_R from (5.39), we define the *right density bundle* by

$$\Omega_R := \pi_R^*(\Omega M) \tag{5.44}$$

Thus, integration in the second variable is a well-defined map $\mathcal{C}_c^\infty(M^2; \Omega_R) \to \mathcal{C}_c^\infty(M)$. More generally, the following map is well-defined:

$$\mathscr{D}'(M^2; \Omega_R) \times \mathcal{C}_c^\infty(M) \ni (K, u) \mapsto \int_M K(\cdot, y) u(y) \in \mathscr{D}'(M). \tag{5.45}$$

By the Schwartz kernel theorem, every continuous map $\mathcal{C}_c^\infty(M) \to \mathscr{D}'(M)$ is of this type! Thus, (5.43) is well-defined as an element

$$K_I \in \mathscr{D}'(M^2; \Omega_R). \tag{5.46}$$

Remark 5.20 (Schwartz kernel of the identity map)*.* As a check, recall that K_I acts on elements of[7]

$$\mathcal{C}_c^\infty(M^2; \Omega(M^2) \otimes (\Omega_R)^*) = \mathcal{C}_c^\infty(M^2; \Omega_L), \tag{5.47}$$

and indeed maps $u \in \mathcal{C}_c^\infty(M^2; \Omega_L)$ into $\int_M u(x, x)$, defined by Proposition 5.14. (Note that restriction to the diagonal gives a map $\mathcal{C}_c^\infty(M^2; \Omega_L) \to \mathcal{C}_c^\infty(M; \Omega M)$ by Lemma 5.12.)

[7] This uses that $\Omega(M^2) = \Omega_L \otimes \Omega_R$.

Given $A \in \mathrm{Diff}^m(M)$, its Schwartz kernel $K_A \in \mathscr{D}'(M^2; \Omega_R)$ is then given by

$$K_A = (\pi_L^* A) K_I, \tag{5.48}$$

where $\pi_L^* A$ denotes the lift of A to the first factor, i.e. differentiating only in the first factor of M^2. Check that this is well-defined in the following general context: if $\pi\colon E \to M$ is a vector bundle, then

$$(\pi_L^* A) K \in \mathscr{D}'(M^2; \pi_R^* E), \quad ((\pi_L^* A) K)(\cdot, y) = (AK)(\cdot, y), \quad y \in M,\ K \in \mathscr{D}'(M^2; \pi_R^* E), \tag{5.49}$$

is well-defined.

5.4. **Definition of $\Psi^m(M)$.** We continue to denote by M a smooth n-dimensional manifold, and use the notation (5.39). The following definition captures what we would like pseudodifferential operators on a manifold (not necessarily compact) to be: their Schwartz kernels should, near the diagonal, be Schwartz kernels of ps.d.o.s on $\mathbb{R}^n$ in local coordinates, while away from the diagonal they are simply smooth.

Definition 5.21 (Pseudodifferential operators on manifolds)**.** Let M be a smooth n-dimensional manifold. Let $m \in \mathbb{R}$. Then $\Psi^m(M)$ is the space of linear operators

$$A\colon \mathcal{C}_c^\infty(M) \to \mathcal{C}^\infty(M) \tag{5.50}$$

with the following properties:

(1) if $\phi, \psi \in \mathcal{C}^\infty(M)$ have $\operatorname{supp}\phi \cap \operatorname{supp}\psi = \emptyset$, then there exists $K \in \mathcal{C}^\infty(M^2; \Omega_R)$ such that
$$\phi A(\psi u) = \int_M K(\cdot, y) u(y), \quad u \in \mathcal{C}_c^\infty(M). \tag{5.51a}$$
(2) if $F\colon U \to \mathbb{R}^n$ is a diffeomorphism from an open set $\emptyset \neq U \subset M$ to $F(U)$, and if $\psi \in \mathcal{C}_c^\infty(U)$, then there exists $B \in \Psi_c^m(F(U)) \subset \Psi^m(\mathbb{R}^n)$ (see Definition 5.1) such that on U, we have
$$\psi A(\psi u) = F^*\Bigl(B\bigl((F^{-1})^*(\psi u)\bigr)\Bigr), \quad u \in \mathcal{C}_c^\infty(M). \tag{5.51b}$$

Remark 5.22 (Ps.d.o.s on the manifold $\mathbb{R}^n$). Taking as the smooth manifold $M = \mathbb{R}^n$, the space $\Psi^m(M)$ defined here is much larger than the space $\Psi^m(\mathbb{R}^n)$ of uniform pseudodifferential operators defined in §4. (One reason is that we do not constrain the size of Schwartz kernels away from the diagonal $\Delta_M = \{(p,p)\colon p \in M\}$.). To avoid confusion, one should denote the latter space by $\Psi^m_\infty(\mathbb{R}^n)$. When working on $\mathbb{R}^n$, we shall, in these notes, only ever employ operators in the uniform algebra, hence we shall right away drop the '∞' subscript again!

Remark 5.23 (Residual operators). Directly from the definition, the space $\Psi^{-\infty}(M)$ consists of all operators which have a Schwartz kernel in $\mathcal{C}^\infty(M^2; \Omega_R)$. Equivalently, $\Psi^{-\infty}(M)$ is the space of all bounded linear operators $\mathscr{E}'(M) \to \mathcal{C}^\infty(M)$.

Ps.d.o.s act on distributions with compact support. We give a direct proof here, and defer a 'better' proof in the spirit of (4.32) to later; see Corollary 5.42.

Proposition 5.24 (Boundedness on distributions)**.** *Let $A \in \Psi^m(M)$. Then A extends by continuity from $\mathcal{C}_c^\infty(M)$ to a bounded linear operator*

$$A\colon \mathscr{E}'(M) \to \mathscr{D}'(M). \tag{5.52}$$

Proof. Fix a cover of M by coordinate systems $F_i\colon U_i \to F_i(U_i) \subset \mathbb{R}^n$ with $\overline{U_i}$ compact, and let $\{\phi_i\}$, $\phi_i \in \mathcal{C}^\infty_c(U_i)$, be a subordinate partition of unity. Fix $\tilde\phi_i \in \mathcal{C}^\infty_c(U_i)$ with $\tilde\phi_i = 1$ near $\operatorname{supp}\phi_i$. By (5.51b), we can write

$$\tilde\phi_i A\phi_i = \tilde\phi_i A\tilde\phi_i\phi_i = F_i^* B_i (F_i^{-1})^*\phi_i, \quad B_i \in \Psi^m_c(F_i(U_i)). \tag{5.53}$$

Let now $u \in \mathscr{E}'(M)$, then $\phi_i u \neq 0$ only for finitely many i. We then set

$$\tilde A u := \sum_i F_i^* B_i (F_i^{-1})^*(\phi_i u) + \sum_i (1-\tilde\phi_i)A\phi_i u. \tag{5.54}$$

Each one of the finitely many non-zero summands in the first sum is a pullback from $\mathbb{R}^n$ of a tempered distribution with compact support, hence lies in $\mathscr{E}'(M)$. The second (also finite) sum lies in $\mathcal{C}^\infty(M)$ by (5.51a).

For $u \in \mathcal{C}^\infty_c(M)$, we clearly have $\tilde A u = Au$. Since $\mathcal{C}^\infty_c(M) \subset \mathscr{E}'(M)$ is dense, (5.54) defines the unique continuous extension of A to $\mathscr{E}'(M)$ (which, of course, we call A simply, rather than $\tilde A$). $\square$

To get a more manageable characterization of $\Psi^m(M)$, we first prove:

Lemma 5.25 (Ps.d.o.s defined in a chart)**.** *Let M be an n-dimensional manifold, and let $F\colon U \to F(U) \subset \mathbb{R}^n$ be a coordinate patch. If $B \in \Psi^m_c(F(U)) \subset \Psi^m(\mathbb{R}^n)$, then the operator $A\colon \mathcal{C}^\infty_c(M) \to \mathcal{C}^\infty(M)$ defined by*

$$Au = F^* B (F^{-1})^*(u|_U), \quad u \in \mathcal{C}^\infty_c(M), \tag{5.55}$$

on U, and $Au = 0$ in $M \setminus U$, defines an element $A \in \Psi^m(M)$.

Proof. We first check (5.51a): given $\phi, \psi \in \mathcal{C}^\infty(M)$ with $\operatorname{supp}\phi \cap \operatorname{supp}\psi = \emptyset$, we have for $u \in \mathcal{C}^\infty_c(M)$

$$\phi A(\psi u) = F^* B' (F^{-1})^*(u|_U), \quad B' := ((F^{-1})^*\phi) B ((F^{-1})^*\psi) \in \Psi^{-\infty}_c(F(U)) \subset \Psi^{-\infty}(\mathbb{R}^n), \tag{5.56}$$

where we used that $\operatorname{supp}((F^{-1})^*\phi) \cap \operatorname{supp}((F^{-1})^*\psi) = \emptyset$. Since the Schwartz kernel of B' is smooth, we obtain (5.51a). (In more detail, if $K_{B'} \in \mathcal{C}^\infty_c(F(U) \times F(U))$ denotes the Schwartz kernel of B', then (5.51a) holds for $K(x,y) := F^*(K_{B'}(x,y)|dy|)$.)

As for (5.51b), suppose $G\colon V \to G(V) \subset \mathbb{R}^n$ is another coordinate patch, and let $\chi \in \mathcal{C}^\infty_c(M)$, $\operatorname{supp}\chi \subset V$. Then

$$B_1 := ((F^{-1})^*\chi) B ((F^{-1})^*\chi) \in \Psi^m_c(F(U \cap V)). \tag{5.57}$$

Denote the change of coordinates by $\kappa = F \circ G^{-1}\colon G(U \cap V) \to F(U \cap V)$, then

$$B_2 := (B_1)_\kappa = \kappa^* B_1 (\kappa^{-1})^* \in \Psi^m_c(G(U \cap V)) \tag{5.58}$$

by Theorem 5.2. Therefore,

$$\chi A(\chi u) = F^* B_1 (F^{-1})^* u|_{U\cap V} = G^*\kappa^* B_1 (\kappa^{-1})^* (G^{-1})^* u|_{U\cap V} = G^* B_2 (G^{-1})^* u|_{U\cap V}, \tag{5.59}$$

as desired. $\square$

This already implies that there are lots of pseudodifferential operators on M, given by locally finite sums of operators of the form (5.55). This gives almost (namely, up to operators with smooth integral kernels) all of $\Psi^m(M)$:

Theorem 5.26 (Characterization of ps.d.o.s via charts)**.** *Let M be an n-dimensional manifold, and let $M = \bigcup_i U_i$ be a locally finite open cover by coordinate charts $F_i\colon U_i \to F_i(U_i) \subset \mathbb{R}^n$ with $\overline{U_i}$ compact. Let $A\colon \mathcal{C}^\infty_c(M) \to \mathscr{D}'(M)$ be a linear operator. Then $A \in \Psi^m(M)$ if and only if there exist operators $B_i \in \Psi^m_c(F_i(U_i))$ and a section $K \in \mathcal{C}^\infty(M^2;\Omega_R)$ such that*

$$A = K + \sum_i F_i^* B_i (F_i^{-1})^*. \tag{5.60}$$

Proof. If A is of the form (5.60), then $A \in \Psi^m(M)$ by the previous lemma. Conversely, suppose $A \in \Psi^m(M)$. Let $\{\phi_i\}$, $\phi_i \in \mathcal{C}^\infty_c(U_i)$, be a partition of unity subordinate to $\{U_i\}$, i.e. $\operatorname{supp}\phi_i \subset U_i$, and $\sum_i \phi_i(x) = 1$ for all $x \in M$. Choose $\tilde\phi_i \in \mathcal{C}^\infty_c(U_i)$ with $\operatorname{supp}\tilde\phi_i \subset U_i$ and $\tilde\phi_i \equiv 1$ near $\operatorname{supp}\phi_i$. For $u \in \mathcal{C}^\infty_c(M)$, we then have

$$Au = \sum_i \tilde\phi_i A\phi_i u + \sum_i (1-\tilde\phi_i)A\phi_i u. \tag{5.61}$$

By definition, $(1-\tilde\phi_i)A\phi_i$ has a smooth Schwartz kernel $K_i \in \mathcal{C}^\infty(M^2;\Omega_R)$; since $\operatorname{supp} K_i$ is locally finite, we can define

$$K := \sum_i K_i \in \mathcal{C}^\infty(M^2;\Omega_R). \tag{5.62}$$

Considering a term $\tilde\phi_i A\phi_i$ in the first sum in (5.61), note that

$$\tilde\phi_i A(\phi_i u) = \tilde\phi_i A\tilde\phi_i(\phi_i u). \tag{5.63}$$

But $\tilde\phi_i A\tilde\phi_i = F_i^* B_i'(F_i^{-1})^*\tilde\phi_i$ for some $B_i' \in \Psi^m_c(F_i(U_i))$, and therefore

$$\tilde\phi_i A\phi_i = F_i^* B_i (F_i^{-1})^*, \quad B_i u := B_i'((F_i^*\phi_i)u),\ u \in \mathcal{C}^\infty_c(M), \tag{5.64}$$

with $B_i \in \Psi^m_c(F_i(U_i))$, as desired. □

When M is not compact, one can in general not compose two ps.d.o.s, even when both are of order $-\infty$, since they only act on $\mathcal{C}^\infty_c(M)$, but not on $\mathcal{C}^\infty(M)$ in general, the problem being potential growth of the Schwartz kernel away from the diagonal. The simplest cure is to place an additional assumption on the Schwartz kernels:

Definition 5.27 (Properly supported ps.d.o.s)**.** We say that $A \in \Psi^m(M)$, with Schwartz kernel $K \in \mathscr{D}'(M^2;\Omega_R)$, is *properly supported* if the projection maps $\pi_L\colon \operatorname{supp} K \to M$ and $\pi_R\colon \operatorname{supp} K \to M$ are *proper*, i.e. preimages of compact sets are compact.

Every ps.d.o. is the sum of a properly supported operator and a residual operator. (See Exercise 5.10.) In other words, in situations where one does not care about order $-\infty$ errors, one can work entirely with properly supported operators.

Thus, properly supported operators are bounded on $\mathcal{C}^\infty_c(M)$ and $\mathscr{E}'(M)$. Using partition of unity arguments, one can show that they are also bounded on $\mathcal{C}^\infty(M)$, $\mathscr{D}'(M)$. For a proof using a duality argument, see Corollary 5.42 below.

Remark 5.28 (Properly supported and uniform ps.d.o.s on $\mathbb{R}^n$)*.* Complementing Remark 5.22, the subspace of $\Psi^m(M)$, $M = \mathbb{R}^n$, consisting of *properly supported* operators does not have a simple relationship with $\Psi^m_\infty(\mathbb{R}^n)$: on the one hand, Schwartz kernels of elements of $\Psi^m_\infty(\mathbb{R}^n)$ may even have full support in $\mathbb{R}^n \times \mathbb{R}^n$, hence are not properly supported; on the other hand, properly supported elements of $\Psi^m(M)$ may not be elements of $\Psi^m_\infty(\mathbb{R}^n)$

since membership in the latter space requires *uniform* bounds off the diagonal, see e.g. Exercise 4.2.

Theorem 5.29 (Composition of ps.d.o.s on manifolds)**.** *Let $A \in \Psi^m(M)$ and $B \in \Psi^{m'}(M)$, and assume at least one of A and B is properly supported. Then $A \circ B \colon \mathcal{C}^\infty_c(M) \to \mathcal{C}^\infty(M)$ is a ps.d.o., $A \circ B \in \Psi^{m+m'}(M)$. If both A and B are properly supported, then so is $A \circ B$.*

We will use the description of Theorem 5.26 for a particular kind of open cover:

Lemma 5.30 (Special covers)**.** *Let M be a smooth manifold. There exists a locally finite open cover $\{U_i\}$ of M such that whenever $U_i \cap U_j \neq \emptyset$, then there exists a local coordinate chart $F \colon U \to F(U) \subset \mathbb{R}^n$ with $U \supset U_i \cup U_j$.*

Proof. M is metrizable; this follows either by Urysohn's metrization theorem, or from basic Riemannian geometry. Denote a fixed metric on M by d, and denote metric balls by $B(p, r) = \{q \in M : d(p, q) < r\}$. For each $p \in M$, let

$$r_0(p) := \sup\{r \in [0, 1] \colon B(p, r) \text{ is contained in a coordinate chart}\}. \tag{5.65}$$

Since M is a manifold, we have $r_0(p) > 0$ for all $p \in M$. For $p \in M$, define the open set

$$V_p := B\Big(p, \frac{r_0(p)}{10}\Big). \tag{5.66}$$

Suppose $V_p \cap V_q \neq 0$; then $d(p, q) \leq \frac{1}{10}(r_0(p) + r_0(q)) \leq \frac{1}{5}\max(r_0(p), r_0(q))$. By symmetry, we may assume $r_0(p) \geq r_0(q)$. If $z \in V_p \cup V_q$, then

$$d(p, z) < \max\Big(\frac{r_0(p)}{10}, d(p, q) + \frac{r_0(q)}{10}\Big) \leq \max\Big(\frac{r_0(p)}{10}, \frac{r_0(p)}{5} + \frac{r_0(p)}{10}\Big) < \frac{1}{2} r_0(p). \tag{5.67}$$

Therefore, $V_p \cup V_q \subset B(p, \frac{r_0(p)}{2})$ is contained in a coordinate chart. Any locally finite refinement $\{U_i\}$ of the cover $\{V_p \colon p \in M\}$ of M satisfies the conditions of the lemma. ☐

Proof of Theorem 5.29. By the previous lemma, we can fix an open cover $M = \bigcup_i U_i$ of M by coordinate charts $F_i \colon U_i \to F_i(U_i) \subset \mathbb{R}^n$, with $\overline{U_i}$ compact, and so that for any i, j with $U_i \cap U_j \neq \emptyset$, the union $U_i \cup U_j$ is contained in a coordinate chart $F_{ij} \colon U_{ij} \to F_{ij}(U_{ij}) \subset \mathbb{R}^n$.

Let us assume that A is properly supported. (The case that B is properly supported is handled similarly.) Write

$$\begin{aligned} A &= K + \sum_i F_i^* A_i (F_i^{-1})^*, \qquad A_i \in \Psi^m_c(F_i(U_i)), \\ B &= K' + \sum_i F_i^* B_i (F_i^{-1})^*, \qquad B_i \in \Psi^m_c(F_i(U_i)),\ K' \in \mathcal{C}^\infty(M^2; \Omega_R). \end{aligned} \tag{5.68}$$

Since A and all the $F_i^* A_i (F_i^{-1})^*$ are properly supported, so is $K \in \mathcal{C}^\infty(M^2; \Omega_R)$.

We consider the composition $A \circ B$ term by term and keep track of the support of the Schwartz kernels of the various pieces.

We first prove that $K \circ K' \in \Psi^{-\infty}(M)$. We have $(K \circ K')(x, y) = \int K(x, z) K'(z, y)\, dz$; for any compact set $K_1 \subset M$ there exists $K_2 \subset M$ such that in fact

$$(K \circ K')(x, y) = \int_{K_2} K(x, z) K'(z, y)\, dz, \quad x \in K_1. \tag{5.69}$$

Indeed, this holds for $K_2 = \pi_L(\operatorname{supp} K \cap \pi_R^{-1}(K_1))$. Thus, the Schwartz kernel of $K \circ K'$ lies in $\mathcal{C}^\infty(M^2; \Omega_R)$.

Consider next the composition $K_i := K \circ F_i^* B_i (F_i^{-1})^*$. This maps $u \in \mathscr{E}'(M)$ into $\mathcal{C}^\infty(M)$ (in fact, into $\mathcal{C}^\infty_c(M)$); and if $\operatorname{supp} u \cap U_i = \emptyset$, then $K_i u = 0$. Thus, by Remark 5.23,

$$K_i \in \mathcal{C}^\infty(M^2; \Omega_R), \quad \operatorname{supp} K_i \subset M \times U_i. \tag{5.70}$$

(In fact, $\operatorname{supp} K_i \subset \pi_L(\operatorname{supp} K \cap \pi_R^{-1}(U_i)) \times U_i$ is compact, but we do not need this information.) Similarly, one shows that

$$K_i' := F_i^* A_i (F_i^{-1})^* \circ K' \in \mathcal{C}^\infty(M^2; \Omega_R), \quad \operatorname{supp} K_i' \subset U_i \times M. \tag{5.71}$$

(Note that its Schwartz kernel is not compactly supported since K' is not properly supported.) Finally, we need to consider the composition

$$C_{ij} := F_i^* A_i (F_i^{-1})^* \circ F_j^* B_j (F_j^{-1})^* \colon \mathcal{C}^\infty_c(M) \to \mathcal{C}^\infty(M). \tag{5.72}$$

When $U_i \cap U_j = \emptyset$, this composition is the 0 operator. When $U_i \cap U_j \neq \emptyset$, we can use Lemma 5.25 and write (5.72) equivalently as

$$F_{ij}^* A_{ij} (F_{ij}^{-1})^* \circ F_{ij}^* B_{ij} (F_{ij}^{-1})^* = F_{ij}^* (A_{ij} \circ B_{ij}) (F_{ij}^{-1})^*, \tag{5.73}$$

where $A_{ij} \in \Psi^m_c(F_{ij}(U_i))$, $B_{ij} \in \Psi^{m'}_c(F_{ij}(U_j))$. But then $A_{ij} \circ B_{ij} \in \Psi^{m+m'}_c(F_{ij}(U_i \cup U_j))$, hence (5.73) lies in $\Psi^{m+m'}(M)$, with Schwartz kernel supported in $U_{ij} \times U_{ij}$.

The proof is complete once we show that the supports of the Schwartz kernels of K_i, K_i', and C_{ij} are locally finite. Take a point $(p, q) \in M^2$, and choose i_0, j_0 such that $p \in U_{i_0}$ and $q \in U_{j_0}$. Then $U := U_{i_0} \times U_{j_0}$ has non-trivial intersection with only finitely many of these supports, as follows immediately from the local finiteness of $\{U_i\}$. □

Since every operator on a *compact* manifold is properly supported, we deduce:

Corollary 5.31 (Composition on compact manifolds)**.** *If M is a compact manifold, then $\Psi^m(M) \circ \Psi^{m'}(M) \subset \Psi^{m+m'}(M)$.*

5.5. **Principal symbol.** Motivated by Theorem 5.2, in particular formula (5.2), we want to define the principal symbol of $A \in \Psi^m(M)$ as an equivalence class of symbols on T^*M.

Definition 5.32 (Symbol spaces on vector bundles)**.** Let M be a manifold and $\pi \colon E \to M$ a rank k vector bundle. For $m \in \mathbb{R}$, we define $S^m(E) \subset \mathcal{C}^\infty(E)$ as the subspace of all $a \in \mathcal{C}^\infty(E)$ having the following property: for each coordinate chart $F \colon U \to F(U) \subset \mathbb{R}^n$ on M on which E is trivial with trivialization $\tau \colon \pi^{-1}(U) \to U \times \mathbb{R}^k \xrightarrow{F \times \mathrm{Id}} F(U) \times \mathbb{R}^k$, set

$$b(x, v) := (\tau^{-1})^* (a|_{\pi^{-1}(U)})(x, v) = a(\tau^{-1}(x, v)) \in \mathcal{C}^\infty(F(U) \times \mathbb{R}^k). \tag{5.74}$$

Then for any $\phi \in \mathcal{C}^\infty_c(F(U)) \subset \mathcal{C}^\infty_c(\mathbb{R}^n)$, we have $\phi(x) b(x, v) \in S^m(\mathbb{R}^n; \mathbb{R}^k)$.

The key to making this a useful definition is the following analogue of Lemma 5.25.

Lemma 5.33 (Symbols defined in a chart)**.** *In the notation of Definition 5.32, suppose $\phi \in \mathcal{C}^\infty_c(F(U))$, $b \in S^m(\mathbb{R}^n; \mathbb{R}^k)$. Then $a := \tau^*(\phi b) \in S^m(E)$.*

Proof. The expression for a in another coordinate system $F' \colon U' \to F'(U') \subset \mathbb{R}^n$ on M and a trivialization of E on U' is

$$b'(x, v) = \phi(\kappa(x)) b(\kappa(x), \Phi(x) v), \quad x \in F'(U \cap U') \tag{5.75}$$

for some diffeomorphism $\kappa\colon F'(U\cap U')\to F(U\cap U')$ and a smooth map $\Phi\colon U'\to U$. Let $\psi\in\mathcal{C}^\infty_c(F'(U'))$. Then $\chi(x):=\psi(x)\phi(\kappa(x))\in\mathcal{C}^\infty_c(F'(U')\cap\kappa^{-1}(F(U)))=\mathcal{C}^\infty_c(F'(U\cap U'))$, and we need to check that

$$\chi(x)b(\kappa(x),\Phi(x)v)\in S^m(\mathbb{R}^n;\mathbb{R}^k). \tag{5.76}$$

This however follows from the same type of calculation as (5.14). □

In analogy with Theorem 5.26, we have:

Corollary 5.34 (Characterization of symbols via charts)**.** *Let $M=\bigcup_i U_i$ be a locally finite open cover by coordinate charts $F_i\colon U_i\to F_i(U_i)\subset\mathbb{R}^n$, with $\overline{U_i}$ compact, over which E has a trivialization $\tau_i\colon\pi^{-1}(U_i)\to F_i(U_i)\times\mathbb{R}^k$. Let $a\in\mathcal{C}^\infty(E)$. Then $a\in S^m(E)$ if and only if there exist symbols $b_i\in S^m(\mathbb{R}^n;\mathbb{R}^k)$ and $\chi_i\in\mathcal{C}^\infty_c(F_i(U_i))$ such that*

$$a=\sum_i(\tau_i^{-1})^*(\chi_i b_i). \tag{5.77}$$

Now, given an operator $A\in\Psi^m(M)$, we expect its principal symbol to be an element of the quotient space $S^m(T^*M)/S^{m-1}(T^*M)$. We first define it locally. Let $F\colon U\to F(U)\subset\mathbb{R}^n$ be a coordinate chart with $\bar U$ compact, and let $V\subset U$ be open with $\bar V\subset U$. Denote by $\tau\colon T^*_U M\to F(U)\times\mathbb{R}^n$ the trivialization induced by F. Let $\chi\in\mathcal{C}^\infty_c(U)$ be such that $\chi=1$ on $\bar V$. Then we put

$$a_V:=\tau^*\big(\sigma_L\big((F^{-1})^*\chi A\chi F^*\big)\big)|_{T^*_V M}\in S^m(T^*_V M). \tag{5.78}$$

By the local ($\mathbb{R}^n$) theory, and in particular by Theorem 5.2, the equivalence class

$$[a_V]\in S^m(T^*_V M)/S^{m-1}(T^*_V M) \tag{5.79}$$

is independent of the choice of χ, and of the coordinate system F covering a neighborhood of $\bar V$. Moreover, if $V'\subset V$, then restriction to V' gives $[a_V]|_{V'}=[a_{V'}]$.

Definition 5.35 (Principal symbol of ps.d.o.s on manifolds)**.** The *principal symbol* of $A\in\Psi^m(M)$ is the unique element

$$\sigma^m(A)\in S^m(T^*M)/S^{m-1}(T^*M) \tag{5.80}$$

with the following property: if $a\in S^m(T^*M)$ is any representative of $\sigma^m(A)$, and V is as above, then $[a|_{T^*_V M}]=[a_V]\in S^m(T^*_V M)/S^{m-1}(T^*_V M)$.

We start by proving existence. (Effectively, we are proving that $U\mapsto S^m(T^*_U M)/S^{m-1}(T^*_U M)$ is a sheaf.) This follows easily from the properties of the $[a_V]$. Indeed, taking a locally finite subcover $\{V_i\}$ of the cover of M by all sets V as above, and a subordinate partition of unity $\{\phi_i\}$, we have $a=\sum_i\phi_i a_{V_i}\in S^m(T^*M)$ by Corollary 5.34; we then put

$$\sigma^m(A):=[a]. \tag{5.81}$$

We check that this satisfies the property required in Definition 5.35. Given V open as above, it suffices to show that for $\phi\in\mathcal{C}^\infty_c(V)$, we have $[\phi a|_V]=[\phi a_V]$. Now $\phi\phi_i a_{V_i}=\phi\phi_i a_V+e_i$ for some $e_i\in S^{m-1}(T^*M)$ with support in $T^*_{V_i\cap V}M$. Let $\tilde\phi_i\in\mathcal{C}^\infty_c(V_i)$ be equal to 1 on $\operatorname{supp}\phi_i$, and with $\operatorname{supp}\tilde\phi_i$ locally finite; then

$$\phi a|_V=\sum_i\tilde\phi_i(\phi_i\phi a_{V_i})=\sum_i\tilde\phi_i(\phi_i\phi a_V+e_i)=\phi a_V+\sum_i\tilde\phi_i e_i, \tag{5.82}$$

as desired (since $\sum_i\tilde\phi_i e_i\in S^{m-1}(T^*M)$).

We now turn to the uniqueness part of Definition 5.35; it suffices to show that if $a \in S^m(T^*M)$ is such that $a|_{T^*_V M} \in S^{m-1}(T^*_V M)$ for open sets $V \subset M$ as above, then $a \in S^{m-1}(T^*M)$. But this follows by writing $a = \sum_i \phi_i a|_{T^*_{V_i} M}$, where ϕ_i, V_i are as above.

Proposition 5.36 (Principal symbol short exact sequence)**.** *The principal symbol map gives a short exact sequence*

$$0 \to \Psi^{m-1}(M) \to \Psi^m(M) \xrightarrow{\sigma^m} S^m(T^*M)/S^{m-1}(T^*M) \to 0. \tag{5.83}$$

Proof. We only prove surjectivity of σ^m. Take a partition of unity $\{\phi_i\}$ subordinate to a locally finite cover of M by coordinate charts $F_i \colon U_i \to F_i(U_i) \subset \mathbb{R}^n$ with $\overline{U_i}$ compact. Writing any $a \in S^m(T^*M)$ as $a = \sum_i \phi_i a$, it suffices to show that there exists an operator $A_i \in \Psi^m(M)$ with Schwartz kernel supported in $U_i \times U_i$ such that $\sigma^m(A_i) = [\phi_i a]$, as we can then take $A = \sum_i A_i$ (which is a locally finite sum). This is easy: if $\tilde\phi_i \in \mathcal{C}^\infty_c(U_i)$, $\tilde\phi_i = 1$ on $\operatorname{supp}\phi_i$, then simply take

$$A_i = F_i^* \operatorname{Op}\Big(\big((F_i^{-1})^*\phi_i\big)(x) a(x,\xi) \big((F_i^{-1})^*\tilde\phi_i\big)(y)\Big)(F_i^{-1})^*. \tag{5.84}$$

□

An immediate consequence of the $\mathbb{R}^n$ result, Proposition 4.21, is:

Proposition 5.37 (Multiplicativity of the principal symbol)**.** *If $A \in \Psi^m(M)$, $B \in \Psi^{m'}(M)$, with at least one of them properly supported, then*

$$\sigma^{m+m'}(A \circ B) = \sigma^m(A)\sigma^{m'}(B). \tag{5.85}$$

The analogue of Proposition 4.22 concerning the principal symbol of commutators will be discussed in §5.13.

5.6. **Quantization.** There is no completely natural way, in general, to quantize symbols on T^*M to pseudodifferential operators on M. However, we do have the following useful construction: fix a locally finite open cover $M = \bigcup_i U_i$ by coordinate charts $F_i \colon U_i \to F_i(U_i) \subset \mathbb{R}^n$ with $\overline{U_i}$ compact, and let $\{\phi_i\}$, $\phi_i \in \mathcal{C}^\infty_c(U_i)$, be a partition of unity subordinate to $\{U_i\}$. Fix $\tilde\phi_i \in \mathcal{C}^\infty_c(U_i)$ with $\tilde\phi_i = 1$ near $\operatorname{supp}\phi_i$. Given $a \in S^m(T^*M)$, define then

$$\operatorname{Op}(a) := \sum_i F_i^* \operatorname{Op}_L\big((F_i^{-1})^*(\phi_i a)\big)\tilde\phi_i (F_i^{-1})^*, \tag{5.86}$$

where $\operatorname{Op}_L \colon S^m(T^*\mathbb{R}^n) \to \Psi^m(\mathbb{R}^n)$ is the left quantization map. By Theorem 5.26, the formula (5.86) defines a map

$$\operatorname{Op}\colon S^m(T^*M) \to \Psi^m(M). \tag{5.87}$$

Proposition 5.38 (Properties of the quantization map)**.** *The map* Op *in* (5.87) *is continuous, linear, and takes values in the subspace of properly supported operators. We have* $\operatorname{Op}(1) = I$ *(the identity operator). Moreover,* $\operatorname{Op}\colon S^m(T^*M) \to \Psi^m(M)$ *is surjective modulo* $\Psi^{-\infty}(M)$*; that is,* $\Psi^m(M) = \operatorname{Op}(S^m(T^*M)) + \Psi^{-\infty}(M)$.

Proof. We only sketch a proof of the final claim. It follows from Theorem 5.26. Indeed, in the notation of equation (5.60), we can write any $A \in \Psi^m(M)$ in the form $A_m := A - K = \sum_i F_i^* B_i (F_i^{-1})^*$. Using a partition of unity, we can combine the symbols of the operators $B_i \in \Psi^m(\mathbb{R}^n)$ into a symbol $a_m \in S^m(T^*M)$; by the coordinate invariance of the principal symbol, we then have $A_{m-1} := A_m - \operatorname{Op}(a_m) \in \Psi^{m-1}(M)$. We may then apply Theorem 5.26 to A_{m-1}. An inductive argument thus produces $a_{m-j} \in S^{m-j}(T^*M)$, $j \in \mathbb{N}$, so that $A_{m-j-1} := A_{m-j} - \mathrm{Op}(a_{m-j}) \in \Psi^{m-j}(M)$. Letting $a \in S^m(T^*M)$ be an asymptotic sum of the a_{m_j}, $j \in \mathbb{N}_0$, we therefore have $A = A_m + K = \mathrm{Op}(a) + K + R$ where $K \in \mathcal{C}^\infty(M^2; \Omega_R)$ and $R \in \Psi^{-\infty}(M)$ are residual operators, and thus so is their sum $K + R$. □

5.7. **Operators acting on sections of vector bundles.** The reader might ask why we have not discussed adjoints of $A \in \Psi^m(M)$ (or even $A \in \mathrm{Diff}^m(M)$) yet. Since we do not have an invariant way of integrating *functions* on M, the only sensible way to define A^* is by

$$\int_M (Au)(x)\overline{v(x)} = \int_M u(x)\overline{A^*v(x)}, \quad u \in \mathcal{C}^\infty_c(M),\ v \in \mathcal{C}^\infty_c(M; \Omega M), \tag{5.88}$$

that is, A^* should be an operator acting on sections of ΩM. We leave it to the reader to define the space of m-th order differential operators $\mathrm{Diff}^m(M; E, F)$ mapping sections of E to section of F, and go straight for the pseudodifferential version.

Definition 5.39 (Ps.d.o.s acting on sections of vector bundles)**.** Let M be a smooth manifold, and let $\pi_E \colon E \to M$, $\pi_F \colon F \to M$ denote two real vector bundles of rank k_E, k_F. Then $\Psi^m(M; E, F)$ is the space of linear operators

$$A \colon \mathcal{C}^\infty_c(M; E) \to \mathcal{C}^\infty(M; F) \tag{5.89}$$

with the following properties:

(1) if $\phi, \psi \in \mathcal{C}^\infty(M)$ have $\operatorname{supp}\phi \cap \operatorname{supp}\psi = \emptyset$, then there exists $K \in \mathcal{C}^\infty(M^2; \pi_L^* F \otimes \pi_R^*(E^* \otimes \Omega M))$ such that $\phi A \psi = K$.
(2) Let $U \subset M$ be any open set, $G \colon U \to G(U) \subset \mathbb{R}^n$ a diffeomorphism, and let $\tau_E \colon \pi_E^{-1}(U) \to G(U) \times \mathbb{R}^{k_E}$, $\tau_F \colon \pi_F^{-1}(U) \to G(U) \times \mathbb{R}^{k_F}$ local trivializations of E, F. Using τ_E, identify smooth sections of E over U with k_E-tuples of smooth functions on U, likewise for sections of F. If $\psi \in \mathcal{C}^\infty_c(U)$, then there exists a $k_F \times k_E$ matrix $B = (B_{ij})$ of ps.d.o.s $B_{ij} \in \Psi^m_c(G(U))$ such that on U

$$\psi A(\psi u)(x)_i = \sum_{j=1}^{k_E} G^*\Big(B_{ij}\big((G^{-1})^* u\big)\big)_j\Big), \quad u \in \mathcal{C}^\infty_c(M; E),\ i = 1, \ldots, k_F. \tag{5.90}$$

In the special case $F = E$, we write $\Psi^m(M; E) = \Psi^m(M; E, E)$.

In local coordinates and trivializations, the symbol of $A \in \Psi^m(M; E, F)$ is a symbol with values in linear maps from $\mathbb{R}^{k_E}$ to $\mathbb{R}^{k_F}$. The invariant definition is as follows. Denote by $\pi \colon T^*M \to M$ the projection. Given a vector bundle $G \to M$, we can consider its pullback $\pi^*G \to T^*M$; a trivialization of G over an open set $U \subset M$, so $G|_U \cong U \times \mathbb{R}^{k_G}$, then induces a trivialization of π^*G over $T^*_U M$ which is 'constant in the fibers of T^*M', namely

$$(\pi^*G)|_{T^*_U M} \cong T^*U \times \mathbb{R}^{k_G}, \tag{5.91}$$

by identifying $(\pi^*G)_{(x,\xi)} = G_x \cong \mathbb{R}^{k_G}$ using the local trivialization. We then denote by

$$S^m(T^*M; \pi^*G) \subset \mathcal{C}^\infty(T^*M; \pi^*G) \tag{5.92}$$

the space of all smooth functions which in local coordinates and in a trivialization of G (which induces a trivialization of π^*G as in (5.91)) are k_G-vectors of symbols on $\mathbb{R}^n$ of order

m. Invariantly then,

$$\sigma^m(A) \in (S^m/S^{m-1})(T^*M; \pi^* \operatorname{Hom}(E,F)), \quad \pi\colon T^*M \to M. \tag{5.93}$$

This means that a representative of $\sigma^m(A)$ is a map assigning to $(x,\xi) \in T^*M$ an element of $\operatorname{Hom}(E_x, F_x)$. We have a short exact sequence

$$0 \to \Psi^{m-1}(M; E, F) \to \Psi^m(M; E, F) \xrightarrow{\sigma^m} (S^m/S^{m-1})(T^*M; \pi^* \operatorname{Hom}(E,F)) \to 0. \tag{5.94}$$

The results from §§5.4–5.5 carry over; moreover, one can define adjoints:

Proposition 5.40 (Compositions and adjoints)**.** *Let M be a smooth manifold, and let $E, F, G \to M$ denote three vector bundles.*

(1) *Let $A \in \Psi^m(M; F, G)$ and $B \in \Psi^{m'}(M; E, F)$, with at least one of them properly supported. Then $A \circ B \in \Psi^{m+m'}(M; E, G)$, and $\sigma^{m+m'}(A \circ B) = \sigma^m(A) \circ \sigma^{m'}(B)$.*[8]
(2) *Let $A \in \Psi^m(M; E, F)$. Then the (real) adjoint A^T, defined by*

$$\int_M (Au)v = \int_M u(A^T v), \quad u \in \mathcal{C}^\infty_c(M; E),\ v \in \mathcal{C}^\infty_c(M; F^* \otimes \Omega M), \tag{5.95}$$

is a pseudodifferential operator,

$$A^T \in \Psi^m(M; F^* \otimes \Omega M, E^* \otimes \Omega M). \tag{5.96}$$

It is properly supported if A is.

Remark 5.41 (Bundles with extra structure).
(1) If E, F are complex vector bundles with a anti-linear involution ('complex conjugation'), then one can define the adjoint A^* similarly to (5.95), but with complex conjugation of the second factor; one then has

$$\sigma^m(A^*) = \sigma^m(A)^*. \tag{5.97}$$

This in particular applies to the case that $E = F = M \times \mathbb{C}$, so sections of E, F are simply complex-valued functions on M, which we discussed in (5.95).
(2) An inner product on E induces an isomorphism $E^* \cong E$ (anti-linear when the inner product is sesquilinear). If one moreover chooses a trivialization of ΩM, e.g. from a semi-Riemannian metric, then $A^T \in \Psi^m(M; F, E)$ (and $A^* \in \Psi^m(M; E, F)$ in the complex case).

A consequence of (5.96) is the following extension of Proposition 5.24:

Corollary 5.42 (Action of ps.d.o.s on distributions and smooth functions)**.** *Let $A \in \Psi^m(M; E, F)$. Then A extends to a bounded linear operator $A\colon \mathscr{E}'(M; E) \to \mathscr{D}'(M; F)$. If A is properly supported, then A also maps $\mathscr{D}'(M; E) \to \mathscr{D}'(M; F)$, and by restriction $\mathcal{C}^\infty(M; E) \to \mathcal{C}^\infty(M; F)$.*

Proof. A^T is a bounded map $\mathcal{C}^\infty_c(M; F^* \otimes \Omega M) \to \mathcal{C}^\infty(M; E^* \otimes \Omega M)$. Therefore we can define $A\colon \mathscr{E}'(M; E) \to \mathscr{D}'(M; F)$ by duality using (5.96); this agrees with the original operator A when restricted to $\mathcal{C}^\infty_c(M; E)$.

If A is properly supported, then A^T maps $\mathcal{C}^\infty_c(M; F^* \otimes \Omega M) \to \mathcal{C}^\infty_c(M; E^* \otimes \Omega M)$, hence we can now define $A\colon \mathscr{D}'(M; E) \to \mathscr{D}'(M; F)$ by duality. Now $\mathcal{C}^\infty(M; E) \subset \mathscr{D}'(M; E)$. It

[8]Note that for operators acting between bundles, composition is no longer commutative on the level of principal symbols.

remains to check that $Au \in \mathcal{C}^\infty(M;F)$ when $u \in \mathcal{C}^\infty(M;E)$; but using a partition of unity $1 = \sum_i \phi_i$, with $\operatorname{supp}\phi_i$ compact, we have

$$Au = A\left(\sum_i \phi_i u\right) = \sum_i A(\phi_i u), \tag{5.98}$$

with convergence in $\mathscr{D}'(M;F)$. But since A is properly supported, the final sum is a *locally finite sum* of smooth terms, hence smooth. □

5.8. **Special classes of operators.** Let M be a manifold, and let $E, F \to M$ denote two vector bundles of rank k_E, k_F.

