SNAP 2019: SCATTERING THEORY, PART 1/2

PETER HINTZ

Contents

These are lecture notes for part 1 (of 2) of the course in scattering theory at the Summer Northwestern Analysis Program (SNAP) in the summer of 2019. The course website http://math.mit.edu/~phintz/snap19/ has further details as well as movies and exercise sheets. These notes mainly draw from the book [DZ19] by Dyatlov and Zworski, specifically chapters 2 and 3. The interested reader may also want to consult Melrose's little red book [Mel95] for a fast-paced development of the basic scattering theory discussed here (as well as further topics in inverse problems and geometric scattering).

Part 2 of 2 will be taught by Prof. Maciej Zworski. See the SNAP website https://sites.northwestern.edu/snap2019/ for details and further links. For an in-depth review of the current state of art in the theory of resonances, see [Zwo17].

1. Motivation

In its simplest form, the term 'scattering' typically refers to the following situation: a particle travelling in $\mathbb{R}^n$ is sent in by an experimentalist from far away, interacts with an object or encounters a potential in a neighborhood of the origin, and travels out into the distance, where the experimentalist measures its direction. One step up from classical mechanics is electromagnetism and the theory of wave equations, which is what we will focus on: one replaces the initial particle by an ingoing wave, and one measures (amplitude and phase of) the outgoing wave.

Date: August 12, 2019.

Thus, let $V(x) \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ denote a potential, and consider a wave $u(t,x)$ described by the wave equation[1]

$$(\partial_t^2 - \Delta_{\mathbb{R}^n} + V(x))u(t,x) = \delta(t)\varphi(x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.1}$$

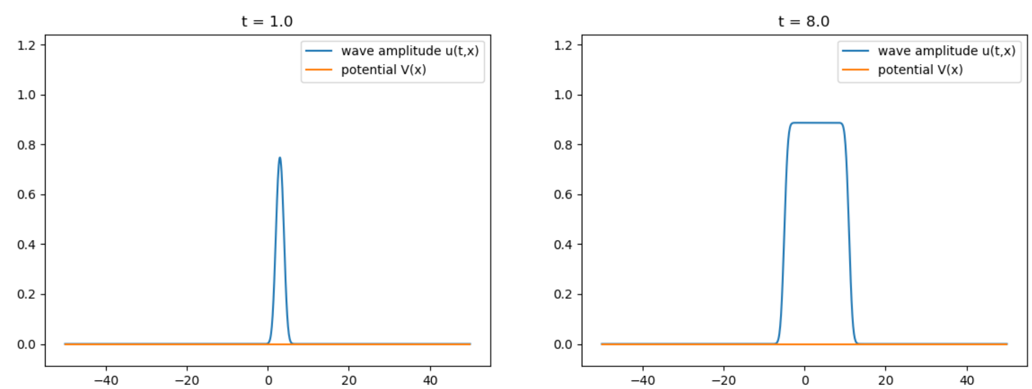
where we set[2]

$$\Delta_{\mathbb{R}^n} = \sum_{j=1}^n \partial_{x_j}^2, \tag{1.2}$$

and where $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is set up by the experimenter.

Let us look at a few examples of solutions of the wave equation (1.1) in $n = 1$ spatial dimensions. (See the course website for movies rather than snapshots.)

(1) $V = 0$: the free wave equation. We take $\varphi(x)$ to be a Gaussian. Figure 1.1 shows a snapshot of the solution at two different times. As $t \to \infty$, the solutions tends to a constant.



(A) $u(t,x)$ at time $t = 1$. (B) $u(t,x)$ at time $t = 8$.

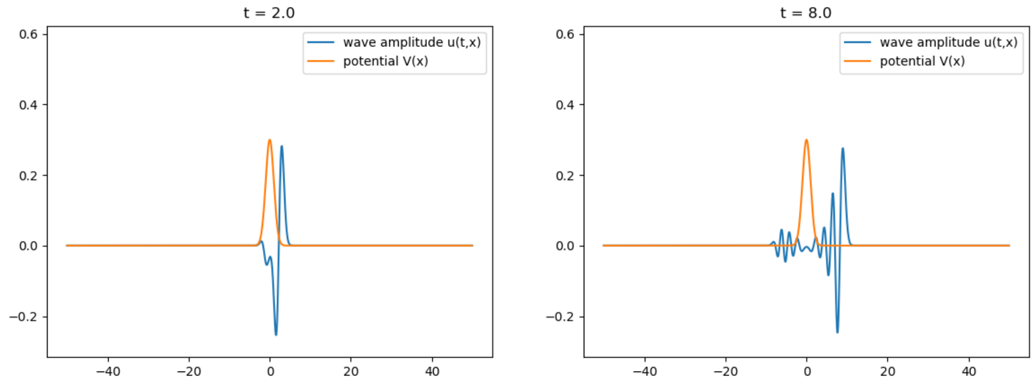
FIGURE 1.1. Snapshots of the solution $u(t,x)$ (in blue) of the free 1-d wave equation.