Definition 5.43 (Classical ps.d.o.s on manifolds)**.** Let $m \in \mathbb{R}$. The subspace $\Psi^m_{\mathrm{cl}}(M;E,F) \subset \Psi^m(M;E,F)$ of *classical pseudodifferential operators* consists of those operators whose *full* symbol in a local coordinate chart and in local trivializations of E, F is a $k_F \times k_E$ matrix of classical symbols of order m. The principal symbol map on $\Psi^m_{\mathrm{cl}}(M;E,F)$ records the leading order homogeneous part,

$$\sigma^m \colon \Psi^m_{\mathrm{cl}}(M;E,F) \to S^m_{\mathrm{hom}}(T^*M \setminus o; \pi^* \operatorname{Hom}(E,F)). \tag{5.99}$$

The reason this is a sensible definition is that classicality is preserved under local coordinate transformations; this follows from the proof of Theorem 5.2, in particular equation (5.13). Using the $\mathbb{R}^n$ results such as Proposition 4.24, one easily checks that the composition of two classical ps.d.o.s (at least one of which is properly supported) is again a classical ps.d.o., and that taking adjoints preserves classicality as well. A class of a classical ps.d.o.s is of course given by differential operators:

$$\operatorname{Diff}^m(M;E,F) \subset \Psi^m_{\mathrm{cl}}(M;E,F). \tag{5.100}$$

Moreover, parametrices of classical operators are again classical.

Often, operators arising in geometric problems are Laplace operators to leading order, such as the Hodge Laplacian (5.120). A very useful generalization of this is the following.

Definition 5.44 (Principally scalar operators)**.** Let $m \in \mathbb{R}$ and $A \in \Psi^m(M;E)$. Then A is *principally scalar* if its principal symbol is multiplication by scalars on the fibers of E, that is, if there exists a symbol $a \in S^m(T^*M)$ such that $\sigma^m(A)(x,\xi) = a(x,\xi)\operatorname{Id}_{E_x}$.

Principally scalar operators behave similarly to operators acting on scalar functions; we shall see examples of this in §8.

5.9. **Elliptic operators on compact manifolds, Fredholm theory.** Let M be a *compact* n-dimensional manifold (without boundary), and let $E, F \to M$ denote two vector bundles.

Definition 5.45 (Ellipticity)**.** We say that $A \in \Psi^m(M;E,F)$, with principal symbol $a = \sigma^m(A)$, is *elliptic* if there exists a symbol $b \in S^{-m}(T^*M;\pi^*\operatorname{Hom}(F,E))$ such that $ab - 1 \in S^{-1}(T^*M;\pi^*\operatorname{End}(F))$, $ba - 1 \in S^{-1}(T^*M;\pi^*\operatorname{End}(E))$.

Remark 5.46 (Equivalent definition of ellipticity). By 'abstract group theory' as in (4.89), the following seemingly more general definition is in fact equivalent to the ellipticity of A: there exist symbols $b, b' \in S^{-m}(T^*M;\pi^*\operatorname{Hom}(F,E))$ with the property that $ab - 1 \in S^{-1}(T^*M;\pi^*\operatorname{End}(F))$ and $b'a - 1 \in S^{-1}(T^*M;\pi^*\operatorname{End}(E))$.

Theorem 5.47 (Fredholm properties and generalized inverses of elliptic ps.d.o.s)**.** *Let $A \in \Psi^m(M;E,F)$ be an elliptic operator.*

1. *Then $A\colon \mathcal{C}^\infty(M;E) \to \mathcal{C}^\infty(M;F)$ is Fredholm, that is,* $\ker A$ *is finite-dimensional, and* $\operatorname{ran} A$ *is closed and has finite codimension. Furthermore,*
$$\operatorname{ran} A = \{f \in \mathcal{C}^\infty(M;F)\colon \langle f^*, f\rangle = 0\ \forall\, f^* \in \mathcal{C}^\infty(M;F^*\otimes\Omega M),\ A^*f^* = 0\}, \tag{5.101}$$
where $A^* \in \Psi^m(M;F^*\otimes\Omega M, E^*\otimes\Omega M)$.
2. *Let $0 < \nu \in \mathcal{C}^\infty(M;\Omega M)$ be a volume density on M, and fix positive definite fiber inner products on E,F. A linear operator $G\colon \mathcal{C}^\infty(M;F) \to \mathcal{C}^\infty(M;E)$ is uniquely determined by*
$$\begin{aligned} Gf &= u \quad \text{if } f \in \operatorname{ran} A,\ Au = f,\ u \perp \ker A \text{ in } L^2(M;E;\nu), \\ Gf &= 0 \quad \text{if } f \perp \operatorname{ran} A \text{ in } L^2(M;F;\nu). \end{aligned} \tag{5.102}$$
It is called the generalized inverse *of A. It satisfies $G \in \Psi^{-m}(M;F,E)$ and*
$$GA = I - \pi_N, \quad AG = I - \pi_R, \tag{5.103}$$
where $\pi_N\colon L^2(M;E;\nu) \to L^2(M;E;\nu)$ is the orthogonal projection onto $\ker A$ *and $\pi_R\colon L^2(M;F;\nu) \to L^2(M;F;\nu)$ is the orthogonal projection onto* $\ker A^* = (\operatorname{ran} A)^\perp$. *We have $\pi_N \in \Psi^{-\infty}(M;E)$ and $\pi_R \in \Psi^{-\infty}(M;F)$. In particular, if A is invertible, then $G = A^{-1} \in \Psi^{-m}(M;F,E)$.*

Proof. The elliptic parametrix construction, see Theorem 4.26, works in this setting as well (see Exercise 5.14). Thus, there exists $B \in \Psi^{-m}(M;F,E)$ such that
$$R_1 = AB - I \in \Psi^{-\infty}(M;F), \quad R_2 = BA - I \in \Psi^{-\infty}(M;E). \tag{5.104}$$
We show that $\dim\ker A < \infty$. First, note that
$$u \in \mathscr{D}'(M;E),\ Au = 0 \implies u = (BA - R_2)u = -R_2 u \in \mathcal{C}^\infty(M;E). \tag{5.105}$$
Let us look at this from the point of view that the identity map on $\ker A \subset L^2(M;E;\nu)$ can be written as $I = BA - R_2 = -R_2$. Now R_2 maps $L^2(M;E;\nu) \to \mathcal{C}^\infty(M;E)$, and hence is compact as a map $L^2(M;E;\nu) \to L^2(M;E;\nu)$ by Arzelà–Ascoli. Therefore, the unit ball in the closed subspace $\ker A \subset L^2(M;E;\nu)$ is compact, thus $\ker A \subset L^2(M;E;\nu)$ is finite-dimensional.

Next, we show that $\operatorname{ran} A$ is closed. Suppose $f_j = Au_j \to f \in \mathcal{C}^\infty(M;F)$, $u_j \in \mathcal{C}^\infty(M;E)$. We may change u_j by an element of $\ker A$ to ensure that $u_j \perp \ker A$. We have
$$u_j = BAu_j - R_2 u_j = Bf_j - R_2 u_j. \tag{5.106}$$
Suppose that, along some subsequence, $\|u_j\|_{L^2} \to \infty$. Then
$$\frac{u_j}{\|u_j\|} = B\left(\frac{f_j}{\|u_j\|}\right) - R_2\left(\frac{u_j}{\|u_j\|}\right). \tag{5.107}$$
This is bounded in $\mathcal{C}^\infty(M;E)$, hence we can pass to a subsequence which converges in $L^2(M;E;\nu)$, say $u_j/\|u_j\| \to u \in L^2(M;E;\nu)$. Then $Au = \lim_{j\to\infty} f_j/\|u_j\| = 0$, so $u \in \ker A$, but also $u \perp \ker A$ by construction. Since $\|u\|_{L^2} = 1$, this is a contradiction.

Therefore, $\|u_j\|_{L^2}$ is bounded. Equation (5.106) then shows that u_j is bounded in $\mathcal{C}^\infty(M;E)$, hence has a subsequence converging to $u \in \mathcal{C}^\infty(M;E)$, and
$$Au = \lim_{j\to\infty} Au_j = \lim_{j\to\infty} f_j = f. \tag{5.108}$$

Since A^* is elliptic, the finite-codimensionality of $\operatorname{ran} A$ is a consequence of (5.101). Now, (5.101) is a consequence of two facts. First, since $R = \operatorname{ran} A \subset \mathcal{C}^\infty(M;F)$ is a closed subspace of a Fréchet spaces, we have $R = {}^\perp(R^\perp) := \{f \in \mathcal{C}^\infty(M;E) \colon \langle f^*, f\rangle = 0\ \forall\, f^* \in R^\perp\}$ where

$$R^\perp = \{f^* \in \mathscr{D}'(M; F^* \otimes \Omega M) \colon \langle f^*, f\rangle = 0\ \forall\, f \in R\} \tag{5.109}$$

is the annihilator of R; indeed, the inclusion $R \subset {}^\perp(R^\perp)$ is clear, while for $f_0 \in \mathcal{C}^\infty(M;F) \setminus R$ the Hahn–Banach theorem, using the closedness of R, produces a continuous linear functional $f^* \colon \mathcal{C}^\infty(M;F) \to \mathbb{C}$, i.e. $f^* \in \mathscr{D}'(M; F^* \otimes \Omega M)$, which vanishes on R—so $f^* \in R^\perp$—but with $\langle f^*, f_0\rangle = 1$. Second, $f^* \in R^\perp$ if and only if for all $u \in \mathcal{C}^\infty(M;E)$ we have $0 = \langle f^*, Au\rangle = \langle A^* f^*, u\rangle$, i.e. if and only if $f^* \in \ker A^*$. Since A^* is elliptic, this implies that $R^\perp \subset \mathcal{C}^\infty(M; F^* \otimes \Omega M)$, and (5.101) follows.

Fixing an orthonormal basis $\{u_1, \ldots, u_J\} \subset \mathcal{C}^\infty(M;E)$ of $\ker A$, the orthogonal projection π_N onto $\ker A$ is given by

$$\pi_N = \sum_{j=1}^J u_j \langle -, u_j\rangle_{L^2(M;E;\nu)}. \tag{5.110}$$

Therefore, it has a smooth Schwartz kernel. An analogous argument shows that the orthogonal projection $\pi_R \colon L^2(M;F;\nu) \to L^2(M;F;\nu)$ onto $(\operatorname{ran} A)^\perp = \ker A^* \subset \mathcal{C}^\infty(M;F)$ has a smooth Schwartz kernel. Therefore, $\pi_N, \pi_R \in \Psi^{-\infty}$.

By (5.101), we have $\operatorname{ran}(I - \pi_R) = (\ker A^*)^\perp = \operatorname{ran} A$, and therefore G is uniquely determined by (5.102). The statement $G \in \Psi^{-m}(M;F,E)$ for the generalized inverse (5.102) now follows by writing

$$\begin{aligned} G &= G(AB - R_1) \\ &= (I - \pi_N)B - GR_1 \\ &= (I - \pi_N)B - (BA - R_2)GR_1 \\ &= (I - \pi_N)B - B(I - \pi_R)R_1 + R_2 G R_1. \end{aligned} \tag{5.111}$$

Indeed, the first summand lies in $\Psi^m(M;F,E)$, the second in $\Psi^{-\infty}(M;F,E)$, and the last one is a smoothing operator, hence lies in $\Psi^{-\infty}(M;F,E)$ as well. □

We revisit the proof of the existence of the generalized inverse using L^2-techniques in §5.11.

As a typical example, we discuss the *Laplace operator* on a compact n-dimensional manifold M, which we assume to be connected for convenience. Denote by S^2T^*M the second symmetric tensor product of T^*M with itself. Let $g \in \mathcal{C}^\infty(M; S^2T^*M)$ be a Riemannian metric, so in local coordinates

$$g = \sum_{i,j=1}^n g_{ij}(x) \cdot \frac{1}{2}(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i), \qquad g_{ij}(x) = g_{ji}(x). \tag{5.112}$$

Write $g^{ij}(x) = g^{-1}(x)_{ij}$ and $|g| = |\det(g_{ij})|$. Then the (scalar) Laplace operator is

$$\Delta_g u = \sum_{i,j=1}^n |g|^{-1/2} D_{x_i}\big(|g|^{1/2} g^{ij}(x) D_{x_j} u\big) \tag{5.113}$$

in local coordinates. Thus, $\Delta_g \in \Psi^2(M)$, with

$$\sigma^2(\Delta_g)(x,\xi) = \sum_{i,j=1}^n g^{ij}(x)\xi_i\xi_j =: |\xi|^2_{g^{-1}(x)}. \tag{5.114}$$

Thus, Δ_g is elliptic. By Theorem 5.47, the kernel and cokernel of Δ_g are finite-dimensional.

Moreover, Δ_g is a symmetric operator with respect to the inner product on $L^2(M;|\mathrm{d}g|)$, where $|\mathrm{d}g| \in \mathcal{C}^\infty(M;\Omega M)$ is defined in local coordinates by

$$|\mathrm{d}g| = |g(x)|^{1/2}\mathrm{d}x. \tag{5.115}$$

Thus, $\ker\Delta_g = (\operatorname{ran}\Delta_g)^\perp$; and if $u \in \ker\Delta_g$, then

$$0 = \int_M (\Delta_g u)\bar{u}\,|\mathrm{d}g| = \int_M |\nabla u|^2_g\,|\mathrm{d}g|, \tag{5.116}$$

so u is constant. In the notation of Theorem 5.47, we thus have

$$\pi_N = \frac{1}{\mathrm{vol}(M)}\langle\cdot,1\rangle 1 = \pi_R \tag{5.117}$$

(projection onto constants).

Let us study

$$\Delta_g u = f, \quad f \in \mathscr{D}'(M). \tag{5.118}$$

Let $G \in \Psi^{-2}(M)$ denote the generalized inverse of Δ_g. In the notation of Theorem 5.47, we then have

$$u = (G\Delta_g + \pi_N)u = Gf + \pi_N u. \tag{5.119}$$

This solves (5.118) if and only if $f = \Delta_g u = \Delta_g Gf + \Delta_g\pi_N u = (I - \pi_R)f$. This shows:

Proposition 5.48 (Laplace equation on compact manifolds)**.** *The equation* (5.118) *has a solution $u \in \mathscr{D}'(M)$ if and only if $\langle f,1\rangle = 0$, and in this case u is unique up to additive constants. If $f \in \mathcal{C}^\infty(M)$, then $u \in \mathcal{C}^\infty(M)$.*

Example 5.49. For the operator $A = \Delta_g + 1 \in \Psi^2(M)$ on a compact Riemannian manifold, one finds $\ker A = 0 = (\operatorname{ran} A)^\perp$, thus one can always solve $Au = f$ for $f \in \mathcal{C}^\infty(M)$ or $f \in \mathscr{D}'(M)$, with solution $u \in \mathcal{C}^\infty(M)$ or $u \in \mathscr{D}'(M)$.

Example 5.50. One can define natural generalizations of Δ_g which act on vector bundles rather than functions. Let $\mathrm{d}_k \in \mathrm{Diff}^1(M;\Lambda^k T^*M;\Lambda^{k+1}T^*M)$ denote the exterior derivative, and denote by $\delta_k \in \mathrm{Diff}^1(M;\Lambda^k T^*M;\Lambda^{k-1}T^*M)$ the adjoint of d_{k-1}. Let $\mathrm{d}_n = 0$ and $\delta_0 = 0$. Then the *Hodge Laplacian* in degree k is

$$\Delta_k := \delta_{k+1}\mathrm{d}_k + \mathrm{d}_{k-1}\delta_k \in \mathrm{Diff}^2(M;\Lambda^k T^*M). \tag{5.120}$$

Its principal symbol is *scalar*, i.e. at each $(x,\xi) \in T^*M$ a multiple of the identity operator on $(\pi^*\Lambda^k T^*M)_{(x,\xi)}$; in fact $\sigma^2(\Delta_k)(x,\xi) = |\xi|^2_{g^{-1}(x)}\,\mathrm{Id}$. (The expression (5.113) is the special case $k = 0$.) Again Δ_k is symmetric with respect to the fiber inner product and volume density induced by g. Its kernel and orthocomplement of the range are finite-dimensional, and can be identified with the singular cohomology group $H^k(M;\mathbb{C})$ by Hodge theory. For a general version of this, see Exercise 5.28.

5.10. **Sobolev spaces on manifolds.** We need two key facts about Sobolev spaces $H^s(\mathbb{R}^n)$ for the generalization of Sobolev spaces to manifolds. For an open set $\Omega \Subset \mathbb{R}^n$, we define

$$H^s_{\mathrm{c}}(\Omega) := \{u \in H^s(\mathbb{R}^n) \colon \operatorname{supp} u \subset \Omega\}. \tag{5.121}$$

Lemma 5.51 (Sobolev spaces under localizations and coordinate changes)**.** *Sobolev spaces on $\mathbb{R}^n$ have the following properties.*

(1) *Let $a \in \mathcal{C}^\infty_{\mathrm{b}}(\mathbb{R}^n)$. Then multiplication by a is a bounded linear map $H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}^n$.*
(2) *Suppose $\kappa\colon \Omega \to \Omega'$ is a diffeomorphism of precompact open subsets $\Omega, \Omega' \Subset \mathbb{R}^n$. Then $\kappa^*\colon H^s_{\mathrm{c}}(\Omega') \to H^s_{\mathrm{c}}(\Omega)$. Here, the pullback of a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ with support in Ω' is defined via duality using the formula*

$$\langle \kappa^* u, \phi\rangle = \langle u, |\det(\kappa^{-1})'|(\kappa^{-1})^*\phi\rangle, \qquad \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega). \tag{5.122}$$

Proof. The 'standard' proof of the first claim proceeds by proving it for $s \in \mathbb{N}_0$ using the Leibniz rule, then for all real $s \geq 0$ by complex interpolation, and then for all $s \in \mathbb{R}$ by duality. With the machinery of §4 at hand, we can instead just observe that $a \in \Psi^0(\mathbb{R}^n)$, and appeal to Corollary 4.34.

The second claim is clear for $s = 0$. We shall prove it for general $s \in \mathbb{R}$ using our ps.d.o. machinery. Indeed, given $u \in H^s_{\mathrm{c}}(\Omega') \subset \mathscr{E}'(\Omega')$, we certainly have $\kappa^* u \in \mathscr{E}'(\Omega)$. Let $A \in \Psi^s(\mathbb{R}^n)$ be elliptic, and let $\phi, \tilde\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ be such that $\phi = \tilde\phi = 1$ on $\operatorname{supp}(\kappa^* u)$, and such that $\tilde\phi = 1$ in a neighborhood of $\operatorname{supp}\phi$. By choosing A carefully (localizing its Schwartz kernel near the diagonal—which does not affects its ellipticity property), we may arrange that

$$A(\kappa^* u) = \tilde\phi A \phi \kappa^* u. \tag{5.123}$$

Note that $\tilde\phi A\phi \in \Psi^s_{\mathrm{c}}(\Omega)$. Therefore, by Theorem 5.2,

$$A(\kappa^* u) = \kappa^*(A' u), \quad A' = (\kappa^{-1})^* \tilde\phi A \phi \kappa^* \in \Psi^s_{\mathrm{c}}(\Omega'). \tag{5.124}$$

Therefore $A' u \in L^2(\Omega')$, hence $\kappa^*(A'u) \in L^2(\Omega)$, so $A(\kappa^* u) \in L^2(\mathbb{R}^n)$. Since A is elliptic, Corollary 4.36 implies that $\kappa^* u \in H^s(\mathbb{R}^n)$, as desired. □

The 'local coordinate' definition of Sobolev spaces on manifolds is then:

Definition 5.52 (Sobolev spaces on manifolds)**.** Let M be an n-dimensional manifold, $s \in \mathbb{R}$. Then:

(1) We define $H^s_{\mathrm{loc}}(M)$ as the space of all $u \in \mathscr{D}'(M)$ such that for all coordinate charts $F\colon U \to F(U) \subset \mathbb{R}^n$ on M, and all $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(U)$, the distribution $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \ni \phi \mapsto \langle u, \chi F^*(\phi|\mathrm{d}x|)\rangle$ is an element of $H^s(\mathbb{R}^n)$.
(2) We define $H^s_{\mathrm{c}}(M) = \{u \in H^s_{\mathrm{loc}}(M) \colon \operatorname{supp} u \subset M \text{ is compact}\}$.

If M is compact, we write

$$H^s(M) = H^s_{\mathrm{loc}}(M) = H^s_{\mathrm{c}}(M). \tag{5.125}$$

Lemma 5.51 shows that if $u \in H^s_{\mathrm{c}}(F(U))$ for some coordinate chart $F\colon U \to F(U) \subset \mathbb{R}^n$ on M, then $F^* u \in H^s_{\mathrm{c}}(M)$.

Remark 5.53 (Topology on $H^s_{\mathrm{loc}}(M)$). One can equip $H^s_{\mathrm{loc}}(M)$ with the structure of a Fréchet space by using the seminorms $\|(F_i^{-1})^*\phi_i u\|_{H^s(\mathbb{R}^n)}$ for any fixed countable cover of M by coordinate charts $F_i\colon U_i \to F_i(U_i) \subset \mathbb{R}^n$ and a subordinate partition of unity $\{\phi_i\}$, $\phi_i \in \mathcal{C}^\infty_{\mathrm{c}}(U_i)$. The resulting topology is independent of the cover and the partition of unity.

The proof of Lemma 5.51 suggests a more intrinsic definition of Sobolev spaces on M. Note first that the spaces $L^2_{\mathrm{loc}}(M)$ and $L^2_{\mathrm{c}}(M)$ are well-defined, independently of a choice of integration measure on M. (On the other hand, the space $L^2(M)$, even as a set, is *not* well-defined when M is non-compact without specified integration measure.)

Proposition 5.54 (Boundedness of ps.d.o.s on Sobolev spaces: I)**.** *Let $u \in \mathscr{D}'(M)$.*

(1) *Suppose $u \in H^s_{\mathrm{c}}(M)$. Then $Au \in L^2_{\mathrm{loc}}(M)$ for all $A \in \Psi^s(M)$. If A is properly supported, then $A\colon H^s_{\mathrm{c}}(M) \to L^2_{\mathrm{c}}(M)$, $H^s_{\mathrm{loc}}(M) \to L^2_{\mathrm{loc}}(M)$.*
(2) *If $Au \in L^2_{\mathrm{loc}}(M)$ for some properly supported elliptic operator $A \in \Psi^s(M)$, then $u \in H^s_{\mathrm{loc}}(M)$.*

Proof. Suppose that $F\colon U \to F(U) \subset \mathbb{R}^n$ is a coordinate system on M, and let $\phi, \tilde\phi \in \mathcal{C}^\infty_{\mathrm{c}}(U)$ with $\tilde\phi = 1$ in a neighborhood of $\operatorname{supp}\phi$.

The first claim follows by writing

$$\phi A u = \phi A \tilde\phi u + \phi A(1-\tilde\phi)u. \tag{5.126}$$

Indeed, the second summand lies in $\mathcal{C}^\infty(M) \subset L^2_{\mathrm{loc}}(M)$. The first summand can be evaluated in local coordinates, and lies in $L^2_{\mathrm{c}}(M)$ by Theorem 4.32.

Turning to the second claim, we need to show that $(F^{-1})^*(\phi u) \in \mathscr{E}'(F(U))$ lies in $H^s(\mathbb{R}^n)$. Let $B \in \Psi^s(\mathbb{R}^n)$ be elliptic. We can arrange for its Schwartz kernel to be supported so close to the diagonal that

$$(1-(F^{-1})^*\tilde\phi)B((F^{-1})^*\phi) = 0. \tag{5.127}$$

By elliptic regularity, we need to establish $B(F^{-1})^*\phi u \in L^2(\mathbb{R}^n)$, which by (5.127) is equivalent to

$$B' u \in L^2_{\mathrm{c}}(M), \quad B' := \tilde\phi F^* B (F^{-1})^*\phi \in \Psi^s(M). \tag{5.128}$$

Since A is elliptic, there exists a properly supported parametrix $Q \in \Psi^{-s}(M)$ with $I = QA + R$, where $R \in \Psi^{-\infty}(M)$ is then also properly supported. Therefore,

$$B' u = B'(QA+R)u = (B'Q)(Au) + B'Ru. \tag{5.129}$$

Now $B'Q \in \Psi^0(M)$ is bounded on $L^2_{\mathrm{loc}}(M)$, so $(B'Q)(Au) \in L^2_{\mathrm{loc}}(M)$, while $B'R \in \Psi^{-\infty}(M)$, so $B'Ru \in \mathcal{C}^\infty(M)$. Therefore, $B'u \in L^2_{\mathrm{loc}}(M)$. $\square$

Corollary 5.55 (Boundedness of ps.d.o.s on Sobolev spaces: II)**.** *Let $A \in \Psi^m(M)$. Then A is a bounded linear operator*

$$A\colon H^s_{\mathrm{c}}(M) \to H^{s-m}_{\mathrm{loc}}(M). \tag{5.130}$$

If A is properly supported, then $A\colon H^s_{\mathrm{c}}(M) \to H^{s-m}_{\mathrm{c}}(M)$, $H^s_{\mathrm{loc}}(M) \to H^{s-m}_{\mathrm{loc}}(M)$.

Proof. We only prove (5.130). Let $\Lambda \in \Psi^{s-m}(M)$ be properly supported and elliptic. By the second part of Proposition 5.54, it suffices to show that $\Lambda \circ A\colon H^s_{\mathrm{c}}(M) \to L^2_{\mathrm{loc}}(M)$; but this follows from $\Lambda \circ A \in \Psi^s(M)$ and the first part of Proposition 5.54. $\square$

Proposition 5.56 (Compactness of negative order ps.d.o.s)**.** *Let M be a compact n-dimensional manifold. Let $A \in \Psi^m(M)$, $m < 0$. Then $A\colon L^2(M) \to L^2(M)$ is compact.*

Proof. Decomposing A into finitely many terms as in (5.60), this follows from the compactness on $L^2(\mathbb{R}^n)$ of negative order ps.d.o.s on $\mathbb{R}^n$ with compactly supported Schwartz kernels (Proposition 4.33) and the compactness of residual operators with Schwartz kernels $K \in \mathcal{C}^\infty(M^2; \Omega_R)$; the latter follows from the fact that such operators are continuous $L^2(M) \to \mathcal{C}^\infty(M)$, while $\mathcal{C}^\infty(M) \subset \mathcal{C}^1(M) \to \mathcal{C}^0(M) \hookrightarrow L^2(M)$ is compact by Arzelà–Ascoli. □

On a compact manifold M, the space $H^s(M)$ can be given the structure of a Hilbert space:

Proposition 5.57 ($H^s(M)$ as a Hilbert space)**.** *Let M be compact, and let $s \in \mathbb{R}$. Fix a volume density on M. Then there exists $A \in \Psi^s(M)$ such that*

$$\langle u, v\rangle_{H^s(M)} := \langle Au, Av\rangle_{L^2(M)}, \quad \|u\|^2_{H^s(M)} := \langle u, u\rangle_{H^s(M)}, \tag{5.131}$$

gives $H^s(M)$ the structure of a Hilbert space. The topology on $H^s(M)$ is equal to the norm topology of $(H^s(M), \|\cdot\|_{H^s(M)})$.

Proof. Let $s \geq 0$. Fix a smooth fiber metric $\|\cdot\|$ on T^*M, and let $\Lambda' \in \Psi^{s/2}(M)$ be an operator with $\sigma^{s/2}(\Lambda')(x,\xi) = \|\xi\|^{s/2}$.[9] Then Λ' is elliptic, and so is

$$\Lambda_s := I + (\Lambda')^*\Lambda' \in \Psi^s(M). \tag{5.132}$$

By Theorem 5.47, $\Lambda_s\colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is Fredholm. We claim that Λ_s is invertible on $\mathcal{C}^\infty(M)$. Indeed, $\Lambda_s u = 0$ implies $\|u\|^2_{L^2(M)} + \|\Lambda' u\|^2_{L^2(M)} = 0$, hence $u = 0$. Since Λ_s is symmetric (that is, $\langle \Lambda_s u, f\rangle_{L^2(M)} = \langle u, \Lambda_s f\rangle_{L^2(M)}$ for $u, f \in \mathcal{C}^\infty(M)$), this also shows that Λ_s is surjective. The second part of Theorem 5.47 then implies that

$$\Lambda_{-s} := \Lambda_s^{-1} \in \Psi^{-s}(M). \tag{5.133}$$

Using Proposition 5.54, we conclude that $\Lambda_s\colon H^s(M) \to L^2(M)$ and $\Lambda_{-s}\colon H^{-s}(M) \to L^2(M)$ are isomorphisms.

For $s \in \mathbb{R}$, we can thus take $A = \Lambda_s$. □

Remark 5.58 (The case $H^{2k}(M)$, $k \in \mathbb{N}_0$). For $s = 2k$, $k \in \mathbb{N}$, one can take $\Lambda_{2k} = (\Delta_g + 1)^k$ for any Riemannian metric g on M. (In fact, this is true for any $k \in \mathbb{R}$ by a theorem of Seeley which states, as a special case, that $(\Delta_g + 1)^s \in \Psi^{2s}(M)$ for any $s \in \mathbb{R}$. This operator is defined using the functional calculus for self-adjoint operators.)

Adding vector bundles to this discussion requires only notational changes. Namely, if $E \to M$ is a real/complex rank k vector bundle, we say that $u \in \mathscr{D}'(M; E)$ lies in $H^s_{\mathrm{loc}}(M; E)$ if and only if in local trivializations of E over coordinate charts on M, u is a k-vector of real-valued/complex-valued elements of $H^s(\mathbb{R}^n)$. We let $H^s_c(M; E) = H^s_{\mathrm{loc}}(M; E) \cap \mathscr{E}'(M; E)$ as usual. We leave the statements and proofs of the generalizations of Proposition 5.54, Corollary 5.55, and Proposition 5.57 to the reader.

Example 5.59. If M is n-dimensional and $p \in M$, then $\delta_p \in H^s(M; \Omega M)$ for all $s < -n/2$; cf. Example (5.16).

[9] Strictly speaking, one should smooth the right hand side out near $\xi = 0$ to get a smooth symbol; but principal symbols only care about behavior for large ξ, hence we do not do this here.

5.11. **Elliptic operators on compact manifolds, revisited.** Throughout this section, we denote by M a *compact* manifold.

Proposition 5.60 (Rellich compactness theorem)**.** *Let $s' < s$. Then the inclusion*

$$H^s(M) \hookrightarrow H^{s'}(M) \tag{5.134}$$

is compact.

Proof. One can prove this by localizing in coordinate charts and using a *suitable* analogue on $\mathbb{R}^n$—in fact, one can use a special case of the first part of Exercise 4.15. (Beware however that the inclusion $H^s(\mathbb{R}^n) \hookrightarrow H^{s'}(\mathbb{R}^n)$ is not compact.) In the spirit of using ps.d.o. techniques to establish properties of Sobolev spaces, one can alternatively argue as follows. Fixing invertible ps.d.o.s $\Lambda_\sigma \in \Psi^\sigma(M)$ for $\sigma = s, s'$ as in Proposition 5.57, we can factor the inclusion (5.134) as

$$H^s(M) \xrightarrow{\Lambda_s} L^2(M) \xrightarrow{\Lambda_{s'}\circ\Lambda_s^{-1}} L^2(M) \xrightarrow{\Lambda_{s'}^{-1}} H^{s'}(M), \tag{5.135}$$

where the first and last arrows are isomorphisms. The middle arrow is $\Lambda_{s'}\circ\Lambda_s^{-1} \in \Psi^{s'-s}(M)$, i.e. a ps.d.o. of negative order. The result then follows from Proposition 5.56. □

We can now refine Theorem 5.47:

Theorem 5.61 (Fredholm properties of elliptic ps.d.o.s on Sobolev spaces)**.** *Let $A \in \Psi^m(M; E, F)$ be an elliptic operator. Then for any $s \in \mathbb{R}$,*

$$A\colon H^s(M; E) \to H^{s-m}(M; F) \tag{5.136}$$

is Fredholm. Its kernel $\ker A$ *is independent of s, and* $\ker A \subset \mathcal{C}^\infty(M; E)$*. Moreover, if we fix a volume density on M and positive definite fiber inner products on E, F, the cokernel* $\operatorname{coker} A$ *can be identified with the subset* $\ker A^* \subset \mathcal{C}^\infty(M; F)$ *which is independent of s; that is, $f \in H^{s-m}(M; F)$ lies in* $\operatorname{ran} A$ *if and only if $\langle f, g\rangle_{L^2(M;F)} = 0$ for all $g \in \ker A^*$.*

Proof. If $B \in \Psi^m(M; F, E)$ denotes an elliptic parametrix, then $AB = I + R_1$ and $BA = I + R_2$ with $R_1, R_2 \in \Psi^{-\infty}$ as in (5.104). By Proposition 5.60, the errors $R_1\colon H^s(M; F) \to \mathcal{C}^\infty(M; F) \hookrightarrow H^s(M; F)$ and $R_2\colon H^{s-m}(M; E) \to \mathcal{C}^\infty(M; E) \hookrightarrow H^{s-m}(M; E)$ are compact operators. Therefore, A is Fredholm. The regularity statement $\ker A \subset \mathcal{C}^\infty(M; E)$ is a special case of (5.105). The solvability claim follows from Theorem 5.47 and elliptic regularity. □

In fact, this theorem has a converse: if $A \in \Psi^m(M; E, F)$ is such that (5.136) is Fredholm for some $s \in \mathbb{R}$, then A is elliptic. A special case of this result is the subject of Exercise 5.23.

Remark 5.62 (Fredholm index)*.* Theorem 5.61 also shows that the index $\operatorname{ind} A = \dim\ker A - \dim\operatorname{coker} A$ is independent of s. Simple functional analytic arguments show that $\operatorname{ind} A = \operatorname{ind}(A + B)$ for any $B \in \Psi^{m-1}(M; E, F)$; thus, $\operatorname{ind} A$ only depends on the principal symbol $\sigma^m(A)$. The Atiyah–Singer index theorem gives a formula to compute $\operatorname{ind} A$ in terms of $\sigma^m(A)$.

We can now give a more transparent perspective on the generalized inverse of Theorem 5.47(2). Indeed, by Theorem 5.61, $\operatorname{ran}_{H^m(M;E)} A := A(H^m(M; E)) \subset L^2(M; F)$ is a closed subspace, and therefore $L^2(M; F) = \operatorname{ran}_{H^m(M;E)} A \oplus (\operatorname{ran}_{H^m(M;E)} A)^\perp$. Thus, (5.102) for $f \in L^2(M;F)$ defines a linear operator $G\colon L^2(M;F) \to H^m(M;E)$. The membership $G \in \Psi^m(M;F,E)$ now again follows from (5.111).

Example 5.63. Expanding upon Example 5.49, the operator $A = \Delta_g + \lambda \in \Psi^2(M)$ on a compact Riemannian manifold (M,g) is invertible as a map $H^s(M) \to H^{s-2}(M)$, $s \in \mathbb{R}$, whenever $\lambda \notin (-\infty,0]$. Indeed, for $u \in \ker A$ one has $u \in \mathcal{C}^\infty(M)$ by elliptic regularity, and therefore $0 = \langle Au,u\rangle_{L^2(M;|dg|)} = \int_M |\nabla u|_g^2\,|dg| + \lambda\|u\|^2_{L^2(M;|dg|)} = 0$ which implies $u = 0$; one similarly shows that $\ker A^* = \ker(\Delta_g + \bar\lambda)$ is trivial.

An interesting application concerns the spectral theory of symmetric ps.d.o.s.

Theorem 5.64 (Self-adjointness of elliptic symmetric ps.d.o.s)**.** *Fix a volume density on M, and a positive definite fiber inner product on $E \to M$. Let $m > 0$, and let $A \in \Psi^m(M;E)$ be elliptic and symmetric, that is, $\langle Au,v\rangle = \langle u,Av\rangle$ for $u,v \in \mathcal{C}^\infty(M;E)$, where $\langle\cdot,\cdot\rangle$ is the inner product on $L^2(M;E)$. Then A is an unbounded self-adjoint operator on $L^2(M;E)$ with domain $H^m(M;E)$. Its spectrum* $\operatorname{spec} A \subset \mathbb{R}$ *is discrete and accumulates only at ∞. There exists an orthonormal basis of $L^2(M;E)$ consisting of eigenfunctions of A, all of which are smooth.*

Proof. By [RS72, Theorem VIII.3], we need to show that $A \pm i\colon H^m(M;E) \to L^2(M;E)$ is surjective. By Theorem 5.61, its range is closed, and any element $u \in (\operatorname{ran}(A \pm i))^\perp$ lies in $\ker(A \mp i) \subset \mathcal{C}^\infty(M;E)$, so

$$0 = \operatorname{Im}\langle (A \mp i)u,u\rangle = \mp i\|u\|^2_{L^2(M;E)} \quad\Longrightarrow\quad u = 0. \tag{5.137}$$

This proves self-adjointness.

One can also argue directly: if A is given the domain $\mathcal{D}(A) = H^m(M;E)$, then $v \in L^2(M;E)$ lies in $\mathcal{D}(A^*)$ if and only if $\mathcal{D}(A) \ni u \mapsto \langle Au,v\rangle$ satisfies a bound $|\langle Au,v\rangle| \le C\|u\|_{L^2}$ for some C. But $\langle Au,v\rangle = \langle u,A^*v\rangle$, hence we conclude that $A^*v \in L^2(M;E)$, and by elliptic regularity $v \in H^m(M;E)$; thus $\mathcal{D}(A^*) \subset \mathcal{D}(A)$. The converse is clear since A is symmetric.

To prove the discreteness of the spectrum, note first that $(A+i)^{-1}\colon L^2(M;E) \to H^m(M;E) \hookrightarrow L^2(M;E)$ is a compact operator, and hence its spectrum is discrete and can only accumulate at 0. Therefore, there exists a complex number $\lambda \in \mathbb{C}$ so that $i - \lambda^{-1} \in \mathbb{R}$ and $(A+i)^{-1} - \lambda$ is invertible on $L^2(M;E)$; but since

$$(A+i)^{-1} - \lambda = -\lambda(A+i)^{-1}\bigl(A + i - \lambda^{-1}\bigr), \tag{5.138}$$

this implies that $A - \mu\colon H^m(M;E) \to L^2(M;E)$ is invertible where $\mu = \lambda^{-1} - i \in \mathbb{R}$. Therefore, $(A-\mu)^{-1}\colon L^2(M;E) \to L^2(M;E)$ is compact *and self-adjoint*, and the spectral theorem produces an orthonormal basis of $L^2(M;E)$ consisting of eigenfunctions of $(A-\mu)^{-1}$ corresponding to a sequence of eigenvalues tending to 0. But $(A-\mu)^{-1}\phi = \lambda\phi$ implies $A\phi = (\mu + \lambda^{-1})\phi$, and hence $\operatorname{spec} A$ accumulates only at ∞. □

Example 5.65. This applies to the Laplacian Δ_g on any compact Riemannian manifold (M,g), acting on functions or differential forms.

Example 5.66. There exist elliptic non-selfadjoint operators whose spectrum is the entire complex plane. In fact, there exists an elliptic operator $A \in \Psi^1(\mathbb{S}^1)$ with index 1 (or any other integer). By Remark 5.62, $A - \lambda$ is never invertible for any $\lambda \in \mathbb{C}$.

5.12. **A simple nonlinear example.** As a simple (and naive, weak, and wasteful, but instructive) nonlinear application of the elliptic theory developed thus far, we shall solve a non-linear elliptic equation on a compact 2-dimensional manifold M. If g is a Riemannian metric on M, we denote the Gauss curvature of M by $K_g \in \mathcal{C}^\infty(M)$. If $\phi \in \mathcal{C}^\infty(M)$, then the metric $g'(x) = e^{2\phi(x)}g(x)$ is said to be *conformal* to g. The Gauss curvature of g' is given by

$$K_{g'} = e^{-2\phi}(K_g + \Delta_g \phi). \tag{5.139}$$

We recall the expression for K_g in local coordinates $(x_1, x_2) \in \mathbb{R}^2$: writing $\partial_i := \partial_{x_i}$ and $g_{ij} = g(\partial_i, \partial_j)$, further g^{ij} for the components of the inverse matrix $(g_{ij})^{-1}$, and $\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ for the Christoffel symbols of g, it is

$$K_g = \frac{1}{2}\sum_{i,j,k} g^{jk} R^i{}_{jik}, \qquad R^i{}_{jik} := \partial_i \Gamma^i_{jk} - \partial_k \Gamma^i_{jk} + \Gamma^i_{il}\Gamma^l_{jk} - \Gamma^i_{kl}\Gamma^l_{ij}. \tag{5.140}$$

Proposition 5.67 (Local version of the uniformization theorem in negative curvature)**.** *Suppose (M, g) has constant Gauss curvature $K_g \equiv -1$. Let $\tilde g \in \mathcal{C}^\infty(M; S^2 T^*M)$ be a Riemannian metric with $\|g - \tilde g\|_{H^4(M;S^2T^*M)} < \epsilon$, $\epsilon > 0$ small. (Here we use g to define the fiber inner product on S^2T^*M.) Then there exists $\phi \in \mathcal{C}^\infty(M)$ such that $e^{2\phi}\tilde g$ has constant Gauss curvature -1.*

This is a local version of the uniformization theorem; the conclusion holds for *any* metric $\tilde g$, not necessarily close to g. The assumptions require that M is a manifold of genus at least 2 (that is, a doughnut with at least two holes). For $M \cong \mathbb{S}^2$, one can always find a conformal multiple with constant curvature $+1$, and for $M \cong \mathbb{T}^2$, one can always find one with constant curvature 0.

In the proof, we will use algebra properties of Sobolev spaces on manifolds which are the subject of Exercise 5.19: since M is 2-dimensional, $H^s(M)$ is an algebra under pointwise multiplication for $s > 1$, with

$$\|uv\|_{H^s(M)} \le C_s \|u\|_{H^s(M)} \|v\|_{H^s(M)} \tag{5.141}$$

for some constant C_s; and we recall that $H^s(M) \hookrightarrow \mathcal{C}^k(M)$ when $s > 1 + k$.

Proof of Proposition 5.67. Metrics $g \in H^4(M; S^2T^*M)$ are twice continuously differentiable, and therefore their Gauss curvature is a continuous function on M. More precisely, in smooth local coordinates $(x_1, x_2) \in U$ on M, we have $g_{ij} \in H^4_{\mathrm{loc}}(U)$, therefore $\det(g_{ij}) \in H^4_{\mathrm{loc}}(U)$ as well and thus also $g^{ij} \in H^4_{\mathrm{loc}}(U)$. This implies $\Gamma^k_{ij} \in H^3_{\mathrm{loc}}(U)$ and $R^i{}_{jik} \in H^2_{\mathrm{loc}}(U)$. By (5.140), K_g lies in $H^2_{\mathrm{loc}}(U)$ in local coordinates. Globally on M, we have thus shown that $K_g \in H^2(M)$; and the map

$$H^4(M; S^2T^*M) \ni g \mapsto K_g \in H^2(M; S^2T^*M) \tag{5.142}$$

is continuous.