(2) We take V to be a bump function, and φ a Gaussian as before; see Figure 1.2. For x in a compact set, and as $t \to \infty$, $u(t,x)$ decays exponentially fast like a damped harmonic oscillator, roughly a superposition of $e^{-(\operatorname{Re}\lambda_0)t}\cos((\operatorname{Im}\lambda_0)t)a(x)$, with a complex frequency $\lambda_0 \in \mathbb{C}$, $\operatorname{Im}\lambda_0 < 0$, and some spatial profile $a(x)$. Note that $\operatorname{Re}\lambda_0$ determines the frequency of oscillation, while $-\operatorname{Im}\lambda_0$ determines the exponential rate of decay.

(3) Finally, we take V to be the sum of two offset bump functions, and φ a Gaussian as before; see Figure 1.3. The well in V 'traps' waves for some amount of time, but they still decay exponentially fast, though at a slower rate (and with a smaller

[1]One can also study the Schrödinger equation, with a more immediate connection to quantum mechanics, but we will not do so here.

[2]This is the *non-positive* Laplacian; this is the standard convention in the mathematics and physics literature on scattering theory.

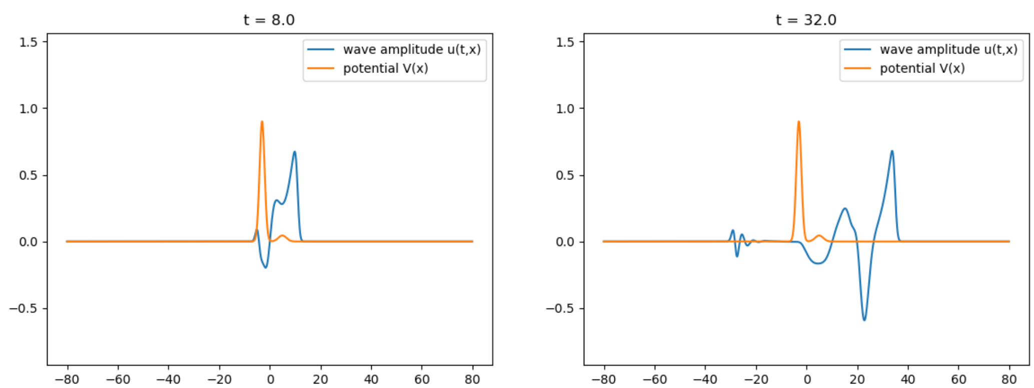


(A) $u(t,x)$ at time $t=2$.

(B) $u(t,x)$ at time $t=8$.

FIGURE 1.2. Snapshots of the solution $u(t,x)$ (in blue) of the 1-d wave equation with bump function potential V (in orange).

frequency). Near the large bump on the other hand, the picture is similar to that in Figure 1.2.



(A) $u(t,x)$ at time $t=8$.

(B) $u(t,x)$ at time $t=32$.

FIGURE 1.3. Snapshots of the solution $u(t,x)$ (in blue) of the 1-d wave equation with potential V (in orange) equal to the sum of two bump functions, with a small well in between.

In all these examples, the solution $u(t,x)$ can be well approximated by a superposition of *resonant states* for x in a compact set, say $|x|<R$:

$$u(t,x) \sim \sum e^{-i\lambda_j t} a_j(x), \quad |x|<R,\ t\to\infty. \tag{1.3}$$

The complex numbers $\lambda_j \in \mathbb{C}$ are called *resonances*, and the profiles $a_j \in \mathcal{C}^\infty(\mathbb{R})$ are called *resonant states*. The λ_j are independent of the choice of initial data, and so are the a_j up to an overall scalar factor: they only depend on the operator $-\Delta_{\mathbb{R}} + V$.

Let us illustrate (1.3) by plotting the local energy

$$E_{[-L,L]}(t) := \frac{1}{2}\int_{-L}^{L} |\partial_t u(t,x)|^2 + |\partial_x u(t,x)|^2\,dx \tag{1.4}$$

of the wave for some reasonable value of L, as well as the normalized wave

$$u_N(t,x) := u(t,x)/\sqrt{E_{[-L,L]}(t)}. \tag{1.5}$$

According to (1.3), $E_{[-L,L]}(t)$ should be exponentially decaying, and $u_N(t,x)$ should 'converge' to a purely oscillatory (or constant) function. See Figures 1.4 and 1.5 for two snapshots.