We want to solve the equation

$$-1 = K_{e^{2\phi}\tilde g} = e^{-2\phi}(K_{\tilde g} + \Delta_{\tilde g}\phi), \tag{5.143}$$

or equivalently

$$\Delta_{\tilde g}\phi + e^{2\phi} - 1 = -(K_{\tilde g} + 1). \tag{5.144}$$

Since $\|K_{\tilde g}+1\|_{H^2(M)} = \|K_{\tilde g}-K_g\|_{H^2(M)}$ is small (namely: smaller than a constant times ϵ), we expect $\phi\in H^4(M)$ to be small. (We remark that for $\phi\in H^4(M)$, the series $\sum_{j=0}^\infty \frac{1}{j!}(2\phi)^j$ converges in $H^4(M)$ in view of (5.141), so $e^{2\phi}\in H^4(M)$ is well-defined.) This suggests Taylor expanding around $\phi=0$, which gives

$$\begin{aligned} &A\phi = E(\phi) - N(\phi), \\ &\qquad A=\Delta_g+2,\quad E(\phi) = -(K_{\tilde g}+1) - (\Delta_{\tilde g}-\Delta_g)\phi,\quad N(\phi) = e^{2\phi}-1-2\phi. \end{aligned} \tag{5.145}$$

We solve this using the contraction mapping principle, i.e. by iterating the map

$$T\colon H^4(M)\ni\phi\mapsto A^{-1}(E(\phi)-N(\phi))\in H^4(M). \tag{5.146}$$

Recall from Example 5.63 that A is invertible as a map $H^s(M)\to H^{s-2}(M)$ for all $s\in\mathbb{R}$.

Now the local coordinate expression (5.113) for the Laplace operator, which can be rewritten as $\Delta_g u = -\sum_{i,j}(g^{ij}\partial_i\partial_j - \sum_k \Gamma_{ij}^k\partial_k)u$, shows that $\|\Delta_{\tilde g}-\Delta_g\|_{\mathcal{L}(H^4(M),H^2(M))}\leq C\epsilon$ for some constant C, thus

$$\|E(\phi)\|_{H^2}\leq C\epsilon(1+\|\phi\|_{H^4(M)}). \tag{5.147}$$

Moreover,

$$\|N(\phi)\|_{H^2(M)}\leq\|N(\phi)\|_{H^4(M)}\leq\sum_{j=2}^\infty\frac{1}{j!}\|(2\phi)^j\|_{H^2(M)}\leq C\|\phi\|_{H^4(M)}^2 \tag{5.148}$$

for $\|\phi\|_{H^4(M)}\leq 1$. Therefore, if $\|\phi\|_{H^4(M)}\leq\delta$ where $\delta\in(0,1]$, then

$$\|T\phi\|_{H^4(M)}\leq C'(C\epsilon(1+\delta)+C\delta^2),\quad C' = \|A^{-1}\|_{\mathcal{L}(H^2(M),H^4(M))}. \tag{5.149}$$

Fix $\delta_0>0$ so that $C'C\delta_0\leq\frac{1}{2}$; then for $0<\delta\leq\min(\delta_0,1)$, we obtain $\|T\phi\|_{H^4(M)}\leq\delta$ provided $\epsilon>0$ is sufficiently small. Thus, T maps the δ-ball in $H^4(M)$ into itself.

The map T is moreover a contraction on the δ-ball in $H^4(M)$, since

$$\begin{aligned} \|T\phi-T\psi\|_{H^4(M)} &\leq C'\big(C\epsilon\|\phi-\psi\|_{H^4(M)} + C\|\phi-\psi\|_{H^4(M)}(\|\phi\|_{H^4(M)}+\|\psi\|_{H^4(M)})\big) \\ &\leq C'(C\epsilon+C\delta)\|\phi-\psi\|_{H^4(M)} \\ &\leq\frac{1}{2}\|\phi-\psi\|_{H^4(M)} \end{aligned} \tag{5.150}$$

for small enough $\delta,\epsilon>0$. Here we use $N(\phi)-N(\psi) = \sum_{j=2}^\infty\frac{1}{j!}2^j(\phi-\psi)\sum_{k=0}^{j-1}\phi^k\psi^{j-1-k}$ and the triangle inequality.

Let now $\phi\in H^4(M)$, $\|\phi\|_{H^4(M)}\leq\delta$, denote the unique fixed point of T; then ϕ solves (5.144). We rewrite this one last time as

$$\Delta_{\tilde g}\phi = -K_{\tilde g} - e^{2\phi}. \tag{5.151}$$

Suppose we already know $\phi\in H^k(M)$, $k\geq 4$. Then the right hand side of this equation lies in $H^k(M)$, so by elliptic regularity we conclude that $\phi\in H^{k+2}(M)$. Therefore, $\phi\in\bigcap_k H^k(M) = \mathcal{C}^\infty(M)$, finishing the proof. □

5.13. **Commutators and symplectic geometry.** We tie up a loose end and describe, invariantly, the principal symbol of the commutator of two ps.d.o.s. Key is the *symplectic* structure of the cotangent bundle T^*M.

Definition 5.68 (Canonical 1-form and symplectic form on T^*M)**.** Let M be an n-dimensional manifold. The *canonical 1-form* on T^*M is the section $\alpha\in\mathcal{C}^\infty(T^*M;T^*(T^*M))$ defined by

$$\alpha_{(x,\xi)}(v) := \xi(\pi_* v),\quad x\in M,\ \xi\in T^*_x M,\ v\in T_{(x,\xi)}(T^*M), \tag{5.152}$$

where $\pi\colon T^*M\to M$ is the projection. The *canonical symplectic form* on T^*M is

$$\omega := -\mathrm{d}\alpha\in\mathcal{C}^\infty(T^*M;\Lambda^2(T^*M)). \tag{5.153}$$

In local coordinates $x\in\mathbb{R}^n$ and corresponding canonical coordinates $\xi\in\mathbb{R}^n$ on the fibers of T^*M, we have $\pi_*(\sum_k a_k\partial_{x_k}+b_k\partial_{\xi_k})=\sum_k a_k\partial_{x_k}$, and therefore

$$\alpha=\sum_{k=1}^n \xi_k\,\mathrm{d}x_k,\quad \omega=\sum_{k=1}^n \mathrm{d}x_k\wedge\mathrm{d}\xi_k. \tag{5.154}$$

This is a non-degenerate 2-form: contraction $T(T^*M)\ni v\mapsto v\lrcorner\,\omega=\omega(v,-)\in T^*(T^*M)$ is an isomorphism, and identifies vector fields and 1-forms on T^*M:

$$\sum_{k=1}^n a_k\partial_{x_k}+b_k\partial_{\xi_k}\quad\overset{\lrcorner\omega}{\longmapsto}\quad\sum_{k=1}^n -b_k\,\mathrm{d}x_k+a_k\,\mathrm{d}\xi_k. \tag{5.155}$$

Definition 5.69 (Hamiltonian vector field)**.** Let $p\in\mathcal{C}^\infty(T^*M)$. Then the *Hamiltonian vector field* of p is the unique $H_p\in\mathcal{V}(T^*M)$ such that

$$H_p\lrcorner\,\omega=\mathrm{d}p. \tag{5.156}$$

The *Poisson bracket* of $p,q\in\mathcal{C}^\infty(T^*M)$ is defined as

$$\{p,q\} := H_p q = -H_q p. \tag{5.157}$$

In local coordinates, we deduce from (5.155) that

$$H_p=\sum_{k=1}^n(\partial_{\xi_k}p)\partial_{x_k}-(\partial_{x_k}p)\partial_{\xi_k}. \tag{5.158}$$

Thus, the 'ad hoc' definition (4.79) in fact makes invariant sense on T^*M. As a consequence of the local $\mathbb{R}^n$ theory, Proposition 4.22, we thus deduce:

Corollary 5.70 (Principal symbols of commutators)**.** *Let $A\in\Psi^m(M)$, $B\in\Psi^{m'}(M)$, at least one of which is properly supported. Then $[A,B]\in\Psi^{m+m'-1}(M)$, and*

$$\sigma^{m+m'-1}(i[A,B])=\{\sigma^m(A),\sigma^{m'}(B)\}. \tag{5.159}$$

5.14. **Exercises.**

Exercise 5.1 (Tangent vectors as directional derivatives). In the notation of Example 5.7, prove that the map (5.22) is well-defined, i.e. does not depend on the choice of coordinate system.

Exercise 5.2 (Cotangent bundle as the dual of the tangent bundle). Prove that the definition of the isomorphism (5.25) given in the subsequent paragraph is independent of the choice of the local coordinate chart.

Exercise 5.3 (Derivatives and differentials). Let $V \in \mathcal{C}^\infty(M; TM)$ denote a vector field.

(1) For a smooth function $f \in \mathcal{C}^\infty(M)$, define $(Vf)(p) := V(p)f$ for $p \in M$ as the directional derivative of f along $V(p)$ (see Example 5.7). Show that $Vf \in \mathcal{C}^\infty(M)$. Show moreover that the map $f \mapsto Vf$ is a derivation, i.e. it satisfies the Leibniz rule
$$V(fg) = fV(g) + gV(f). \tag{5.160}$$
(2) Given $f \in \mathcal{C}^\infty(M)$, note that $\mathrm{d}f \in \mathcal{C}^\infty(M, T^*M)$. Show that $\mathrm{d}f(V(p)) = V(p)f$, where the left hand side is the dual pairing between T_p^*M and T_pM (see (5.25)).

Exercise 5.4 (Quotient vector bundle). Let $E \to M$ be a vector bundle. Let $F \to M$ be a subbundle of E, i.e. a vector bundle over M with the property that $F_x \subset E_x$ for all $x \in M$. Give a construction of the quotient vector bundle $E/F \to M$ whose fibers are the quotient vector spaces E_x/F_x.

Exercise 5.5 (Normal and conormal bundles). Given a smooth submanifold $S \subset M$ of a manifold M, define its normal bundle as the quotient bundle $NS = T_SM/TS$. Show that the duality between TM and T^*M induces a duality between NS and the conormal bundle $N^*S \subset T_S^*M$ consisting of all covectors which annihilate TS.

Exercise 5.6 (Properties of density bundles). Prove Lemma 5.11. (*Hint.* One method of proof is to analyze the transition functions of the various bundles. The idea of another method is to prove the analogous statement for the vector spaces $\Omega^\alpha V$ of Remark 5.10.)

Exercise 5.7 (Pullback of densities). Prove Lemma 5.12 and the statement in Remark 5.13.

Exercise 5.8 (Distributions on manifolds).

(1) Let M be a manifold (compact or non-compact), and let $u \in \mathscr{E}'(M)$. Show that there exists $s \in \mathbb{R}$ so that $u \in H_{\mathrm{c}}^s(M)$. (Thus, $\mathscr{E}'(M) = \bigcup H_{\mathrm{c}}^s(M)$.)
(2) Suppose M is non-compact. Show that there exists $u \in \mathscr{D}'(M)$ so that $u \notin H_{\mathrm{loc}}^s(M)$ for any $s \in \mathbb{R}$. (Thus, $\mathscr{D}'(M) \supsetneq \bigcup H_{\mathrm{loc}}^s(M)$.)

Exercise 5.9 (Schwartz kernel theorem: manifold case). Prove the following generalization of Theorem 5.17: if $E \to M$ and $F \to N$ are vector bundles over the smooth manifolds M and N, then there is a one-to-one correspondence between continuous linear operators $A\colon \mathcal{C}_{\mathrm{c}}^\infty(M; E) \to \mathscr{D}'(N; F)$ and distributional Schwartz kernels $K \in \mathscr{D}'(N \times M; \pi_L^*F \otimes \pi_R^*(E^* \otimes \Omega M))$, where $\pi_L\colon N \times M \to N$ and $\pi_R\colon N \times M \to M$ are the projection maps.

Exercise 5.10 (Ps.d.o.s are properly supported modulo residual operators). Let $A \in \Psi^m(M)$. Show that there exists a properly supported operator $A_0 \in \Psi^m(M)$ with $A - A_0 \in \Psi^{-\infty}(M)$.

Exercise 5.11 (Operators on half-densities). Let M be a smooth manifold. Let $A \in \Psi^m(M; \Omega^{\frac12}M)$.

(1) What bundle is the Schwartz kernel of A a section of? Show that $A^* \in \Psi^m(M; \Omega^{\frac12}M)$.
(2) Suppose A is a classical operator. Write $a \sim \sum_{j=0}^\infty a_{m-j}$, $a_{m-j} \in S_{\mathrm{hom}}^{m-j}$, for the left symbol $a = a(x,\xi)$ of A in a local coordinate chart and corresponding local trivialization of $\Omega^{\frac12}M$. Show that not only the principal symbol a_m, but also the *subprincipal symbol*
$$a_{m-1}(x,\xi) - \frac{1}{2i}\sum_{j=1}^n \partial_{x_j}\partial_{\xi_j} a_m(x,\xi) \tag{5.161}$$

is well-defined (i.e. independent of the choice of coordinates) as a function on $T^*M \setminus o$.

Exercise 5.12 (Asymptotic summation on manifolds). Let M be a manifold, and let $m \in \mathbb{R}$.

(1) Given a sequence of symbols $a_j \in S^{m-j}(T^*M)$, $j \in \mathbb{N}_0$, show that there exists a symbol $a \in S^m(T^*M)$ so that for all $N \in \mathbb{N}_0$, we have $a - \sum_{j=0}^{N-1} a_j \in S^{m-N}(T^*M)$.
(2) Given a sequence of operators $A_j \in S^{m-j}(M)$, $j \in \mathbb{N}_0$, show that there exists a properly supported operator $A \in \Psi^m(M)$ so that for all $N \in \mathbb{N}_0$, we have $A - \sum_{j=0}^{N-1} A_j \in \Psi^{m-N}(M)$.

Exercise 5.13 (Fractional Laplacians). Let $n \in \mathbb{N}$, and let $\alpha \in \mathbb{R}$, $\alpha > -n/2$. Set $\Delta^\alpha := \mathcal{F}^{-1}|\xi|^{2\alpha}\mathcal{F}$.

(1) Show that Δ^α is a well-defined operator $\mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$.
(2) Show that Δ^α is a classical pseudodifferential operator of order 2α on $\mathbb{R}^n$ in the sense of Definition 5.21 with $M = \mathbb{R}^n$. Compute its principal symbol and show that Δ^α is elliptic.
(3) Show that Δ^α is not a uniform ps.d.o. on $\mathbb{R}^n$ in the sense of Definition 4.7 unless $\alpha \in \mathbb{N}_0$.

Exercise 5.14 (Elliptic parametrix). Give a detailed proof of the existence of elliptic parametrices on manifolds. That is, if $A \in \Psi^m(M)$ is elliptic, show that there exists a properly supported operator $B \in \Psi^{-m}(M)$ so that $A \circ B - I$, $B \circ A - I \in \Psi^{-\infty}(M)$.

Exercise 5.15 (Exterior derivative on k-forms). Let M denote a smooth manifold, and denote by

$$\mathrm{d}\colon \mathcal{C}^\infty(M;\Lambda^k T^*M) \to \mathcal{C}^\infty(M;\Lambda^{k+1}T^*M) \tag{5.162}$$

the exterior derivative. Show that $\mathrm{d} \in \mathrm{Diff}^1(M;\Lambda^k T^*M, \Lambda^{k+1}T^*M)$, and compute its principal symbol.

Exercise 5.16. Let (M, g) denote a smooth Riemannian manifold, and denote by

$$\nabla\colon \mathcal{C}^\infty(M;TM) \to \mathcal{C}^\infty(M;T^*M \otimes TM), \qquad V \mapsto (\nabla V\colon X \mapsto \nabla_X V), \tag{5.163}$$

the covariant derivative on vector fields. Show that ∇ is a first order differential operator, and compute its principal symbol.

Exercise 5.17 (A classical ps.d.o.). Let $\Gamma \subset \mathbb{C}$ be a smooth, simple, closed curve. Let $K \in \mathcal{C}^\infty(\Gamma \times \Gamma)$. Prove that

$$Au(t) := \lim_{\epsilon\to 0} \int_{|t-s|\geq\epsilon} \frac{K(t,s)}{t-s} u(s)\,\mathrm{d}s, \quad u \in \mathcal{C}^\infty(\Gamma) \tag{5.164}$$

is well-defined and defines an element $A \in \Psi^0_{\mathrm{cl}}(\Gamma)$. Here, $t, s \in \Gamma \subset \mathbb{C}$ are complex numbers, and the division here is *division by a complex number*. Compute its principal symbol.

Exercise 5.18 (Regularity of the Green's function). Let M be a smooth compact n-dimensional manifold, and fix a volume density on M. Let $A \in \Psi^m(M)$ be an elliptic ps.d.o., and assume that $A\colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is invertible. Let $p \in M$, and put $G(p, -) := A^{-1}(\delta_p)$. Determine the set of all $s \in \mathbb{R}$ so that $G(p, -) \in H^s(M)$.

Exercise 5.19 (Properties of Sobolev spaces). Let M be a smooth n-dimensional manifold, and let $s > \frac{n}{2}$. This exercise builds on Exercises 2.4 and 2.5.

(1) Let $k \in \mathbb{N}_0$ be such that $s - \frac{n}{2} > k$. Show that $H^s_{\mathrm{loc}}(M) \hookrightarrow \mathcal{C}^k(M)$.
(2) If $u, v \in H^s_{\mathrm{loc}}(M)$, show that $uv \in H^s_{\mathrm{loc}}(M)$.
(3) When M is compact, show that for any fixed choice of norm on $H^s(M)$ there exists a constant C_s so that $\|uv\|_{H^s(M)} \leq C_s \|u\|_{H^s(M)} \|v\|_{H^s(M)}$ for all $u, v \in H^s(M)$.

Exercise 5.20 (Fredholm estimates). Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be two Banach spaces, and suppose $A\colon X \to Y$ is a bounded linear map.

(1) Suppose Z is another Banach space, and there is an inclusion (continuous injective map) $X \hookrightarrow Z$ which is compact. Suppose there exists a constant $C > 0$ such that

$$\|u\|_X \leq C\left(\|Au\|_Y + \|u\|_Z\right). \tag{5.165}$$

Show that $\ker A \subset X$ is finite-dimensional, and that $\operatorname{ran} A \subset Y$ is closed.
(2) Suppose that, in addition, to (1), there exists a Banach space $\tilde{Z}$ and an inclusion $Y^* \hookrightarrow \tilde{Z}$ which is compact. Suppose there exists $C > 0$ such that

$$\|v\|_{Y^*} \leq C\left(\|A^* v\|_{X^*} + \|v\|_{\tilde{Z}}\right). \tag{5.166}$$

Show that if $f \in Y$ is such that $v(f) = 0$ for all $v \in \ker A^*$, then there exists $u \in X$ with $Au = f$. Deduce that A is a Fredholm operator.

Exercise 5.21 (Elliptic estimates). Let M be a compact manifold, let $E, F \to M$ denote two vector bundles, and let $A \in \Psi^m(M; E, F)$ be an elliptic operator. Show that for all $s, N \in \mathbb{R}$, there exists a constant $C \in \mathbb{R}$ so that

$$\|u\|_{H^s(M;E)} \leq C\Big(\|Au\|_{H^{s-m}(M;F)} + \|u\|_{H^{-N}(M;E)}\Big), \qquad u \in H^s(M;E). \tag{5.167}$$

Show that this estimate holds in the strong sense that if the terms on the right hand side are well-defined and finite, then the left hand side is finite and the estimate holds.

Exercise 5.22 (Principal symbol via oscillatory testing). Let M be a smooth manifold, and let $A \in \mathrm{Diff}^m(M)$. Let $x_0 \in M$ and $0 \neq \xi_0 \in T^*_{x_0}M$.

(1) Show that there exists a smooth function $u \in \mathcal{C}^\infty(M)$ with $(\mathrm{d}u)(x_0) = \xi_0$.
(2) Prove that $\sigma^m(A)(x_0, \xi_0) = \lim_{\lambda \to \infty} \lambda^{-m} e^{-i\lambda u(x_0)} A(e^{i\lambda u})(x_0)$.
(3) State and prove analogous results for $A \in \mathrm{Diff}^m(M; E, F)$ where $E, F \to M$ are two vector bundles.
(4) Prove analogous results for $A \in \Psi^m_{\mathrm{cl}}(M; E, F)$. (*Hint.* This requires the use of the *stationary phase lemma*, which we do not discuss in these notes.)

Exercise 5.23 (Ellipticity and the Fredholm property). Let M be a smooth manifold, and let $A \in \mathrm{Diff}^m(M)$. Show that A is elliptic if and only if $A\colon H^m(M) \to H^0(M)$ is Fredholm. (*Hint.* If A is Fredholm, prove the validity of an estimate (5.167) for A. Plug in highly oscillatory functions, as in the previous exercise, multiplied with cutoff functions to neighborhoods of points in M, into this estimate to conclude that $\sigma^m(A)$ is injective. Argue similarly for A^*.)

Exercise 5.24 (Over- and underdetermined elliptic operators: I). Let M be a compact manifold, let $E, F \to M$ denote two vector bundles, and let $A \in \Psi^m(M; E, F)$.

(1) Suppose there exists a symbol $b \in S^{-m}(T^*M; \mathrm{Hom}(F, E))$ such that $b\sigma^m(A) - 1 \in S^{-1}(T^*M; \mathrm{End}(E))$. (If A is not elliptic, one says in this case that A is *overdetermined elliptic*.) Show that $A\colon H^s(M;E) \to H^{s-m}(M;F)$ has finite-dimensional kernel and closed range.

(2) Suppose $\sigma^m(A)$ there exists $b \in S^{-m}(T^*M; \mathrm{Hom}(F,E))$ such that $\sigma^m(A)b - 1 \in S^{-1}(T^*M; \mathrm{End}(F))$. (If A is not elliptic, one says in this case that A is *underdetermined elliptic*.) Show that $A\colon H^s(M;E) \to H^{s-m}(M;F)$ has closed range and finite-dimensional cokernel.

(3) Show that if A has a homogeneous principal symbol $\sigma^m(A)$ (so in particular when A is a classical operator), the assumption in part (1) is equivalent to the injectivity of $\sigma^m(A)$ on $T^*M \setminus o$, and the assumption in part (2) to the surjectivity.

Exercise 5.25 (Over- and underdetermined elliptic operators: II). Let M be a compact manifold, let $E, F \to M$ denote two vector bundles, and let $A \in \Psi^m(M;E,F)$ be elliptic, or over- or underdetermined elliptic. Fix a positive smooth density on M and fiber inner products on E, F.

(1) Establish the $L^2(M;E)$-orthogonal splitting
$$L^2(M;E) = \ker_{L^2(M;E)} A \oplus A^*(H^m(M;F)). \tag{5.168}$$

(2) For $s \in \mathbb{R}$, show that $H^s(M;E) = \ker_{H^s(M;E)} A \oplus A^*(H^{s+m}(M;F))$. (*Hint.* When A is underdetermined elliptic, use the decomposition of the first part. When A is overdetermined elliptic, work with the elliptic operator A^*A.)

(3) Show that $\mathcal{C}^\infty(M;E) = \ker_{\mathcal{C}^\infty(M;E)} A \oplus A^*(\mathcal{C}^\infty(M;F))$. That is, show that every $u \in \mathcal{C}^\infty(M;E)$ can be written as $u = u_0 + A^*u_1$ for smooth u_0, u_1, with u_0 and A^*u_1 unique, and prove that $A^*(\mathcal{C}^\infty(M;F)) \subset \mathcal{C}^\infty(M;E)$ is closed.

Exercise 5.26 (Underdetermined elliptic operators: I). Let M be a compact manifold, let $E, F \to M$ denote two vector bundles, and let $A \in \mathrm{Diff}^m(M;E,F)$ be underdetermined elliptic, i.e. $\sigma^m(A)$ is surjective but not injective.

(1) Show that there exists a constant $C \in \mathbb{R}$ so that for all $u \in A^*(H^{s+m}(M;F))$ one has $\|u\|_{H^s(M;E)} \le C\|Au\|_{H^{s-m}(M;F)}$.

(2) Show that $\ker_{H^s(M;E)} A$ is infinite-dimensional. (*Hint.* If this were false, show the validity of an estimate
$$\|u\|_{H^s(M;E)} \le C(\|Au\|_{H^{s-m}(M;F)} + \|u\|_{H^{-N}(M;E)}) \tag{5.169}$$
and use this to deduce that $\sigma^m(A)$ is injective.)

(3) Show that $\ker_{\mathcal{C}^\infty(M;E)} A$ is infinite-dimensional. (*Hint.* Use the last part of the previous exercise and, assuming that $\ker_{\mathcal{C}^\infty(M;E)} A$ is finite-dimensional, prove the validity of the estimate (5.169) for smooth u.)

Exercise 5.27 (Helmholtz decomposition). Let (M,g) be a compact Riemannian manifold, denote by $\mathrm{d}\colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M;T^*M)$ the exterior derivative acting on functions, and denote by $\delta_g = \mathrm{d}^*$ its adjoint. Let $\omega \in H^s(M;T^*M)$ be a 1-form. Prove that there exist $u \in H^{s+1}(M)$ and $\eta \in H^s(M;T^*M)$ such that
$$\omega = \mathrm{d}u + \eta, \quad \delta_g\eta = 0. \tag{5.170}$$
(Note that d and δ_g are first order differential operators with smooth coefficients, and hence they do act on distributions valued in the appropriate bundles.)

Exercise 5.28 (Elliptic complexes). Let M be a compact manifold, let $E_i \to M$, $i = 0, \ldots, N$, be complex vector bundles, and suppose $\mathrm{d}_i \in \mathrm{Diff}^1(M;E_i,E_{i+1})$, $i = 0, \ldots, N-1$. Suppose they form a *complex of differential operators*
$$\mathcal{C}^\infty(M;E_0) \xrightarrow{\mathrm{d}_0} \mathcal{C}^\infty(M;E_1) \xrightarrow{\mathrm{d}_1} \cdots \xrightarrow{\mathrm{d}_{N-1}} \mathcal{C}^\infty(M;E_N); \tag{5.171}$$

that is, for each $i < N$,

$$\mathrm{d}_{i+1} \circ \mathrm{d}_i = 0 \in \mathrm{Diff}^2(M; E_i, E_{i+2}). \tag{5.172}$$

Assume moreover that this complex is *elliptic*, meaning that the symbol complex

$$\mathcal{C}^\infty(T^*M \setminus o; \pi^* E_0) \xrightarrow{\sigma^1(\mathrm{d}_0)} \mathcal{C}^\infty(T^*M \setminus o; \pi^* E_1) \xrightarrow{\sigma^1(\mathrm{d}_1)} \cdots \xrightarrow{\sigma^1(\mathrm{d}_{N-1})} \mathcal{C}^\infty(T^*M \setminus o; \pi^* E_N) \tag{5.173}$$

is exact (that is, $\mathrm{ran}\, \sigma^1(\mathrm{d}_{i-1})(x,\xi) = \ker \sigma^1(\mathrm{d}_i)$ for all $i < N$). The goal of this exercise is to study the cohomology groups

$$H^i(E_\bullet) := (\ker \mathrm{d}_i)/(\mathrm{ran}\, \mathrm{d}_{i-1}), \quad i = 1, \ldots, N-1, \tag{5.174}$$

using PDE theory.

(1) Equip M with a volume density and the E_i with Hermitian fiber inner products; define $\delta_i \in \mathrm{Diff}^1(M; E_i, E_{i-1})$ to be the adjoint of d_{i-1}. Show that the 'Laplacian'

$$\Delta_i := \mathrm{d}_{i-1} \circ \delta_i + \delta_{i+1} \circ \mathrm{d}_i \in \mathrm{Diff}^2(M; E_i), \quad 1 \leq i \leq N-1, \tag{5.175}$$

is elliptic and symmetric.

(2) Show that

$$\ker \Delta_i = \{u \in \mathcal{C}^\infty(M; E_i) \colon \mathrm{d}_i u = 0,\ \delta_i u = 0\}. \tag{5.176}$$

(3) Show that the inclusion $\ker \Delta_i \hookrightarrow \ker \mathrm{d}_i$ induces an isomorphism of vector spaces

$$\ker \Delta_i \cong H^i(E_\bullet). \tag{5.177}$$

(4) Prove the *Hodge theorem*: if (M, g) is a compact Riemannian manifold, and $\Delta_k \in \mathrm{Diff}^2(M; \Lambda^k T^*M)$ is the Hodge Laplacian on k-forms, then $\ker \Delta_k \cong H^k(M)$, where $H^k(M)$ denotes the k-th de Rham cohomology group (with complex coefficients) of M.

6. Microlocalization

We now turn to the second part of these lecture notes: finer properties of distributions, and, closely related, non-elliptic phenomena. We develop the notion of distributional wave front set, following [Hör71b], from the observation about the local nature of full symbolic expansions that we made e.g. after the statement of Theorem 4.16.

From now on, all ps.d.o.s shall either be properly supported, or elements of the uniform ps.d.o. algebra on $\mathbb{R}^n$.

6.1. Operator wave front set.

Recall from (4.59) and (4.61) the full symbols, modulo $S^{-\infty}$, for adjoints and compositions: if $A = \mathrm{Op}_L(a)$, $B = \mathrm{Op}_L(b)$ are ps.d.o.s on $\mathbb{R}^n$, then

$$\begin{aligned} \sigma_L(A^*)(x,\xi) &\sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar{a}(x,\xi), \\ \sigma_L(A \circ B)(x,\xi) &\sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi) \cdot D_x^\alpha b(x,\xi). \end{aligned} \tag{6.1}$$

A key feature, which so far we have only exploited at the principal symbol level, is that these formulas are *local* in (x, ξ). We would like to say that if A is 'trivial' at or near (x, ξ), in the sense that if a vanishes there, then A^* and $A \circ B$ (for any B) are trivial there as well. Unfortunately, since the expressions (6.1) are *asymptotic* sums, thus have no content at any fixed point (x,ξ), the meaning of this is not immediately clear.

The correct notion of 'triviality' must depend on the behavior of symbols as $|\xi|\to\infty$, and must be insensitive to modifications by symbols of order $-\infty$. This leads to the following definition:

Definition 6.1 (Essential support)**.** Let $a\in S^m(\mathbb{R}^n;\mathbb{R}^N)$. Then a point $(x_0,\xi_0)\in\mathbb{R}^n\times(\mathbb{R}^N\setminus\{0\})$ does *not* lie in the *essential support*

$$\operatorname{ess\,supp} a\subset\mathbb{R}^n_x\times(\mathbb{R}^N_\xi\setminus\{0\}) \tag{6.2}$$

if and only if a is a symbol of order $-\infty$ near x_0 and in a conic neighborhood of ξ_0; that is, there exists $\epsilon>0$ such that for all $\alpha\in\mathbb{N}_0^n$, $\beta\in\mathbb{N}_0^N$, $k\in\mathbb{R}$, we have

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\le C_{\alpha\beta k}\langle\xi\rangle^{-k}\quad\forall\ (x,\xi),\ |\xi|\ge 1,\ |x-x_0|+\left|\frac{\xi}{|\xi|}-\frac{\xi_0}{|\xi_0|}\right|<\epsilon. \tag{6.3}$$

It suffices, in fact, to assume (6.3) only for $\alpha=\beta=0$; the estimates for general α,β are then automatic. See Exercise 6.1.

Remark 6.2 (Simple properties of the essential support)*.* By definition, $\operatorname{ess\,supp} a$ is a *closed* subset of $\mathbb{R}^n\times(\mathbb{R}^N\setminus\{0\})$. Moreover, $\operatorname{ess\,supp} a$ is *conic* in ξ, that is, $(x,\xi)\in\operatorname{ess\,supp} a$ implies $(x,\lambda\xi)\in\operatorname{ess\,supp} a$ for all $\lambda>0$.

Definition 6.3 (Operator wave front set: Euclidean case)**.** Let $A=\mathrm{Op}_L(a)$. Then we define the *operator wave front set of* A as the closed, conic set

$$\mathrm{WF}'(A):=\operatorname{ess\,supp}(a)\subset\mathbb{R}^n_x\times(\mathbb{R}^n_\xi\setminus\{0\}). \tag{6.4}$$

The following follows immediately from (6.1), the formulas for left/right reductions in (4.35)–(4.36), as well as formula (5.13) in the proof of the local coordinate invariance of ps.d.o.s:

Proposition 6.4 (Properties of the operator wave front set for ps.d.o.s on $\mathbb{R}^n$)**.** *The operator wave front set for operators $A,B\in\Psi(\mathbb{R}^n)$ has the following properties:*

(1) *Suppose A has compactly supported Schwartz kernel. Then $\mathrm{WF}'(A)=\emptyset$ if and only if $A\in\Psi^{-\infty}(\mathbb{R}^n)$.*[10]
(2) *Let $A=\mathrm{Op}_R(a')$. Then $\mathrm{WF}'(A)=\operatorname{ess\,supp} a'$.*
(3) $\mathrm{WF}'(A+B)\subset\mathrm{WF}'(A)\cup\mathrm{WF}'(B)$.
(4) $\mathrm{WF}'(A\circ B)\subset\mathrm{WF}'(A)\cap\mathrm{WF}'(B)$.
(5) $\mathrm{WF}'(A^*)=\mathrm{WF}'(A)$.
(6) *If $\Omega,\Omega'\Subset\mathbb{R}^n$, $\kappa\colon\Omega\to\Omega'$ is a diffeomorphism, $A\in\Psi_c(\Omega')$, and $A_\kappa=\kappa^*A(\kappa^{-1})^*$, then*
$$\mathrm{WF}'(A_\kappa)=\kappa^*\mathrm{WF}'(A), \tag{6.5}$$
where we define $\kappa^*(x,\xi)=(\kappa^{-1}(x),\kappa'(x)^T\xi)$.

[10]We make the assumption on the Schwartz kernel merely to exclude scenarios where $A=\mathrm{Op}(a)$ has empty wave front set, but the constants in the estimate (6.3) blow up as $|x_0|$ gets large. An example is given by $a(x,\xi)=\chi(\langle x\rangle\langle\xi\rangle^{-1})$ where $\chi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$. Indeed, a is a uniform symbol of order 0 on $\mathbb{R}^n$, but $\mathrm{WF}'(\mathrm{Op}(a))=\emptyset$ since a is locally in x a symbol of order $-\infty$.

Of these, properties (4) and (5) were our motivation for the introduction of ess supp above. Property (6) implies that the operator wave front set can be defined invariantly for operators on manifolds:

Definition 6.5 (Operator wave front set: manifold case)**.** Let M be a manifold and $A \in \Psi^m(M)$. Then $\mathrm{WF}'(A) \subset T^*M \setminus o$ is the closed conic subset (i.e. invariant under dilations in the fibers of T^*M) given near T^*_pM, $p \in M$, by $\mathrm{WF}'(A_0)$ where $A_0 \in \Psi^m(\mathbb{R}^n)$ is the expression for A in a local coordinate system near p.

Properties (1) and (3)–(5) in Proposition 6.4 thus hold for ps.d.o.s on manifolds as well; since on general manifolds we do not impose growth restrictions on symbols outside of compact sets in the base, property (1) in fact holds without any assumption on the Schwartz kernel of A. We leave the details of the definition of $\mathrm{WF}'(A)$ for operators $A \in \Psi^m(M; E, F)$ acting between sections of vector bundles over M to the reader; in local coordinates and trivializations, a point is in $\mathrm{WF}'(A)$ if it is in the operator wave front set of at least one entry of the matrix of ps.d.o.s on $\mathbb{R}^n$ representing A locally.

One thinks of $\mathrm{WF}'(A) \subset T^*M \setminus o$ as the set in phase space where A is *microlocally non-trivial*. This is a much weaker notion than having a (microlocally) elliptic principal symbol, see §6.2.

Example 6.6. If $A \in \Psi^m(M)$ is elliptic, then $\mathrm{WF}'(A) = T^*M \setminus o$.

Example 6.7. Let $A = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha \in \mathrm{Diff}^m(\mathbb{R}^n)$. Then

$$\mathrm{WF}'(A) = \left(\bigcup_{|\alpha|\leq m} \operatorname{supp} a_\alpha \right) \times (\mathbb{R}^n \setminus \{0\}). \tag{6.6}$$

Thus, *differential* operators never have 'interesting' operator wave front set.

Example 6.8. Let $\chi \in S^m(\mathbb{R}^n_\xi)$, and consider the Fourier multiplier $A = \chi(D) := \mathrm{Op}(\chi)$. Then $\mathrm{WF}'(A) = \mathbb{R}^n_x \times \operatorname{ess\,supp} \chi$.

Example 6.9. We combine Examples 6.7 (for $m = 0$) and 6.8. Let $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ and $\chi \in S^0(\mathbb{R}^n_\xi)$. Then

$$A := \chi(D) \circ \phi(x) = \mathrm{Op}_R(\chi(\xi)\phi(y)) \in \Psi^0(\mathbb{R}^n) \tag{6.7}$$

has $\mathrm{WF}'(A) = (\operatorname{supp} \phi) \times (\operatorname{ess\,supp} \chi)$.

Working with conic sets is a bit tedious. In most circumstances, one can simplify notation by working on the *cosphere bundle*

$$S^*M := (T^*M \setminus o)/\mathbb{R}^+, \tag{6.8}$$

with fibers given by $S^*_pM = (T^*_pM \setminus o)/\mathbb{R}^+$, where $\mathbb{R}^+$ acts by dilations in the fibers. Thus, S^*M is a fiber bundle with typical fiber $\mathbb{S}^{n-1}$. We can identify conic subsets of $T^*M \setminus o$ with their image in S^*M. For instance, if $A \in \Psi^m(M)$, then for $\alpha \in S^*_pM$, the condition $\alpha \in \mathrm{WF}'(A)$ means that $(p, \xi) \in \mathrm{WF}'(A)$ where $\alpha = [\xi]$ (i.e. $\alpha = \mathbb{R}_+\xi$). Note that a compact subset of S^*M is identified with a conic subset of $T^*M \setminus o$ whose cross section (i.e. intersection with $|\xi| = 1$ for some choice of fiber metric on T^*M) is compact. The projection map is denoted

$$\pi \colon S^*M \to M. \tag{6.9}$$

The following technical result states that one can construct partitions of unity *microlocally*:

Lemma 6.10 (Microlocal partitions of unity). *Suppose $S^*M = \bigcup_i U_i$ is an open cover. Then there exist operators $A_i \in \Psi^0(M)$ such that*

(1) *the supports of the Schwartz kernels of A_i are locally finite,*
(2) $\mathrm{WF}'(A_i) \subset U_i$,
(3) $\sum_i A_i = I$.

Proof. It suffices to prove this for a locally finite refinement of the cover, which we shall denote by $\{U_i\}$ still, for which moreover each U_i lies over a coordinate chart, i.e. $U_i \subset S^*_{V_i}M$ with $F_i\colon V_i \to F_i(V_i) \subset \mathbb{R}^n$ a chart, and with $\{V_i\}$ locally finite.

Pick a partition of unity $\{\chi_i\}$ subordinate to the cover $\{U_i\}$; fix $\psi_i, \tilde\psi_i \in \mathcal{C}^\infty_c(F_i(V_i))$ such that $\psi_i \equiv 1$ near $F_i(\pi(\operatorname{supp}\chi_i))$, and $\tilde\psi_i \equiv 1$ near $\operatorname{supp}\psi_i$. We then put

$$A_i' := F_i^*\left(\tilde\psi_i \operatorname{Op}(\chi_i)\psi_i\right)(F_i^{-1})^*. \tag{6.10}$$

Then (1) and (2) are satisfied for A_i', but rather than (3) we only have $\sum_i A_i' = I - R'$, $R' \in \Psi^{-1}(M)$. Thus, simply let $B \sim \sum_{j=0}^\infty (R')^j$ and put $A_i := A_i' B$. (This still satisfies (1) and (2), in the former case since B is properly supported, and in the latter case by part (4) of Proposition 6.4.) then $\sum_i A_i = I - R$, $R \in \Psi^{-\infty}(M)$. Replacing any single one of the A_i by $A_i + R$, we are done. □

Corollary 6.11 (Microlocalizers). *Let $K \Subset U \subset S^*M$, with U open. Then there exists $A \in \Psi^0(M)$ such that $\mathrm{WF}'(A) \subset U$ and $\mathrm{WF}'(I - A) \cap K = \emptyset$.*

We say that A is *microlocally equal to I on K*.

Proof of Corollary 6.11. $S^*M = U \cup (S^*M \setminus K)$ is an open cover of S^*M, hence there exists a partition of unity $I = A + B$ with $\mathrm{WF}'(A) \subset U$ and $\mathrm{WF}'(I - A) \cap K = \mathrm{WF}'(B) \cap K = \emptyset$. □

6.2. **Elliptic set, characteristic set.** We next refine the notion of ellipticity of operators and symbols in a microlocal manner analogous to ess supp and WF′.

Definition 6.12 (Elliptic set). Let $A \in \Psi^m(M)$. Then the *elliptic set of A*,

$$\mathrm{Ell}(A) \subset T^*M \setminus o \tag{6.11}$$

(or $\mathrm{Ell}_m(A)$ if one wants to make the order explicit), consists of all $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ in a conic neighborhood of which $\sigma^m(A)$ is elliptic; that is, in local coordinates and picking a representative of $\sigma^m(A)$, there exist $c, C > 0$ and $\epsilon > 0$ such that

$$|\sigma^m(A)(x,\xi)| \geq c|\xi|^m, \quad |\xi| \geq C,\ |x - x_0| + \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| < \epsilon. \tag{6.12}$$

The complement of $\mathrm{Ell}(A)$ is the *characteristic set*

$$\mathrm{Char}(A) := (T^*M \setminus o) \setminus \mathrm{Ell}(A). \tag{6.13}$$

An equivalent definition of $\mathrm{Ell}(M)$, closer to Definition 3.8, is that there exists $b \in S^{-m}(T^*M)$ such that $\sigma^m(A)b - 1$ is a symbol of order -1 in a conic neighborhood of (x_0, ξ_0). Note that $\mathrm{Ell}(A)$ is automatically *open.* Moreover,

$$\mathrm{Ell}_{m+m'}(A \circ B) = \mathrm{Ell}_m(A) \cap \mathrm{Ell}_{m'}(B), \quad A \in \Psi^m(M),\ B \in \Psi^{m'}(M). \tag{6.14}$$

Example 6.13. Elliptic operators on M have elliptic set equal to $T^*M \setminus o$, and empty characteristic set.

Example 6.14. On $\mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x$, with canonical momentum variables $\sigma \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, the wave operator $\Box = D_t^2 - D_x^2$ has principal symbol $\sigma^2(\Box) = \sigma^2 - \xi^2$, hence its characteristic set is the double cone

$$\mathrm{Char}(\Box) = \{(t, x, \sigma, \xi) \colon \sigma^2 - |\xi|^2 = 0,\ (\sigma, \xi) \neq (0, 0)\}. \tag{6.15}$$

The elliptic parametrix construction, Theorem 4.26, can be microlocalized:

Proposition 6.15 (Microlocal elliptic parametrix)**.** *Let $A \in \Psi^m(M)$, and suppose $K \subset \mathrm{Ell}(A)$ is a closed subset. Then there exists a* microlocal parametrix *for A on K, namely, an operator $B \in \Psi^{-m}(M)$ such that*

$$K \cap \mathrm{WF}'(AB - I) = \emptyset, \quad K \cap \mathrm{WF}'(BA - I) = \emptyset. \tag{6.16}$$

Proof. By Corollary 6.11, we can pick $Q, \tilde{Q} \in \Psi^0(M)$ with

$$\mathrm{WF}'(I - Q) \cap K = \emptyset,\ \mathrm{WF}'(Q), \mathrm{WF}'(\tilde{Q}) \subset \mathrm{Ell}(A),\ \mathrm{WF}'(I - \tilde{Q}) \cap \mathrm{WF}'(Q) = \emptyset. \tag{6.17}$$

Let then $B_0 \in \Psi^{-m}(M)$, $\sigma^{-m}(B_0) = \sigma^0(\tilde{Q})/\sigma^m(A)$, and write

$$AB_0 = I - R = I - QR - (I - Q)R, \quad R \in \Psi^0(M). \tag{6.18}$$

Now $\mathrm{WF}'((I - Q)R) \cap K = \emptyset$, while $\sigma^0(QR) = \sigma^0(Q)(1 - \sigma^0(\tilde{Q})) = 0$, so $QR \in \Psi^{-1}(M)$. We then improve the situation near K using a Neumann series argument, cf. Lemma 4.27; that is, let

$$B' \sim \sum_{j=0}^{\infty} (QR)^j \in \Psi^0(M), \tag{6.19}$$

and put $B := B_0 B' \in \Psi^{-m}(M)$. Then

$$AB = I - R' - (I - Q)RB', \quad R' \in \Psi^{-\infty}(M),\ \mathrm{WF}'((I - Q)RB') \cap K = \emptyset, \tag{6.20}$$

as desired.