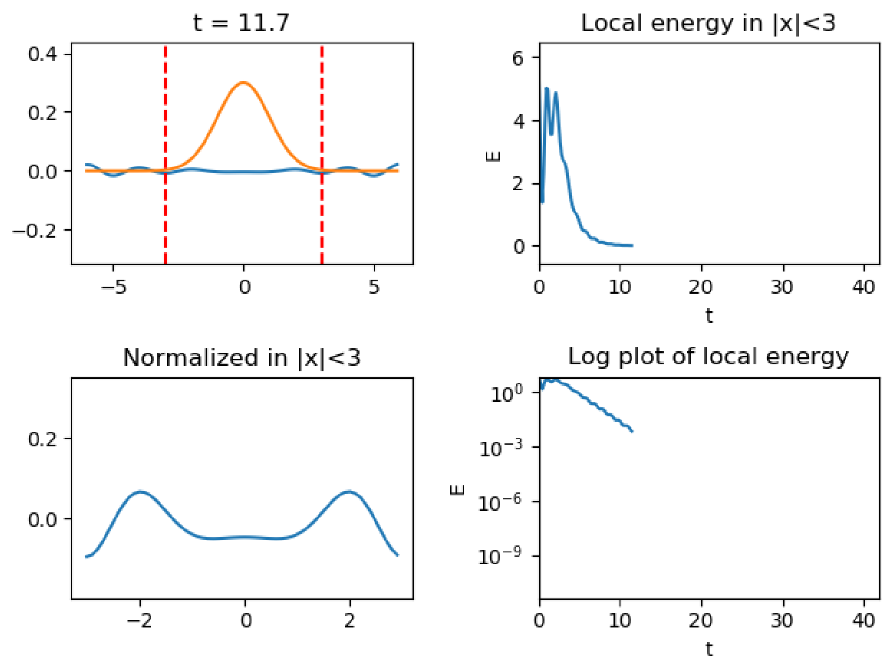


FIGURE 1.4. *Top left:* the wave (blue, exponentially small) and potential (orange) as in Figure 1.2. *Top right:* local energy calculated between the red dashed lines on the left. *Bottom left:* renormalized wave between the red dashed lines; this converges to (a superposition of) resonant states oscillating at the frequency given by the real part of the corresponding resonance(s). *Bottom right:* log plot of the local energy.

Using code by David Bindel, available at `http://www.cs.cornell.edu/~bindel/blurbs/matscat.html`, we can compute the resonances of the two potentials used above; see Figures 1.6 and 1.7.

The fundamental reason that local energy can decay (and does decay), i.e. energy can 'escape' from compact sets, is that $\mathbb{R}$ is *non-compact*: energy can escape to infinity. This

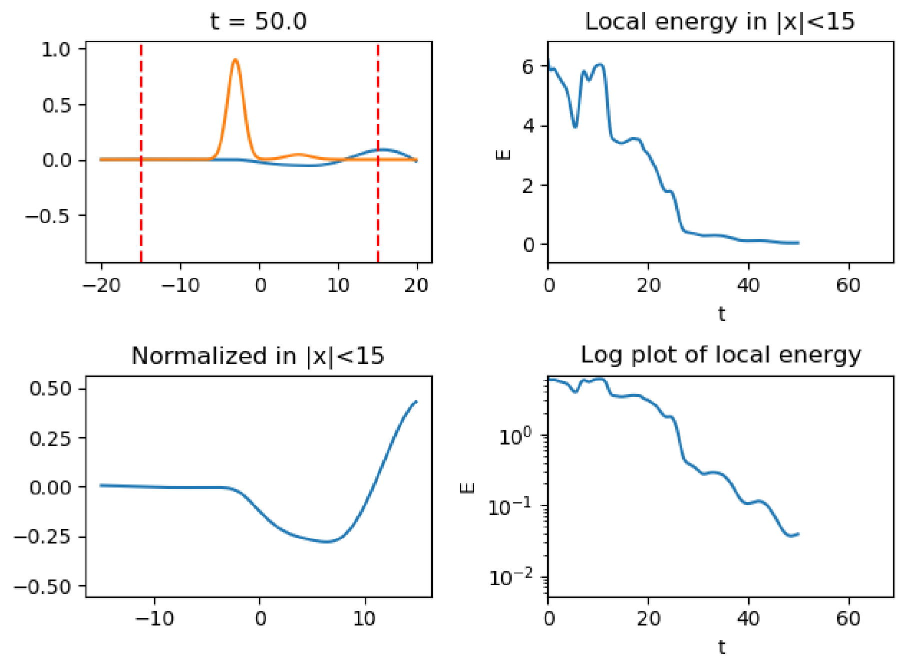


FIGURE 1.5. The same as Figure 1.4, but now for the two-bump potential as in Figure 1.3.

is in marked contrast to the case of waves on *compact* manifolds: for example, solutions $v(t,x)$ of the free wave equation on the unit circle $\mathbb{S}^1_{2\pi}$ can be given in terms of Fourier series as

$$v(t,\theta) = \sum_{k\in\mathbb{Z}} e^{ik\theta}(e^{ikt}a_+ + e^{-ikt}a_-); \tag{1.6}$$

they do *not* decay. In terms of energy: $E[v](t) := \frac{1}{2}\int_0^{2\pi} |\partial_t v(t,\theta)|^2 + |\partial_\theta v(t,\theta)|^2\, d\theta$ is constant in time.