A microlocal left parametrix, say $\tilde{B}$, can be constructed similarly. Then, modulo operators with WF' disjoint from K, we have

$$B \equiv (\tilde{B}A)B = \tilde{B}(AB) \equiv \tilde{B}. \tag{6.21}$$

Since (6.16) is invariant under addition to B of an operator with WF' disjoint from K, this proves that any microlocal left parametrix is also a right parametrix, and vice versa. $\Box$

One would like to use this to sharpen elliptic regularity theory, Proposition 4.28, by saying that if $Au = f$, then on $\mathrm{Ell}(A)$, u is smooth when f is. This leads to the notion of wave front set, which we discuss next.

6.3. **Wave front set of distributions.** Let $u \in \mathscr{S}'(\mathbb{R}^n)$ denote a distribution, and $A \in \Psi^m(\mathbb{R}^n)$. Then $Au \in \mathscr{S}'(\mathbb{R}^n)$ is 'trivial' outside of $\mathrm{WF}'(A)$: all information about singularities of u is lost. Indeed, if $B \in \Psi^0(\mathbb{R}^n)$ is such that $\mathrm{WF}'(B) \cap \mathrm{WF}'(A) = \emptyset$, we have $B(Au) \in \mathcal{C}^\infty(\mathbb{R}^n)$ by part (1) of Proposition 6.4. The precise notion of 'triviality' here is, directly stated on manifolds:

Definition 6.16 (Wave front set)**.** Let $u \in \mathscr{D}'(M)$. Then $\alpha \in S^*M$ does *not* lie in the *wave front set*,

$$\alpha \notin \mathrm{WF}(u) \subset S^*M \tag{6.22}$$

if and only if there exists a neighborhood $U \subset S^*M$ of α such that

$$A \in \Psi^0(M),\ \mathrm{WF}'(A) \subset U \implies Au \in \mathcal{C}^\infty(M). \tag{6.23}$$

By definition, $\mathrm{WF}(u) \subset S^*M$ is closed. We leave to the reader the simple verification that for $M = \mathbb{R}^n$, $\mathrm{WF}(u)$ for $u \in \mathscr{S}'(\mathbb{R}^n)$ can be defined equivalently by testing with uniform ps.d.o.s $A \in \Psi^0(\mathbb{R}^n)$.

We will give equivalent conditions which are easier to verify. We begin by reducing the number of operators for which one needs to check (6.23) to *one*.

Lemma 6.17 (Wave front set: equivalent definition #1)**.** *Let $u \in \mathscr{D}'(M)$. Then $\alpha \notin \mathrm{WF}(u)$ if and only if there exists $A \in \Psi^0(M)$, elliptic at α, such that $Au \in \mathcal{C}^\infty(M)$.*

Proof. The direction '⇒' is obvious. To prove '⇐', we take $U := \mathrm{Ell}(A)$, which by assumption is a neighborhood of α. Let $B \in \Psi^0(M)$, $\mathrm{WF}'(B) =: K \subset U$; we claim that $Bu \in \mathcal{C}^\infty(M)$. By Proposition 6.15, there exists a microlocal parametrix $Q \in \Psi^0(M)$ of A with $QA = I - R$, $R \in \Psi^0(M)$, $\mathrm{WF}'(R) \cap K = \emptyset$. Therefore,

$$Bu = B(QA + R)u = (BQ)(Au) + BRu \in \mathcal{C}^\infty(M). \tag{6.24}$$

Indeed, $Au \in \mathcal{C}^\infty(M)$, hence the first summand is smooth; and $\mathrm{WF}'(BR) \subset \mathrm{WF}'(B) \cap \mathrm{WF}'(R) \subset K \cap \mathrm{WF}'(R) = \emptyset$, hence $BR \in \Psi^{-\infty}(M)$ and so $BRu \in \mathcal{C}^\infty(M)$ as well. □

Corollary 6.18 (Wave front set: equivalent definition #2)**.** *Let $u \in \mathscr{D}'(M)$. Then*

$$\mathrm{WF}(u) = \bigcap_{\substack{A \in \Psi^0(M) \\ Au \in \mathcal{C}^\infty(M)}} \mathrm{Char}(A). \tag{6.25}$$

Proof. If $\alpha \notin \mathrm{WF}(u)$, then $Au \in \mathcal{C}^\infty(M)$ for some $A \in \Psi^0(M)$ with $\alpha \in \mathrm{Ell}(A)$, so $\alpha \notin \mathrm{Char}(A)$. Conversely, if $\alpha \in \mathrm{Ell}(A)$ for some $A \in \Psi^0(M)$ with $Au \in \mathcal{C}^\infty(M)$, then $\alpha \notin \mathrm{WF}(u)$ by Lemma 6.17. □

This leads to the following very concrete description of the wave front set, which we state directly on $\mathbb{R}^n$; it is the same on manifolds upon localizing in a chart and transferring to $\mathbb{R}^n$.

Proposition 6.19 (Wave front set: equivalent definition #3)**.** *Let $u \in \mathscr{D}'(\mathbb{R}^n)$, and let $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Then $(x_0, \xi_0) \notin \mathrm{WF}(u)$ if and only if there exist $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, $\phi(x_0) \neq 0$, and $\epsilon > 0$ such that for all $N \in \mathbb{R}$ we have*

$$|\widehat{\phi u}(\xi)| \leq C_N |\xi|^{-N}, \quad \xi \in \mathbb{R}^n,\ |\xi| \geq 1,\ \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \epsilon. \tag{6.26}$$

Remark 6.20 (A mild but useful strengthening)*.* The converse direction can be strengthened slightly: to show that $(x_0, \xi_0) \in \mathrm{WF}(u)$ *does* lie in the wave front set, it suffices to show that for any $\epsilon > 0$ there *exists* $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, $\phi(x_0) \neq 0$, $\operatorname{supp} \phi \subset B(x_0, \epsilon)$, such that the estimate (6.26) fails. (That is, as witnesses one can take any convenient cutoffs ϕ with support arbitrarily close to x_0.) Indeed, this follows from the definition of WF and (the proof of) Lemma 6.17.

An advantage of our invariant approach to WF is that it implies 'for free' that the hands-on condition (6.26) gives a well-defined notion of wave front set as a subset of the *cotangent bundle*. We encourage the reader to try and give a direct proof of this fact, based on the characterization (6.26).

Proof of Proposition 6.19. Suppose (6.26) holds. Let $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ have support in an ϵ-ball around $\xi_0/|\xi_0|$, with $\psi(\frac{\xi_0}{|\xi_0|}) \neq 0$. Let moreover $\chi \in \mathcal{C}^\infty(\mathbb{R}^n_\xi)$, $\chi(\xi) = 0$ for $|\xi| \leq 1$ and $\chi(\xi) = 1$ for $|\xi| \geq 2$. Then

$$a(\xi, y) := \chi(\xi)\psi\left(\frac{\xi}{|\xi|}\right)\phi(y) \in S^0(\mathbb{R}^n; \mathbb{R}^n) \tag{6.27}$$

and $|\mathcal{F}(\mathrm{Op}_R(a)u)(\xi)| \leq C_N\langle\xi\rangle^{-N}$; therefore $\mathrm{Op}_R(a)u \in \mathcal{C}^\infty(\mathbb{R}^n)$.[11] Since a is elliptic at (x_0, ξ_0), this implies $(x_0, \xi_0) \notin \mathrm{WF}(u)$.

Conversely, if $(x_0, \xi_0) \notin \mathrm{WF}(u)$, pick $B \in \Psi^0(\mathbb{R}^n)$, elliptic at (x_0, ξ_0), such that $Bu \in \mathcal{C}^\infty(\mathbb{R}^n)$. We can then choose $\phi \in \mathcal{C}^\infty_\mathrm{c}(\mathbb{R}^n)$, $\phi(x_0) \neq 0$, and $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, $\psi(\frac{\xi_0}{|\xi_0|}) \neq 0$, such that for $a(\xi, y)$ defined in (6.27), and $A := \mathrm{Op}_R(a)$, we have

$$\mathrm{WF}'(A) \subset \mathrm{Ell}(B). \tag{6.28}$$

(Cf. Example 6.9.) By the proof of Lemma 6.17, we thus have $Au \in \mathcal{C}^\infty(\mathbb{R}^n)$. We claim that in fact

$$Au \in \mathscr{S}(\mathbb{R}^n), \tag{6.29}$$

which proves (6.26) upon taking the inverse Fourier transform of Au. To prove (6.29), let $\tilde\phi \in \mathcal{C}^\infty_\mathrm{c}(\mathbb{R}^n)$ be identically 1 near $\operatorname{supp}\phi$. Then $\tilde\phi(Au) \in \mathcal{C}^\infty_\mathrm{c}(\mathbb{R}^n)$, while $(1-\tilde\phi)Au = \mathrm{Op}(a')u$ where

$$a'(x, y, \xi) = (1 - \tilde\phi(x))a(\xi, y) \in S^0(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n). \tag{6.30}$$

But $a'(x, y, \xi) = 0$ near $x = y$! In fact, a' is a scattering symbol of order $(0, 0, 0)$ vanishing near the diagonal, hence $\mathrm{Op}(a') \in \Psi^{-\infty,-\infty}_\mathrm{sc}(\mathbb{R}^n)$ has Schwartz kernel in $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ by Exercises 4.9–4.10, which implies $\mathrm{Op}(a')u \in \mathscr{S}(\mathbb{R}^n)$.[12] □

Another important consequence of Lemma 6.17 and Corollary 6.18 is the following result which shows that WF is a significant refinement of sing supp:

Theorem 6.21 (Wave front set and singular support)**.** *Let $u \in \mathscr{D}'(M)$, and denote by $\pi\colon T^*M \to M$ the projection. Then*

$$\pi(\mathrm{WF}(u)) = \operatorname{sing\,supp} u. \tag{6.31}$$

Proof. If $x_0 \notin \operatorname{sing\,supp} u$, then there exists $\chi \in \mathcal{C}^\infty_\mathrm{c}(M)$ with $\chi(x_0) \neq 0$ such that $\chi u \in \mathcal{C}^\infty_\mathrm{c}(M)$. But χ is elliptic at (x_0, ξ_0) for any $0 \neq \xi_0 \in T^*_{x_0}M$; hence $T^*_{x_0}M \cap \mathrm{WF}(u) = \emptyset$.

[11] In fact, the Fourier transform of $\phi u \in \mathscr{E}'(\mathbb{R}^n)$ is analytic and polynomially bounded. Using Cauchy's integral formula, or Exercise 6.1 if one wants to stick to real methods, one then shows that the estimate (6.26) holds for all derivatives $\partial_\xi^\alpha\widehat{\phi u}(\xi)$, $\alpha \in \mathbb{N}_0^n$, as well. Thus $\mathrm{Op}_R(a)u \in \mathscr{S}(\mathbb{R}^n)$.

[12] A direct proof proceeds by writing $\mathrm{Op}(a') = \mathrm{Op}(|x-y|^{-2N}\Delta_\xi^N a')$ and noting that $|x-y|^{-2N} \lesssim \langle x\rangle^{-N}\langle y\rangle^{-N}$ on $\operatorname{supp} a'$ as well as $\Delta_\xi^N a' \in S^{-2N}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. Thus, mimicking the proof of Proposition 4.10 gives $\mathrm{Op}(a') \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$.

To prove the converse, suppose $x_0 \notin \pi(\mathrm{WF}(u))$. Then for each $\xi \in S^*_{x_0}M$, there exists $A_\xi \in \Psi^0(M)$, elliptic at ξ, such that $Au \in \mathcal{C}^\infty(M)$. Let $U_\xi := \mathrm{Ell}(A_\xi) \cap S^*_{x_0}M$. Then U_ξ is an open cover of the *compact* set $S^*_{x_0}M$; thus we can pick a finite subcover,

$$S^*_{x_0}M = \bigcup_{i=1}^N U_{\xi_i}, \quad \xi_i \in S^*_{x_0}M,\ i = 1, \ldots, N. \tag{6.32}$$

But then the operator

$$A := \sum_{i=1}^N A^*_{\xi_i} A_{\xi_i} \in \Psi^0(M) \tag{6.33}$$

is elliptic on $S^*_{x_0}M$, and satisfies $Au \in \mathcal{C}^\infty(M)$. If $\chi \in \mathcal{C}^\infty_c(M)$, $\chi(x_0) \neq 0$, is chosen to have support so close to x_0 such that $S^*_{\operatorname{supp}\chi}M \subset \mathrm{Ell}(A)$, then $\chi u \in \mathcal{C}^\infty(M)$ by the proof of Lemma 6.17. □

Corollary 6.22 (Wave front set and smoothness)**.** *Let $u \in \mathscr{D}'(M)$. Then $\mathrm{WF}(u) = \emptyset$ if and only if $u \in \mathcal{C}^\infty(M)$.*

It is now time to give some examples:

Example 6.23. Let $\delta \in \mathscr{D}'(\mathbb{R}^n)$. We claim that $\mathrm{WF}(\delta) = \{(0, \xi) : \xi \neq 0\} = N^*\{0\} \setminus o$. There are several ways to see this. For instance:

1. Using Proposition 6.19: since $\operatorname{supp}\delta = \{0\}$, $\mathrm{WF}(\delta)$ can *at most* be equal to $\{(0, \xi)\}$. But given a cutoff $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi(0) \neq 0$, we have $\widehat{\phi\delta}(\xi) = \phi(0)$, which is not rapidly decreasing in the conic neighborhood of any $\xi_0 \neq 0$; hence the claim.
2. Ad hoc argument: since $\operatorname{sing\,supp}\delta = \{0\}$, $\mathrm{WF}(\delta)$ can *at most* be equal to $\{(0, \xi)\}$ by Corollary 6.22. But $\delta \notin \mathcal{C}^\infty(\mathbb{R}^n)$, hence $\mathrm{WF}(\delta) \neq \emptyset$. But δ is rotationally symmetric, hence $(0, \xi) \in \mathrm{WF}(\delta)$ implies $(0, R\xi) \in \mathrm{WF}(\delta)$ for all $R \in SO(n-1)$, and we are done.

Example 6.24. Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain. Then $\mathrm{WF}(1_\Omega) = N^*\partial\Omega \setminus o$. (See Exercise 6.2.)

Example 6.25. Consider $(x + i0)^{-1} = \lim_{\epsilon\to 0}(x + i\epsilon)^{-1} \in \mathscr{D}'(\mathbb{R})$. Then $\mathrm{WF}((x + i0)^{-1}) = \{(0, \xi) : \xi > 0\}$. This can be proved very explicitly using $\mathcal{F}((x+i0)^{-1}) = (2\pi i)^{-1}H$, where H is the Heaviside function. (This equality is proved easily by calculating the inverse Fourier transform of H as the $\mathscr{S}'(\mathbb{R})$-limit of that of $H(x)e^{-\epsilon x}$ as $\epsilon \searrow 0$.) See also Exercise 6.3.

Example 6.26. If $A \in \Psi^m(\mathbb{R}^n)$ is a ps.d.o. with Schwartz kernel K, then

$$\mathrm{WF}(K) = \{(x, x, \xi, -\xi) : (x, \xi) \in \mathrm{WF}'(A)\}. \tag{6.34}$$

See Exercise 6.6.

We next study the relationship of wave front sets and PDE. We first prove:

Proposition 6.27 (Microlocality of pseudodifferential operators)**.** *Let $A \in \Psi^m(M)$ and $u \in \mathscr{D}'(M)$. Then*

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}(u). \tag{6.35}$$

In view of Theorem 6.21, this is a significant strengthening of the pseudolocality property of ps.d.o.s, see Proposition 4.17 (which holds on manifolds as well).

Proof of Proposition 6.27. Suppose $\alpha \notin \mathrm{WF}'(A)$, then there exists $B \in \Psi^0(M)$, elliptic at α, but with $\mathrm{WF}'(B) \cap \mathrm{WF}'(A) = \emptyset$. Thus $B(Au) = (BA)u \in \mathcal{C}^\infty(M)$ since $BA \in \Psi^{-\infty}(M)$.

If $\alpha \notin \mathrm{WF}(u)$, then there exists $B \in \Psi^0(M)$, elliptic at α, such that $Bu \in \mathcal{C}^\infty(M)$. Let $\tilde{B} \in \Psi^0(M)$ be elliptic at α and with $\mathrm{WF}'(\tilde{B}) \subset \mathrm{Ell}(B)$. Let $Q \in \Psi^0(M)$ be a microlocal parametrix of B on $\mathrm{WF}'(\tilde{B})$, that is, $QB = I - R$, $R \in \Psi^0(M)$, $\mathrm{WF}'(\tilde{B}) \cap \mathrm{WF}'(R) = \emptyset$. Then

$$\tilde{B}(Au) = \tilde{B}A(QB + R)u = \tilde{B}AQ(Bu) + (\tilde{B}AR)u. \tag{6.36}$$

The first summand is smooth since Bu is; the second summand is smooth since $\tilde{B}AR \in \Psi^{-\infty}(M)$. □

Moreover, we have the following regularity result, which substantially sharpens Proposition 4.28:

Proposition 6.28 (Microlocal elliptic regularity). *Let $u \in \mathscr{D}'(M)$ and $A \in \Psi^m(M)$. Then*

$$\mathrm{WF}(u) \subset \mathrm{WF}(Au) \cup \mathrm{Char}(A). \tag{6.37}$$

In particular, if A is elliptic, then $\mathrm{WF}(u) = \mathrm{WF}(Au)$.

Proof. Suppose $\alpha \notin \mathrm{WF}(Au)$ and $\alpha \in \mathrm{Ell}(A)$. Then there exists $B \in \Psi^0(M)$, elliptic at α, such that $B(Au) \in \mathcal{C}^\infty(M)$; but $\alpha \in \mathrm{Ell}(BA)$ by (6.14), hence $\alpha \notin \mathrm{WF}(u)$ by Corollary 6.18.

The claim about elliptic A follows from (6.35) and (6.37) since $\mathrm{Char}(A) = \emptyset$. □

The wave front set studied above is more specifically the *smooth wave front set* or $\mathcal{C}^\infty$ *wave front set*, as it measures the lack of smoothness of a distribution. In applications, a more refined notion is much more useful:

Definition 6.29 (H^s wave front set). Let $s \in \mathbb{R}$, $u \in \mathscr{D}'(M)$. Then the H^s *wave front set* of u is

$$\mathrm{WF}^s(u) := \bigcap_{\substack{A \in \Psi^0(M) \\ Au \in H^s_{\mathrm{c}}(M)}} \mathrm{Char}(A). \tag{6.38}$$

That is, its complement is the set of all $\alpha \in S^*M$ for which there exists $A \in \Psi^0(M)$, elliptic at α, such that $Au \in H^s_{\mathrm{c}}(M)$.

This is equivalent to the alternative definition paralleling Definition 6.16. We collect results analogous to those for the $\mathcal{C}^\infty$ wave front set; the proofs are left to the reader. (They are the same as those for the $\mathcal{C}^\infty$ wave front set, except one now one needs to keep track of Sobolev orders.) We have

$$\mathrm{WF}^s(u) = \emptyset \iff u \in H^s_{\mathrm{loc}}(M). \tag{6.39}$$

The analogue of Proposition 6.19 is the following:

Proposition 6.30 (H^s wave front set: equivalent definition). *Let $u \in \mathscr{D}'(\mathbb{R}^n)$, $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Then $(x_0, \xi_0) \notin \mathrm{WF}^s(u)$ if and only if there exists $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$, $\phi(x_0) \neq 0$, and $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, $\psi(\xi_0/|\xi_0|) \neq 0$, such that*

$$\langle \xi \rangle^s \psi\Big(\frac{\xi}{|\xi|}\Big) \widehat{\phi u}(\xi) \in L^2(\mathbb{R}^n_\xi). \tag{6.40}$$

The sharpening of Propositions 6.27 and (6.28) is:

Proposition 6.31 (Microlocality and microlocal elliptic regularity: H^s version)**.** *Let $A \in \Psi^m(M)$, $u \in \mathscr{D}'(M)$. Then*

$$\mathrm{WF}^{s-m}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}^s(u), \tag{6.41}$$

$$\mathrm{WF}^s(u) \subset \mathrm{WF}^{s-m}(Au) \cup \mathrm{Char}(A). \tag{6.42}$$

In particular, if A is elliptic, then $\mathrm{WF}^s(u) = \mathrm{WF}^{s-m}(Au)$.

Proof. See Exercise 6.8. □

The more precise way of stating the *qualitative* statement (6.42) of microlocal elliptic regularity is the following *quantitative estimate*,[13] stated on a compact manifold for convenience: if $B, G \in \Psi^0(M)$ are such that

$$\mathrm{WF}'(B) \subset \mathrm{Ell}(G), \quad \mathrm{WF}'(B) \subset \mathrm{Ell}(A), \tag{6.43}$$

then for any $N \in \mathbb{R}$, there exists $C > 0$ such that

$$\|Bu\|_{H^s(M)} \leq C \left(\|GAu\|_{H^{s-m}(M)} + \|u\|_{H^{-N}(M)} \right); \tag{6.44}$$

and this estimate holds in the strong sense that if $u \in \mathscr{D}'(M)$ is such that the right hand side is finite, then so is the left hand side, and the estimate holds.[14] (Thus, this is *better* than an a priori estimate, as microlocal H^s-membership of u is *concluded*, with estimates—rather than merely assumed and estimated.)

We end with recording the relationship between H^s and $\mathcal{C}^\infty$ wave front set:

Proposition 6.32 ($\mathcal{C}^\infty$ and H^s wave front sets)**.** *Let $u \in \mathscr{D}'(M)$. Then*

$$\mathrm{WF}(u) = \overline{\bigcup_{s \in \mathbb{R}} \mathrm{WF}^s(u)}. \tag{6.45}$$

It is easy to see that $\bigcup_{s \in \mathbb{R}} \mathrm{WF}^s(u)$ is, in general, a proper subset of $\mathrm{WF}(u)$.

Proof of Proposition 6.32. Clearly $\mathrm{WF}^s(u) \subset \mathrm{WF}(u)$, implying '$\supseteq$'. For the converse, suppose $\alpha \in S^*M$ has an open neighborhood $U \subset S^*M$ such that $U \cap \mathrm{WF}^s(u) = \emptyset$ for all $s \in \mathbb{R}$. Then if $A \in \Psi^0(M)$, $\mathrm{WF}'(A) \subset U$, is elliptic at α and has compactly supported Schwartz kernel, then $Au \in \bigcap_{s \in \mathbb{R}} H^s_{\mathrm{c}}(M) = \mathcal{C}^\infty_{\mathrm{c}}(M)$, hence $\alpha \notin \mathrm{WF}(u)$. □

6.4. **Pairings, products, restrictions.** The wave front set allows one to give fairly precise answers to questions such as: when is the product of two distributions well-defined? When can distributions be restricted to submanifolds? *For notational simplicity, we work on $\mathbb{R}^n$*, but all results have analogues on manifolds.

We first consider generalizations of the $L^2(\mathbb{R}^n)$ inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^n} u(x) \overline{v(x)} \, \mathrm{d}x. \tag{6.46}$$

[13] One can in fact recover the estimate from (6.42) using the closed graph theorem, though this loses the (in principle) explicit nature of the constant C as depending on seminorms of A.

[14] On a non-compact manifold, this holds if one takes B, G with Schwartz kernels supported in $K \times K$, $K \Subset M$, and upon replacing the final, error term by $\|\chi u\|_{H^{-N}}$ where $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(M)$ is identically 1 near K.

Proposition 6.33 (L^2 pairings with wave front set conditions)**.** *Suppose* $u, v \in \mathscr{E}'(\mathbb{R}^n)$ *satisfy* $\mathrm{WF}(u) \cap \mathrm{WF}(v) = \emptyset$*. If* $A \in \Psi^0(\mathbb{R}^n)$ *is such that*

$$\mathrm{WF}(u) \cap \mathrm{WF}'(A) = \emptyset, \quad \mathrm{WF}(v) \cap \mathrm{WF}'(I - A) = \emptyset, \tag{6.47}$$

then the sesquilinear form

$$\langle u, v\rangle := \langle Au, v\rangle + \langle u, (I - A^*)v\rangle \tag{6.48}$$

is independent of the choice of A*.*

Proof. Note that (6.48) is well-defined since $Au, (I - A^*)v \in \mathcal{C}^\infty(\mathbb{R}^n)$ by microlocality, Proposition 6.27.

Suppose $B \in \Psi^0(\mathbb{R}^n)$ satisfies the conditions on A in (6.47). Then

$$\langle u, v\rangle' := \langle Bu, v\rangle + \langle u, (I - B^*)v\rangle \tag{6.49}$$

is well-defined, too, and we want to show that the difference

$$\langle u, v\rangle' - \langle u, v\rangle = \langle (A - B)u, v\rangle - \langle u, (A^* - B^*)v\rangle \tag{6.50}$$

vanishes. If u, v were in $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$, this would be clear by integration by parts. Since u, v are merely distributions, we need to be more careful and use an approximation argument.

Thus, choose $v_j \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ such that $v_j \to v$ in $\mathscr{E}'(\mathbb{R}^n)$; then

$$\langle (A - B)u, v\rangle = \lim_{j\to\infty} \langle (A - B)u, v_j\rangle = \lim_{j\to\infty} \langle u, (A^* - B^*)v_j\rangle. \tag{6.51}$$

We have $(A^* - B^*)v_j \to (A^* - B^*)v$ in $\mathscr{D}'(\mathbb{R}^n)$; but since $u \in \mathscr{E}'(\mathbb{R}^n)$, this is not enough to naively take the limit in (6.51). Pick thus $Q \in \Psi^0(\mathbb{R}^n)$ with compactly supported Schwartz kernel, and with $\mathrm{WF}'(Q) \cap \mathrm{WF}(u) = \emptyset$ and $\mathrm{WF}'(I - Q) \cap (\mathrm{WF}'(A) \cup \mathrm{WF}'(B)) = \emptyset$, then we can further write

$$\langle u, (A^* - B^*)v_j\rangle = \langle u, (I - Q)(A^* - B^*)v_j\rangle + \langle u, Q(A^* - B^*)v_j\rangle. \tag{6.52}$$

Since $(I - Q)(A^* - B^*) \in \Psi^{-\infty}(\mathbb{R}^n)$, we have $(I - Q)(A^* - B^*)v_j \to (I - Q)(A^* - B^*)v$ with convergence in $\mathcal{C}^\infty(\mathbb{R}^n)$, hence the first pairing converges to $\langle u, (I - Q)(A^* - B^*)v\rangle$. In the second pairing, we can integrate Q by parts, and then

$$\langle Q^*u, (A^* - B^*)v_j\rangle \to \langle Q^*u, (A^* - B^*)v\rangle, \quad j \to \infty \tag{6.53}$$

since $Q^*u \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Since $(A^* - B^*)v \in \mathcal{C}^\infty(\mathbb{R}^n)$, we can move Q^* back to the second factor.

Altogether, we have proved that the limit in (6.51) is indeed equal to $\langle u, (A^* - B^*)v\rangle$, hence (6.50) vanishes, as desired. □

We state a more precise form of Proposition 6.33 which will be useful in positive commutator arguments in §§8–9. First, note that the L^2-pairing (6.46) extends to a sesquilinear pairing

$$H^s(\mathbb{R}^n) \times H^{-s}(\mathbb{R}^n) \ni (u, v) \mapsto \langle u, v\rangle, \tag{6.54}$$

defined by $\langle u, v\rangle := \langle \langle D\rangle^s u, \langle D\rangle^{-s} v\rangle$. The following is proved similarly to Proposition 6.33:

Lemma 6.34 (L^2 pairings with H^s wave front set conditions)**.** *Let* $s \in \mathbb{R}$*. Suppose* $u \in H^s(\mathbb{R}^n)$*,* $v \in \mathscr{E}'(\mathbb{R}^n)$*, and suppose that* $\mathrm{WF}(u) \cap \mathrm{WF}^{-s}(v) = \emptyset$*. Let* $A \in \Psi^0(\mathbb{R}^n)$ *be such that*

$$\mathrm{WF}(u) \cap \mathrm{WF}'(A) = \emptyset, \quad \mathrm{WF}^{-s}(v) \cap \mathrm{WF}'(I - A) = \emptyset. \tag{6.55}$$

Then the sesquilinear form $(u,v)\mapsto\langle Au,v\rangle+\langle u,(I-A^*)v\rangle$ *is independent of* A.

Remark 6.35 (L^2 pairings: manifold version). The manifold version of (6.54) is the following: fixing a smooth density μ on M, we have a pairing

$$L^2_{\mathrm{c}}(M)\times L^2_{\mathrm{loc}}(M)\ni(u,v)\mapsto\langle u,v\rangle=\int_M u(x)\overline{v(x)}\,\mathrm{d}\mu(x). \tag{6.56}$$

For $s\in\mathbb{R}$ then, fix an elliptic operator $\Lambda\in\Psi^s(M)$ with parametrix $\Lambda_-\in\Psi^{-s}(M)$, so $I=\Lambda_-\Lambda+R$ with $R\in\Psi^{-\infty}(M)$. Then the pairing

$$(u,v)\mapsto\langle\Lambda u,\Lambda_-^*v\rangle+\langle Ru,v\rangle \tag{6.57}$$

agrees with (6.56) for $u,v\in\mathcal{C}^\infty_{\mathrm{c}}(M)$, and extends by continuity to a sesquilinear pairing $H^s_{\mathrm{c}}(M)\times H^{-s}_{\mathrm{loc}}(M)\to\mathbb{C}$.

For a distribution $u\in\mathscr{D}'(\mathbb{R}^n)$, we have

$$\mathrm{WF}(u)=-\mathrm{WF}(\bar u):=\{(x,-\xi)\colon(x,\xi)\in\mathrm{WF}(u)\}. \tag{6.58}$$

We thus deduce from Proposition 6.33 that we can define a pairing

$$(u,v)=\langle u,\bar v\rangle,\quad u,v\in\mathscr{E}'(\mathbb{R}^n),\ \mathrm{WF}(u)\cap(-\mathrm{WF}(v))=\emptyset. \tag{6.59}$$

Corollary 6.36 (Product of distributions)**.** *Let* $u,v\in\mathscr{E}'(\mathbb{R}^n)$*, and suppose that*

$$\mathrm{WF}(u)\cap(-\mathrm{WF}(v))=\emptyset. \tag{6.60}$$

Then the product $uv\in\mathscr{E}'(\mathbb{R}^n)$ *given, in terms of* (6.59)*, by*

$$\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)\ni\phi\mapsto(u,\phi v), \tag{6.61}$$

is well-defined.

The condition (6.60) is of course much more precise than the condition $\operatorname{sing\,supp}u\cap\operatorname{sing\,supp}v=\emptyset$, under which the product uv can be defined easily using a partition of unity on $\mathbb{R}^n$.

Proof of Corollary 6.36. If $A\in\Psi^0(\mathbb{R}^n)$ has $\mathrm{WF}(u)\cap\mathrm{WF}'(A)=\emptyset$ and $(-\mathrm{WF}(v))\cap\mathrm{WF}'(I-A)=\emptyset$, then

$$|(u,\phi v)|\le|(Au,\phi v)|+|(u,(I-A^T)(\phi v))|. \tag{6.62}$$

Since $Au\in\mathcal{C}^\infty(\mathbb{R}^n)$, the first summand is clearly continuous in ϕ. For the second summand, choose $B\in\Psi^0(\mathbb{R}^n)$ such that $\mathrm{WF}'(B)\cap\mathrm{WF}'(I-A^T)=\emptyset$ and $\mathrm{WF}'(I-B)\cap\mathrm{WF}(v)=\emptyset$, then

$$v=Bv+w,\quad w=(I-B)v\in\mathcal{C}^\infty(\mathbb{R}^n), \tag{6.63}$$

hence $|(u,(I-A^T)w\phi)|\le C\|\phi\|_{\mathcal{C}^k(\mathbb{R}^n)}$ by the continuity of u and the ps.d.o. $(I-A^T)w\in\Psi^0(\mathbb{R}^n)$. Furthermore,

$$(I-A^T)\phi Bv\in\mathcal{C}^\infty(\mathbb{R}^n) \tag{6.64}$$

depends continuously on $\phi\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ since $(I-A^T)\phi B\in\Psi^{-\infty}(\mathbb{R}^n)$ does. □

Remark 6.37 (Topology). One can put a complete locally convex topology on the space $\mathscr{D}'_\Lambda(\mathbb{R}^n):=\{u\in\mathscr{D}'(\mathbb{R}^n)\colon\mathrm{WF}(u)\subset\Lambda\}$, where $\Lambda\subset S^*\mathbb{R}^n$ is closed, such that $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)\subset\mathscr{D}'_\Lambda(\mathbb{R}^n)$ is dense, and such that the pairing (u,v) for $u,v\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ extends by continuity to $u\in(\mathscr{D}'_\Lambda\cap\mathscr{E}')(\mathbb{R}^n)$, $v\in\mathscr{D}'_{\Lambda'}(\mathbb{R}^n)$ when $\Lambda\cap(-\Lambda')=\emptyset$.

Remark 6.38 (Sobolev refinements). One can substantially refine Corollary 6.36, e.g. by working with Sobolev spaces and assumptions on H^s wave front sets for various s. Such refinements are useful in the study of nonlinear PDE.

Lastly, we consider restrictions to submanifolds, starting with the local model

$$Y = \{(x,0)\colon x\in\mathbb{R}^k\} \subset \mathbb{R}^n = \mathbb{R}^k_x \times \mathbb{R}^{n-k}_y. \tag{6.65}$$

Denote by $\iota\colon Y \hookrightarrow \mathbb{R}^n$ the inclusion map. For $u \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, its restriction to Y is the distribution $\iota^* u$ defined by

$$\mathcal{C}^\infty_c(\mathbb{R}^k) \ni \phi \mapsto (\iota^* u)(\phi) := (u\cdot\delta(y))(\tilde\phi), \quad \tilde\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n),\ \iota^*\tilde\phi = \phi. \tag{6.66}$$

If $u \in \mathscr{E}'(\mathbb{R}^n)$ is such that

$$\mathrm{WF}(u) \cap \{(x,0,0,\eta)\colon x\in\mathbb{R}^k,\ \eta\in\mathbb{R}^{n-k}\} = \emptyset, \tag{6.67}$$

then the product $u\delta(y) \in \mathscr{E}'(\mathbb{R}^n)$ is well-defined by Corollary 6.36, and hence we get the first part of the following result:

Proposition 6.39 (Restriction of distributions to linear subspaces)**.** *Suppose $u \in \mathscr{E}'(\mathbb{R}^n)$ satisfies (6.67). Then (6.66) defines a linear restriction map, and*

$$\mathrm{WF}(\iota^* u) \subset \{(x,\xi) \in \mathbb{R}^k \times (\mathbb{R}^k\setminus\{0\})\colon \exists\, \eta\in\mathbb{R}^{n-k},\ (x,0,\xi,\eta)\in \mathrm{WF}(u)\}. \tag{6.68}$$

Proof. We have $\iota^*(\chi u) = \iota^* u$ for any $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^{n-k})$ which is identically 1 near 0. The assumption (6.67) implies that if χ has sufficiently small support, then $v = \chi u$ satisfies that

$$\hat v(\xi,\eta) \text{ is rapidly decreasing in a cone around } \{0\}\times(\mathbb{R}^{n-k}\setminus\{0\}). \tag{6.69}$$

When $u = u(x,y) \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, the Fourier inversion formula gives

$$\widehat{\iota^* u}(\xi) = (\mathcal{F}_1 u)(\xi,0) = (2\pi)^{-(n-k)} \int_{\mathbb{R}^{n-k}} \hat u(\xi,\eta)\,\mathrm{d}\eta, \tag{6.70}$$

where $\mathcal{F}_1$ denotes the Fourier transform in the first argument of u. More generally then, the property (6.69) ensures that the integral in (6.70) converges, and it computes $\widehat{\iota^* u}$ even for distributional u subject to (6.67) by a density argument.

To prove (6.68), we apply (6.70) to a localized version of u. Indeed, suppose $(x_0,\xi_0) \in \mathbb{R}^k \times (\mathbb{R}^k\setminus\{0\})$ is such that for all $\eta\in\mathbb{R}^{n-k}$, we have $(x_0,0,\xi_0,\eta)\notin \mathrm{WF}(u)$. Then for $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^k)$ with support close to x and $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^{n-k})$ with support close to 0, the Fourier transform of $\psi(x)\chi(y)u(x,y)$ is rapidly decreasing for (ξ,η) in a conic neighborhood of (ξ_0,η) for all $\eta\in\mathbb{R}^{n-k}$, as well as for (ξ,η) in a conic neighborhood of $(0,\eta)$ by (6.67). Therefore, there exists $\epsilon > 0$ such that

$$|\widehat{\psi\chi u}(\xi,\eta)| \le C_N \langle\xi\rangle^{-N}\langle\eta\rangle^{-N}, \quad \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| < \epsilon, \tag{6.71}$$

for all N. Using (6.70) for χu (which satisfies $\iota^*(\chi u) = \iota^* u$), we conclude that

$$|\widehat{\psi\iota^* u}(\xi)| \le C'_N \langle\xi\rangle^{-N}, \quad \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| < \epsilon, \tag{6.72}$$

for all N, proving that $(x_0,\xi_0)\notin \mathrm{WF}(\iota^* u)$. □

Corollary 6.40 (Restriction of distributions to submanifolds)**.** *If $\iota\colon Y\subset M$ is a smooth submanifold, then there exists a linear restriction map*

$$\iota^*\colon \{u\in\mathscr{D}'(M)\colon \mathrm{WF}(u)\cap N^*Y=\emptyset\}\to\mathscr{D}'(Y), \tag{6.73}$$

and $\mathrm{WF}(\iota^*u)\subset T^*Y\setminus o$ *is the image of* $\mathrm{WF}(u)\cap T^*_Y M$ *in* $T^*Y\cong T^*_Y M/N^*Y$.

Example 6.41. On $\mathbb{R}^2_{x,y}$, consider $u_a=\delta(y-ax)$, $a\in\mathbb{R}$. Then the restriction of u_a to $Y=\{y=0\}$ is well-defined for $a\neq 0$. We have $\iota^*(u_a)=|a|^{-1}\delta(x)$, where $\iota\colon Y\hookrightarrow\mathbb{R}^2$ is the inclusion.

One can similarly analyze the wave front sets of general pullbacks and pushforwards of distributions, and analyze the relationship between $\mathrm{WF}(Au)$ and $\mathrm{WF}(u)$ in terms of the wave front set of the Schwartz kernel of $A\colon\mathcal{C}^\infty_c(\mathbb{R}^n)\to\mathscr{D}'(\mathbb{R}^m)$. See e.g. [Hör71b, §2.5].

We have now developed the main aspects of the pseudodifferential calculus. For a partial summary of the calculus on compact manifolds, see [Wun13, §3.4].

6.5. **Exercises.**

Exercise 6.1 (Equivalent definition of essential support). Let $a\in S^m(\mathbb{R}^n;\mathbb{R}^N)$. Show that $(x_0,\xi_0)\in\mathbb{R}^n\times(\mathbb{R}^N\setminus\{0\})$ does *not* lie in $\operatorname{ess\,supp} a$ if and only if there exist $\epsilon>0$ such that for all $k\in\mathbb{R}$, we have

$$|a(x,\xi)|\le C_k\langle\xi\rangle^{-k}\quad\forall\ (x,\xi),\ |\xi|\ge 1,\ |x-x_0|+\left|\frac{\xi}{|\xi|}-\frac{\xi_0}{|\xi_0|}\right|<\epsilon. \tag{6.74}$$

(*Hint.* To obtain an estimate for derivatives of a, say $\partial_x a(x,\xi)$ in the case $n=1$, write $a(x+h,\xi)=a(x,\xi)+h\partial_x a(x,\xi)+\frac{h^2}{2}\partial_x^2 a(x+\theta h,\xi)$ where $\theta\in[0,1]$. Rewrite this as an expression for $\partial_x a(x,\xi)$ and select h suitably, depending on ξ, to prove an upper bound $|\partial_x a(x,\xi)|\le C'_l\langle\xi\rangle^{-l}$ for all l.)

Exercise 6.2 (Wave front set of characteristic functions). Let $\Omega\subset\mathbb{R}^n$ be a smoothly bounded domain, and denote by

$$1_\Omega(x)=\begin{cases}1, & x\in\Omega\\ 0, & x\notin\Omega\end{cases} \tag{6.75}$$

its characteristic function. Compute (with proof) $\mathrm{WF}(1_\Omega)$. (*Hint.* Straighten out $\partial\Omega$ locally in suitable local coordinates. Then use the characterization of Proposition 6.19 and Remark 6.20 for cleverly chosen cutoffs; alternatively, use Lemma 6.17 for some well-chosen test operators A.)

Exercise 6.3 ($\pm i0$ distributions). Let $s\in\mathbb{C}$, $s\notin\mathbb{N}_0$. Recall that the distribution $(x\pm i0)^s\in\mathscr{D}'(\mathbb{R})$ is defined as the limit

$$\langle(x\pm i0)^s,\phi\rangle:=\lim_{\epsilon\to 0}\langle(x\pm i\epsilon)^s,\phi\rangle, \tag{6.76}$$

where $(x\pm i\epsilon)^s=\exp(s\log(x\pm i\epsilon))$. (The logarithm here is the principal branch, i.e. it is real-valued for real arguments, and its branch cut is along $(-\infty,0]$.) Prove that

$$\mathrm{WF}((x\pm i0)^s)=\{(0,\xi)\colon \pm\xi>0\}. \tag{6.77}$$

(*Hint.* Show that it suffices to prove this for s with (large) negative real part. When $\operatorname{Re} s<0$, shift the contour of integration in the Fourier transform $\int_{\mathbb{R}}(x+i\epsilon)^s e^{-ix\xi}\,\mathrm{d}x$ to a line $\operatorname{Im} x=C$ and analyze what happens when you let $\epsilon\to 0$ and then $C\to\infty$.)

Exercise 6.4 (Distribution with special wave front set). Let $n \in \mathbb{N}$, $n \geq 2$. Give an example of a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ whose wave front set $\mathrm{WF}(u) \subset S^*\mathbb{R}^n$ consists of a single point.

Exercise 6.5 (Conormal distributions). Let $a \in S^m(\mathbb{R}^k_x; \mathbb{R}^{n-k}_\zeta)$.

(1) Make sense of the oscillatory integral

$$u(x,z) := (2\pi)^{-(n-k)} \int_{\mathbb{R}^{n-k}} e^{iz\cdot\zeta} a(x,\zeta)\, \mathrm{d}\zeta \tag{6.78}$$

as a distribution on $\mathbb{R}^n = \mathbb{R}^k_x \times \mathbb{R}^{n-k}_z$ in such a way that for $m < -(n-k)$, your definition agrees with the Riemann integral.

(2) Show that $\mathrm{WF}(u) \subset \{(x,z,\xi,\zeta) \colon z = 0,\ \xi = 0\}$.

(3) Prove that $\mathrm{WF}(u) = \{(x,0,0,\zeta) \colon (x,\zeta) \in \operatorname{ess\,supp} a\}$.

Exercise 6.6 (Wave front sets of Schwartz kernels of ps.d.o.s). Let $A \in \Psi^m(M)$ be a pseudodifferential operator, and denote its Schwartz kernel by K. Prove that

$$\mathrm{WF}(K) = \{(x,x,\xi,-\xi) \in T^*(M \times M) \colon (x,\xi) \in \mathrm{WF}'(A)\} \tag{6.79}$$

Exercise 6.7 (Holomorphic functions in a half space). The following is a generalization of Exercise 6.3. Denote by $\Omega = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$ the upper half plane, and let $F \colon \Omega \to \mathbb{C}$ be holomorphic. Suppose that for each $C > 0$ there exist $C', N \in \mathbb{R}$ so that $|F(z)| < C'|\operatorname{Im} z|^{-N}$ for $z \in \Omega$, $|\operatorname{Re} z| < C$, $\operatorname{Im} z \in (0,1]$.

(1) Show that the functions $F_\epsilon = F(\cdot + i\epsilon) \in \mathcal{C}^\infty(\mathbb{R})$ converge in $\mathscr{D}'(\mathbb{R})$ as $\epsilon \searrow 0$. The limit is denoted $f := F(\cdot + i0) \in \mathscr{D}'(\mathbb{R})$. (*Hint.* Write F_ϵ in terms of F_1 using the fundamental theorem of calculus for F in the imaginary direction. Using the Cauchy–Riemann equations, show in this manner that for $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ with support in $(-C,C)$, one can write $\langle F_\epsilon, \phi\rangle = \langle F^{(1)}_\epsilon, \phi'\rangle$ where $F^{(1)}$ is holomorphic in Ω and satisfies $|F^{(1)}(z)| < C'|\operatorname{Im} z|^{-N+1}$ for $|\operatorname{Re} z| < C$, $\operatorname{Im} z \in (0,1]$ when $N > 1$, or with $F^{(1)}$ continuous down to the real line when $N < 1$. Starting with general N, proceed iteratively.)

(2) Show that $\mathrm{WF}(F(\cdot + i0)) \subset \{(x,\xi) \colon \xi > 0\}$.

Exercise 6.8 (H^s wave front sets and estimates). (1) Prove Proposition 6.31.

(2) Prove the estimate (6.44).

7. Hyperbolic evolution equations

As a neat application of the ps.d.o. machinery, we now study first order systems of evolution equations; our presentation is inspired by [Tay11, §§7.7–7.8]. We work on $\mathbb{R}^n$, but all results have analogues on compact manifolds.

7.1. **Existence and uniqueness.** Consider

$$\begin{cases} D_t u = a(t,x,D_x)u + g(t,x), & t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ u(0,x) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{7.1}$$

where

$$f \in H^s(\mathbb{R}^n; \mathbb{C}^K), \quad g \in \mathcal{C}^0\left(\mathbb{R}; H^s(\mathbb{R}^n; \mathbb{C}^K)\right). \tag{7.2}$$

We let $A(t) = \mathrm{Op}(a(t,x,\xi))$, and assume that $a(t,x,\xi) \in \mathcal{C}^\infty(\mathbb{R}_t; \mathrm{Mat}^{K\times K}(S^1(\mathbb{R}^n;\mathbb{R}^n)))$ is a $K \times K$ matrix of first order symbols with smooth dependence on t; we simply write $a(t) \in S^1$. We further assume that (7.1) is *symmetric hyperbolic*, meaning

$$a(t,x,\xi) - a(t,x,\xi)^* \in S^0. \tag{7.3}$$

Theorem 7.1 (Existence and uniqueness for first order symmetric hyperbolic systems)**.** *The equation* (7.1) *with data* (7.2) *has a unique solution*

$$u \in \mathcal{C}^0\left(\mathbb{R}; H^s(\mathbb{R}^n;\mathbb{C}^K)\right) \cap \mathcal{C}^1\left(\mathbb{R}; H^{s-1}(\mathbb{R}^n;\mathbb{C}^K)\right). \tag{7.4}$$

Proof. We drop the 'bundle' $\mathbb{C}^K$ from the notation. We shall obtain u as a limit of solution u_ϵ to a regularized equation

$$\begin{cases} D_t u_\epsilon = J_\epsilon A J_\epsilon u_\epsilon + g, \\ u_\epsilon(0) = f, \end{cases} \tag{7.5}$$

where we use a *Friedrichs mollifier*

$$J_\epsilon = \phi(\epsilon D_x), \quad \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n),\ \phi(0) = 1. \tag{7.6}$$

Note that $J_\epsilon \in \Psi^{-\infty}(\mathbb{R}^n)$ for $\epsilon > 0$, and $J_\epsilon \in \Psi^0(\mathbb{R}^n)$ is uniformly bounded for $\epsilon \in (0,1]$.

For $\epsilon > 0$, $J_\epsilon A J_\epsilon$ is a smooth family of bounded operators on $H^s(\mathbb{R}^n)$, hence solvability of (7.5) with $u_\epsilon \in \mathcal{C}^1(\mathbb{R}; H^s(\mathbb{R}^n))$ follows from ODE theory. We need to establish uniform estimates on u_ϵ. Let $\Lambda^s = \langle D_x\rangle^s$. Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\epsilon(t)\|^2_{H^s} = \mathrm{Re}\langle \Lambda^s J_\epsilon i A J_\epsilon u_\epsilon, \Lambda^s u_\epsilon\rangle + \mathrm{Re}\langle \Lambda^s g, \Lambda^s u_\epsilon\rangle \tag{7.7}$$

$$= \mathrm{Re}\langle iA\Lambda^s J_\epsilon u_\epsilon, \Lambda^s J_\epsilon u_\epsilon\rangle + \mathrm{Re}\langle [\Lambda^s, iA] J_\epsilon u_\epsilon, \Lambda^s J_\epsilon u_\epsilon\rangle + \mathrm{Re}\langle \Lambda^s g, \Lambda^s u_\epsilon\rangle. \tag{7.8}$$

Since $B(t) = A(t) - A(t)^* \in \Psi^0$, the first term is equal to

$$\langle B(t)\Lambda^s J_\epsilon u_\epsilon, \Lambda^s J_\epsilon u_\epsilon\rangle \le C\|J_\epsilon u_\epsilon\|^2_{H^s} \le C\|u_\epsilon\|^2_{H^s}. \tag{7.9}$$

Since $[\Lambda^s, A] \in \Psi^s$, the second term in (7.8) is bounded by $C\|u_\epsilon\|^2_{H^s}$ as well. Applying Cauchy–Schwarz to the third term in (7.8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\epsilon(t)\|^2_{H^s} \le C\|u_\epsilon(t)\|^2_{H^s} + C\|g(t)\|^2_{H^s}. \tag{7.10}$$

By Grönwall's inequality, this implies the ϵ-independent estimate

$$\|u_\epsilon(t)\|^2_{H^s} \le C(t)\left(\|f\|^2_{H^s} + \|g\|^2_{\mathcal{C}^0([0,t];H^s(\mathbb{R}^n))}\right). \tag{7.11}$$

Therefore, for any $T > 0$ and $I = [-T, T]$,

$$u_\epsilon \in \mathcal{C}^0(I; H^s(\mathbb{R}^n)) \cap \mathcal{C}^1(I; H^{s-1}(\mathbb{R}^n)) \tag{7.12}$$

is uniformly bounded. (Boundedness in the second space follows boundedness in the first space and the equation (7.5).)

We can extract a subsequential limit of u_ϵ very easily by using the continuous injection $\mathcal{C}^1(I; H^{s-1}(\mathbb{R}^n)) \hookrightarrow H^1(I; H^{s-1}(\mathbb{R}^n))$; the latter space is a Hilbert space, so there exists a weak subsequential limit

$$u \in H^1(I; H^{s-1}(\mathbb{R}^n)) \hookrightarrow \mathcal{C}^0(I; H^{s-1}(\mathbb{R}^n)) \tag{7.13}$$

of u_ϵ. Thus, u is a weak solution of (7.1), and thus also $u \in \mathcal{C}^1(I; H^{s-2}(\mathbb{R}^n))$. Since $\delta(t) \in H^{-1/2-\epsilon}(\mathbb{R})$, we also have $u(0) = \langle u, \delta\rangle = \lim\langle u_\epsilon, \delta\rangle = f$. Uniqueness of u follows using estimates similar to (7.11) for the difference of two putative solutions.

To prove the correct regularity of u, we approximate $f \in H^s(\mathbb{R}^n)$, $g \in \mathcal{C}^0(\mathbb{R}; H^s(\mathbb{R}^n))$ in these topologies by $f_j \in H^{s+1}(\mathbb{R}^n)$, $g_j \in \mathcal{C}^0(\mathbb{R}; H^{s+1}(\mathbb{R}^n))$. Then we have just constructed a solution $u_j \in \mathcal{C}^0(I; H^s(\mathbb{R}^n)) \cap \mathcal{C}^1(I; H^{s-1}(\mathbb{R}^n))$ of (7.1). Moreover,

$$v_{jk} := u_j - u_k \tag{7.14}$$

solves (7.1) with initial data $f_j - f_k$ and forcing $g_j - g_k$. An estimate similar to (7.11) thus implies that $v_{jk} \to 0$ in $\mathcal{C}^0(I; H^s(\mathbb{R}^n))$ as $j, k \to \infty$. Therefore, u_j is Cauchy in $\mathcal{C}^0(I; H^s(\mathbb{R}^n))$, hence its limit u satisfies (7.4), as desired. □

As a simple example, we solve the wave equation on $\mathbb{R}^n$,

$$\begin{cases} \Box u := (D_t^2 - \Delta)u = g, & t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ u(0,x) = f_0(x), & x \in \mathbb{R}^n, \\ D_t u(0,x) = f_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{7.15}$$

where

$$g \in \mathcal{C}^0(\mathbb{R}; H^{s-1}(\mathbb{R}^n)), \quad (f_0, f_1) \in H^s(\mathbb{R}^n) \oplus H^{s-1}(\mathbb{R}^n). \tag{7.16}$$

Corollary 7.2 (Solving the wave equation)**.** *The wave equation* (7.15) *with data* (7.16) *has a unique solution*

$$u \in \mathcal{C}^0(\mathbb{R}; H^s(\mathbb{R}^n)) \cap \mathcal{C}^1(\mathbb{R}; H^{s-1}(\mathbb{R}^n)). \tag{7.17}$$

Proof. Let $\Lambda = \langle D\rangle$. We write

$$\begin{aligned} &U := (\Lambda u, D_t u), \\ &F := (\Lambda f_0, f_1) \in H^{s-1}(\mathbb{R}^n; \mathbb{C}^2), \\ &G := (0, g) \in \mathcal{C}^0(\mathbb{R}; H^{s-1}(\mathbb{R}^n; \mathbb{C}^2)), \end{aligned} \tag{7.18}$$

then the equation (7.15) is equivalent to the first order system

$$\begin{cases} D_t U = AU + G, \\ U(0) = F, \end{cases} \tag{7.19}$$

where the operator A is given by

$$A = \begin{pmatrix} 0 & \Lambda \\ \Lambda^{-1}\Delta & 0 \end{pmatrix}. \tag{7.20}$$

Note that $A(t) - A(t)^* \in \Psi^0$. Therefore, by Theorem 7.1, the equation (7.19) has a solution $U \in \bigcap_{j=0}^1 \mathcal{C}^j(\mathbb{R}; H^{s-1-j}(\mathbb{R}^n; \mathbb{C}^2))$. The function $u := \Lambda^{-1} U_0$ is the desired solution of (7.15). □

7.2. **Egorov's theorem; propagation of singularities.** We now study the microlocal behavior of solutions of *scalar* hyperbolic equations

$$D_t u = A(t, x, D_x)u, \tag{7.21}$$

where we assume that $A \in \mathcal{C}^\infty(\mathbb{R}_t; \Psi^1_{\mathrm{cl}}(\mathbb{R}^n))$. Denote by $a \in \mathcal{C}^\infty(\mathbb{R}_t; S^1_{\mathrm{hom}}(\mathbb{R}^n; \mathbb{R}^n \setminus \{0\}))$ the homogeneous principal symbol of A; we assume that

$$a(t, x, \xi) \text{ is real-valued.} \tag{7.22}$$

We denote the solution operator for (7.21) by

$$S(t, s) \colon u(s) \mapsto u(t). \tag{7.23}$$

By Theorem 7.1, $S(t, s) \in \mathcal{L}(H^\sigma(\mathbb{R}^n))$ for all $\sigma \in \mathbb{R}$; moreover, $S(t, s)$ is invertible with $S(t, s)^{-1} = S(s, t)$.

Theorem 7.3 (Egorov's theorem)**.** *Let $P_0 = \mathrm{Op}(p_0) \in \Psi^m(\mathbb{R}^n)$ be a 'test operator', and define*

$$P(t) := S(t, 0) \circ P_0 \circ S(0, t), \quad t \in \mathbb{R}. \tag{7.24}$$

Then $P(t) \in \Psi^m(\mathbb{R}^n)$ modulo a smoothing operator: there exists $R \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{C}^\infty(\mathbb{R}^{2n}))$ such that $P(t) - R(t) \in \Psi^m(\mathbb{R}^n)$. The principal symbol of $P(t)$ is given by

$$\sigma^m(P(t))\big(\mathcal{C}(t)(x_0, \xi_0)\big) = p_0(x_0, \xi_0), \tag{7.25}$$

where $\mathcal{C}(t)$ is the time t flow (from 0 to t) of the time-dependent Hamiltonian vector field $H_{a(t,x,\xi)}$; that is, $\mathcal{C}(t)(x_0, \xi_0) = \gamma(t)$ where $\gamma(0) = (x_0, \xi_0)$ and $\gamma'(s) = H_{a(s)}|_{\gamma(s)}$.

Proof. Differentiating (7.24) in t gives the equation

$$P'(t) = i[A(t, x, D), P(t)], \quad P(0) = P_0. \tag{7.26}$$

Using the symbol calculus and an asymptotic summation, we will first construct an approximate solution $Q(t) = \mathrm{Op}(q(t))$, $q(t, x, \xi) \in S^m$, of this, so

$$Q'(t) = i[A(t, x, D), Q(t)] + R_1(t), \quad Q(0) = P_0, \qquad R_1 \in \mathcal{C}^\infty(\mathbb{R}_t; \Psi^{-\infty}(\mathbb{R}^n)). \tag{7.27}$$

We make the ansatz

$$q(t) \sim \sum_{k=0}^\infty q_k(t), \quad q_k(t) \in S^{m-k}. \tag{7.28}$$

Taking the principal symbol of (7.27) then gives

$$\big(\partial_t - H_{a(t)}\big)\, q_0(t, x, \xi) = 0, \quad q_0(0, x, \xi) = p_0(x, \xi). \tag{7.29}$$

Thus $q_0(t, \mathcal{C}(t)(x, \xi)) = p_0(x, \xi)$. We leave it to the reader to check that $q_0(t) \in S^m$. Proceeding iteratively, we take $q_j(t) \in S^{m-j}$, $j \geq 1$, to be the solution of a transport equation

$$\big(\partial_t - H_{a(t)}\big)\, q_j(t, x, \xi) = e_j(t, x, \xi), \tag{7.30}$$

where $e_j(t) \in S^{m-j}$ is computed from the full symbol of A and $q_0, \ldots, q_{j-1}$.

Having thus arranged (7.27), we now prove that for any $N \in \mathbb{R}$, the difference $R(t) = P(t) - Q(t)$ maps any $f \in H^{-N}(\mathbb{R}^n)$ into $H^\infty(\mathbb{R}^n)$. Equivalently, we will show

$$v(t) - w(t) \in H^\infty(\mathbb{R}^n), \quad v(t) := S(t, 0)P_0 f,\ w(t) := Q(t)S(t, 0)f. \tag{7.31}$$

Note that $v(t)$ and $w(t)$ solve the equations

$$\begin{aligned} D_t v &= A(t,x,D_x)v, & v(0) &= f, \\ D_t w &= A(t,x,D_x)w - iR_1 S(t,0)f, & w(0) &= f. \end{aligned} \tag{7.32}$$

Therefore, putting $g(t) = iR_1 S(t,0)f \in \mathcal{C}^\infty(\mathbb{R}_t; H^\infty(\mathbb{R}^n))$, we have

$$D_t(v-w) = g, \quad (v-w)(0) = 0. \tag{7.33}$$

By Theorem 7.1, (7.31) follows, finishing the proof. (This argument shows that the smoothing error in fact lies in the space $\Psi^{-\infty}(\mathbb{R}^n) + H^\infty(\mathbb{R}^{2n})$, with smooth dependence on t.) □

As a simple consequence, we can track the wave front set of a solution of a scalar evolution equation (7.21).

Theorem 7.4 (Propagation of wave front sets)**.** *Suppose A is as in* (7.21)–(7.22)*. Let $u_0 \in H^{-N}(\mathbb{R}^n)$, and let u denote the solution*

$$\begin{cases} D_t u = A(t,x,D_x)u, \\ u(0) = u_0. \end{cases} \tag{7.34}$$

Then, with $\mathcal{C}(t)$ as in the statement of Theorem 7.3, we have

$$\mathrm{WF}(u(t)) = \mathcal{C}(t)\mathrm{WF}(u_0). \tag{7.35}$$

Proof. It suffices to prove the inclusion '$\subseteq$' (since switching the time direction then proves '$\supseteq$'). Thus, suppose $\alpha \notin \mathrm{WF}(u_0)$. Take an operator $P_0 \in \Psi^0(\mathbb{R}^n)$, elliptic at α, such that $P_0 u_0 \in \mathcal{C}^\infty(\mathbb{R}^n)$. Then, in the notation of Theorem 7.3,

$$P_0 u_0 = S(0,t)P(t)S(t,0)u_0 \in \mathcal{C}^\infty(\mathbb{R}^n), \tag{7.36}$$

so $P(t)u(t) \in \mathcal{C}^\infty(\mathbb{R}^n)$. But $P(t)$ is elliptic at $\mathcal{C}(t)\alpha$, hence $\mathcal{C}(t)\alpha \notin \mathrm{WF}(u(t))$. □

Remark 7.5 (Generalization to weighted Sobolev spaces). Theorem 7.1, and thus also Theorem 7.4, can be generalized easily to the case of initial data and forcing terms in *weighted* Sobolev spaces. In particular, by Theorem 2.14, we can allow u_0 in Theorem 7.4 to be any tempered distribution $u_0 \in \mathscr{S}'(\mathbb{R}^n)$.

Example 7.6. For $A = D_x \in \Psi^1(\mathbb{R})$, the solution operator is $(S(t,0)u)(x) = u(t+x)$, and $\mathcal{C}(t)(x_0,\xi_0) = (x_0+t,\xi_0)$. And indeed $P(t) = \mathrm{Op}(p(t))$, $p(t,x,\xi) = p_0(x+t,\xi)$.

Example 7.7. Consider the half Klein–Gordon equation

$$D_t u = \langle D_x \rangle u \quad \text{on } \mathbb{R}_t \times \mathbb{R}^n_x. \tag{7.37}$$

(This arises from the factorization $D_t^2 - (\Delta+1) = (D_t - \langle D_x\rangle)(D_t + \langle D_x\rangle)$.) In this case, $a(x,\xi) = |\xi|$, which has Hamiltonian vector field $H_a = |\xi|^{-1}\xi\cdot\partial_x$. Thus, the operator P gets 'transported' along straight lines with direction determined by the momentum variable ξ. Explicitly, (7.37) is solved by $u(t) = e^{it\langle D\rangle}u(0)$, and the wave front set statement can be checked explicitly from this; however, the true power of Theorems 7.3 and 7.4 of course lies in the fact that they apply to equations with non-constant coefficients as well.

7.3. Exercises.

Exercise 7.1 (Necessity of a symmetry assumption)*.* On $\mathbb{R}_t \times \mathbb{R}_x$, consider the equation $D_t u = -iD_x u$. (This is the Cauchy–Riemann equation for $u = u(t,x)$, regarded as a function of the complex variable $t+ix$.) Suppose that u is a solution of this equation with $u \in \mathcal{C}^0(\mathbb{R}; H^s(\mathbb{R}^n))$. What can you say about $u(0,x)$? Conclude that Theorem 7.1 fails for $a(t,x,D_x) = -iD_x$.

Exercise 7.2 (Solutions of the Klein–Gordon equation and their singularities)*.* Consider the initial value problem for the Klein–Gordon equation,

$$\begin{cases} (D_t^2 - \Delta - 1)u = g, & t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ u(0,x) = f_0(x), & x \in \mathbb{R}^n, \\ D_t u(0,x) = f_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{7.38}$$

Here $(f_0, f_1) \in H^s(\mathbb{R}^n) \oplus H^{s-1}(\mathbb{R}^n)$ for some s, and $g \in \mathcal{C}^\infty(\mathbb{R}; H^\infty(\mathbb{R}^n))$ is smooth.

(1) Show that (7.38) has a unique solution $u \in \bigcap_{j=0}^\infty \mathcal{C}^j(\mathbb{R}; H^{s-j}(\mathbb{R}^n))$.
(2) Show that $\mathrm{WF}(u(t)) \subset T^*\mathbb{R}^n \setminus o$ is contained in the set

$$\bigcup_{\pm} \Big\{ \Big(x_0 \pm t\frac{\xi_0}{|\xi_0|}, \xi_0\Big) \colon (x_0,\xi_0) \in \mathrm{WF}(u_0) \cup \mathrm{WF}(u_1) \Big\}. \tag{7.39}$$

(*Hint.* Factor equation (7.38) as $(D_t - \langle D\rangle)(D_t + \langle D\rangle)u = g$.) Can you make a more precise statement?

Exercise 7.3 (Symmetrizable hyperbolic systems)*.* Suppose the $K \times K$ system of first order evolution equations

$$D_t u = L(t,x,D_x)u + g, \quad u(0) = f \in H^s(\mathbb{R}^n), \qquad g \in \mathcal{C}^0\big(\mathbb{R}; H^s(\mathbb{R}^n; \mathbb{C}^K)\big), \tag{7.40}$$

is a *symmetrizable hyperbolic system*: there exists a $K\times K$-matrix-valued symbol $S(t,x,\xi) \in S^0$ which is positive definite and such that $S(t,x,\xi)L(t,x,\xi)$ is symmetric modulo S^0. Prove that (7.40) has a unique solution

$$u \in \mathcal{C}^0\big(\mathbb{R}; H^s(\mathbb{R}^n;\mathbb{C}^K)\big) \cap \mathcal{C}^1\big(\mathbb{R}; H^{s-1}(\mathbb{R}^n;\mathbb{C}^K)\big). \tag{7.41}$$

(*Hint.* Construct a positive definite operator $S(t) \in \Psi^0$ with principal symbol $S(t,x,\xi)$. Mimic the proof of Theorem 7.1 and prove an ϵ-independent estimate for the quantity $\frac{\mathrm{d}}{\mathrm{d}t}\langle \Lambda^s u_\epsilon(t), S(t)\Lambda^s u_\epsilon\rangle_{L^2}$ instead of $\frac{\mathrm{d}}{\mathrm{d}t}\|u_\epsilon(t)\|_{H^s}^2$.)

Exercise 7.4 (Strictly hyperbolic systems)*.* A $K \times K$ system of the form (7.40) is *strictly hyperbolic* if the principal symbol $L_1(t,x,\xi)$ of L is positively homogeneous of degree 1 in ξ, and if for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus \{0\}$, $L_1(t,x,\xi)$ has K distinct real eigenvalues. Show that a strictly hyperbolic system is symmetrizable.

Exercise 7.5 (Higher order strictly hyperbolic equations)*.* On $\mathbb{R}_t \times \mathbb{R}^n_x$, consider an m-th order operator

$$L = D_y^m + \sum_{j=0}^{m-1} a_j(y,x,D_x) D_y^j \tag{7.42}$$

where $a_j(y,x,D_x) \in \mathrm{Diff}^{m-j}(\mathbb{R}^n)$. We study the system

$$\begin{cases} Lu = g \in \mathcal{C}^0(\mathbb{R}_t; H^{s-m+1}), \\ (u, D_y u, \ldots, D_y^{m-1}u) = (u_0, u_1, \ldots, u_{m-1}), \end{cases} \tag{7.43}$$

where $u_j \in H^{s-j}(\mathbb{R}^n)$ for $j = 0, \ldots, m-1$.

(1) Reduce (7.43) to an $m \times m$ system of first order evolution equations for the $\mathbb{C}^m$-valued function $(\Lambda^{m-1}u, \Lambda^{m-2}D_y u, \ldots, D_y^{m-1}u)$ where $\Lambda = \langle D_x \rangle$.
(2) Give a condition on the principal symbol of L which is equivalent to the strict hyperbolicity (see Exercise 7.4) of this $m \times m$ system.
(3) Under the strict hyperbolicity assumption of part (2), prove existence and uniqueness of a solution $u \in \bigcap_{j=0}^{\infty} \mathcal{C}^j(\mathbb{R}_t; H^{s-j}(\mathbb{R}^n))$ of (7.43).

8. Real principal type propagation of singularities

We now free ourselves from the restrictive setting[15] of equations which are explicitly given in evolution form, and consider the propagation of singularities (wave front set)/regularity for solutions of rather general non-elliptic (pseudo)differential equations

$$Pu = f, \quad P \in \Psi^m(M), \tag{8.1}$$

where $M = \mathbb{R}^n$ or some other manifold, and $m \in \mathbb{R}$; we assume that the principal symbol

$$p(x,\xi) = \sigma^m(P)(x,\xi) \tag{8.2}$$

is (positively) *homogeneous* of degree m and *real*.

Definition 8.1 (Null-bicharacteristics)**.** A *null-bicharacteristic* of P is an integral curve in $\mathrm{Char}(P) \subset T^*M \setminus o$ of the Hamiltonian vector field H_p.

Note that $H_p p = 0$; hence an integral curve of H_p with initial condition $\alpha \in \mathrm{Char}(P)$ is automatically a null-bicharacteristic (and all null-bicharacteristics arise in this fashion).

Theorem 8.2 (Propagation of singularities: smooth case)**.** *Suppose $P \in \Psi^m(M)$ has a real-valued homogeneous principal symbol, and $u \in \mathscr{D}'(M)$ is such that $Pu \in \mathcal{C}^\infty(M)$. Then* $\mathrm{WF}(u) \subset \mathrm{Char}(P)$ *is a union of maximally extended null-bicharacteristics of P.*

The statement $\mathrm{WF}(u) \subset \mathrm{Char}(P)$ is just microlocal elliptic regularity, Proposition 6.28. The theorem asserts that within $\mathrm{Char}(P)$, the wave front set of a 'microlocal solution' u is invariant under the H_p-flow. Rather than calling Theorem 8.2 a result on the propagation of *singularities*, one often (and more usefully) regards it as a result on the propagation of *regularity*, since $\alpha \notin \mathrm{WF}(u) \cap \mathrm{Char}(P)$ implies that the entire maximally extended null-bicharacteristic of P through α is disjoint from $\mathrm{WF}(u)$. The first proof of Theorem 8.2 was given in [DH72] using the machinery of Fourier integral operators.

Remark 8.3 (Real principal type operators). Suppose $\alpha \in T^*M \setminus o$ is such that $H_p|_\alpha = 0$. Then the null-bicharacteristic through α is the constant curve α, i.e. Theorem 8.2 does not give any information at α. One says P is of *real principal type* if $\mathrm{d}p \neq 0$ on $\mathrm{Char}(P)$; in this case, H_p never vanishes on $\mathrm{Char}(P)$.

Remark 8.4 (Radial points). Denote by V the generator of dilations in the fibers of T^*M, so $V = \xi\partial_\xi$ in local coordinates. Since $\mathrm{WF}(u)$ is conic, Theorem 8.2 is trivial also at *radial points*: these are points $\alpha \in T^*M \setminus o$ where $H_p|_\alpha = cV$, $c \in \mathbb{R}$. We shall discuss interesting classes of radial points in §9.

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[15] The theory of Fourier integral operators provides tools to 'microlocally conjugate' every real principal type operator into the operator D_t on $\mathbb{R}_t \times \mathbb{R}^{n-1}$, see [Hör71b, DH72, Hör09], thus this setting, with $L = 0$, in fact captures the general situation. We shall however not develop this theory here.

Proof of Theorem 8.2. This can easily be reduced to a local result. Indeed, if $\chi, \tilde\chi \in \mathcal{C}_c^\infty(M)$ are two cutoffs and $\tilde\chi \equiv 1$ near $\operatorname{supp}\chi$, then

$$\chi P \tilde\chi u = \chi P u + \chi[P, \tilde\chi]u \in \mathcal{C}_c^\infty(M). \tag{8.3}$$

Thus, if $\phi \in \mathcal{C}_c^\infty(M)$ is identically 1 near $\operatorname{supp}\tilde\chi$, we can replace P by $\chi P \tilde\chi$ and u by ϕu, and we still have $Pu \in \mathcal{C}^\infty(M)$; moreover, these replacements do not alter null-bicharacteristics of P and the wave front set of u in $\chi^{-1}(1)$. Localizing in this fashion to a coordinate patch, we can thus assume

$$P \in \Psi^m(\mathbb{R}^n),\ u \in \mathscr{E}'(\mathbb{R}^n),\ Pu = f \in \mathcal{C}_c^\infty(\mathbb{R}^n). \tag{8.4}$$

We normalize this using $\Lambda = \langle D\rangle^{m-1}$ by replacing (P, u) by $(P\Lambda^{-1}, \Lambda u)$; thus we can assume $m = 1$. After these replacements, the principal symbol of P is still homogeneous (of degree 1) and real.

We now add an artificial time variable t and set

$$\tilde u(t, x) = u(x), \quad \tilde f(t, x) = f(x). \tag{8.5}$$

Then $\tilde u$ solves the equation

$$D_t \tilde u = P\tilde u - \tilde f. \tag{8.6}$$

A simple extension of the proof of Theorem 7.4 (taking into account the presence of $\tilde f \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{C}_c^\infty(\mathbb{R}^n))$) implies that

$$\mathrm{WF}(u) = \mathrm{WF}(\tilde u(t)) = \mathcal{C}(t)\mathrm{WF}(\tilde u(0)) = \mathcal{C}(t)\mathrm{WF}(u), \tag{8.7}$$

where $\mathcal{C}(t)$ is the time t flow of H_p. Thus, $\mathrm{WF}(u)$ is invariant under the H_p-flow, proving the theorem. □

One of the main drawbacks of the above proof (apart from being rather ad hoc) is that it ultimately rests on the solvability theory for the (auxiliary) equation (8.6). But solving PDE is difficult, hence one should try to solve as few as possible! We shall thus present another proof of Theorem 8.2 which is longer and looks more complicated, but is at its core very simple, and the prototypical example of a *positive commutator argument* which in this form first appeared in [Hor71a]. It has the technical benefit of only utilizing pseudodifferential operators (rather than Fourier integral operators as in [DH72]).

We will prove the following sharpening of Theorem 8.2:

Theorem 8.5 (Propagation of singularities: Sobolev case)**.** *Suppose $P \in \Psi^m(M)$ has a real-valued homogeneous principal symbol. Let $u \in \mathscr{D}'(M)$ and $f := Pu \in \mathscr{D}'(M)$. Let $s \in \mathbb{R}$. Then*

$$\mathrm{WF}^s(u) \subset \mathrm{Char}(P) \cup \mathrm{WF}^{s-m}(f). \tag{8.8}$$

Moreover, within $\mathrm{Char}(P) \setminus \mathrm{WF}^{s-m+1}(f)$,

$$\mathrm{WF}^s(u) \setminus \mathrm{WF}^{s-m+1}(f) \tag{8.9}$$

is a union of maximally extended null-bicharacteristics of P.

Remark 8.6 (Loss of one derivative). Note that in order to obtain the propagation of microlocal H^s-regularity of u in $\mathrm{Char}(P)$, one needs to assume that f lies in H^{s-m+1} microlocally. This is one degree more smoothness than what microlocal elliptic regularity requires. Thus, on $\mathrm{Char}(P)$, u 'loses' one derivative relative to elliptic regularity.

From now on, all ps.d.o.s will have Schwartz kernels supported in a fixed compact subset of $M \times M$, and all distributions are supported in a fixed compact subset $K \subset M$. This is not a restriction in view of the arguments at the beginning of the proof of Theorem 8.2. Note that $\{u \in H^s_c(M) : \operatorname{supp} u \subset K\}$ has the structure of a Hilbert space. We will really prove *quantitative estimates* somewhat similar to those that arose in the discussion of microlocal elliptic regularity, see (6.44). Namely, we will show:

Theorem 8.7 (Propagation of singularities: estimate version)**.** *We use the notation of Theorem 8.5. Let $\gamma\colon [0, s_0] \to \mathrm{Char}(P)$ be a null-bicharacteristic of P; let $\mathcal{U}_0 \subset S^*M$ and $\mathcal{U} \subset S^*M$ be arbitrary neighborhoods of $\gamma(0)$ and $\gamma([0, s_0])$, respectively. Then there exist*

$$B, G, E \in \Psi^0(M) \tag{8.10}$$

so that

(1) $\mathrm{Ell}(B) \supset \gamma([0, s_0])$,
(2) $\mathrm{WF}'(B) \subset \mathrm{Ell}(G) \subset \mathcal{U}$,
(3) $\mathrm{WF}'(E) \subset \mathcal{U}_0$

so that the following estimate holds for any $N \in \mathbb{R}$ and a constant $C = C(N) > 0$:

$$\|Bu\|_{H^s} \leq C\left(\|Eu\|_{H^s} + \|GPu\|_{H^{s-m+1}} + \|u\|_{H^{-N}}\right). \tag{8.11}$$

Moreover, this holds in the strong sense that if $u \in \mathscr{E}'(M)$ is such that the right hand side is finite, then so is the left hand side, and the estimate holds.

One reads the estimate (8.11) as follows: *assuming* a priori microlocal H^s control of u on $\mathrm{Ell}(E)$, we *conclude* microlocal H^s control of u on $\mathrm{Ell}(B)$ by propagation along null-bicharacteristics (provided Pu remains microlocally in H^{s-m+1} along the way). See Exercise 8.2 for a semiglobal version of Theorem 8.7.

Theorem 8.5 is an immediate consequence of Theorem 8.7. Indeed, if $\alpha \in \mathrm{Char}(P) \setminus \mathrm{WF}^s(u)$, then this implies that the *forward* null-bicharacteristic of P with initial condition α remains disjoint from $\mathrm{WF}^s(u)$ as long as it does not intersect $\mathrm{WF}^{s-m+1}(f)$. Applying the theorem to $-P$ gives the *backward* propagation of regularity. (Away from $\mathrm{Char}(P)$, the estimate (8.11) follows from microlocal elliptic regularity in view of condition (2) in Theorem 8.7, though in a weak form since we are assuming microlocal H^{s-m+1} control on Pu).

8.1. **Positive commutator argument I: sketch.** Let us consider a basic example: $P = D_{x_1}$ on $\mathbb{R}^n_x$, so $p = \xi_1$ and $H_p = \partial_{x_1}$. Let us take B, E, G to be cutoff functions, say smoothed out versions of the characteristic functions of $Q_E := [-2, -1]_{x_1} \times [-2, 2]^{n-1}$ (for E), $Q_B := [1, 2] \times [-1, 1]^{n-1}$ (for B) and $[-3, 3] \times [-3, 3]^{n-1}$ (for G). Take $s = 0$; assume

$$D_{x_1} u = f \in L^2(\mathbb{R}^n). \tag{8.12}$$

Then (8.11) asserts that $u|_{Q_B} \in L^2$, provided $u|_{Q_E} \in L^2$. But this is obvious! Indeed, one can solve (8.12) explicitly, and do simple estimates. (The 'high brow' proof of Theorem 8.7, see [DH72], reduces to this situation (modulo smoothing operator) using Fourier integral operators.)

A much better proof does not require the explicit solution of (8.12). (This 'better proof' does use the fundamental theorem of calculus, but in the form of integration by parts, and in a way that generalizes readily to general operators.) To explain it, let us take $n = 1$,

$x = x_1$, for simplicity, and let us estimate $|u(1)|^2$ in terms of $|u(-1)|^2$ and $D_x u$. Let $\chi = 1_{[-1,1]}$ denote the characteristic function of $[-1,1]$. Then

$$\begin{aligned} -|u(1)|^2 + |u(-1)|^2 &= \int i[D_x, \chi]u \cdot \bar{u}\, \mathrm{d}x \\ &= i^{-1}\left(\int D_x u \cdot \chi\bar{u}\, \mathrm{d}x - \int \chi u \cdot \overline{D_x u}\, \mathrm{d}x\right) \\ &= 2\,\mathrm{Im}\langle D_x u, \chi u\rangle. \end{aligned} \tag{8.13}$$

If $D_x u = 0$, we see from the first line that what *really* provides control of $|u(1)|^2$ is the fact that the cutoff χ has negative (the 'good' sign) derivative along H_p at 1. Since we are proving a localized estimate, the function χ must be compactly supported, and hence it must have a positive (the 'bad' sign) derivative somewhere, here at -1, which necessitates a priori control of u there.

If $D_x u \neq 0$, one needs more 'negativity' of the commutator $i[D_x, \chi]$; one can e..g take

$$2\,\mathrm{Im}\langle D_x u, e^{-x}\chi u\rangle = -\int_{-1}^{1} e^{-x}|u|^2\, \mathrm{d}x - e^{-1}|u(1)|^2 + e|u(-1)|^2, \tag{8.14a}$$

and estimate the left hand side using Cauchy–Schwarz by

$$2\,\mathrm{Im}\langle D_x u, e^{-x}\chi u\rangle \geq -\|\chi e^{-x/2} D_x u\|_{L^2}^2 - \|e^{-x/2}u\|_{L^2([-1,1])}^2. \tag{8.14b}$$

Combining (8.14a)–(8.14b) gives $|u(1)|^2 \leq C(|u(-1)|^2 + \|D_x u\|_{L^2}^2)$, as desired. This is a typical positive commutator argument; the function $e^{-x}\chi$ is called the *commutant*.[16]

The proof of Theorem 8.7 will be based on similar considerations. The rough, formal sketch goes as follows.[17] We formally compute for $A = \mathrm{Op}(a) \in \Psi^{2s-m+1}$, $A = A^*$,

$$2\,\mathrm{Im}\langle Pu, Au\rangle = i\big(\langle Au, Pu\rangle - \langle Pu, Au\rangle\big) = \big\langle (i[P,A] + i(P^* - P)A)u, u\big\rangle. \tag{8.15}$$

Let $p = \sigma^m(P)$ and $p_1 = \sigma^{m-1}(i(P^* - P))$. Then

$$\sigma^{2s}\big(i[P,A] + i(P^* - P)A\big) = H_p a + p_1 a. \tag{8.16}$$

Suppose we can arrange

$$H_p a + p_1 a = -b^2 + e' \tag{8.17}$$

where $b \in S^s$ is elliptic in the desired conclusion region, and $e' \in S^{2s}$ has essential support contained in the a priori control region. Taking $B = \mathrm{Op}(b)$, $E' = \mathrm{Op}(e')$, we then have

$$i[P,A] + i(P^* - P)A = -B^*B + E' + R, \quad R \in \Psi^{2s-1}, \tag{8.18}$$

hence (8.15) implies

$$\|Bu\|^2 = -2\,\mathrm{Im}\langle Pu, Au\rangle + \langle E'u, u\rangle + \langle Ru, u\rangle. \tag{8.19}$$

If $Pu = 0$, this controls the microlocal H^s norm of u on $\mathrm{Ell}(B)$ by that on $\mathrm{Ell}(E')$. (Note that, using Lemma 6.34, $\langle E'u, u\rangle$ is finite by the a priori H^s control on u on $\mathrm{WF}'(E')$.) The term $\langle Ru, u\rangle$ is lower order, and finite provided we already have proved $H^{s-1/2}$ control of u. Thus, starting with $s = -N + 1/2$, we can iteratively improve the control on u by half

[16]Strictly speaking, one should call it a negative commutator argument, which can be turned into a positive commutator argument by switching the sign of the commutant. However, people typically use positive commutants, and we will do the same here.

[17]We encourage the reader to assume at first reading that $P = P^*$.

a derivative, until after finitely many iterations we reach the desired level of regularity. (If $Pu \neq 0$, we arrange for more negativity in (8.17) and estimate the first term in (8.19) using Cauchy–Schwarz, similarly to (8.14a)–(8.14b) above.)

To make this into an honest positive commutator argument, we need to

(1) construct the commutant (this is the 'interesting' part of the argument), i.e. construct a satisfying (8.17);
(2) regularize the argument (this is the 'technical' but straightforward part of the argument): we need to ensure that the integrations by parts in (8.15) and (8.19) as well as various norms are well-defined.

8.2. **Positive commutator argument II: construction of the commutant.** We first do some preliminary simplifications. Using a partition of unity argument, it suffices to work near a single null-bicharacteristic segment

$$\gamma\colon [0,s_0] \ni s \mapsto \gamma(s) \in \mathrm{Char}(P), \quad s_0 > 0. \tag{8.20}$$

We will show that $\gamma(0) \notin \mathrm{WF}^s(u)$ and $\gamma([0,s_0]) \cap \mathrm{WF}^{s-m+1}(Pu) = \emptyset$ implies $\gamma(s_0) \notin \mathrm{WF}^s(u)$, with estimates.

We further simplify notation by passing to the cosphere bundle.