Let us try to see what is going on in Figures 1.1–1.5. By using the Fourier transform in the time variable t, the solution of (1.1) is given by

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda t}(-\Delta_{\mathbb{R}^n} + V - \lambda^2)^{-1}\varphi(x)\, d\lambda. \tag{1.7}$$

Thus, one is naturally led to the study of the *resolvent family*

$$R_V(\lambda) := (-\Delta_{\mathbb{R}^n} + V - \lambda^2)^{-1}, \tag{1.8}$$

where of course we will have to specify precisely the spaces on which we invert the operator in parentheses. Roughly speaking, $R_V(\lambda)\varphi$, as a fixed frequency part of the *scattered* wave, should be *outgoing*, meaning that $e^{-i\lambda t}R_V(\lambda)\varphi(x)$ is, for large $r = |x|$, a function of $t - r$ only; this is guaranteed provided that

$$R_V(\lambda)\varphi(x) \sim e^{i\lambda r}, \quad r \gg 1. \tag{1.9}$$

We will show that $R_V(\lambda)$ exists as an operator on $L^2(\mathbb{R})$ when $\operatorname{Im}\lambda \gg 1$ (for real-valued V, this follows from the spectral theorem), and that it can be continued to a *meromorphic*

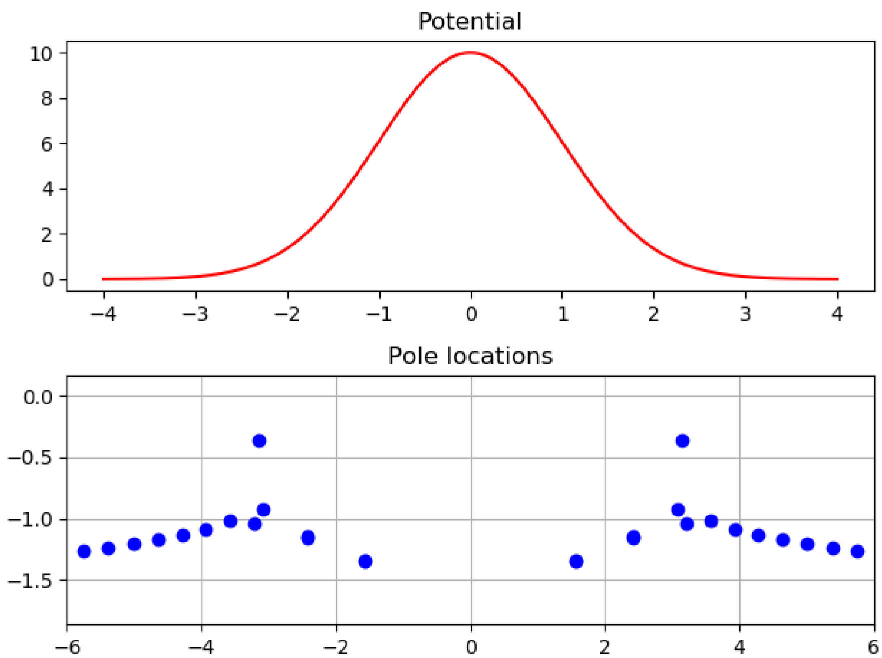


FIGURE 1.6. The bottom panel shows the first few (numerically computed) resonances of the bump potential shown on the top.

family of operators (on suitable function spaces) to the entire complex plane in $\lambda \in \mathbb{C}$. The resonances of $-\Delta_{\mathbb{R}} + V$ are then precisely the poles of $R_V(\lambda)$. (The expansion (1.3) then follows from the residue theorem upon shifting the contour of integration in (1.7).)

For instance, for $V = 0$, the *free resolvent* is given by

$$R_0(\lambda)\varphi_0(x) = \int_{\mathbb{R}} \frac{i}{2\lambda} e^{i\lambda|x-y|}\varphi_0(y)\,dy, \tag{1.10}$$

and thus has a pole at $\lambda = 0$, corresponding to the constant asymptotics observed in Figure 1.1.

For the real-valued potential in Figure 1.2, there are two (dominant) resonances $\lambda_\pm = \pm\alpha - i\beta$ (this is due to V being real-valued) with corresponding 'resonant states' $a_\pm(x) \sim e^{i\lambda_\pm|x|}$, and their superposition is

$$e^{-i\lambda_+ t}a_+(x) + e^{-i\lambda_- t}a_-(x) \sim e^{-\beta t}\cos(\alpha(|x| - t)). \tag{1.11}$$

And indeed, this function qualitatively looks like what we observe in the bottom left panel of Figure 1.4. (Similarly for Figure 1.5, except here the resonant state is very small for $x < 0$, hence not visible graphically.)

Here are two 'more' real life examples of resonances:

(1) The ringing of bell (or an empty coffee mug, or even better: a crystal glass) upon striking it with a hammer (or spoon): there will be a (dominant) pitch which does not depend on where or how hard one strikes the bell, and the note one hears decays

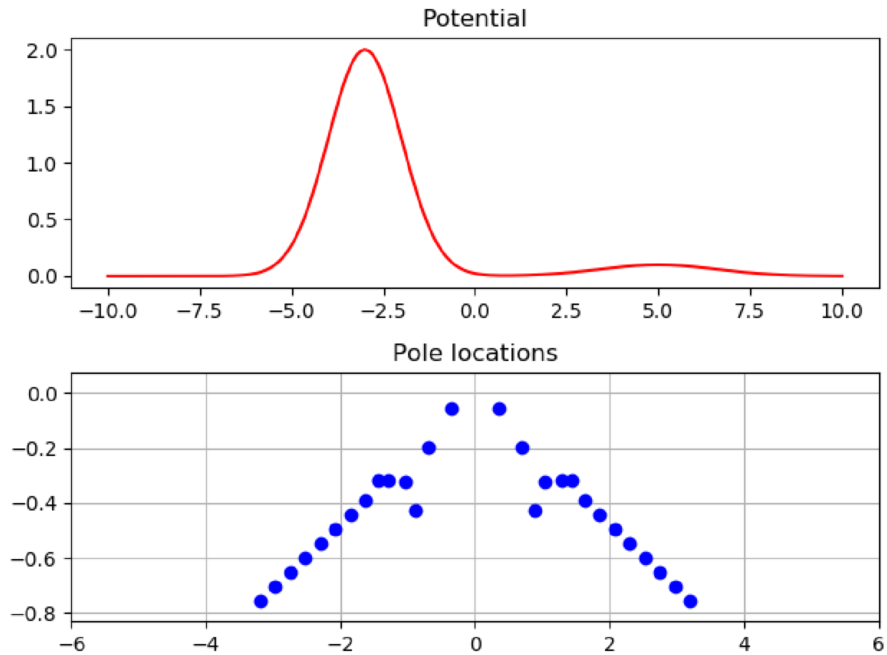


FIGURE 1.7. The bottom panel shows the first few (numerically computed) resonances of the bump potential shown on the top. Note that the resonance closest to the real axis has significantly smaller imaginary and real parts than that in Figure 1.6, corresponding to a slower decay rate and slower oscillation, as seen in the wave evolutions.

exponentially in time. For big church bells, one can often quite easily discern at least two pitches whose volume decays at different rates.

(2) Black holes in Einstein's theory of General Relativity are certain stationary (time-independent) solutions of Einstein's field equation. Their perturbations (i.e. black holes which are slightly out of equilibrium) settle down to a stationary black hole under emission of gravitational waves; these waves have resonance expansions of the form (1.3). (See [BH08, Dya12, HV18] for mathematical results.)

In these lectures, we will mostly focus on scattering theory in *three* spatial dimensions; the one-dimensional case will be discussed in the exercise sheets. Thus, the goal of the first few lectures is:

- the study of the free resolvent $R_0(\lambda) = (-\Delta_{\mathbb{R}^3} - \lambda^2)^{-1}$ and its meromorphic (in fact: analytic) continuation, see §2;
- the meromorphic continuation of the resolvent $R_V(\lambda) = (-\Delta_{\mathbb{R}^3} + V - \lambda^2)^{-1}$ for scattering by compactly supported potentials, see §3;
- proof of the resonance expansion of waves, thus making (1.3) precise and rigorous, see §4;
- properties of resonances and the 'outgoing' property of the resolvent, see §5.

From now on, we work in $n = 3$ dimensions and write $\Delta = \Delta_{\mathbb{R}^3}$.

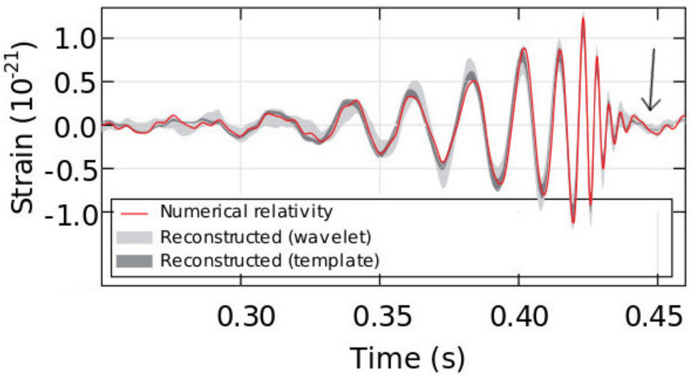


FIGURE 1.8. Gravitational waves signal measured by LIGO on September 14, 2015 [LIG16] emitted by the inspiral and merger of two black holes. The amplitude starting at the arrow (namely, shortly after the merger) is (roughly) given by an resonance expansion like (1.3) (evaluated at a single point, the location of the detector).

2. The free resolvent

For $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda > 0$, we can formally define the *free resolvent*

$$R_0(\lambda) := (-\Delta - \lambda^2)^{-1}, \quad \operatorname{Im}\lambda > 0, \tag{2.1}$$

using the Fourier transform:

$$R_0(\lambda)\varphi(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{e^{ix\xi}}{|\xi|^2-\lambda^2}\widehat{\varphi}(\xi)\,d\xi, \quad \widehat{\varphi}(\xi) := \int_{\mathbb{R}^3} e^{-ix\xi}\varphi(x)\,dx. \tag{2.2}$$

Lemma 2.1. *For $\operatorname{Im}\lambda > 0$, the free resolvent is a bounded map $R_0(\lambda)\colon L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3)$. Furthermore,*

$$\|R_0(\lambda)\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \leq \frac{C}{|\lambda|\operatorname{Im}\lambda}. \tag{2.3}$$

Proof. This follows from the characterization of $L^2(\mathbb{R}^3)$ and $H^2(\mathbb{R}^3)$ via the Fourier transform. Indeed, for $\varphi \in L^2(\mathbb{R}^3)$, we have

$$\|R_0(\lambda)\varphi\|_{H^2(\mathbb{R}^3)} \leq C\left\|\frac{1+|\xi|^2}{|\xi|^2-\lambda^2}\widehat{\varphi}(\xi)\right\|_{L^2(\mathbb{R}^3_\xi)} \leq C(\lambda)\|\widehat{\varphi}\|_{L^2(\mathbb{R}^3_\xi)} \leq C(\lambda)\|\varphi\|_{L^2(\mathbb{R}^3)}. \tag{2.4}$$

(It is easy to show that $R_0(\lambda)\colon L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3)$ is in fact an isomorphism.)