Lemma 8.8 (Homogeneity of the Hamiltonian vector field)**.** *Let $p \in S^m_{\mathrm{hom}}(T^*M \setminus o)$. Then H_p is homogeneous of degree $m-1$. That is, denoting by $M_\lambda\colon (x,\xi) \mapsto (x,\lambda\xi)$, $\lambda > 0$, the dilation in the fibers of T^*M, we have*

$$M_\lambda^* H_p = \lambda^{m-1} H_p. \tag{8.21}$$

Proof. We work in local coordinates. Since $p(x,\lambda\xi) = \lambda^m p(x,\xi)$, differentiation in ξ shows that $\partial_{\xi_i} p \in S^{m-1}_{\mathrm{hom}}$. Let now $f \in \mathcal{C}^\infty(T^*M)$ and $(x_0,\xi_0) \in T^*M \setminus o$, then

$$\begin{aligned} (M_\lambda^* H_p)|_{(x_0,\xi_0)}(f) &= H_p|_{(x_0,\lambda\xi_0)}(f \circ M_\lambda^{-1}) \\ &= (\partial_\xi p)(x_0,\lambda\xi_0) \cdot \big(\partial_x (f \circ M_\lambda^{-1})\big)(x_0,\lambda\xi_0) \\ &\qquad - (\partial_x p)(x_0,\lambda\xi_0) \cdot \big(\partial_\xi (f \circ M_\lambda^{-1})\big)(x_0,\lambda\xi_0) \\ &= \lambda^{m-1}(\partial_\xi p)(x_0,\xi_0) \cdot (\partial_x f)(x_0,\xi_0) \\ &\qquad - \lambda^m (\partial_x p)(x_0,\xi_0) \cdot \lambda^{-1}(\partial_\xi f)(x_0,\xi_0) \\ &= \lambda^{m-1} H_p|_{(x_0,\xi_0)} f, \end{aligned} \tag{8.22}$$

as claimed. □

Fix an elliptic symbol

$$|\xi| \in S^1_{\mathrm{hom}}(T^*M \setminus o) \tag{8.23}$$

and define

$$\tilde H_p := |\xi|^{-m+1} H_p \in \mathcal{V}(T^*M \setminus o). \tag{8.24}$$

This is homogeneous of degree 0 and hence descends to a smooth vector field $\tilde H_p' \in \mathcal{V}(S^*M)$ on the cosphere bundle.[18] (Indeed, for $f \in \mathcal{C}^\infty(S^*M)$, one defines $\tilde H_p' f$ by $q^*(\tilde H_p' f) :=$

[18] The passage from $\tilde H_p \in \mathcal{V}(T^*M \setminus o)$ to $\tilde H_p \in \mathcal{V}(S^*M)$ does lose information, namely the fiber-radial component of $\tilde H_p$. For example, the vector field $\xi\partial_\xi \in \mathcal{V}(T^*\mathbb{R}^n)$, which is homogeneous of degree 0, descends to the 0 vector field on $S^*\mathbb{R}^n$. Keeping track of the radial component will be crucial in §9.

$\tilde{H}_p(q^*f)$ where $q\colon T^*M\setminus o\to S^*M$ is the quotient map.) We immediately simplify notation and denote $\tilde{H}'_p$ by $\tilde{H}_p$ simply. An integral curve of $\tilde{H}_p$ is the image in $T^*M\setminus o$ of an integral curve of H_p, up to reparameterization.

Working in S^*M, we thus take γ in (8.20) to be an integral curve of $\tilde{H}_p$; we may rescale to arrange that $s_0=1$. Note that there is nothing to prove if γ is stationary, cf. Remarks 8.3–8.4. Otherwise, $\tilde{H}_p$ is non-zero along γ, and by basic ODE theory, we can straighten out $\tilde{H}_p$ locally: there exist local coordinates on S^*M

$$(z_1,z'),\quad z_1\in[-2,2],\ z'\in\mathbb{R}^{2n-2},\ |z'|<1, \tag{8.25}$$

near $\gamma([0,1])$ such that $\gamma(0)=(0,0)$, $\tilde{H}_p=\partial_{z_1}$, and $\gamma(1)=(1,0)$.

We now construct a commutant a; we need to ensure that a is supported in any pre-specified neighborhood of $\gamma([0,1])$, and that e' is supported in any pre-specified neighborhood of $\gamma(0)$. Suppose $\epsilon>0$ is such that

$$\begin{split}\bigl([-2\epsilon,1+2\epsilon]\times\{|z'|<2\epsilon\}\bigr)\cap\mathrm{WF}^{s-m+1}(Pu)=\emptyset,\\ \bigl([-2\epsilon,2\epsilon]\times\{|z'|<2\epsilon\}\bigr)\cap\mathrm{WF}^{s}(u)=\emptyset.\end{split} \tag{8.26}$$

Fix a cutoff (in the transverse directions)

$$\psi\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{2n-2}),\quad \operatorname{supp}\psi\subset\{|z'|<2\epsilon\},\quad \psi(z')=1\text{ for }|z'|<\epsilon, \tag{8.27}$$

and a 'turn-on' function (in the z_1 direction)

$$\chi_1\in\mathcal{C}^\infty(\mathbb{R}),\quad \operatorname{supp}\chi_1\subset(-\epsilon,\infty),\ \operatorname{supp}(1-\chi_1)\subset(-\infty,\epsilon). \tag{8.28}$$

The main term of the commutant arises from

$$\chi_0(x):=\begin{cases}e^{-F/x}, & x>0,\\ 0, & x\leq 0\end{cases}, \tag{8.29}$$

where the constant $F>1$ will be chosen below. Note that

$$\chi_0'(x)=Fx^{-2}\chi_0(x), \tag{8.30}$$

so for x in any fixed compact subset $I\Subset[0,\infty)$, and for any given $C>0$, we can choose $F\gg 1$ so that $\chi_0'\geq C\chi_0$ for $x\in I$. That is, *the derivative of χ_0 can be made to dominate any multiple of χ_0*; this is an important mantra in the commutant construction business. We then set

$$\chi(z_1):=\chi_0(1+\epsilon-z_1),\quad \tilde{a}:=\chi(z_1)\chi_1(z_1)^2\psi(z')^2, \tag{8.31}$$

which is supported in a 2ϵ-neighborhood of $\gamma([0,1])$. Setting $\tilde{p}_1=|\xi|^{-m+1}p_1\in\mathcal{C}^\infty(S^*M)$ (using the notation $p_1=\sigma^{m-1}(i(P^*-P))$ from (8.16)), we then compute

$$\begin{split}\tilde{H}_p\tilde{a}+\tilde{p}_1\tilde{a}&=2\chi(z_1)\chi_1(z_1)\chi_1'(z_1)\psi(z')^2+\chi'(z_1)\chi_1(z_1)^2\psi(z')^2+\tilde{p}_1\chi(z_1)\chi_1(z_1)^2\psi(z')^2\\ &=-\tilde{b}^2+\tilde{e}',\end{split} \tag{8.32}$$

where

$$\begin{split}\tilde{e}'&=2\chi(z_1)\chi_1(z_1)\chi_1'(z_1)\psi(z')^2,\\ \tilde{b}&=\chi_1(z_1)\psi(z')\sqrt{-\chi'(z_1)-\tilde{p}_1\chi(z_1)}\\ &=\chi_1(z_1)\psi(z')\sqrt{\chi(z_1)}\sqrt{F(1+\epsilon-z_1)^{-2}-\tilde{p}_1}.\end{split} \tag{8.33}$$

Note that for F sufficiently large, $\tilde b \in \mathcal{C}^\infty(S^*M)$. Moreover, $\operatorname{supp}\tilde e' \subset \{|z_1|,\ |z'| < 2\epsilon\}$, and $\tilde b$ is positive on $[\epsilon, 1]\times\{0\}$.

This almost arranges (8.17): we need to put the differential order back. Thus, we set

$$a := |\xi|^{2s-m+1}\tilde a \in S^{2s-m+1}_{\rm hom}(T^*M\setminus o) \tag{8.34}$$

and compute

$$H_p a + p_1 a = |\xi|^{2s}\big(\tilde H_p\tilde a + \tilde p_2\tilde a\big), \quad \tilde p_2 := \tilde p_1 + \big(|\xi|^{-2s+m-1}\tilde H_p|\xi|^{2s-m+1}\big). \tag{8.35}$$

Therefore, giving ourselves some extra room (to deal with $Pu \neq 0$), we have

$$H_p a + p_1 a = -|\xi|^{2m-2s-2}a^2 - b^2 + e' \tag{8.36}$$

if we set

$$e' = |\xi|^{2s}\tilde e', \tag{8.37}$$

$$b = |\xi|^s\chi_1(z_1)\psi(z')\sqrt{-\chi'(z_1) - \tilde p_2\chi(z_1) - \chi(z_1)^2\chi_1(z_1)^2\psi(z')^2} \tag{8.38}$$

$$= |\xi|^s\chi_1(z_1)\psi(z')\sqrt{\chi(z_1)}\sqrt{F(1+\epsilon-z_1)^{-2} - \tilde p_2 - \chi(z_1)\chi_1(z_1)^2\psi(z')^2}; \tag{8.39}$$

we have $e' \in S^{2s}$ and $b \in S^s$ for sufficiently large $F > 1$.

8.3. **Positive commutator argument III: a priori estimate.** Let us quantize these symbols as in (8.18), giving $A \in \Psi^{2s-m+1}$, $B \in \Psi^s$, $E' \in \Psi^{2s}$; we can also arrange $\mathrm{WF}'(A) = \operatorname{ess\,supp} a$ etc. Assuming $u \in \mathcal{C}^\infty(M)$, integrations by parts are never a concern, and we then have the following slight improvement over (8.19):

$$\|Bu\|^2 + \|\Lambda Au\|^2 = -2\operatorname{Im}\langle Pu, Au\rangle + \langle E'u, u\rangle + \langle Ru, u\rangle, \quad R \in \Psi^{2s-1}, \tag{8.40}$$

where $\Lambda \in \Psi^{m-s-1}(M)$ is elliptic with principal symbol $|\xi|^{m-s-1}$; and $\mathrm{WF}'(R) \subset \mathrm{WF}'(A)$. Let $\Lambda_- \in \Psi^{-m+s+1}(M)$ denote an elliptic parametrix of Λ, with $I = \Lambda_-\Lambda + \tilde R$, $\tilde R \in \Psi^{-\infty}(M)$. Fix an operator $G \in \Psi^0(M)$ with $\mathrm{WF}'(I-G)\cap\mathrm{WF}'(A) = \emptyset$; in particular, G is elliptic on $\mathrm{WF}'(A)$. We then have

$$\begin{aligned} 2|\operatorname{Im}\langle Pu, Au\rangle| &\le 2|\operatorname{Im}\langle GPu, (\tilde R + \Lambda_-\Lambda)Au\rangle| + 2|\operatorname{Im}\langle Pu, (I-G^*)Au\rangle| \\ &\le \|\Lambda_-^* GPu\|^2 + \|\Lambda Au\|^2 + C\|u\|^2_{H^{-N}}. \end{aligned} \tag{8.41}$$

Let $E \in \Psi^0(M)$ be elliptic on $\mathrm{WF}'(E')$. Plugging (8.41) into (8.40), we then get the estimate

$$\|Bu\|_{L^2} \le C\big(\|GPu\|_{H^{s-m+1}} + \|Eu\|_{H^s} + \|Gu\|_{H^{s-1/2}} + \|u\|_{H^{-N}}\big). \tag{8.42}$$

If $s - 1/2 \le -N$, we simply estimate $\|Gu\|_{H^{s-1/2}} \le C\|u\|_{H^{-N}}$, obtaining the desired estimate (8.11). For $s > -N + 1/2$, we can control $\|Gu\|_{H^{s-1/2}}$ inductively. Indeed, if $\mathrm{WF}'(G)$ lies in an $\epsilon 2^{-|s|-2}$ neighborhood of $\mathrm{WF}'(A)$, one can control $\|Gu\|_{H^{s-1/2}}$ by the right hand side of (8.42) with E, G replaced by operators $\tilde E, \tilde G$ elliptic on $\mathrm{WF}'(E), \mathrm{WF}'(G)$ and with operator wave front set in an $\epsilon 2^{-|s|-1}$ neighborhood of $\mathrm{WF}'(E), \mathrm{WF}'(G)$. After finitely many iterations, we thus obtain the desired estimate

$$\|Bu\|_{L^2} \le C\big(\|\tilde GPu\|_{H^{s-m+1}} + \|\tilde Eu\|_{H^s} + \|u\|_{H^{-N}}\big), \tag{8.43}$$

where $\tilde E, \tilde G \in \Psi^0$, with $\mathrm{WF}'(\tilde E)$ in a 3ϵ-neighborhood of $\gamma(0)$, and $\mathrm{WF}'(\tilde G)$ in a 3ϵ-neighborhood of $\gamma([0,1])$. Starting out with ϵ replaced by $\frac{2}{3}\epsilon$, we have the desired a priori estimate.

Remark 8.9 (A priori estimate). While these arguments required the a priori membership $u \in \mathcal{C}^\infty(M)$ (or at least for u to have sufficiently high regularity), the estimate (8.43) is highly non-trivial as an a priori estimate, as it gives quantitative control on the microlocal H^s-mass of u along $\gamma([0,1])$.

8.4. **Positive commutator argument IV: regularization.** We now regularize the argument so that $u \in H^{-N}$ together with some microlocal regularity is sufficient. By an inductive argument as above, we may moreover assume that $\mathrm{WF}^{s-1/2}(u)$ is disjoint from a 2ϵ-neighborhood of $\gamma([0,1])$. The a priori assumption is that $\mathrm{WF}^s(u)$ is disjoint from a 2ϵ-neighborhood of $\gamma(0)$.

The regularization argument replaces a, b, e' by symbols a_r, b_r, e'_r, $r \in (0,1]$, of (much) lower symbolic order, which converge to a, b, e' as $r \to 0$ (or rather, to a, b, e' multiplied by a cutoff which cuts away the singularity at $\xi = 0$) in slightly weakened symbol classes. We first deal with the symbolic construction.

For $K > 1$, define

$$\phi_r(\tau) = (1 + r\tau)^{-K/2}, \quad r \in (0,1]. \tag{8.44}$$

Thus, $\phi_r(|\xi|^2) \in S^{-K}(T^*M)$ is uniformly bounded in $S^0(T^*M)$, and converges to $\phi_0 \equiv 1$ in the topology of $S^\delta(T^*M)$ for any $\delta > 0$. Note moreover that

$$\tau\phi_r'(\tau) = f_r(\tau)\phi_r(\tau), \quad f_r(\tau) = -(K/2)\frac{r\tau}{1 + r\tau}, \tag{8.45}$$

so in particular $|f_r(\tau)| \leq K/2$. With $\eta \in \mathcal{C}^\infty(\mathbb{R})$, vanishing near 0 and identically 1 outside $[-1,1]$, we then define the *regularized commutant*

$$\begin{aligned} a_r &= \phi_r(|\xi|^2)\eta(|\xi|) \cdot a \\ &= \phi_r(|\xi|^2)\eta(|\xi|) \cdot |\xi|^{2s-m+1}\chi(z_1)\chi_1(z_1)^2\psi(z')^2. \end{aligned} \tag{8.46}$$

Thus, $a_r \in L^\infty((0,1]_r, S^{2s-m+1}(T^*M))$, and $a_r \in S^{2s-m+1-K}(T^*M)$ for $r > 0$. In addition to the terms in (8.36), the computation of $H_p a_r$ produces two extra terms: for H_p falling on ϕ_r, we get a term involving

$$\tilde H_p\bigl(\phi_r(|\xi|^2)\bigr) = \tilde f_r \phi_r(|\xi|^2), \quad \tilde f_r := (|\xi|^{-2}\tilde H_p|\xi|^2) f_r(|\xi|^2); \tag{8.47}$$

note that $\eta(|\xi|)\tilde f_r \in L^\infty((0,1]_r, S^0(T^*M))$ is uniformly bounded. When H_p falls on $\eta(|\xi|)$, we get a symbol with compact support in ξ, which is hence of order $-\infty$.

Using the notation of (8.32) and (8.35), we then compute

$$\begin{aligned} H_p a_r + p_1 a_r &= |\xi|^{2s}\phi_r(|\xi|^2)\eta(|\xi|)\bigl(\tilde H_p\tilde a + \tilde p_2 \tilde a\bigr) \\ &\qquad + \tilde f_r \phi_r(|\xi|^2)\eta(|\xi|)|\xi|^{m-1} a + \phi_r(|\xi|^2)(H_p\eta(|\xi|))a \\ &= -|\xi|^{2m-2s-2}a_r^2 - b_r^2 + e'_r \end{aligned} \tag{8.48}$$

where

$$\begin{aligned} e'_r &= \phi_r(|\xi|^2)\bigl(\eta(|\xi|) \cdot 2|\xi|^{2s}\chi(z_1)\chi_1(z_1)\chi_1'(z_1)\psi(z')^2 + (H_p\eta(|\xi|))a\bigr), \\ b_r &= |\xi|^s\sqrt{\phi_r(|\xi|^2)\eta(|\xi|)}\chi_1(z_1)\psi(z')\sqrt{\chi(z_1)} \\ &\qquad \times \sqrt{F(1+\epsilon-z_1)^{-2} - \tilde p_2 - \tilde f_r - \phi_r(|\xi|^2)\eta(|\xi|)\chi(z_1)\chi_1(z_1)^2\psi(z')^2}. \end{aligned} \tag{8.49}$$

Since $\tilde f_r$ is uniformly bounded, the extra term $\tilde f_r$ here is harmless: choosing $F > 1$ sufficiently large makes the square root well-defined. Indeed, we have

$$b_r \in L^\infty((0,1]_r; S^s(T^*M)), \quad e'_r \in L^\infty((0,1]_r; S^{2s}(T^*M)). \tag{8.50}$$

Moreover, by construction, $\operatorname{supp} a_r, \operatorname{supp} b_r$ and $\operatorname{supp} e'_r$ are contained in 2ϵ-neighborhoods of $\gamma([0,1])$ and $\gamma(0)$, respectively.

The quantization of (8.49) requires a bit of care since we need more precision than that afforded by a quantization which only respects principal symbols. Recalling the construction in §5.6, we thus fix a linear continuous quantization map

$$\mathrm{Op}\colon S^m(T^*M) \to \Psi^m(M) \tag{8.51}$$

by $\mathrm{Op}(a) = \sum \tilde\phi_i \,\mathrm{Op}(a_i)\phi_i$, where ϕ_i is a partition of unity on M subordinate to a cover by coordinate systems, $\tilde\phi_i = 1$ near $\operatorname{supp}\phi_i$, and $a_i \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ is the local coordinate expression for a. Thus, Op is a quantization map in the sense that $\sigma^m(\mathrm{Op}(a)) = [a]$ for $a \in S^m(T^*M)$, and Op is surjective modulo $\Psi^{-\infty}(M)$. This definition also ensures that Op is continuous, and $\mathrm{WF}' \circ \mathrm{Op} = \operatorname{ess\,supp}$.

Let then

$$\begin{aligned} A_r &= \mathrm{Op}(a_r) \in L^\infty((0,1]_r; \Psi^{2s-m+1}(M)),\\ B_r &= \mathrm{Op}(b_r) \in L^\infty((0,1]_r; \Psi^{s}(M)),\\ E'_r &= \mathrm{Op}(e'_r) \in L^\infty((0,1]_r; \Psi^{2s}(M)). \end{aligned} \tag{8.52}$$

Letting $\Lambda = \mathrm{Op}(\langle\xi\rangle^{m-s-1})$, we then have

$$\begin{aligned} i[P, A_r] + i(P^* - P)A_r = -(\Lambda A_r)^*(\Lambda A_r) - B_r^* B_r + E'_r + R_r,&\\ R_r \in L^\infty((0,1]_r, \Psi^{2s-1}(M)).& \end{aligned} \tag{8.53}$$

The orders of A_r, B_r and E'_r, R_r are lower by K and $2K$, respectively, for $r > 0$. Thus, if we take K large enough (depending on s and N), we can safely compute

$$\begin{aligned} 2\operatorname{Im}\langle Pu, A_r u\rangle &= \langle (i[P,A_r] + i(P^*-P)A_r)u, u\rangle \\ &= -\|\Lambda A_r u\|^2 - \|B_r u\|^2 + \langle E'_r u, u\rangle + \langle R_r u, u\rangle. \end{aligned} \tag{8.54}$$

We need to show that the final two terms are uniformly bounded for $r \in (0,1]$. The crucial insight is that we have uniform control on A_r etc. in the following sense:

Definition 8.10 (Uniform wave front set)**.** Suppose $\mathcal{A} = \{A_r\} \in L^\infty((0,1]_r; \Psi^N(M))$ is a bounded family (for some $N \in \mathbb{R}$) of ps.d.o.s on M. Then $\alpha \in S^*M$ does *not* lie in the *uniform wave front set* $\mathrm{WF}'_{L^\infty}(\mathcal{A}) \subset S^*M$ if and only if there exists an operator $B \in \Psi^0(M)$, elliptic at α, such that BA_r is bounded in $\Psi^{-\infty}(M)$.

(This generalizes WF': if $A_r = A$ is r-independent, then $\mathrm{WF}'_{L^\infty}(\mathcal{A}) = \mathrm{WF}'(A)$.) We then have the following extension of microlocal elliptic regularity:

Lemma 8.11 (Uniform microlocal elliptic regularity)**.** *Let $\mathcal{A} = \{A_r\} \in L^\infty((0,1]_r; \Psi^m(M))$. Suppose $B \in \Psi^0(M)$ is such that $\mathrm{WF}'_{L^\infty}(\mathcal{A}) \subset \mathrm{Ell}(B)$. Let $s, N \in \mathbb{R}$. Then there exists a constant C (independent of r) such that*

$$\|A_r u\|_{H^{s-m}} \le C\big(\|Bu\|_{H^s} + \|u\|_{H^{-N}}\big). \tag{8.55}$$

Proof. Writing $I = QB + R$ with $Q, R \in \Psi^0(M)$, $\mathrm{WF}'(R) \cap \mathrm{WF}'_{L^\infty}(\mathcal{A}) = \emptyset$, we have

$$A_r u = A_r Q B u + A_r R u. \tag{8.56}$$

But $A_r Q \in L^\infty((0,1]_r; \Psi^m(M))$ and $A_r R \in L^\infty((0,1]_r; \Psi^{-\infty}(M))$ are uniformly bounded; this implies (8.55) in view of the quantitative version of Corollary 4.34. □

By construction, we have $\mathrm{WF}'_{L^\infty}(\{A_r\}) \subset \operatorname{ess\,supp} a$ etc. Let us thus take $G \in \Psi^0(M)$, elliptic near $\mathrm{WF}'_{L^\infty}(\{A_r\})$ and with $\mathrm{WF}'(G)$ contained in a 2ϵ-neighborhood of $\gamma([0,1])$, and $E \in \Psi^0(M)$ elliptic near $\mathrm{WF}'_{L^\infty}(\{E'_r\})$ and with $\mathrm{WF}'(E)$ contained in a 2ϵ-neighborhood of $\gamma(0)$; we then conclude that

$$\begin{aligned} |\langle E'_r u, u\rangle| &\le C\bigl(\|Eu\|_{H^s}^2 + \|u\|_{H^{-N}}^2\bigr), \\ |\langle R_r u, u\rangle| &\le C\bigl(\|Gu\|_{H^{s-1/2}}^2 + \|u\|_{H^{-N}}^2\bigr). \end{aligned} \tag{8.57}$$

Plugging this into (8.54) and arguing as in (8.41)–(8.42), we thus obtain a *uniform* estimate

$$\|B_r u\|_{L^2} \le C\bigl(\|Eu\|_{H^s} + \|GPu\|_{H^{s-m+1}} + \|Gu\|_{H^{s-1/2}} + \|u\|_{H^{-N}}\bigr). \tag{8.58}$$

Since the unit ball in L^2 is compact, $B_r u$ has a weakly convergent subsequence with limit $v \in L^2$. On the other hand, $B_r u \to B_0 u$ in $\mathscr{D}'(M)$; hence $B_0 u = v \in L^2$. Therefore, $\mathrm{WF}^s(u) \cap \mathrm{Ell}(B_0) = \emptyset$, proving microlocal H^s-regularity of u at $\gamma(1)$, and at the same time giving an estimate for $\|B_0 u\|_{L^2} \le \liminf \|B_r u\|_{L^2}$ by the right hand side of (8.58).

The proof of Theorem 8.7 is complete.

Remark 8.12 (Generalization to operators on vector bundles)*.* Theorems 8.5 and 8.7 also hold for operators $P \in \Psi^m(M; E)$ acting on sections of a vector bundle E, provided P has a scalar, homogeneous principal symbol, see Definition 5.44. To extend the proof to this case, one fixes an arbitrary smooth fiber inner product on E. The main change is that the 'subprincipal' symbol p_1 is now endomorphism-valued, and hence so is $\tilde p_1 \in \mathcal{C}^\infty(S^*M; \mathrm{End}(\pi^* E))$. This is inconsequential however since the square root in (8.33) is still well-defined (using the power series expansion for $\sqrt{1-S}$ for $S \in \mathrm{End}(E_x)$ with $\|S\| \le \frac{1}{2}$).

8.5. **Exercises.**

Exercise 8.1 (Hands-on propagation of singularities)*.* Prove the following statements *without* using any of the machinery developed in this section. We work on $\mathbb{R}^n = \mathbb{R}_x \times \mathbb{R}^{n-1}_y$ and study the equation $D_x u(x,y) = f(x,y)$.

(1) Write covectors as $\xi\,\mathrm{d}x + \eta\,\mathrm{d}y$. Compute $\mathrm{Char}(D_x)$.
(2) Suppose $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. Show that $\mathrm{WF}(u) \subset \mathrm{Char}(D_x)$.
(3) Suppose $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $(x,y,0,\eta) \in \mathrm{WF}(u)$. Show that $(x+s,y,0,\eta) \in \mathrm{WF}(u)$ for all s.
(4) Let $f \in \mathscr{D}'(\mathbb{R}^n)$ and $(x,y,0,\eta) \in \mathrm{WF}(u)$. Suppose that $s_1 < 0 < s_2$ are such that $(x+s,y,0,\eta) \notin \mathrm{WF}(f)$ for all $s \in (s_1,s_2)$. Show that $(x+s,y,0,\eta) \in \mathrm{WF}(u)$ for $s \in (s_1,s_2)$.
(5) Let $u(x,y) = 1$ for $(x,y) \in [0,1]^2$ and $u(x,y) = 0$ otherwise. Compute $\mathrm{WF}(u)$ and $\mathrm{WF}(f)$ for $f = D_x u$. Describe the wave front set of u over $y = 1$ and its relationship to $\mathrm{WF}(f)$.

Exercise 8.2 (Semiglobal propagation of regularity)*.* Prove the estimate (8.11), in the strong sense, given any three operators $B, G, E \in \Psi^0(M)$ whose Schwartz kernels supported in a fixed compact subset of $M \times M$ and which satisfy the following two conditions:

(1) $\mathrm{WF}'(B) \subset \mathrm{Ell}(G)$,
(2) all backwards null-bicharacteristics of P starting from a point in $\mathrm{WF}'(B) \cap \mathrm{Char}(P)$ enter $\mathrm{Ell}(E)$ in finite time while remaining in $\mathrm{Ell}(G)$.

(*Hint.* Control u on $\mathrm{WF}'(B) \setminus \mathrm{Char}(P)$ using elliptic estimates. Near $\mathrm{Char}(P)$ on the other hand, exploit the localization properties of the specific operators B, G, E in Theorem 8.7 and piece together finitely many propagation estimates for such specific operators.)

Exercise 8.3 (Keldysh equation). Suppose $u \in \mathscr{D}'(\mathbb{R}^2)$ solves the *Keldysh equation*

$$(xD_x^2 + D_y^2)u = f \in \mathcal{C}^\infty(\mathbb{R}^2). \tag{8.59}$$

Assume that

$$\mathrm{WF}(u) \cap N^*\{x = 0\} = \emptyset. \tag{8.60}$$

Show that $u \in \mathcal{C}^\infty(\mathbb{R}^2)$. Show also that there exist solutions of the equation (8.59) which are not smooth (and which thus necessarily violate (8.60)).

Exercise 8.4 (Tricomi equation). Suppose $u \in \mathscr{D}'(\mathbb{R}^2)$ solves the *Tricomi equation* $(D_x^2 + xD_y^2)u = f \in \mathcal{C}^\infty(\mathbb{R}^2)$. Assume that $u = u(x, y)$ is smooth for $x < -1$. Show that $u \in \mathcal{C}^\infty(\mathbb{R}^2)$.

Exercise 8.5 (A simple system of equations). Suppose $u \in \mathscr{D}'(\mathbb{R}^2)$ satisfies $xu \in \mathcal{C}^\infty(\mathbb{R}^2)$ and $yu \in \mathcal{C}^\infty(\mathbb{R}^2)$.

(1) Show that $\mathrm{WF}(u) \subset T_0^*\mathbb{R}^2 \setminus o = \{(x, y, \xi, \eta) \colon (x, y) = (0, 0),\ (\xi, \eta) \neq (0, 0)\}$.
(2) Suppose that there exists $\alpha \in T_0^*\mathbb{R}^2 \setminus o$ with $\alpha \notin \mathrm{WF}(u)$. Show that $\mathrm{WF}(u) = \emptyset$.

9. Propagation of singularities at radial points

The propagation theorem proved in §8 is a general purpose tool for analyzing the regularity of solutions of general linear PDE $Pu = f$ when $P \in \Psi^m(M)$ has real homogeneous principal symbol, *assuming* one has information on u *somewhere* to begin with. In particular, in view of the (necessary) a priori control assumption of microlocal regularity of u (encoded by the term Eu in the estimate (8.11)), one *cannot*, in general, control u globally *only in terms of* f.

What is needed for global control of u is the existence of a subset of phase space S^*M where one can get unconditional control of u. There are two main situations in which this happens:

(1) initial value problems. Consider, as the simplest example, the *forcing problem* for the wave equation on $\mathbb{R}^n$,

$$\begin{cases} \Box u(t, x) = g(t, x), & t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ u(0, x) = D_t u(0, x) = 0, & x \in \mathbb{R}^n, \end{cases} \tag{9.1}$$

and assume that $t \geq 1$ on $\operatorname{supp} g$. By Corollary 7.2, equation (9.1) has a unique solution u, which is necessarily equal to 0 for $t < 1$. A fortiori, u is smooth there, and we can then analyze the regularity of u for later times using Theorem 8.5. This gives more information than Corollary 7.2, since we can precisely study situations where the forcing term g is smooth in some places but singular at others.

We remark that our discussion of hyperbolic evolution equations in §7 was based on a product decomposition of $\mathbb{R}^{n+1}$ into $\mathbb{R}_t \times \mathbb{R}^n_x$, starting already with the function space we used for g in (7.2); this is a sensible setting for the study of the operator $D_t - A(t)$ there, which is *not* a ps.d.o. in general, unless $A(t)$ is a differential operator. The wave operator in (9.1) can be analyzed both from this product perspective (§7) and from the 'spacetime' perspective (§8) in which one simply views $\Box$ as an operator $\Box \in \Psi^2(\mathbb{R}^{n+1})$.

(2) radial points (or other degeneracies) of P.

Let us give a simple example of an operator with radial points. Let $P = x \in \Psi^0(\mathbb{R}^n)$, $\mathbb{R}^n = \mathbb{R}_x \times \mathbb{R}_y^{n-1}$, be the multiplication operator, with principal symbol $p(x, y, \xi, \eta) = x$, characteristic set

$$\Sigma := \mathrm{Char}(P) = \{(x, y, \xi, \eta) \colon x = 0,\ (\xi, \eta) \neq (0, 0)\} \subset T^*\mathbb{R}^n \setminus o, \tag{9.2}$$

and Hamiltonian vector field $H_p = -\partial_\xi$. Suppose $u \in \mathscr{D}'(\mathbb{R}^n)$ solves the 'PDE'

$$Pu = xu = f \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{9.3}$$

Elliptic regularity (Proposition 6.27) or common sense imply that $\mathrm{WF}(u) \subset \Sigma$. By the propagation of singularities (Theorem 8.2), $\mathrm{WF}(u)$ is a union of maximally extended null-bicharacteristics of P. Note that at $(x, 0, \xi, 0) \in \Sigma$, $H_p = -\partial_\xi$ is radial; the null-bicharacteristic remains in the half-line $\{(x, 0, c\xi, 0) \colon c > 0\}$, hence the propagation theorem is vacuous there. Let us thus define the following two sets of radial points:

$$\mathcal{R}_\pm = \{(0, y, \xi, 0) \colon \pm\xi > 0\} \subset \Sigma. \tag{9.4}$$

Now, the general solution of the PDE (9.3) is of the form

$$u(x, y) = u_+(y)(x + i0)^{-1} + u_-(y)(x - i0)^{-1} + \tilde{u}(x, y), \tag{9.5}$$

where $u_\pm \in \mathscr{D}'(\mathbb{R}^{n-1})$, $u_+(y) + u_-(y) = f(0, y)$ (note that $u_\pm$ do *not* need to be smooth!), and $\tilde{u} \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Proposition 9.1 (Multiplication by x)**.** *Suppose $u \in \mathscr{D}'(\mathbb{R}^n)$, $xu = f \in \mathcal{C}^\infty(\mathbb{R}^n)$, and $\mathrm{WF}^{s_0}(u) \cap \mathcal{R}_+ = \emptyset$ for some $s_0 > -\frac{1}{2}$. Then $\mathrm{WF}(u) \subset \mathcal{R}_-$. Moreover, $\mathrm{WF}^s(u) = \emptyset$ for all $s < -\frac{1}{2}$.*

Proof. The key observation is that $(x \pm i0)^{-1} \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$ if and only if $s < -\frac{1}{2}$. The assumption thus implies that $u_+ \equiv 0$; thus $u_-(y) = f(0, y)$ is smooth, and the first conclusion follows from the fact that $\mathrm{WF}(u_+(y)(x - i0)^{-1}) \subset \mathcal{R}_-$, see Example 6.25. The second conclusion then follows again from the fact that $(x - i0)^{-1} \in H^{-1/2-\epsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $\epsilon > 0$. $\square$

This can be broken down into a concatenation of three arguments:

(1) if $\mathrm{WF}^{s_0}(u) \cap \mathcal{R}_+ = \emptyset$, then $\mathrm{WF}(u) \cap \mathcal{R}_+ = \emptyset$, hence u is microlocally smooth in a neighborhood of $\mathcal{R}_+$;
(2) by propagation of regularity, $\mathrm{WF}(u) \subset \mathcal{R}_-$;
(3) if $\mathrm{WF}(u)$ is disjoint from a punctured neighborhood of $\mathcal{R}_-$, then u is microlocally in H^s at $\mathcal{R}_-$ for all $s < -\frac{1}{2}$.

Parts (1) and (3) are special cases of a general result on the propagation of singularities/regularity at radial points proved below. A key feature is that there is a *threshold regularity*: if the microlocal regularity of u exceeds a threshold (here $-\frac{1}{2}$) at $\mathcal{R}_+$, then u is microlocally smooth at $\mathcal{R}_+$ (provided f is) and we can propagate H^s regularity out of $\mathcal{R}_+$ for $s > -\frac{1}{2}$; on the other hand, one can conclude microlocal regularity of u *below* this threshold when propagating *into* $\mathcal{R}_-$. The first microlocal radial point estimate appeared in the context of scattering theory on asymptotically Euclidean spaces in [Mel94] and takes place in scattering Sobolev spaces (see Exercises 4.8–4.15). The version we present here takes place in standard Sobolev spaces on precompact domains and originates in [Vas13]; see also [DZ19, Appendix E] for another presentation.

9.1. **Intermezzo: radial compactification of phase space.** We pause to describe a convenient and intuitive point of view for understanding qualitative properties of null-bicharacteristic flows.

Definition 9.2 (Radial compactification)**.** The *radial* (or *projective*) *compactification* of $\mathbb{R}^n$ is the set $\overline{\mathbb{R}^n} = \mathbb{R}^n \sqcup \mathbb{S}^{n-1}$, equipped with the structure of a manifold with boundary as follows: writing $0 \neq x \in \mathbb{R}^n$ in polar coordinates as $x = r\omega$, $r > 0$, $\omega \in \mathbb{S}^{n-1}$, then

$$\overline{\mathbb{R}^n} = \Big(\mathbb{R}^n \cup \big([0,\infty)_\rho \times \mathbb{S}^{n-1}\big)\Big) / \sim, \tag{9.6}$$

where $\mathbb{R}^n \ni r\omega \sim (r^{-1}, \omega)$. Thus, $\rho^{-1}(0) \cong \mathbb{S}^{n-1}$ is the 'sphere at infinity', and $\mathbb{R}^n \subset \overline{\mathbb{R}^n}$ is the interior.

Remark 9.3 (Smooth functions on $\overline{\mathbb{R}^n}$)*.* We have $\mathcal{C}^\infty(\overline{\mathbb{R}^n}) = S^0_{\rm cl}(\mathbb{R}^n)$: being smooth on $\overline{\mathbb{R}^n}$ precisely means having a Taylor expansion in $\rho = r^{-1}$ at $\rho = 0$. More generally, $S^\mu_{\rm cl}(\mathbb{R}^n) = \rho^{-\mu}\mathcal{C}^\infty(\overline{\mathbb{R}^n})$, in the sense that the space of restrictions of elements of $\rho^{-\mu}\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ is equal to $S^\mu_{\rm cl}(\mathbb{R}^n)$.

Convenient local coordinates near $\partial\overline{\mathbb{R}^n}$ are projective coordinates: write $x = (x_1, \ldots, x_n)$, and let us work in the subset of $\mathbb{R}^n$ where $x_1 > \epsilon \max(|x_2|, \ldots, |x_n|)$. We then let

$$\rho_1 := \frac{1}{x_1}, \quad \hat{x}_j := \frac{x_j}{x_1},\ j = 2, \ldots, n. \tag{9.7}$$

Then $(\rho_1, \hat{x}_2, \ldots, \hat{x}_n)$ (with $|\hat{x}_j| < \epsilon^{-1}$) is system of local coordinates on $\mathbb{R}^n$ which by continuity extends to a local coordinate system

$$[0,\infty)_{\rho_1} \times \{(\hat{x}_2, \ldots, \hat{x}_n) \colon |\hat{x}_j| < \epsilon^{-1},\ j = 2, \ldots, n\} \tag{9.8}$$

on $\overline{\mathbb{R}^n}$. Together with the standard coordinate system on $\mathbb{R}^n$, such coordinate systems (upon permuting indices and taking $\epsilon > 0$ small enough) cover $\overline{\mathbb{R}^n}$.

Lemma 9.4 (Invertible linear maps and radial compactifications)**.** *Let $A \in GL(n, \mathbb{R})$. Then matrix-vector multiplication $\mathbb{R}^n \ni x \mapsto Ax \in \mathbb{R}^n$ extends, by continuity, to a diffeomorphism $A \colon \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$.*

Proof. This is an easy verification in projective coordinate systems. □

This lemma allows us to define radial compactifications of vector bundles:

Definition 9.5 (Radial compactification of vector bundles)**.** Let $E \to M$ be a real rank k vector bundle. Then the radial compactification $\overline{E} \to M$ is the fiber bundle obtained by radially compactifying each fiber of E. (In local trivializations of E, the transition maps of $\overline{E}$ are the continuous extensions of those of E using Lemma 9.4.)

As microlocal analysts, we are interested in the radially compactified cotangent bundle

$$\overline{T^*}M \to M. \tag{9.9}$$

Note that for $p \in M$, we can identify $S^*_p M$ with the sphere at infinity of $\overline{T^*_p}M$; this embeds

$$S^*M \subset \overline{T^*}M \tag{9.10}$$

as a submanifold, called *fiber infinity*. We can now make the relationship between homogeneous vector fields and vector fields on S^*M more precise.

Lemma 9.6 (Homogeneous vector fields and compactifications)**.** *Suppose $V \in \mathcal{V}(T^*M \setminus o)$ is homogeneous of degree 0. Then V extends by continuity to a smooth vector field*

$$V \in \mathcal{V}(\overline{T^*}M \setminus o) \tag{9.11}$$

which is tangent to S^*M.

Proof. Indeed, in local coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ on T^*M, this means that

$$V = \sum_{j=1}^n a_j(x,\xi)\partial_{x_j} + b_{jk}(x,\xi)\xi_k\partial_{\xi_j}, \tag{9.12}$$

where $a_j(x,\lambda\xi) = a_j(x,\xi)$ and $b_{jk}(x,\lambda\xi) = b_{jk}(x,\xi)$ for all $\lambda > 0$. Let us work in projective coordinates

$$\rho = \frac{1}{\xi_1}, \quad \hat\xi_j = \frac{\xi_j}{\xi_1},\ j = 2,\ldots,n \tag{9.13}$$

in $\xi_1 > \epsilon\max(|\xi_2|,\ldots,|\xi_n|)$. Then $a_j(x,\xi) = a_j(x,(1,\hat\xi_2,\ldots,\hat\xi_n))$ is smooth down to $\rho = 0$, and so is b_{jk}. Moreover, $\partial_{x_j} \in \mathcal{V}(\overline{T^*}M)$. It remains to compute for $2 \le i,j \le n$:

$$\begin{aligned} \xi_1\partial_{\xi_1} &= -\rho\partial_\rho - \sum_{k=2}^n \hat\xi_k\partial_{\hat\xi_k}, \\ \xi_1\partial_{\xi_j} &= \partial_{\hat\xi_j}, \\ \xi_i\partial_{\xi_1} &= -\hat\xi_i\rho\partial_\rho - \sum_{k=2}^n \hat\xi_i\hat\xi_k\partial_{\hat\xi_k}, \\ \xi_i\partial_{\xi_j} &= \hat\xi_i\partial_{\hat\xi_j}. \end{aligned} \tag{9.14}$$

This proves the lemma. □

9.2. **Radial point estimates: a simple example.** In the coordinates used in (9.2), consider again the equation $Pu := xu = f$ and the Hamiltonian vector field $H_p = -\partial_\xi$. In projective coordinates

$$\big(\rho = \xi^{-1},\ \hat\eta = \frac{\eta}{\xi}\big), \qquad \xi > \epsilon|\eta|, \tag{9.15}$$

let us rescale this to the homogeneous degree 0 vector field

$$V = \xi H_p = -\xi\partial_\xi = \rho\partial_\rho + \hat\eta\partial_{\hat\eta}, \quad \rho > 0,\ |\hat\eta| < \epsilon^{-1}. \tag{9.16}$$

Restricting this to a vector field on $S^*\mathbb{R}^n$, the first term disappears, and we see that H_p being radial means that $V|_{S^*\mathbb{R}^n}$ vanishes on

$$\partial\mathcal{R}_+ := \{(x,y,\rho,\hat\eta) \colon x = 0,\ \hat\eta = 0\} \subset S^*\mathbb{R}^n, \tag{9.17}$$

the boundary of $\mathcal{R}_+$ (from (9.4)) at fiber infinity.

In our quest to prove microlocal estimates at $\partial\mathcal{R}_+$ via positive commutators, we therefore need to make use of the first summand in (9.16): we need to exploit that V has non-trivial behavior in the fiber-radial direction, that is, it acts non-trivially on differential weights $\rho^{-s} = \xi^s$. Concretely, consider a commutant

$$a = \rho^{-2s+1}\psi(\hat\eta), \tag{9.18}$$

where $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$ is identically 1 near 0, and satisfies $x\psi'(x) \le 0$ for all $x \in \mathbb{R}$. Then, in our projective coordinate system (9.15), we have

$$H_p a = \rho V a = \rho^{-2s}\big(-(2s+1)\psi(\hat\eta) + \hat\eta\psi'(\hat\eta)\big). \tag{9.19}$$

Thus, when $s > -\frac{1}{2}$, *both summands have the same (indefinite) sign* (namely, they are ≤ 0). Moreover, crucially, the first summand is *elliptic* at $\partial\mathcal{R}_+$. Thus, quantizing the calculation (9.19) as in §8, we can write

$$i[P, A] = -B^*B + R, \quad R \in \Psi^{2s-1}, \quad s > -\frac{1}{2}, \tag{9.20}$$

with $B \in \Psi^s$ elliptic at $\partial\mathcal{R}_+$. Ultimately, this gives an estimate for $\|Bu\|_{L^2}$, thus a microlocal H^s estimate of u at $\partial\mathcal{R}_+$, *without any a priori control.* Notice on the other hand that the 'positivity' (meaning: the 'good' sign, so negativity...) of the first term in (9.19) is delicate and limited; thus, error terms from the regularization argument can only be absorbed when the amount of regularization is limited, which will be the reason for an a priori regularity assumption at $\partial\mathcal{R}_+$.