To prove (2.3), we again work on the Fourier transform side, and it remains to observe that

$$\sup_{\xi\in\mathbb{R}^3}\left|\frac{1}{|\xi|^2-\lambda^2}\right| = \frac{1}{d(\lambda^2,[0,\infty))} \leq \frac{C}{|\lambda|\operatorname{Im}\lambda}, \tag{2.5}$$

as follows by separating the cases $|\operatorname{Re}\lambda| > \operatorname{Im}\lambda$ and $|\operatorname{Re}\lambda| \leq \operatorname{Im}\lambda$. $\square$

By differentiating under the integral sign, one deduces that $R_0(\lambda) \in \mathcal{L}(L^2(\mathbb{R}^3), H^2(\mathbb{R}^3))$ depends holomorphically on λ.[3] We proceed to show that $R_0(\lambda)$ can be analytically continued:

Remark 2.2. (Connection to spectral theory.) It is easy to see that $R_0(\lambda)\colon L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3)$ is an isomorphism. This is equivalent to the statement that the symmetric operator Δ is an unbounded self-adjoint operator on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$, and $R_0(\lambda)$ is its resolvent.

Theorem 2.3. *The free resolvent $R_0(\lambda)$ has an analytic continuation from $\operatorname{Im}\lambda > 0$ to an analytic function*

$$R_0(\lambda)\colon L^2_{\mathrm{c}}(\mathbb{R}^3) \to H^2_{\mathrm{loc}}(\mathbb{R}^3), \quad \lambda \in \mathbb{C}, \tag{2.6}$$

on the entire complex plane. Its Schwartz kernel is

$$R_0(\lambda; x, y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}. \tag{2.7}$$

Since $L^2_{\mathrm{c}}(\mathbb{R}^3)$ and $H^2_{\mathrm{loc}}(\mathbb{R}^3)$ are not Banach spaces, we clarify that the conclusion means: for every cutoff function $\rho \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$, the *cutoff resolvent*

$$\rho R_0(\lambda)\rho \in \mathcal{L}(L^2(\mathbb{R}^3), H^2(\mathbb{R}^3)) \tag{2.8}$$

has an analytic continuation from $\operatorname{Im}\lambda > 0$ to $\lambda \in \mathbb{C}$.

Proof of Theorem 2.3. The strategy is to compute the Schwartz kernel (or 'integral kernel') of $R_0(\lambda)$ explicitly for $\operatorname{Im}\lambda > 0$ and study what one gets. As an oscillatory integral, the Schwartz kernel $R_0(\lambda; x, y)$ of $R_0(\lambda)$ is given by

$$R_0(\lambda; x, y) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{i(x-y)\xi}}{|\xi|^2 - \lambda^2}\, d\xi =: R_0(\lambda; z), \quad z = x - y. \tag{2.9}$$

Using polar coordinates $\xi = R\omega$, we evaluate

$$R_0(\lambda; z) = \frac{1}{(2\pi)^3} \int_{\mathbb{S}^2} \int_0^\infty \frac{e^{iRz\cdot\omega}}{R^2 - \lambda^2} R^2\, dR\, d\omega. \tag{2.10}$$

We evaluate the spherical integral

$$\int_{\mathbb{S}^2} e^{iRz\cdot\omega}\, d\omega = \int_0^{2\pi} \int_0^\pi e^{iR|z|\cos\theta} \sin\theta\, d\theta\, d\phi = \frac{2\pi(e^{iR|z|} - e^{-iR|z|})}{iR|z|}, \tag{2.11}$$

hence we have

$$\begin{aligned} R_0(\lambda; z) &= \frac{1}{i|z|(2\pi)^2} \int_0^\infty \frac{e^{iR|z|} - e^{-iR|z|}}{R^2 - \lambda^2} R\, dR \\ &= \frac{1}{2i|z|(2\pi)^2} \int_{-\infty}^\infty \left(\frac{e^{iR|z|}}{R^2 - \lambda^2} - \frac{e^{-iR|z|}}{R^2 - \lambda^2} \right) R\, dR. \end{aligned} \tag{2.12}$$

[3] The definition of holomorphicity of a Banach space valued function is exactly the same as that of a complex-valued function, and all the standard theorems of complex analysis go through with the same proofs, including Cauchy's theorem, expansion into power series (with norm convergence), and the residue theorem.

In the first summand, we shift the integration contour to $\operatorname{Im} R = +\infty$ and encounter a single pole at $R = \lambda$ with residue $e^{i\lambda|z|}/2$; in the second summand, we shift to $\operatorname{Im} R = -\infty$ and encounter a single pole at $R = -\lambda$ with residue $e^{i\lambda|z|}/2$. Altogether, this gives

$$R_0(\lambda; z) = 2\pi i \cdot e^{i\lambda|z|} \cdot \frac{1}{2i|z|(2\pi)^2} = \frac{e^{i\lambda|z|}}{4\pi|z|}, \tag{2.13}$$

as claimed.

Let $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^3)$. The boundedness

$$\rho R_0(\lambda)\rho\colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \tag{2.14}$$

follows using Schur's criterion, namely the fact that its Schwartz kernel $\rho(x)R_0(\lambda; x, y)\rho(y)$ has uniformly bounded (in x, resp. y) $L^1(\mathbb{R}^3_x)$, resp. $L^1(\mathbb{R}^3_y)$ norm. This in turn is a consequence of the observation that the singularity $|z|^{-1}$ of $R_0(\lambda; z)$ is integrable; to see this, use polar coordinates in z.

Moreover, we have $(-\Delta - \lambda^2) \circ R_0(\lambda)\rho = \rho$ as an operator on $L^2(\mathbb{R}^3)$ for $\operatorname{Im}\lambda > 0$, hence for all $\lambda \in \mathbb{C}$ by analytic continuation. Therefore, for $\varphi \in L^2_c(B(0, R))$ and $u = R_0(\lambda)\varphi$, we have

$$(-\Delta - \lambda^2)u = \varphi \in L^2(\mathbb{R}^3), \tag{2.15}$$

hence by elliptic regularity $u \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ with an estimate

$$\|\rho u\|_{H^2} \leq C\bigl(\|\varphi\|_{L^2} + \|\widetilde{\rho} u\|_{L^2}\bigr), \quad \widetilde{\rho} \in \mathcal{C}_c^\infty(\mathbb{R}^3),\ \widetilde{\rho} \equiv 1 \text{ on } \operatorname{supp}\varphi. \tag{2.16}$$

Since we already proved that $\|\widetilde{\rho} u\|_{L^2} \leq C(R)\|\varphi\|_{L^2}$, this implies

$$\|\rho u\|_{H^2} \leq C(R)\|\varphi\|_{L^2}, \quad \varphi \in L^2_c(B(0, R)), \tag{2.17}$$

proving the boundedness (2.6). □

We stress that for $\operatorname{Im}\lambda < 0$, the operator $R_0(\lambda)$ is not bounded on $L^2(\mathbb{R}^3)$. Indeed, acting on $\mathcal{C}_c^\infty(\mathbb{R}^3)$, it typically produces functions which grow like $r^{-1}e^{(\operatorname{Im}\lambda)_- r}$, where

$$x_- = \min(0, -x)$$

denotes the negative part.

Remark 2.4. While the formula (2.2) gives a well-defined operator on $L^2(\mathbb{R}^3)$ also when $\operatorname{Im}\lambda < 0$, it does *not* produce the analytic continuation $R_0(\lambda)$ constructed above. (This is clear since, by contrast, for $\operatorname{Im}\lambda < 0$, $R_0(\lambda)$ is *not* bounded on $L^2(\mathbb{R}^3)$.) Note that for real $\lambda \neq 0$, the formula (2.2) is *not* well-defined (due to division by $|\xi|^2 - \lambda^2$ not being well-defined).

We next demonstrate in what sense $R_0(\lambda)$ produces *outgoing* distributions, cf. (1.9).

Theorem 2.5. *Let $\varphi \in \mathscr{E}'(\mathbb{R}^3)$, and let $\lambda \in \mathbb{C}$. Then, for $r \gg 1$, $\omega \in \mathbb{S}^2$,*

$$\begin{aligned} R_0(\lambda)\varphi(r\omega) &= \frac{e^{i\lambda r}}{4\pi r} h(r^{-1}, \omega), \\ &\qquad h(\rho, \omega) \in \mathcal{C}^\infty([0, 1)_\rho \times \mathbb{S}^2_\omega),\ h(0, \omega) = \widehat{\varphi}(\lambda\omega). \end{aligned} \tag{2.18}$$

This says that h has a full Taylor expansion in terms of powers of r^{-1}.

Proof of Theorem 2.5. This follows from Taylor expanding $|x-y|$ into powers of $|x|$. The first two terms are

$$\begin{aligned} |x-y| &= |x|\big|x/|x| - y/|x|\big| \\ &= |x|\big(1 - 2\langle x/|x|, y\rangle/|x| + |y|^2/|x|^2\big)^{1/2} \\ &= |x| - \langle x/|x|, y\rangle + \mathcal{O}(|y|^2|x|^{-1}). \end{aligned} \tag{2.19}$$

Plugging this (with $x = r\omega$) into

$$R_0(\lambda)\varphi(r\omega) = \int_{\mathbb{R}^3} \frac{e^{i\lambda|r\omega-y|}}{4\pi|r\omega-y|}\varphi(y)\,dy, \tag{2.20}$$

the integrand becomes to leading order $(4\pi r)^{-1}e^{i\lambda r}e^{-i\lambda\omega\cdot y}\varphi(y)$, which upon integration gives the stating leading order term of the expansion. For the full Taylor series in $r^{-1} = |x|^{-1}$, one uses the full Taylor expansion of $|x-y|$. (See [DZ19, §3.1.3] for details.) $\square$

To connect this to the very rough discussion of wave asymptotics mentioned in §1, note that the free resolvent does not have any poles, i.e. the free Laplacian $\Delta_{\mathbb{R}^3}$ does not have any resonances. Thus, one expects solutions to the wave equation

$$(\partial_t^2 - \Delta)u(t,x) = \delta(t)\varphi(x), \quad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^3), \tag{2.21}$$

in compact sets in x to decay to 0 as $t \to \infty$ faster than any exponential. And indeed, the sharp Huygens principle states that $u(t,x)$ is in fact *identically* 0 for large t when x is restricted to a compact set.