Conversely, if $s < -\frac{1}{2}$, then the two terms in (9.19) have *opposite* signs, but the first summand is still elliptic at $\partial\mathcal{R}_+$. Thus, *assuming* H^s control of u on the support of the second summand (which is contained in a punctured neighborhood of $\partial\mathcal{R}_+$), we can *conclude* H^s regularity of u at $\partial\mathcal{R}_+$. (The situation at $\partial\mathcal{R}_-$ is completely analogous of course.)

9.3. **Radial point estimates: general setup.** We now set up the general theorem on the propagation of singularities/regularity at (generalized) radial points.

Thus, suppose $P \in \Psi^m_{\mathrm{cl}}(M)$ is a classical operator with real homogeneous principal symbol p.[19] Fix an elliptic symbol $0 \ne \rho \in S^{-1}_{\mathrm{cl}}(T^*M)$ and let

$$\tilde p := \rho^m p \in \mathcal{C}^\infty(\overline{T^*}M \setminus o), \quad \tilde H_p := \rho^{m-1}H_p \in \mathcal{V}(\overline{T^*}M \setminus o). \tag{9.21}$$

Suppose that

$$\mathcal{R} \subset \mathrm{Char}(P) \tag{9.22}$$

is a smooth submanifold to which $\tilde H_p$ is tangent. Suppose that $\mathrm{d}\tilde p \ne 0$ in a neighborhood of $\mathcal{R}$ in S^*M. For the sake of definiteness, we assume that $\mathcal{R}$ is a *source* for the $\tilde H_p$-flow, in the following precise sense:

(1) Suppose $\rho_{1,j} \in \mathcal{C}^\infty(S^*M)$, $j = 1, \ldots, k$, define $\mathcal{R}$ inside $\mathrm{Char}(P)$, in the sense that

$$\mathcal{R} = \{\tilde p = 0,\ \rho_{1,1} = \cdots = \rho_{1,k} = 0\}, \tag{9.23}$$

[19]It suffices to assume that $P \in \Psi^m(M)$, with real homogeneous principal symbol. The only change is that in equation (9.27) below, $\tilde\beta \in S^0$ is not necessarily smooth on $\overline{T^*}M$; what enters in the threshold quantities in Theorems 9.8 and 9.9 below is then the supremum or infimum of $\tilde\beta$, whichever gives the stronger requirement.

and $\mathrm{d}\rho_{1,1}, \ldots, \mathrm{d}\rho_{1,k}$ are linearly independent at $\mathcal{R}$. Let

$$\rho_1 = \sum_{j=1}^{k} \rho_{1,j}^2, \tag{9.24}$$

which is a 'quadratic defining function' of $\mathcal{R}$. Since $\tilde{H}_p$ is tangent to $\mathcal{R}$, the derivatives $\tilde{H}_p\rho_{1,j}$ vanish at $\mathcal{R}$, hence $\tilde{H}_p\rho_1$ vanishes quadratically at $\mathcal{R}$. We then assume that there exists a positive function $0 < \beta_1 \in \mathcal{C}^\infty(S^*M)$ such that

$$\tilde{H}_p\rho_1 = \beta_1\rho_1 + F_2 + F_3, \tag{9.25}$$

where $F_2 \geq 0$, and F_3 vanishes cubically at $\mathcal{R}$. (Thus, $\mathcal{R}$ is a source for the $\tilde{H}_p$-flow *within* $\mathrm{Char}(P) \subset S^*M$ since $|F_3| \leq C\rho_1^{3/2} \leq \frac{1}{2}\beta_1\rho_1$ near $\mathcal{R}$, so $\tilde{H}_p\rho_1 \geq \frac{1}{2}\beta_1\rho_1$; cf. the behavior in the $\hat{\eta}$-variables in (9.16).)

(2) We have

$$\tilde{H}_p\rho = \beta_0\rho, \quad \beta_0|_{\mathcal{R}} > 0. \tag{9.26}$$

Note that since $\tilde{H}_p$ is tangent to S^*M, $\tilde{H}_p\rho$ vanishes there, hence is of the stated form with $\beta_0 \in \mathcal{C}^\infty(\overline{T^*}M)$ near $\mathcal{R}$. (The assumption (9.26) implies that $\mathcal{R}$ is a source for the $\tilde{H}_p$-flow also in the fiber-radial direction.)

The subprincipal part of P at $\mathcal{R}$ now plays a significant role, too:[20]

(3) Let $p_1 := \sigma^{m-1}(\frac{1}{2i}(P - P^*))$ and $\tilde{p}_1 := \rho^{m-1}p_1$. Define $\tilde{\beta} \in \mathcal{C}^\infty(S^*M)$ near $\mathcal{R}$ by

$$\tilde{p}_1 = \beta_0\tilde{\beta}. \tag{9.27}$$

Remark 9.7 (Choices). Condition (1) is independent of choices, and the positivity of β_0 in (9.26) does not depend on the choice of ρ in the case that $\tilde{H}_p$ vanishes at $\mathcal{R}$; in general, when $\tilde{H}_p$ is only tangent to $\mathcal{R}$, the choice of ρ *does* matter (but only through the derivative taken in (9.26), not through the rescaling in (9.21).

We state the main result of this section in two forms, one qualitative (analogous to Theorem 8.5), one quantitative (analogous to Theorem 8.7).

Theorem 9.8 (Microlocal regularity at radial sets: qualitative statement)**.** *Let P and $\mathcal{R} \subset \mathrm{Char}(P) \subset S^*M$ be as above. Let $u \in \mathscr{D}'(M)$, $Pu = f$.*

(1) *(Propagation out of the radial set.) Let $s, s_0 \in \mathbb{R}$, and suppose that $s > s_0 > \frac{m-1}{2} + \tilde{\beta}$ on $\mathcal{R}$. If $\mathrm{WF}^{s_0}(u) \cap \mathcal{R} = \emptyset$ and $\mathrm{WF}^{s-m+1}(f) \cap \mathcal{R} = \emptyset$, then $\mathrm{WF}^s(u) \cap \mathcal{R} = \emptyset$.*

(2) *(Propagation into the radial set.) Let $s \in \mathbb{R}$, and suppose $s < \frac{m-1}{2} + \tilde{\beta}$ on $\mathcal{R}$. If $\mathrm{WF}^s(u)$ is disjoint from a punctured neighborhood of $\mathcal{R}$, and if $\mathrm{WF}^{s-m+1}(f) \cap \mathcal{R} = \emptyset$, then $\mathrm{WF}^s(u) \cap \mathcal{R} = \emptyset$.*

The quantitative version (and also slightly more global, though the difference can be bridged using the propagation estimates of Exercise 8.2) is the following:

[20] For a simple example, consider $P = xD_x - \lambda \in \Psi^1(\mathbb{R})$. Then $Pu = 0$ e.g. for $u = x_+^{i\lambda}$, suggesting that the threshold regularity at the radial sets $T_0^*\mathbb{R} \setminus o$ is $\frac{1}{2} - \mathrm{Im}\,\lambda$ (which is the Sobolev regularity which $x^{i\lambda}$ barely fails to have). And indeed, $\frac{1}{2} - \mathrm{Im}\,\lambda = \frac{1}{2i}(P - P^*)$ is the skew-adjoint part of P. (Any additional terms of even lower order do *not* contribute to the threshold regularity.)

Theorem 9.9 (Microlocal regularity at radial sets, quantitative statement)**.** *Let P and $\mathcal{R} \subset \mathrm{Char}(P) \subset S^*M$ be as above. Let $u \in \mathscr{D}'(M)$, $Pu = f$.*

(1) *(Propagation out of $\mathcal{R}$.) Let $B, G \in \Psi^0(M)$ be such that*
  (a) $\mathrm{WF}'(B) \subset \mathrm{Ell}(G)$*;*
  (b) $\mathrm{Ell}(G)$ *contains a neighborhood of $\mathcal{R}$;*
  (c) *all backward null-bicharacteristics of P from* $\mathrm{WF}'(B) \cap \mathrm{Char}(P)$ *tend to $\mathcal{R}$ (that is, ρ_1 tends to 0 along them) while remaining in* $\mathrm{Ell}(G)$.

  Then for all $s, s_0, N \in \mathbb{R}$ such that $s > s_0 > \frac{m-1}{2} + \tilde{\beta}$ on $\mathcal{R}$, there exists $C > 0$ such that if $\mathrm{WF}^{s_0}(u) \cap \mathcal{R} = \emptyset$, then

$$\|Bu\|_{H^s} \leq C\big(\|GPu\|_{H^{s-m+1}} + \|u\|_{H^{-N}}\big). \tag{9.28a}$$

  This estimate does not *hold in the usual strong sense. However, if $\tilde{B} \in \Psi^0(M)$ is elliptic at $\mathcal{R}$, then the estimate*

$$\|Bu\|_{H^s} \leq C\big(\|GPu\|_{H^{s-m+1}} + \|\tilde{B}u\|_{H^{s_0}} + \|u\|_{H^{-N}}\big) \tag{9.28b}$$

  does *hold in the strong sense that if all quantities on the right are finite, then so is the left hand side, and the estimate holds.*

(2) *(Propagation into $\mathcal{R}$.) Let $B, G, E \in \Psi^0(M)$ be such that*
  (a) $\mathrm{WF}'(B) \subset \mathrm{Ell}(G)$*;*
  (b) *all forward null-bicharacteristics of P from* $\mathrm{WF}'(B) \cap \mathrm{Char}(P)$ *are either contained in $\mathcal{R}$, or enter* $\mathrm{Ell}(E)$ *in finite time, all while remaining in* $\mathrm{Ell}(G)$.

  Then for all $s, N \in \mathbb{R}$ such that $s < \frac{m-1}{2} + \tilde{\beta}$ on $\mathcal{R}$, there exists $C > 0$ such that

$$\|Bu\|_{H^s} \leq C\big(\|GPu\|_{H^{s-m+1}} + \|Eu\|_{H^s} + \|u\|_{H^{-N}}\big). \tag{9.29}$$

  This estimate holds in the usual strong sense.

The proof will require a secondary regularization argument, which will use the following lemma:

Lemma 9.10 (Strong convergence of ps.d.o.s)**.** *Suppose $A_\epsilon \in L^\infty((0,1]_\epsilon; \Psi^m)$ is uniformly bounded, and $A_\epsilon \to A$ in $\Psi^{m+\eta}$ as $\epsilon \to 0$, for all $\eta > 0$. Then A_ϵ converges strongly to A in $\mathcal{L}(H^s; H^{s-m})$; that is, for any $u \in H^s$, we have $A_\epsilon u \to Au$ in H^{s-m} as $\epsilon \to 0$.*

Proof. If $u \in H^{s+1}$, then we certainly have $A_\epsilon u \to Au$ in H^{s-m}. Given $u \in H^s$ and $\rho > 0$, choose $u' \in H^{s+1}$ with $\|u - u'\|_{H^s} < \rho$. Let then $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$, we have $\|A_\epsilon u' - Au'\|_{H^{s-m}} < \rho$. Then for such ϵ,

$$\begin{aligned} \|A_\epsilon u - Au\|_{H^{s-m}} &\leq \|A_\epsilon(u - u')\|_{H^{s-m}} + \|A_\epsilon u' - Au'\|_{H^{s-m}} + \|A(u - u')\|_{H^{s-m}} \\ &\leq C\rho + \rho + C\rho, \end{aligned} \tag{9.30}$$

where $C = \sup \|A_\epsilon\|_{\mathcal{L}(H^s, H^{s-m})}$. □

Proof of Theorems 9.8 and 9.9. We follow the steps of the positive commutator argument in §§8.2–8.4.

• Construction of the commutant for part (1). With $\tilde{p}$ as in (9.21), the quadratic defining function of $\mathcal{R}$, ρ_1, as in (9.24)–(9.25), and the defining function of fiber infinity, ρ, as used in (9.26), we set

$$a := \rho^{-2s+m-1}\phi(\tilde{p})^2\psi(\rho_1)^2. \tag{9.31}$$

Here, $\phi \in \mathcal{C}^\infty_c(\mathbb{R}; [0,1])$, $\phi(0) = 1$, so $\phi(\tilde p)$ localizes near $\mathrm{Char}(P)$; and $\psi \in \mathcal{C}^\infty_c([0,\infty); [0,1])$ is 1 near 0 (so $\psi(\rho_1)$ localizes further near $\mathcal{R}$) and satisfies $\sqrt{-\psi'\psi} \in \mathcal{C}^\infty([0,\infty))$. (The latter assumption only requires a bit of thought near the boundary of $\operatorname{supp}\psi$. Taking ψ to be a variant of $e^{-1/x}H(x)$ there does the job.) Write

$$\tilde H_p \tilde p = \tilde q \tilde p, \quad \tilde q = \rho^{-m}\tilde H_p \rho^m, \tag{9.32}$$

where $\tilde q$ is smooth near $\mathcal{R}$. We then compute the symbol of $i[P,A] + i(P^* - P)A = i[P,A] + 2\frac{P-P^*}{2i}A$ to be

$$\begin{aligned} H_p a + 2p_1 a &= \rho^{-m+1}(\tilde H_p a + 2\tilde p_1 a) \\ &= \rho^{-2s}\Big(\beta_0(-2s + m - 1 + 2\tilde\beta)\phi(\tilde p)^2\psi(\rho_1)^2 \\ &\qquad + 2(\tilde H_p\rho_1)\phi(\tilde p)^2\psi'(\rho_1)\psi(\rho_1) \\ &\qquad + 2\tilde q\tilde p\phi'(\tilde p)\phi(\tilde p)\psi(\rho_1)^2\Big). \end{aligned} \tag{9.33}$$

When the support of ψ and ϕ is sufficiently small, the terms in the parenthesis here play the following roles:

(1) the first is elliptic (and negative) at $\mathcal{R}$ under the assumptions on s;
(2) the second is non-positive as well, and has essential support contained in a punctured neighborhood of $\mathcal{R}$;
(3) the third is supported away from $\mathrm{Char}(P)$, hence can be dealt with using elliptic regularity.

At this point, one can already prove the estimate (9.28a) as in §8.3; we leave this to the reader.

• Regularization of commutant for part (1). We regularize our commutant a as in §8.4, see equations (8.44)–(8.45), though with slightly different notation. Thus, let now $K > 0$ and $r \geq 0$, and put

$$\phi_r(\rho) = (1 + r\rho^{-1})^{-K}; \qquad \rho\phi_r'(\rho) = f_r(\rho)\phi_r(\rho), \quad f_r(\rho) = K\frac{r\rho^{-1}}{1 + r\rho^{-1}}\phi_r, \tag{9.34}$$

so $\phi_r \in L^\infty((0,1]_r; S^0)$, and $\phi_r \in S^{-K}$ for $r > 0$. We then let

$$a_r := \phi_r(\rho)^2 a \in L^\infty((0,1]_r; S^{2s-m+1}), \tag{9.35}$$

and compute

$$\begin{aligned} H_p a_r + 2p_1 a_r = \rho^{-2s}\phi_r(\rho)^2\Big(&\beta_0(-2s + m - 1 + 2\tilde\beta + 2f_r)\phi(\tilde p)^2\psi(\rho_1)^2 \\ &+ 2(\tilde H_p\rho_1)\phi(\tilde p)^2\psi'(\rho_1)\psi(\rho_1) \\ &+ 2\tilde q\tilde p\phi'(\tilde p)\phi(\tilde p)\psi(\rho_1)^2\Big). \end{aligned} \tag{9.36}$$

Note that since $0 \leq f_r \leq K$, the amount K of regularization we can do is limited when propagating out of the radial set $\mathcal{R}$: in order to ensure that the first term is negative at $\mathcal{R}$, we need $s - K > \frac{m-1}{2} + \tilde\beta$, restricting K. Fix such $K > 0$. For $\delta > 0$ chosen so small that still $s - K - \delta\beta_0^{-1} > \frac{m-1}{2} + \tilde\beta$ on $\mathcal{R}$, we then write

$$H_p a_r + 2p_1 a_r = -2\delta\rho^{2s-2m+2}a_r^2 - b_r^2 - b_{1,r}^2 + h_r p, \tag{9.37}$$

where

$$
\begin{aligned}
b_r &:= \rho^{-s}\phi_r(\rho)\phi(\tilde p)\psi(\rho_1)\sqrt{\beta_0\bigl(2s-\bigl(m-1+2\tilde\beta+2f_r+2\delta\beta_0^{-1}\phi_r(\rho)^2\phi(\tilde p)^2\psi(\rho_1)^2\bigr)\bigr)},\\
b_{1,r} &:= \rho^{-s}\phi_r(\rho)\phi(\tilde p)\sqrt{-2(\tilde H_p\rho_1)\psi'(\rho_1)\psi(\rho_1)},\\
h_r &:= 2\rho^{-2s+m}\phi_r(\rho)^2\tilde q\phi'(\tilde p)\phi(\tilde p)\psi(\rho_1)^2.
\end{aligned}
\tag{9.38}
$$

Thus, $b_r, b_{1,r}\in L^\infty((0,1]_r;S^s)$ and $h_r\in L^\infty((0,1]_r,S^{2s-m})$, with orders reduced by K,K, and $2K$, respectively, for $r>0$.

• Quantization of the symbol calculation; conclusion of the proof of part (1). Let $A_r=\mathrm{Op}(a_r)$, $B_r=\mathrm{Op}(b_r)$, $B_{1,r}=\mathrm{Op}(b_{1,r})$, and $H_r=\mathrm{Op}(h_r)$, using a full quantization as in (8.51). Put $\Lambda=\mathrm{Op}(\rho^{s-m+1})$. Then (9.37) and (9.38) imply

$$
i[P,A_r]+i(P^*-P)A_r=-2\delta(\Lambda A_r)^*(\Lambda A_r)-B_r^*B_r-B_{1,r}^*B_{1,r}+H_rP+R_r, \tag{9.39}
$$

where $R_r\in L^\infty((0,1]_r;\Psi^{2s-1})$, with $\mathrm{WF}'_{L^\infty}(\{R_r\})\subset\operatorname{ess\,supp}a$.

Now, recall that we are assuming $\mathrm{WF}^{s_0}(u)\cap\mathcal R=\emptyset$; let $\tilde B\in\Psi^0$ be elliptic at $\mathcal R$ and such that $\tilde Bu\in H^{s_0}$. Fix $K>0$ such that

$$
\frac{m-1}{2}+\tilde\beta<s-K<s_0, \tag{9.40}
$$

and choose the support of our cutoffs so small that $\operatorname{ess\,supp}a\subset\mathrm{Ell}(\tilde B)$. Since $A_r\in\Psi^{2s-m+1-2K}\subset\Psi^{2s_0-m+1}$ for $r>0$, we have $A_ru\in H^{-s_0+m-1}$. We want to compute

$$
\begin{aligned}
2\operatorname{Im}\langle Pu,A_ru\rangle &= i\bigl(\langle Pu,A_ru\rangle-\langle A_ru,Pu\rangle\bigr)\\
&=\bigl\langle\bigl(i[P,A_r]+i(P^*-P)A_r\bigr)u,u\bigr\rangle.
\end{aligned}
\tag{9.41}
$$

All terms make sense individually using Lemma 6.34 (since $\mathrm{WF}^{s_0-m+1}(Pu)\cap\mathrm{WF}'(A_r)=\emptyset$ and since the operator in the second line lies in Ψ^{2s_0}). However, the integration by parts needs to be justified, since for *general* $u\in H^{s_0}$, one only has $Pu\in H^{s_0-m}$, which is in general insufficient to justify the integration by parts. This is easily accomplished by inserting yet another regularizer, $J_\epsilon\in L^\infty((0,1]_\epsilon;\Psi^0)$, with $J_\epsilon=J_\epsilon^*\in\Psi^{-\infty}$ for $\epsilon>0$, and $J_\epsilon\to I$ in the topology of Ψ^η, $\eta>0$, and using Lemma 9.10. Namely,

$$
\begin{aligned}
\langle Pu,A_ru\rangle-\langle A_ru,Pu\rangle &= \lim_{\epsilon\to0}\bigl(\langle Pu,J_\epsilon A_ru\rangle-\langle A_ru,J_\epsilon Pu\rangle\bigr)\\
&=\lim_{\epsilon\to0}\langle(A_rJ_\epsilon P-P^*J_\epsilon A_r)u,u\rangle\\
&=\lim_{\epsilon\to0}\bigl(\langle J_\epsilon(A_rP-P^*A_r)u,u\rangle+\langle([A_r,J_\epsilon]P-[P^*,J_\epsilon]A_r)u,u\rangle\bigr)\\
&=\langle(A_rP-P^*A_r)u,u\rangle.
\end{aligned}
\tag{9.42}
$$

Note here that $[A_r,J_\epsilon]$ is uniformly bounded (in $\epsilon\in(0,1]$, for $r>0$ *fixed*) in Ψ^{2s_0-m} and converges to 0 in $\Psi^{2s_0-m+\eta}$, $\eta>0$, hence $[A_r,J_\epsilon]Pu\to0$ in H^{-s_0}, and therefore $\langle[A_r,J_\epsilon]Pu,u\rangle\to0$ as $\epsilon\to0$; likewise, $\langle[P^*,J_\epsilon]A_ru,u\rangle\to0$ as $\epsilon\to0$ for fixed $r>0$.

We proceed to rewrite the right hand side of the pairing (9.41) by plugging in (9.39). Let $G\in\Psi^0$, $\mathrm{WF}'(I-G)\cap\mathrm{WF}'_{L^\infty}(\{A_r\})=\emptyset$. Then the Peter–Paul inequality and Lemma 8.11

give the estimate

$$\begin{aligned} &\|B_r u\|_{L^2}^2 + \|B_{1,r}u\|_{L^2}^2 + 2\delta\|A_r u\|_{H^{-s+m-1}}^2 \\ &\qquad \le 2\delta\|A_r u\|_{H^{-s+m-1}}^2 + \delta^{-1}\|GPu\|_{H^{s-m+1}}^2 \\ &\qquad\qquad + |\langle Pu, H_r u\rangle| + C\big(\|Gu\|_{H^{s-1/2}}^2 + \|u\|_{H^{-N}}^2\big) \end{aligned} \tag{9.43}$$

for an r-independent constant C. The first term in the last line can be estimated (using $\mathrm{WF}'_{L^\infty}(\{H_r\}) \cap \mathrm{WF}'(I-G) = \emptyset$) by

$$\begin{aligned} |\langle Pu, H_r u\rangle| &\le C\big(\|GPu\|_{H^{s-m+1}}^2 + \|H_r u\|_{H^{-s+m-1}}^2 + \|u\|_{H^{-N}}^2\big) \\ &\le C\big(\|GPu\|_{H^{s-m+1}}^2 + \|Gu\|_{H^{s-1}}^2 + \|u\|_{H^{-N}}^2\big); \end{aligned} \tag{9.44}$$

recall that $H_r \in L^\infty((0,1];\Psi^{2s-m})$. Combined with (9.43), and an iterative argument (improving the regularity by $1/2$ in each step) as usual, we finally obtain the uniform estimate

$$\|B_r u\|_{L^2} \le C\big(\|GPu\|_{H^{s-m+1}} + \|u\|_{H^{-N}}\big). \tag{9.45}$$

(Recall that our proof of this estimate requires that $\tilde B u \in H^s$.) As in §8.4, we thus conclude that $B_0 u \in L^2$, in particular $\mathrm{WF}^s(u) \cap \mathcal{R} = \emptyset$, together with an estimate of $\|B_0 u\|_{L^2}$ by the right hand side of (9.45). This proves the estimate (9.28a).

• Modifications for part (2). The propagation of microlocal regularity *into* a radial point uses the same commutant; now the degree K of regularization is arbitrary. Indeed, in the calculation (9.36), the first term (which is the main term, elliptic at $\mathcal{R}$) is now *positive* (and only gets more positive with more regularization), and thus has the opposite sign of the second term. One thus now writes

$$H_p a_r + 2p_1 a_r = 2\delta\rho^{2s-2m+2}a_r^2 + b_r^2 - b_{1,r}^2 + h_r p, \tag{9.46}$$

where $b_{1,r}$, h_r are as in (9.38), and

$$b_r := \rho^{-s}\phi_r(\rho)\phi(\tilde p)\psi(\rho_1)\sqrt{\beta_0\big(-2s+m-1+2\tilde\beta+2f_r-2\delta\beta_0^{-1}\phi_r(\rho)^2\phi(\tilde p)^2\psi(\rho_1)^2\big)}. \tag{9.47}$$

Upon quantizing this, we get a uniform estimate

$$\|B_r u\|_{L^2} \le C\big(\|GPu\|_{H^{s-m+1}} + \|B_{1,r}u\|_{L^2} + \|Gu\|_{H^{s-1/2}} + \|u\|_{H^{-N}}\big). \tag{9.48}$$

Thus, we now have an a priori control term $\|B_{1,r}u\|_{L^2}$: it is uniformly bounded if $\mathrm{WF}^s(u)$ is disjoint from a punctured neighborhood of $\mathcal{R}$. (Note that $\mathrm{WF}'_{L^\infty}(B_{1,0})$ is some small positive distance away from $\mathcal{R}$, hence this appears stronger than merely assuming $B_{1,0}u \in L^2$; but from $B_{1,0}u \in L^2$, one can conclude that $\mathrm{WF}^s(u)$ is disjoint from a punctured neighborhood of $\mathcal{R}$ using the propagation of regularity, Theorem 8.5.) The study of the limit $r \to 0$ thus gives $B_0 u \in L^2$, hence $\mathrm{WF}^s(u) \cap \mathcal{R} = \emptyset$, and (after an iterative argument improving the regularity by $1/2$ at each step) the uniform estimate

$$\|B_0 u\|_{L^2} \le C\big(\|GPu\|_{H^{s-m+1}} + \|Eu\|_{H^s} + \|u\|_{H^{-N}}\big) \tag{9.49}$$

for $E \in \Psi^0$ with $\mathrm{Ell}(E) \supset \mathrm{WF}'_{L^\infty}(\{B_{1,r}\})$. □

Remark 9.11 (Bundles). Paralleling Remark 8.12, we point out that Theorems 9.8 and 9.9 apply also to ps.d.o.s $P \in \Psi^m_{\mathrm{cl}}(M;E)$ acting between sections of vector bundles, provided P has a real scalar principal symbol. Now, a subprincipal term of P modifies the threshold regularity; and in fact the mere definition of $\tilde\beta$ in equation (9.27) requires the choice of a fiber inner product on E. Thus, in applications, one typically needs to choose this fiber inner product carefully in order to obtain the strongest possible conclusions under the weakest possible assumptions in these theorems. (Note that this is still a purely symbolic calculation, hence straightforward, even if occasionally a bit lengthy in practice.)

10. Asymptotic behavior of linear waves on de Sitter space

We now show, following [Vas13] (see also [Zwo16]) how the tools developed so far can be used for a description of the precise asymptotic (late time) behavior of solutions of wave equations on spacetimes of interest in general relativity. Concretely, we shall consider *de Sitter space*, or rather a subset of it called the *static patch* (or *static model*) of de Sitter space (M, g) which is a solution of Einstein's vacuum equation with cosmological constant $\Lambda > 0$,

$$\mathrm{Ric}(g) + \Lambda g = 0, \tag{10.1}$$

where Ric denotes the Ricci curvature of g.

We first give a quick introduction to Lorentzian metrics and wave equations in §10.1 before studying the wave equation on static de Sitter space in §§10.2–10.3.

10.1. Lorentzian geometry and wave operators.

Definition 10.1 (Lorentzian manifold)**.** Let M be an n-dimensional manifold, $n \geq 2$. Let $g \in \mathcal{C}^\infty(M; S^2\,T^*M)$, so $g_p = g(p)$, $p \in M$, is a bilinear form on T_pM depending smoothly on p. Then g is a *Lorentzian metric* if g_p has signature $(1, n-1)$ (sometimes written $(+, -, \ldots, -)$) for all p. We call (M, g) a *Lorentzian manifold*.

This means that at any $p \in M$, there exists a basis $V_1, \ldots, V_n$ of T_pM such that

$$g(V_1, V_1) = 1; \qquad g(V_j, V_j) = -1,\ j = 2, \ldots, n; \qquad g(V_i, V_j) = 0,\ i \neq j. \tag{10.2}$$

Since a Lorentzian metric g is a *non-degenerate* bilinear form on T_pM, it induces an isomorphism $T_pM \to T_p^*M$ via $V \mapsto g_p(V, -)$. Thus, g induces a signature $(1, n-1)$ bilinear form on T_p^*M, denoted G or $g^{-1} \in \mathcal{C}^\infty(M; S^2TM)$ and called the *dual metric*.

Example 10.2. Let $M = \mathbb{R}^n = \mathbb{R}_t \times \mathbb{R}^{n-1}_x$. Then the *Minkowski metric* on M is

$$g = \mathrm{d}t^2 - \sum_{j=1}^{n-1} \mathrm{d}x_j^2. \tag{10.3}$$

The dual metric is

$$G = \partial_t^2 - \sum_{j=1}^{n-1} \partial_{x_j}^2 = \partial_t \otimes \partial_t - \sum_{j=1}^{n-1} \partial_{x_j} \otimes \partial_{x_j} \tag{10.4}$$

so for instance $G(\mathrm{d}t, \mathrm{d}t) = 1$.

Definition 10.3 (Tangent vectors in spacetimes)**.** Let (M, g) be a Lorentzian manifold.

(1) Let $p \in M$. Then we say that a tangent vector $V \in T_pM$ is
- *timelike* if $g_p(V, V) > 0$,
- *spacelike* if $g_p(V, V) < 0$,
- *null* or *lightlike* if $g_p(V, V) = 0$.

Likewise, one can classify covectors $\zeta \in T_p^*M$ as timelike, spacelike, or null, depending on the sign of $G_p(\zeta, \zeta)$.

(2) Let $S \subset M$ be a smooth hypersurface. Then S is *spacelike* if for all $p \in S$ and $\zeta \in N^*S \setminus o$, the covector ζ is timelike.

Physically, massive observers (like myself) travel along timelike curves in M (curves with timelike tangent vectors), and massless particles (think of photons) travel along lightlike curves.

Example 10.4. In the notation of Example 10.2, the vector $\partial_t + v_1\partial_{x_1}$ is timelike iff $|v_1| < 1$, null iff $|v_1| = 1$, and spacelike iff $|v_1| > 1$. The hypersurface $\{t = 0\}$ is spacelike; indeed, its conormal bundle is spanned by $\mathrm{d}t$, which is timelike. More generally, for $v \in \mathbb{R}^{n-1}$, the hypersurface $\{t = v \cdot x\}$ is spacelike if and only if $|v| < 1$.

Note that the set of timelike vectors is a solid cone with the vertex removed, thus has two connected components. A continuous choice of one of them is called a *time orientation* of (M, g). This does not exist in general. If it does, there exists a smooth timelike vector field V on M; we then say that a timelike/null vector W is *future* timelike/null if $g(V, W) > 0$. (In particular, V is future timelike.)

Given a Lorentzian manifold (M, g), we define the *wave operator* $\Box_g \in \mathrm{Diff}^2(M)$ by the *same* formula as the Laplace operator on a Riemannian manifold: in local coordinates $(z_1, \ldots, z_n)$ on M, we write $g_{ij} = g(\partial_{z_i}, \partial_{z_j})$, $g^{ij} = G(\mathrm{d}z_i, \mathrm{d}z_j)$, and $|g| = |\det(g_{ij})|$; then

$$\Box_g u := \sum_{i,j=1}^{n} |g|^{-1/2} D_{z_i}\big(|g|^{1/2} g^{ij} D_{z_j} u\big). \tag{10.5}$$

Its principal symbol is the *dual metric function*

$$G(\zeta) := \sigma^2(\Box_g)(\zeta) = \sum_{i,j=1}^{n} g^{ij}\zeta_i\zeta_j = |\zeta|^2_{G_p}, \quad \zeta \in T^*_pM. \tag{10.6}$$

Thus, the characteristic set $\mathrm{Char}(\Box_g) = \{\zeta \in T^*M \setminus o \colon G(\zeta) = 0\}$ consists of all lightlike covectors.

Remark 10.5 (Null-bicharacteristic and null geodesics). Integral curves of H_p are the lift to T^*M of geodesics of (M, g). Recall that for a geodesic $\gamma\colon I \subset \mathbb{R} \to M$, the squared length $g_{\gamma(s)}(\gamma'(s), \gamma'(s))$ is constant; we then call a geodesic with squared length 0 (i.e. $\gamma'(s)$ is null for all s) a *null-geodesic*. Correspondingly, singularities of solutions of the wave equation $\Box_g u = f$ propagate along null-geodesics inside of $\mathrm{Char}(\Box_g)$.

Definition 10.6 (Wave-type operators)**.** Let $E \to M$ be a vector bundle over the Lorentzian manifold (M, g). We say that $P \in \mathrm{Diff}^2(M; E)$ is a *wave-type operator* if P is principally scalar with $\sigma^2(P) = G$, where $G(\zeta) = g^{-1}(\zeta, \zeta)$ is the dual metric function.

Typical examples include the scalar wave operator $\Box_g$, or the tensor wave operator $-\operatorname{tr}_g \nabla^2$, or modifications of such operators by first and zeroth order terms.

We record here an existence and uniqueness statement for the wave equation whose proof we omit.

Proposition 10.7 (Solvability of forward problems)**.** *Let $P \in \mathrm{Diff}^2(M; E)$ be a wave-type operator on a Lorentzian manifold (M, g). Suppose $t \in \mathcal{C}^\infty(M)$ is a timelike function, i.e.* $\mathrm{d}t$ *is everywhere timelike. Suppose $\Omega \subset M$ is a domain with spacelike boundary, and suppose $\Omega_0 := \bar{\Omega} \cap t^{-1}([0, \infty))$ is compact. Then, given any $f \in \mathcal{C}^\infty(\Omega_0; E)$ such that*

$\operatorname{supp} f \subset \{t \geq 0\}$, *there exists a unique* $u \in \mathcal{C}^\infty(\Omega; E)$ *with* $\operatorname{supp} u \subset \{t \geq 0\}$ *such that* $Pu = f$ *in* Ω.

This for example applies to wave-type operators on Minkowski space (Example 10.2) for domains $\Omega = \{t < F(x)\}$, where $F \in \mathcal{C}^\infty(\mathbb{R}^{n-1})$ satisfies $|F'(x)| < 1$ for all x, and $F(x) \to -\infty$ when $|x| \to \infty$. By a simple approximation argument, one can also take functions like $F(x) = 1 - c|x|$ when $c < 1$.

10.2. **Waves on the static model of de Sitter space.** From now on, we shall work on a particular 3-dimensional Lorentzian manifold. (This can all be generalized significantly, of course, but we stick to a concrete setting for simplicity of presentation.)

Definition 10.8 (de Sitter space)**.** We define 3-dimensional de Sitter space (M, g) by

$$\begin{aligned} M &:= \mathbb{R}_t \times \{x \in \mathbb{R}^2 \colon |x| < 2\}, \\ g &:= (1 - |x|^2)\mathrm{d}t^2 + (\mathrm{d}t \otimes (x \cdot \mathrm{d}x) + (x \cdot \mathrm{d}x) \otimes \mathrm{d}t) - \mathrm{d}x^2, \end{aligned} \tag{10.7}$$

where we write $x \cdot \mathrm{d}x = x_1\,\mathrm{d}x_1 + x_2\,\mathrm{d}x_2$, and $\mathrm{d}x^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2$.

It is easier to work with polar coordinates (r, θ) on $\mathbb{R}^2$, in which

$$\begin{aligned} g &= (1 - r^2)\mathrm{d}t^2 + (\mathrm{d}t \otimes r\,\mathrm{d}r + r\,\mathrm{d}r \otimes \mathrm{d}t) - \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2, \\ G = g^{-1} &= \partial_t^2 + (\partial_t \otimes r\partial_r + r\partial_r \otimes \partial_t) - (1 - r^2)\partial_r^2 - r^{-2}\partial_\theta^2. \end{aligned} \tag{10.8}$$

We note a few features of this spacetime:

(1) g is *stationary*, that is, $\mathcal{L}_{\partial_t} g = 0$, or more prosaically: the coefficients of g are t-independent.
(2) $\mathrm{d}t$ is timelike (since $|\mathrm{d}t|_G^2 = 1 > 0$), so the level sets $t^{-1}(t_0)$, $t_0 \in \mathbb{R}$, are spacelike; we declare $\mathrm{d}t$ to be *future timelike*;
(3) for any $r_0 > 1$, the level set $r^{-1}(r_0)$ is spacelike (since $|\mathrm{d}r|_G^2 = -(1 - r_0^2) < 0$), and $G(\mathrm{d}r, \mathrm{d}t) = r_0 > 0$, so $\mathrm{d}t$ and $\mathrm{d}r$ are both *future* timelike;
(4) the hypersurface
$$\overline{\mathcal{H}}^+ := r^{-1}(1) \tag{10.9}$$
is null (meaning $\mathrm{d}r$ is null there). It is called the *cosmological horizon.*

Thus, the geometry of (M, g) is quite interesting: consider a point $p \in M$ with $r(p) > 1$, and a future timelike or null vector $V \in T_pM$, $\zeta := g_p(V, -) \in T_p^*M$. Then $Vr = \mathrm{d}r(V) = G_p(\mathrm{d}r, \zeta) > 0$. Therefore, any physical observer or light particle travels even further away from $r = 1$. On the other hand, if $r(p) < 1$, there are no such restrictions.

An application of Proposition 10.7 (with Ω a smoothed out version of $\{(t, x) \in M \colon t \leq T,\ |x| < R\}$ for $R \in (1, 2)$ and any $T > 0$) implies that the wave equation $\Box_g u = f \in \mathcal{C}^\infty(M)$ with $\operatorname{supp} f \subset t^{-1}([0, \infty))$ has a unique solution $u \in \mathcal{C}^\infty(M)$ with $\operatorname{supp} u \subset t^{-1}([0, \infty))$. (This is true more generally for wave-type operators on (M, g).) Our aim is to describe the asymptotic behavior of $u(t, x)$ as $t \to \infty$.

We denote the spatial slices of M by

$$X = \{x \in \mathbb{R}^2 \colon |x| < 2\}, \tag{10.10}$$

which can be identified with $t^{-1}(t_0)$ for any $t_0 \in \mathbb{R}$.

Theorem 10.9 (Resonance expansion for waves on de Sitter space)**.** *Let $P \in \mathrm{Diff}^2(M)$ be a wave-type operator on (M, g) with t-independent coefficients, that is, $[\partial_t, P] \equiv 0$. Then there exists a sequence of numbers (called* resonances*) $\sigma_j \in \mathbb{C}$ with $\operatorname{Im}\sigma_j \to -\infty$, and finite-dimensional spaces (of* (generalized) resonant states *$\mathcal{R}_j \subset \mathcal{C}^\infty(M) \cap \ker P$ consisting of functions of the form $\sum_{k=0}^{k_j-1} e^{-i\sigma_j t} t^k a_k(x)$, $a_k \in \mathcal{C}^\infty(X)$, such that the following holds.*

Let $f \in \mathcal{C}^\infty_{\mathrm{c}}(M)$, $\operatorname{supp} f \subset t^{-1}([0,\infty))$, and let $u \in \mathcal{C}^\infty(M)$ denote the unique solution of

$$Pu = f, \quad \operatorname{supp} u \subset t^{-1}([0,\infty)). \tag{10.11}$$

Let $\alpha \in \mathbb{R}$ be such that $-\operatorname{Im}\sigma_j \neq \alpha$ for all j. Then there exist $u_j \in \mathcal{R}_j$ and a constant $C > 0$ such that for $t \geq 0$

$$u(t,x) = \sum_{\operatorname{Im}\sigma_j > -\alpha} u_j(t,x) + \tilde{u}(t,x), \qquad |\tilde{u}(t,x)| \leq C e^{-\alpha t}. \tag{10.12}$$

That is, modulo an error decaying at any fixed exponential rate α, $u(t,x)$ is equal to a finite sum of terms of the form $e^{-i\sigma_j t} t^k a_{jk}(x)$ with $a_{jk} \in \mathcal{C}^\infty(X)$.

Note that $|e^{-i\sigma_j t}| = e^{(\operatorname{Im}\sigma_j)t}$, which decays when $\operatorname{Im}\sigma_j < 0$.

Remark 10.10 (Comparison with waves on compact manifolds)*.* Compare this with the description of linear waves on a compact Riemannian manifold (X, h), i.e. solutions u of $(D_t^2 - \Delta_h)u = f \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}_t \times X)$: they can be expanded in eigenfunctions ϕ_j, $\Delta_h \phi_j = \lambda_j^2 \phi_j$, $\lambda_j \in \mathbb{R}$, so $u = \sum_{j=0}^\infty (a_{j+} e^{i\lambda_j t} + a_{j-} e^{-i\lambda_j t})$. All frequencies here are real, so this is a sum of oscillating, but non-decaying terms. A strong manifestation of the lack of decay is that $\int_X |\partial_t u(t,x)|^2 + |\partial_x u(t,x)|^2 \,\mathrm{d}x$ is conserved (t-independent).

Morally speaking, the reason for the decay (modulo finitely many terms) of u in Theorem 10.9 is that waves can cross $\overline{\mathcal{H}}^+$, and once they have done so, they continue travelling outwards and leave our (incomplete) spacetime M. The numbers σ_j in Theorem 10.9 are the replacement for eigenvalues in the system of interest here were energy can 'leak' out, and the spaces $\mathcal{R}_j$ of generalized resonant states are the replacements of eigenspaces. (In particular, just like eigenvalues, the resonances σ_j cannot be computed explicitly except in very special situations.)

Remark 10.11 (Explicit formulas for resonances)*.* For the wave operator $P = \Box_g$, the resonances are $\sigma_j = -ij$, $j \in \mathbb{N}_0$, and $k_j = 1$ for $j = 0, 1$, while $k_j = 2$ for $j \geq 2$. For the Klein–Gordon operator $P = \Box_g - m^2$, the resonances are $-i \pm i\sqrt{1-m^2} - i\mathbb{N}_0$. (See [Vas10] and [HV18, Appendix C], the latter also including a calculation for a wave-type operator acting on symmetric 2-tensors, though in $3+1$ dimensions.)

Remark 10.12 (Wave type operators on stationary vector bundles)*.* If $E \to X$ is a vector bundle which, via $\pi\colon M \ni (t,x) \mapsto x \in X$, lifts to a 'stationary' vector bundle $\mathcal{E} := \pi^* E \to M$, then Theorem 10.9 remains valid for wave-type operators $P \in \mathrm{Diff}^2(M; \mathcal{E})$ with t-independent coefficients. (This is a well-defined notion since sections of $\mathcal{E}$ can be invariantly differentiated with respect to t.) In this case, the resonant states are elements of $\mathcal{C}^\infty(X; E)$. Examples include the wave operator on differential forms or symmetric 2-tensors (or other tensor bundles).

For the most part, we shall only sketch the proof of Theorem 10.9; we provide details for the most interesting (and conceptually central) part of the argument. To begin with, it is not hard to show an exponential bound for u: there exists $C_0 > 0$ such that

$$|u(t,x)| \le C_0 e^{C_0 t}, \quad t \ge 0. \tag{10.13}$$

(This follows from the stationarity and linearity of P by a simple energy estimate. Morally, we can see this as follows: we have $Pu = 0$ for $t \ge t_0 \gg 1$ since f has compact support; the estimate (10.13) then follows from an estimate of the energy $E(t) := \|u(t,x)\|_{H^2(X)} + \|\partial_t u(t,x)\|_{H^1(X)}$ of the form $E(t+1) \le CE(t)$ for a constant C which, by stationarity, can be taken to be t-independent.)

The strategy of the proof is to use spectral theory after taking the Fourier transform in t (with a sign change relative to our previous convention, for consistency with the literature): letting

$$\hat{u}(\sigma, x) := \int_{\mathbb{R}} e^{i\sigma t} u(t,x)\,\mathrm{d}t, \tag{10.14}$$

and likewise $\hat{f}(\sigma, x)$, (formally) taking the Fourier transform of (10.11) gives

$$\hat{P}(\sigma)\hat{u}(\sigma) = \hat{f}(\sigma), \tag{10.15}$$

where the operator $\hat{P}(\sigma) \in \mathrm{Diff}^2(X)$ is obtained from $P = P(x, D_t, D_x)$ by replacing D_t by $-\sigma$, so

$$\hat{P}(\sigma) = P(x, -\sigma, D_x). \tag{10.16}$$

Since the leading order part of the wave-type operator P is $D_t^2 + 2D_t r D_r - (1-r^2)D_r^2 - r^{-2}D_\theta$, the leading order part of $-\hat{P}(\sigma)$ is $(1-r^2)D_r^2 + r^{-2}D_\theta^2$; near $r = 0$, this is close to the Laplacian on $\mathbb{R}^2$, and indeed it is elliptic for $r < 1$, but at $r = 1$ it degenerates, and it becomes a hyperbolic operator in $r > 1$ (with r taking the role of a 'time function' there).

Now, since $u(t,x) = 0$ for $t \le 0$, the bound (10.13) implies that $\hat{u}(\sigma, x)$ is well-defined for $\operatorname{Im}\sigma > C_0$; moreover, it implies that all resonances σ_j satisfy $\operatorname{Im}\sigma_j \le C_0$. For $\hat{f}(\sigma, x)$, the situation is even better: since f has compact support in t, $\hat{f}(\sigma, x) \in \mathcal{C}^\infty(X)$ is holomorphic in the full complex plane $\sigma \in \mathbb{C}$.

Thus, the equation (10.15) holds true for $\operatorname{Im}\sigma > C_0$. Suppose now we can invert $\hat{P}(\sigma)$ (on $\mathcal{C}^\infty(X)$, or suitable Sobolev spaces) for such σ; then

$$\hat{u}(\sigma, x) = \hat{P}(\sigma)^{-1}\hat{f}(\sigma, x), \tag{10.17}$$

and we therefore have

$$u(t,x) = (2\pi)^{-1} \int_{\operatorname{Im}\sigma = C_0+1} e^{-i\sigma t}\hat{P}(\sigma)^{-1}\hat{f}(\sigma, x)\,\mathrm{d}\sigma. \tag{10.18}$$

We shall prove is that $\hat{P}(\sigma)^{-1}$ is a *meromorphic* family of operators on $\mathcal{C}^\infty(X)$; the connection to Theorem 10.9 will then be:

(1) the resonances σ_j are then the poles of $\hat{P}(\sigma)^{-1}$;
(2) the integer $k_j + 1$ is the order of the pole at $\sigma = \sigma_j$.

Indeed, in the expression (10.18), we use Cauchy's theorem to shift the integration contour from $\operatorname{Im}\sigma = C_0 + 1$ to $\operatorname{Im}\sigma = -\alpha$, giving

$$\begin{aligned} u(t,x) = \sum_{\operatorname{Im}\sigma_j > -\alpha} &(2\pi)^{-1} \operatorname{Res}_{\sigma=\sigma_j} \bigl(e^{-i\sigma t}\hat P(\sigma)^{-1}\hat f(\sigma,x)\bigr) \\ &+ (2\pi)^{-1} \int_{\operatorname{Im}\sigma=-\alpha} e^{-i\sigma t}\hat P(\sigma)^{-1}\hat f(\sigma,x)\,\mathrm{d}\sigma. \end{aligned} \tag{10.19}$$

In the case $k_j = 0$ and $\hat P(\sigma) = (\sigma-\sigma_j)^{-1}P_1 + \text{holomorphic}$, we have

$$\operatorname{Res}_{\sigma=\sigma_j} e^{-i\sigma t}\hat P(\sigma)^{-1}\hat f(\sigma,x) = e^{-i\sigma_j t}P_1(\hat f(\sigma_j)). \tag{10.20}$$

Thus, in this case, $\mathcal{R}_j = \{e^{-i\sigma_j t}a(x) \colon a(x) \in \operatorname{ran} P_1\}$; the case of higher order poles is similar. The second term in (10.19) is the remainder $\tilde u(t,x)$ in the notation of Theorem 10.9; note that the integrand is pointwise bounded by $e^{-\alpha t}$.

Remark 10.13 (Contour shifting). Justifying (10.19) uses that $\hat P(\sigma)^{-1}\hat f(\sigma,x)$ has suitable decay as $|\operatorname{Re}\sigma| \to \infty$ with $\operatorname{Im}\sigma \in [-\alpha, C_0+1]$. Operator norm bounds on $\hat P(\sigma)^{-1}$ for such σ are called *high energy estimates*, which can be proved by methods from *semiclassical microlocal analysis*. Moreover, such estimates imply that there are only finitely many resonances in any strip $|\operatorname{Im}\sigma| < C$, $C \in \mathbb{R}$.

Remark 10.14 (Black hole spacetimes). The arguments sketched here can be used to describe in a similar manner linear and even nonlinear waves on *black hole spacetimes* such as Schwarzschild–de Sitter and Kerr–de Sitter black holes. For the high energy estimates (briefly mentioned below), one needs an additional ingredient to deal with *trapping effects*. See for instance [Vas13, WZ11, Dya11, HV18], and references therein.

The only statement we shall prove here in detail is that $\hat P(\sigma)^{-1}$ is meromorphic (on suitable function spaces).

10.3. **Analysis of the spectral family $\hat P(\sigma)$.** We begin by defining the relevant function spaces:

Definition 10.15 (Extendible and supported distributions)**.** Suppose M is a manifold, and $X \subset M$ is open. Let $s \in \mathbb{R}$, and let $\mathcal{F}(M)$ denote a space of distributions on M, such as $\mathcal{F}(M) = \mathscr{D}'(M)$ or $\mathcal{F}(M) = H^s_{\mathrm{loc}}(M)$. We then define the space

$$\bar{\mathcal{F}}(X) := \{u|_X \colon u \in \mathcal{F}(M)\} \tag{10.21}$$

of restrictions to X; its elements are called *extendible distributions*. We also define

$$\dot{\mathcal{F}}(\bar X) := \{u \colon u \in \mathcal{F}(M),\ \operatorname{supp} u \subset \bar X\}. \tag{10.22}$$

Its elements are called *supported distributions*.

Note that the kernel of $\mathcal{F}(M) \ni u \mapsto u|_X \in \bar{\mathcal{F}}(X)$ is $\dot{\mathcal{F}}(M \setminus X)$; hence we have

$$\bar{\mathcal{F}}(X) \cong \mathcal{F}(M)/\dot{\mathcal{F}}(M \setminus X). \tag{10.23}$$

Recall from (10.10) that $\hat P(\sigma)$ is an operator on the spatial slice $X = \{|x| < 2\} \subset \mathbb{R}^2$. Taking $\mathcal{F} = H^s(\mathbb{R}^2)$ gives the function spaces

$$\bar H^s(X),\ \dot H^s(\bar X); \tag{10.24}$$

we have $\hat{P}(\sigma)\colon \bar{H}^s(X) \to \bar{H}^{s-2}(X)$ and $\dot{H}^s(\bar{X}) \to \dot{H}^{s-2}(\bar{X})$. Note that $\dot{H}^s(\bar{X})$ is a closed subspace of $H^s(\mathbb{R}^2)$. In view of (10.23), the space $\bar{H}^s(X)$ also carries the structure of a Hilbert space. Note moreover that $\bar{\mathcal{C}}^\infty(X) = \mathcal{C}^\infty(\bar{X}) \subset \bar{H}^s(X)$ and $\dot{\mathcal{C}}^\infty(\bar{X}) \subset \dot{H}^s(\bar{X})$ are dense.

Lemma 10.16 (Duality between extendible and supported Sobolev spaces)**.** *The L^2 pairing $\bar{\mathcal{C}}^\infty(X) \times \dot{\mathcal{C}}^\infty(X) \ni (u, v) \mapsto \langle u, v\rangle = \int u\bar{v}\,\mathrm{d}x \in \mathbb{C}$ extends by continuity to a pairing*

$$\bar{H}^s(X) \times \dot{H}^{-s}(\bar{X}) \to \mathbb{C}. \tag{10.25}$$

It has the property that $\bar{H}^s(X) \ni u \mapsto \langle u, -\rangle \in (\dot{H}^{-s}(\bar{X}))^*$ is an isomorphism.

One says that $\dot{H}^{-s}(\bar{X})$ is the dual space of $\bar{H}^s(X)$ relative to $L^2(X)$.

Proof of Lemma 10.16. We have $\langle u, v\rangle = 0$ for $u \in \dot{\mathcal{C}}^\infty_{\mathrm{c}}(\mathbb{R}^2 \setminus X)$ and $v \in \dot{\mathcal{C}}^\infty(\bar{X})$, hence this holds also for $u \in \dot{H}^s(\mathbb{R}^2 \setminus X)$ and $v \in \dot{H}^{-s}(\bar{X})$. By (10.23), the pairing (10.25) is therefore well-defined.

For the final claim, note that if $u \in \bar{H}^s(X)$ is such that $\langle u, v\rangle = 0$ for all $v \in \dot{H}^{-s}(\bar{X})$, write $u = \tilde{u}|_X$, $\tilde{u} \in H^s(\mathbb{R}^2)$ and conclude that $\operatorname{supp}\tilde{u} \subset \mathbb{R}^2 \setminus X$, therefore $u = 0$. Conversely, given $\ell \in (\dot{H}^{-s}(\bar{X}))^*$, use Hahn–Banach to extend ℓ to a continuous linear functional $\tilde{\ell} \in H^{-s}(\mathbb{R}^2)$; then $\tilde{\ell}(v) = \langle \tilde{u}, v\rangle$ for some $\tilde{u} \in H^s(\mathbb{R}^2)$, and setting $u := \tilde{u}|_X$ completes the proof. □

Given $P \in \mathrm{Diff}^2(M)$ as in Theorem 10.9, there exists a constant $\tilde{\beta} \in \mathbb{R}$ (explicitly computable and given in the course of the proof) such that the following holds:

Theorem 10.17 (Fredholm property of the spectral family)**.** *For $s \in \mathbb{R}$, define the function space*

$$\mathcal{X}^s := \{u \in \bar{H}^s(X)\colon \hat{P}(0)u \in \bar{H}^{s-1}(X)\}. \tag{10.26}$$

Let $\alpha \in \mathbb{R}$. Then, for $s > \frac{1}{2} + \tilde{\beta} + \alpha$,

$$\hat{P}(\sigma)\colon \mathcal{X}^s \to \bar{H}^{s-1}(X), \quad \sigma \in \mathbb{C},\ \operatorname{Im}\sigma > -\alpha, \tag{10.27}$$

is a Fredholm operator. Moreover,

$$\ker \hat{P}(\sigma) \cap \mathcal{X}^s \subset \bar{\mathcal{C}}^\infty(X), \tag{10.28}$$

and $\operatorname{ran}_{\mathcal{X}^s} \hat{P}(\sigma) \subset \bar{H}^{s-1}(X)$ is the annihilator of

$$\ker \hat{P}(\sigma)^* \cap \dot{H}^{-s+1}(X) \subset \dot{H}^{1/2-\tilde{\beta}+\operatorname{Im}\sigma-\epsilon} \quad \forall\,\epsilon > 0. \tag{10.29}$$

Note that $\hat{P}(\sigma) - \hat{P}(0) \in \mathrm{Diff}^1(X)$, hence $\hat{P}(\sigma)$ indeed maps $\mathcal{X}^s \to \bar{H}^{s-1}(X)$. In the final statement, $\hat{P}(\sigma)^*$ is the formal adjoint defined by $\langle \hat{P}(\sigma)^* u, v\rangle = \langle u, \hat{P}(\sigma)v\rangle$ for $u, v \in \mathcal{C}^\infty_{\mathrm{c}}(X)$; it is easy to see that

$$\hat{P}(\sigma)^* = \widehat{P^*}(\bar{\sigma}). \tag{10.30}$$

We prove this theorem below; first, we explain why it is so useful.

Lemma 10.18 (Invertibility of the spectral family when $\operatorname{Im}\sigma \gg 1$)**.** *For $\operatorname{Im}\sigma \gg 1$, $\hat{P}(\sigma)\colon \mathcal{X}^s \to \bar{H}^{s-1}(X)$ is invertible.*

Proof (sketch). An element $u \in \ker \hat{P}(\sigma) \cap \bar{\mathcal{C}}^\infty(X)$ gives rise to a solution $U(t,x) = e^{-i\sigma t}u(x)$ of $PU = 0$. In view of the estimate (10.13), we must have $U \equiv 0$ when $\operatorname{Im}\sigma \geq C_0$, hence $u \equiv 0$. Therefore, $\hat{P}(\sigma)$ is injective for large $\operatorname{Im}\sigma$.

Dually, if $v \in \ker \hat{P}(\sigma)^*$, then $P^*V = 0$ for $V(t,x) = e^{i\bar{\sigma}t}v(x)$. Since $v = 0$ for $r > 1$ (which follows from the fact that v, extended by 0 beyond X, solves the hyperbolic equation $\hat{P}(\sigma)^*v = 0$ in $r > 1$), we have $V = 0$ for $r > 1$ as well. Moreover, for $\operatorname{Im}\sigma \gg 1$, v lies in H^1. One can then again use an energy estimate (for P^* and 'from $t = \infty$') to show that there exists $C_1 \in \mathbb{R}$ such that $V \equiv 0$ when $\operatorname{Im}\sigma > C_1$, hence $v \equiv 0$. By Theorem 10.9, this implies that $\hat{P}(\sigma)$ is surjective. □

Corollary 10.19 (Meromorphic extension). *For $\alpha \in \mathbb{R}$, $s > \frac{1}{2} + \tilde{\beta} + \alpha$, $\operatorname{Im}\sigma > -\alpha$ as in Theorem 10.9, the family $\hat{P}(\sigma)\colon \mathcal{X}^s \to \bar{H}^{s-1}(X)$ is a family of Fredholm operators of index 0. Its inverse extends from $\operatorname{Im}\sigma \gg 1$ to a finite-meromorphic family*

$$\hat{P}(\sigma)^{-1}\colon \bar{H}^{s-1}(X) \to \bar{H}^s(X). \tag{10.31}$$

The first part is clear since the index of a continuous family of Fredholm operators is constant. For the second part, we use the following terminology:

Definition 10.20 (Finite-meromorphic functions). Let X, Y denote two Banach spaces. Let $\Omega \subset \mathbb{C}$ be an open set. Then we say that $B(\sigma)\colon X \to Y$, $\sigma \in \Omega$, is *finite-meromorphic* if there exists a discrete subset $D = \{\sigma_1, \sigma_2, \ldots\} \subset \Omega$ such that:

1. $B(\sigma)$ is holomorphic on $\Omega \setminus D$;
2. near σ_j, there exists $k_j \in \mathbb{N}$ such that
$$B(\sigma) = \sum_{k=1}^{k_j} (\sigma - \sigma_j)^{-k} B_{jk} + \tilde{B}_j(\sigma), \tag{10.32}$$
where $\tilde{B}_j(\sigma)\colon X \to Y$ is holomorphic near $\sigma = \sigma_j$, and $B_{jk}\colon X \to Y$, $1 \leq k \leq k_j$, is a finite rank operator.

Corollary 10.19 is then an immediate consequence of:

Proposition 10.21 (Analytic Fredholm Theorem). *Let X, Y be Banach spaces, let $\Omega \subset \mathbb{C}$ be open and connected, and suppose $A(\sigma)\colon X \to Y$, $\sigma \in \Omega$, is an analytic family of Fredholm operators. Then either $A(\sigma)$ is not invertible for any $\sigma \in \Omega$, or $A(\sigma)^{-1}$ is finite-meromorphic.*

Proof. Suppose the (open) set $\Omega' \subset \Omega$ of σ for which $A(\sigma)$ is invertible is non-empty; then $A(\sigma)$ has index 0 for $\sigma \in \Omega'$, hence for all $\sigma \in \Omega$.

If $\Omega' \neq \Omega$, let $\sigma_0 \in \Omega \cap \partial\Omega'$. Consider $A(\sigma_0)\colon X \to Y$. Let $X_2 = \ker A(\sigma_0)$ and $R_1 = \operatorname{ran} A(\sigma_0)$; pick closed subspaces $X_1 \subset X$ and $Y_2 \subset Y$ with

$$X = X_1 \oplus X_2, \quad Y = Y_1 \oplus Y_2. \tag{10.33}$$

Since $\operatorname{ind} A(\sigma_0) = 0$, $\dim X_2 = \dim Y_2 = N < \infty$. We write $A(\sigma)$ as a block matrix in the decomposition (10.33),

$$A(\sigma) = \begin{pmatrix} P(\sigma) & Q(\sigma) \\ S(\sigma) & T(\sigma) \end{pmatrix}, \tag{10.34}$$

where $P(\sigma_0)\colon X_1 \to Y_1$ is invertible, and $Q, S, T = 0$ at $\sigma = \sigma_0$. Thus, $P(\sigma)\colon X_1 \to Y_1$ is invertible for $|\sigma - \sigma_0| < \epsilon$ for some $\epsilon > 0$; by the Schur complement formula (block-wise inversion of $A(\sigma)$), $A(\sigma)$ is invertible for $|\sigma - \sigma_0| < \epsilon$ if and only if

$$Z(\sigma) := T(\sigma) - S(\sigma)P(\sigma)^{-1}Q(\sigma)\colon X_2 \to Y_2 \tag{10.35}$$

is invertible; in this case, we have

$$A(\sigma)^{-1} = \begin{pmatrix} P^{-1} + P^{-1}QZ^{-1}SP^{-1} & -P^{-1}QZ^{-1} \\ -Z^{-1}SP^{-1} & Z^{-1} \end{pmatrix}. \tag{10.36}$$

But $Z(\sigma)$ is a holomorphic $N \times N$ *matrix* near σ_0, and invertible for some σ arbitrarily close to σ_0. Hence, fixing a basis of X_2 and Y_2, its determinant $\det Z(\sigma)$ is a non-zero holomorphic function which vanishes at $\sigma = \sigma_0$; hence $\det Z(\sigma)^{-1}$ is meromorphic, and so is $Z(\sigma)^{-1}$. Therefore, $A(\sigma)$ is invertible in a punctured neighborhood of σ_0. The conclusion is now immediate from (10.36). □

Returning to the main calculation (10.19) in our sketch of the proof of Theorem 10.9, this justifies (modulo control for large $|\operatorname{Re}\sigma|$) the contour shifting and the use of the residue theorem.

Now, Theorem 10.9 will be an easy consequence of the following result:

Proposition 10.22 (Fredholm estimates for the spectral family)**.** *We have the following* Fredholm estimates *for $\hat P(\sigma)$:*

1. *Let $s > s_0 > \frac{1}{2} + \tilde\beta - \operatorname{Im}\sigma$. Then there exists $C > 0$ such that for $u \in \mathcal{X}^s$,*

$$\|u\|_{\bar H^s(X)} \le C\big(\|\hat P(\sigma)u\|_{\bar H^{s-1}(X)} + \|u\|_{\bar H^{s_0}(X)}\big); \tag{10.37}$$

this holds in the strong sense that if all quantities on the right hand side are finite, then so is the left hand side, and the inequality holds.

2. *Define, analogously to $\mathcal{X}^s$, the space*

$$\mathcal{Y}^{-s+1} := \{v \in \dot H^{-s+1}(\bar X) \colon \hat P(\sigma)^* v \in \dot H^{-s}(\bar X)\} \tag{10.38}$$

Let $N \in \mathbb{R}$ and $s > \frac{1}{2} + \tilde\beta - \operatorname{Im}\sigma$. Then there exists $C > 0$ such that for all $v \in \mathcal{Y}^{-s+1}$,

$$\|v\|_{\dot H^{-s+1}(\bar X)} \le C\big(\|\hat P(\sigma)^* v\|_{\dot H^{-s}(\bar X)} + \|v\|_{\dot H^{-N}(\bar X)}\big); \tag{10.39}$$

this holds in the strong sense.

Proof of Theorem 10.9 assuming Proposition 10.22. The estimate (10.37) together with the compactness of the inclusion $\bar H^s(X) \hookrightarrow \bar H^{s_0}(X)$ (exercise!) imply that $\dim \ker_{\bar H^s(X)} \hat P(\sigma) < \infty$, and that $\operatorname{ran}_{\mathcal{X}^s} \hat P(\sigma) \subset \bar H^{s-1}(X)$ is closed. Moreover, since (10.37) holds in the strong sense, it implies that if $\hat P(\sigma)u = 0$, then we can take s arbitrary and obtain $u \in \bar{\mathcal{C}}^\infty(X)$.

On the other hand, the estimate (10.39) implies $\dim K < \infty$ where

$$K := \ker_{\dot H^{-s}(\bar X)} \hat P(\sigma)^* < \infty. \tag{10.40}$$

Again, since (10.39) holds in the strong sense, we see that $\hat P(\sigma)^* v = 0$ implies $v \in \dot H^{1/2-\tilde\beta+\operatorname{Im}\sigma-\epsilon}(\bar X)$ for all $\epsilon > 0$.

Finally, let $f \in \bar{H}^{s-1}(X)$ be such that $\langle f, v\rangle = 0$ for all $v \in K$. We claim that there exists $u \in \bar{H}^s(X)$ such that

$$\hat{P}(\sigma)u = f \in \bar{H}^{s-1}(X). \tag{10.41}$$

(This implies that $\operatorname{ran}\hat{P}(\sigma)$ has finite codimension, thus finishing the proof.) This solvability follows by a general argument from the (almost) injectivity (10.39) of the adjoint operator.

First of all, fix a closed complementary subspace $L \subset \dot{H}^{-s}(\bar{X})$ of K; then a simple argument by contradiction shows that there exists a constant C' such that

$$\|v\|_{\dot{H}^{-s+1}(\bar{X})} \leq C'\|\hat{P}(\sigma)^* v\|_{\dot{H}^{-s}(\bar{X})}, \quad v \in L. \tag{10.42}$$

Therefore, we have $|\langle v, f\rangle| \lesssim \|\hat{P}(\sigma)^* v\|_{\dot{H}^{-s}(\bar{X})}$ for $v \in L$. Writing a general element $v \in \dot{H}^{-s+1}(\bar{X})$ as $v = v_1 + v_2$, $v_1 \in L$, $v_2 \in K$, we have

$$|\langle v, f\rangle| = |\langle v_1, f\rangle| \lesssim \|\hat{P}(\sigma)^* v_1\|_{\dot{H}^{-s}(\bar{X})} = \|\hat{P}(\sigma)^* v\|_{\dot{H}^{-s}(\bar{X})}. \tag{10.43}$$

Using Hahn–Banach, the (thus well-defined and bounded) functional

$$\dot{H}^{-s}(\bar{X}) \ni \hat{P}(\sigma)^* v \mapsto \langle v, f\rangle, \quad v \in \mathcal{Y}^{-s+1}, \tag{10.44}$$

can be extended to an element of $(\dot{H}^{-s}(\bar{X}))^*$, which is represented by an element $u \in \bar{H}^s(\bar{X})$ by Lemma 10.16. In particular, for all $v \in \mathcal{C}^\infty_c(X)$,

$$\langle v, f\rangle = \langle \hat{P}(\sigma)^* v, u\rangle = \langle v, \hat{P}(\sigma)u\rangle, \tag{10.45}$$

which implies $\hat{P}(\sigma)u = f$, as desired. □

The proof of Proposition 10.22 will, of course, be microlocal. Thus, we need to analyze the characteristic set and null-bicharacteristic flow of $\hat{P}(\sigma)$. Recall the form (10.8) of the dual metric G of de Sitter space; writing covectors on $X = \{|x| < 2\}$ in polar coordinates in $r = |x| \neq 0$ as

$$\xi\,\mathrm{d}r + \eta\,\mathrm{d}\theta, \tag{10.46}$$

we therefore have

$$p(r, \theta, \xi, \eta) = \sigma^2(\hat{P}(\sigma)) = -(1 - r^2)\xi^2 - r^{-2}\eta^2. \tag{10.47}$$

We denote the characteristic set of $\hat{P}(\sigma)$ by

$$\Sigma := p^{-1}(0) \subset T^*X \setminus o. \tag{10.48}$$

Polar coordinates break down at $r = 0$, one can easily calculate in standard coordinates (x_1, x_2) on $\mathbb{R}^2$ (namely: by computing the form of the dual metric of (10.7)) that $p(x_1, x_2, \xi_1, \xi_2) = -(1 - x_1^2)\xi_1^2 - (1 - x_2^2)\xi_2^2 - 2x_1x_2\xi_1\xi_2$, which is clearly elliptic for (x_1, x_2) near $(0, 0)$.

Lemma 10.23 (Properties of the characteristic set)**.** *Σ is a smooth conic submanifold of $T^*X \setminus o$, and $r \geq 1$ on Σ. It has two connected components,*

$$\Sigma = \Sigma_+ \cup \Sigma_-, \quad \Sigma_\pm = \{(r, \theta, \xi, \eta) \in \Sigma\colon \pm\xi > 0\}. \tag{10.49}$$

Proof. Certainly, $p = 0$ requires $r \geq 1$ in view of (10.47). Furthermore, suppose $\zeta \in \Sigma$ is a point at which $r \geq 1$, $p = 0$; we need to show $\mathrm{d}p \neq 0$. If we assume the contrary, $\mathrm{d}p = 0$, then $\partial_\eta p = -2r^{-2}\eta = 0$ implies $\eta = 0$. Then $0 = p = -(1 - r^2)\xi^2$ implies $\xi = 0$ (and thus we are the zero section, hence outside of Σ) unless $r = 1$. If $r = 1$ and $\xi \neq 0$, however, we have $0 = \partial_r p = 2r\xi^2 \neq 0$, a contradiction.

The final claim follows immediately from (10.47). □

In fact, since p is homogeneous, we have $\Sigma_- = -\Sigma_+$. Moreover, the $(-H_p)$-flow in Σ_- is the mirror image (multiplication by -1 in the fibers of T^*X) of the H_p-flow on Σ_+. We thus only study the properties of the H_p-flow in Σ_+.

Note now that ξ^{-1} is elliptic and positive near Σ_+; let us thus work with projective coordinates

$$\rho := \frac{1}{\xi}, \quad \hat{\eta} = \frac{\eta}{\xi}. \tag{10.50}$$

We can then identify Σ_+ with its boundary at fiber infinity inside $\overline{T^*}X$,

$$\Sigma_+ = \{(r, \theta, \hat{\eta}) \colon \hat{\eta}^2 = r^2(1 - r^2)\}. \tag{10.51}$$

(Forgetting about the θ-variable, this thus looks like a parabola in $(r, \hat{\eta})$ with vertex at $r = 1$.)

Next, we compute the Hamiltonian vector field $H_p = -2(1-r^2)\xi\partial_r - 2r^{-2}\eta\partial_\theta - 2(r\xi^2 + r^{-3}\eta^2)\partial_\xi$ and its rescaling

$$\tilde{H}_p := \xi^{-1}H_p = -2(1-r^2)\partial_r - 2r^{-2}\hat{\eta}\partial_\theta + 2(r + r^{-3}\hat{\eta}^2)(\rho\partial_\rho + \hat{\eta}\partial_{\hat{\eta}}), \tag{10.52}$$

which on Σ_+ takes the form

$$\tilde{H}_p = -2(1-r^2)\partial_r - 2r^{-2}\hat{\eta}\partial_\theta + 2r^{-1}(\rho\partial_\rho + \hat{\eta}\partial_{\hat{\eta}}). \tag{10.53}$$

Its only critical points are at $r = 1$, $\hat{\eta} = 0$. We have thus identified the radial set

$$\mathcal{R}_+ := \{(r = 1, \theta, \hat{\eta} = 0)\} \subset \Sigma_+ \subset S^*X. \tag{10.54}$$

Lemma 10.24 (Dynamics of the null-bicharacteristic flow)**.** *Let $s \mapsto \gamma(s) \in \Sigma_+$ be a null-bicharacteristic, i.e. an integral curve of $\tilde{H}_p$, with $\gamma(0) \notin \mathcal{R}_+$. Then:*

1. *in the backward direction, $\gamma(s)$ tends to $\mathcal{R}_+$ as $s \to -\infty$;*
2. *in the forward direction, $\gamma(s)$ crosses $r = 2$ in finite time (in the direction of increasing r).*

Proof. We have $r > 1$ at $\gamma(0)$. Note then that $\tilde{H}_p r = 2(r^2 - 1) > 0$; thus, $r \circ \gamma(s)$ is monotonically increasing in the forward direction, and indeed $\tilde{H}_p r \geq 2(r(\gamma(0))^2 - 1)$ for $s \geq 0$. This implies the second statement. On the other hand, as $s \to -\infty$, $r(\gamma(s)) \to 0$; in view of (10.51), this implies $\gamma(s) \to \mathcal{R}_+$ indeed. □

Thus, the only interesting place is $\mathcal{R}_+$.

Lemma 10.25 (Dynamics of the null-bicharacteristic flow near the radial set)**.** *$\mathcal{R}_+$ is a source for the $\tilde{H}_p$-flow inside of $\overline{T^*}X$. For ρ as in (10.50), we have*

$$\beta_0 := \rho^{-1}\tilde{H}_p\rho = 2 \quad \text{at } \mathcal{R}_+ \tag{10.55}$$

(cf. the definition (9.26)). Moreover, we have

$$\tilde{\beta}(\sigma) := \beta_0^{-1}\xi^{-1}\sigma^1\Big(\frac{\hat{P}(\sigma) - \hat{P}(\sigma)^*}{2i}\Big) = \tilde{\beta} - \operatorname{Im}\sigma \tag{10.56}$$

at $\mathcal{R}_+$ for some (σ-independent) $\tilde{\beta} \in \mathcal{C}^\infty(\mathcal{R}_+)$ (cf. the definition (9.27)).

Proof. The calculation of β_0 is trivial, and shows that $\mathcal{R}_+$ is a source in the fiber-radial direction. Next, the function $\rho_1 = \hat{\eta}^2$ is a quadratic defining function for $\mathcal{R}_+$ inside of Σ_+, and we have $\tilde{H}_p \rho_1 = 4r^{-1}\rho_1$.

For the calculation of $\tilde{\beta}(\sigma)$, note that by inspection of (10.8), we have

$$\hat{P}(\sigma) = \hat{P}(0) + \sigma(-2rD_r + R_0) + \sigma^2 R_1 \tag{10.57}$$

near $r = 1$, where $R_0, R_1 \in \mathcal{C}^\infty(X)$ are lower order terms, and $\hat{P}(0) \in \mathrm{Diff}^2(X)$ has real principal symbol. Thus,

$$\sigma_1\Big(\frac{\hat{P}(\sigma) - \hat{P}(\sigma)^*}{2i}\Big) = \sigma_1\Big(\frac{\hat{P}(0) - \hat{P}(0)^*}{2i}\Big) - 2(\operatorname{Im}\sigma) r\xi. \tag{10.58}$$

This implies (10.56). □

Equipped with this dynamical information, and the calculation (10.56), we are now in a position to prove Proposition 10.22.

Proof of Proposition 10.22. • Proof of the estimate (10.37). The idea is to piece together radial point estimates, real principal type propagation estimates, and microlocal elliptic regularity to control u solving

$$\hat{P}(\sigma)u = f \in \bar{H}^{s-1}(X). \tag{10.59}$$

For clarity and simplicity, we shall not use the semiglobal results (such as Theorems 9.9 and 8.7), but rather work step by step.

For $0 < \delta \ll 1$, let

$$X_\delta = \{|x| < 2 - \delta\} \subset X; \tag{10.60}$$

we assume all Schwartz kernels below to have compact support in $X_\delta \times X_\delta$. For $s > s_0 > \frac{1}{2} + \tilde{\beta} - \operatorname{Im}\sigma$, and for $B \in \Psi^0$ elliptic near $\mathcal{R}_+$, Theorem 9.9 gives the estimate

$$\|Bu\|_{H^s} \lesssim \|f\|_{H^{s-1}} + \|u\|_{H^{s_0}}, \tag{10.61}$$

in the strong sense. (We may replace f by Gf, where $G \in \Psi^0$ microlocalizes near $\mathrm{WF}'(B)$.) By Lemma 10.24, the H^s-regularity of u can now be propagated to all of the characteristic set over X_δ by means of Theorem 8.7; thus, for $B_+ \in \Psi^0$ elliptic near $\Sigma_+ \cap S^*X_\delta$, we have (for any fixed $N \in \mathbb{R}$)

$$\begin{aligned} \|B_+u\|_{H^s} &\lesssim \|Bu\|_{H^s} + \|f\|_{H^{s-1}} + \|u\|_{H^{-N}} \\ &\lesssim \|f\|_{H^{s-1}} + \|u\|_{H^{s_0}}. \end{aligned} \tag{10.62}$$

The same estimate holds, by the same reasoning, for $B_- \in \Psi^0$ elliptic near $\Sigma_- \cap S^*X_\delta$.

On the other hand, for $B_0 \in \Psi^0$ elliptic near $S^*X_\delta \setminus (\mathrm{Ell}(B_+) \cup \mathrm{Ell}(B_-))$, microlocal elliptic regularity (Proposition 6.28, or really the quantitative form (6.44)) gives

$$\|B_0u\|_{H^s} \lesssim \|f\|_{H^{s-2}} + \|u\|_{H^{-N}}. \tag{10.63}$$

But $\mathrm{Ell}(B_0) \cup \mathrm{Ell}(B_-) \cup \mathrm{Ell}(B_+) \supset S^*X_\delta$. Fix cutoffs

$$\chi \in \mathcal{C}_c^\infty(X_\delta),\ \chi \equiv 1 \text{ on } X_{2\delta}, \qquad \tilde{\chi} \in \mathcal{C}_c^\infty(X),\ \tilde{\chi} \equiv 1 \text{ on } X_\delta; \tag{10.64}$$

we have then proved

$$\|\chi u\|_{H^s} \lesssim \|\tilde{\chi} f\|_{H^{s-1}} + \|\tilde{\chi} u\|_{H^{s_0}} \lesssim \|f\|_{\bar{H}^{s-1}(X)} + \|\tilde{\chi} u\|_{H^{s_0}}. \tag{10.65}$$

This gives an estimate of $\|u\|_{\bar H^s(X_\delta)}$, but with an error term (the last term on the right) which is measured on a larger set than u itself (but in a weaker norm as far as the degree of differentiability is concerned), as is typical for any microlocal estimate. To bridge the gap, we use that in $r > 1$, $\hat P(\sigma)$ is a hyperbolic operator (equal to $(r^2-1)D_r^2 - r^{-2}D_\theta^2$ to leading order, so r becomes the 'time' function); note then that if

$$\phi \in \mathcal{C}^\infty_{\mathrm{c}}(X_{2\delta}),\ \phi \equiv 1 \text{ on } X_{3\delta}, \tag{10.66}$$

then for

$$\tilde u := (1-\phi)u, \tag{10.67}$$

which is supported in $r \geq 2-3\delta$, we have

$$\hat P(\sigma)\tilde u = \tilde f, \quad \tilde f := (1-\phi)f - [\hat P(\sigma), \phi]u, \tag{10.68}$$

and the forcing $\tilde f \in \bar H^{s-1}(X)$, with $r \geq 2-3\delta$ on $\operatorname{supp} \tilde f$, satisfies the estimate

$$\|\tilde f\|_{\bar H^{s-1}(X)} \lesssim \|f\|_{\bar H^{s-1}(X)} + \|\chi u\|_{H^s} \lesssim \|f\|_{\bar H^{s-1}(X)} + \|\tilde\chi u\|_{H^{s_0}} \tag{10.69}$$

in view of (10.65). We claim that the unique solution $\tilde u$ (subject to the support condition) of (10.68) satisfies the estimate

$$\|\tilde u\|_{\bar H^s(X)} \lesssim \|\tilde f\|_{\bar H^{s-1}(X)}. \tag{10.70}$$

One way to prove this estimate is the following: using (a slight extension of) the uniqueness and existence theory for hyperbolic equations developed in §7, $\tilde u$ can be estimated on X in some space of distributions by the norm of $\tilde f$ on X; using that $\tilde u$ vanishes, hence is smooth, in $r < 2-3\delta$, the propagation of regularity implies that $\tilde u \in H^s_{\mathrm{loc}}(X)$. This is almost what we are after, except for the loss of uniform control right at ∂X (which is a completely artificial place!); to fix this, one proceeds as follows:

(1) one extends $\tilde f$ to an element of H^{s-1} on a slightly enlarged domain $X_{-\delta}$, and so that the $\bar H^{s-1}(X_{-\delta})$-norm of the extension is bounded by, say, $2 \times \|\tilde f\|_{\bar H^{s-1}(X)}$;
(2) one then solves (10.68) on $X_{-\delta}$, getting $\tilde u \in H^s_{\mathrm{loc}}(X_{-\delta})$ by the arguments described just now;
(3) finally, one restricts back to X, giving $\tilde u \in \bar H^s(X)$ and the estimate (10.70) plus an extra term $\|\tilde u\|_{\bar H^{-N}(X_{-\delta})}$ coming from the use of microlocal propagation estimates; the latter term however is bounded by some (weak) norm of $\tilde f$ by the results of §7.

Putting (10.65) together with (10.69), (10.70), and writing $u = \chi u + (1-\chi)u = \chi u + (1-\chi)\tilde u$, we find

$$\|u\|_{\bar H^s(X)} \lesssim \|f\|_{\bar H^{s-1}(X)} + \|u\|_{\bar H^{s_0}(X)}, \tag{10.71}$$

as desired.

- Proof of the estimate (10.39). We study the equation

$$\hat P(\sigma)^* v = h \in \dot H^{-s}(\bar X). \tag{10.72}$$

The arguments near ∂X are now slightly easier, as we are working with supported distributions which vanish on $\mathbb{R}^2 \setminus X$. Thus, letting $\chi, \tilde\chi, \phi$ be as in (10.64) and (10.66), we have

$$\|(1-\chi)v\|_{\dot H^{-s+1}(\bar X)} \lesssim \|(1-\phi)h\|_{\dot H^{-s}(\bar X)}. \tag{10.73}$$

But this H^{-s+1}-control of v for $2-\delta<r<2$ can be propagated along $\Sigma\cap S^*_{\{r>1\}}X$. A simple calculation shows that the threshold regularity at $\mathcal{R}_\pm$ for $\hat P(\sigma)$ is $\frac12-\tilde\beta(\sigma)$ (i.e. there is a sign switch); since $-s+1<\frac12-\tilde\beta(\sigma)$, we can thus propagate H^{-s+1}-regularity of v into $\mathcal{R}_\pm$. We thus control v microlocally near the full characteristic set Σ; away from Σ, we have microlocal H^{-s+2}-estimates on v by microlocal elliptic regularity. Altogether, the microlocal estimates give

$$\|\chi v\|_{H^{-s+1}}\lesssim\|\tilde\chi h\|_{H^{-s}}+\|(1-\chi)v\|_{\dot H^{-s+1}(\bar X)}+\|\tilde\chi v\|_{H^{-N}}. \tag{10.74}$$

(The first term on the right is the forcing term of the equation (10.72), the second term is the a priori control term needed for real principal type propagation estimates, and the last the term is the usual weak error term in microlocal estimates.) Combined with (10.73), we obtain the estimate

$$\|v\|_{\dot H^{-s+1}(\bar X)}\lesssim\|h\|_{\dot H^{-s}(\bar X)}+\|v\|_{\dot H^{-N}(\bar X)}, \tag{10.75}$$

as desired. The proof is complete. □

We end this section with a general observation which is of critical importance when studying perturbations of linear operators or nonlinear PDE (the two being closely related): the microlocal estimates used above (elliptic regularity, real principal type propagation, radial point estimates) are *stable under perturbations*. Let us explain this ingredient by ingredient for a family of operators $P(a)\in\Psi^m$ depending continuously on a parameter $a\in\mathcal{A}$ (where $\mathcal{A}$ is a normed vector space), $|a|\ll1$. For example, in the notation above, the reader may take $P(0)=\hat P(\sigma)$ for some fixed σ, and $P(a)$ is any perturbation of this.

(1) (Elliptic estimates.) Suppose $B,G\in\Psi^0$ are such that $\mathrm{WF}'(B)\subset\mathrm{Ell}(G)\cap\mathrm{Ell}(P(0))$. Then there exist ϵ,C such that for $|a|<\epsilon$, we have the *uniform estimate*

$$\|Bu\|_{H^s}\le C\big(\|GP(a)u\|_{H^{s-m}}+\|u\|_{H^{-N}}\big) \tag{10.76}$$

(for any fixed $N\in\mathbb{R}$), cf. (6.44). This follows from the fact that ellipticity is an open condition, hence the microlocal parametrix construction for $P(a)$ on $\mathrm{WF}'(B)$ can be performed with uniform control of the ps.d.o. seminorms of all operators arising in the construction.

(2) (Real principal type propagation.) The flow of the Hamiltonian vector field $H_{p(a)}$ of $P(a)$ depends continuously on the parameter a. In particular, if the assumptions on the microlocalizers B,G,E in Theorem 8.7 hold for the operator $P(0)$, then they hold for $P(a)$ as well when a is small, *for the same microlocalizers*. We claim that the estimate (8.11) (with s,N fixed) holds uniformly for small a. The robust way to prove this (which does not involve straightening out $H_{p(a)}$ in a manner that is continuous in a) is to take the commutant used in the positive commutator argument for the operator $P(0)$, and run the argument *with the same commutant*: this works since positivity is an open condition, hence any square roots we took, and any symbols which were elliptic in the arguments for $P(0)$, will remain elliptic for $P(a)$ as well.

(3) (Radial point estimates.) Even if $P(0)$ has a radial set satisfying the hypotheses in §9.3, this is general not true anymore for $P(a)$. However, fixing microlocalizers as in any of the two parts of Theorem 9.9 when applied to $P(0)$, the quantitative *estimates* (9.28a), (9.28b), (9.29) (with s,s_0,N fixed) continue to hold for $P(a)$ when a is sufficiently small, with uniform constants C. This is again due to the stability

of the positive commutator arguments under perturbations: the same commutant that was used for $P(0)$ can be used for $P(a)$ as well.

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