3. Potential scattering, meromorphic continuation

Equipped with precise information about the free resolvent, we next study potential scattering, in the following sense. Denote by

$$V \in L_c^\infty(\mathbb{R}^3;\mathbb{C}) \tag{3.1}$$

a compactly supported complex-valued potential; we then consider the differential operator

$$P_V := -\Delta + V \colon H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \tag{3.2}$$

and study its resolvent

$$R_V(\lambda) = (P_V - \lambda^2)^{-1}, \tag{3.3}$$

concretely, its existence for $\operatorname{Im}\lambda \gg 1$ and meromorphic continuation to $\lambda \in \mathbb{C}$.

Proposition 3.1. *There exists $C > 0$ such that for $\operatorname{Im}\lambda > C$, $R_V(\lambda)\colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is an analytic family of operators, and so that the estimate*

$$\|R_V(\lambda)\|_{L^2\to L^2} \le \frac{C}{|\lambda|\operatorname{Im}\lambda} \tag{3.4}$$

holds. (In fact, $R_V(\lambda)$ maps into $H^2(\mathbb{R}^3)$.)

Proof. An approximate inverse of $P_V - \lambda^2$ is given by the *free* resolvent, so we compute for $\operatorname{Im}\lambda > 0$, as an operator on $L^2(\mathbb{R}^3)$,

$$(P_V - \lambda^2)R_0(\lambda) = I + VR_0(\lambda). \tag{3.5}$$

The estimate (2.3) and the boundedness of the multiplication operator V on $L^2(\mathbb{R}^3)$ imply that

$$\|VR_0(\lambda)\|_{L^2\to L^2} \leq \frac{1}{2}, \quad \operatorname{Im}\lambda \gg 1. \tag{3.6}$$

Therefore, we can invert $I + VR_0(\lambda) \in \mathcal{L}(L^2(\mathbb{R}^3))$ by a Neumann series, with analytic inverse of norm ≤ 2; the estimate (3.4) follows from (2.3). By (3.5), we therefore have proved the existence of a right inverse

$$R_V(\lambda) := R_0(\lambda)(I + VR_0(\lambda))^{-1} \colon L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3) \tag{3.7}$$

of $P_V - \lambda^2$.

In order to show that this is also a left inverse for $\operatorname{Im}\lambda \gg 1$, take $u \in H^2(\mathbb{R}^3)$ and put

$$u' := R_V(\lambda)(P_V - \lambda^2)u \in H^2(\mathbb{R}^3). \tag{3.8}$$

We need to show $u' = u$. Applying $P_V - \lambda^2$ to both sides of equation (3.8), we find

$$(P_V - \lambda^2)w = 0, \quad w := u' - u \in H^2(\mathbb{R}^3). \tag{3.9}$$

Multiplying this by $\bar{w}$, integrating over $\mathbb{R}^3$ and using an integration by parts, this gives

$$\int_{\mathbb{R}^3} |\nabla w|^2 + (V - \lambda^2)|w|^2 \, dx = 0. \tag{3.10}$$

For $\operatorname{Im}\lambda > 0$ sufficiently large, either the real or the imaginary part of this imply $\int |w|^2\,dx = 0$, hence $w = 0$, as desired. $\square$

Remark 3.2. We present a more abstract argument that $R_V(\lambda)$ is also the left inverse, using the theory of Fredholm operators which we will use extensively below.[4] Namely, note that $P_0 - \lambda^2 \colon H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is bounded and invertible (with inverse given by the free resolvent $R_0(\lambda)$); but $P_V - \lambda^2 = P_0 - \lambda^2 + V$. We claim that $V \colon H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is compact; to see this, note that due to V having compact support, say $\operatorname{supp} V \subset B(0,R)$, multiplication by V is the concatenation of $H^2(\mathbb{R}^3) \to H^2(B(0,R)) \to L^2(B(0,R)) \xrightarrow{V} L^2(B(0,R)) \hookrightarrow L^2(\mathbb{R}^3)$, where the second arrow is a compact inclusion by Rellich's theorem. Therefore, $\operatorname{ind}(P_V - \lambda^2) = \operatorname{ind}(P_0 - \lambda^2) = 0$, and the surjectivity of $P_V - \lambda^2$ (with right inverse $R_V(\lambda)$) implies the injectivity, hence invertibility, of $P_V - \lambda^2$. A simple abstract argument (well-known from group theory) shows that the left inverse is necessarily equal to the right inverse, hence $P_V - \lambda^2$ indeed has $R_V(\lambda)$ as its inverse.

Theorem 3.3. *$R_V(\lambda)$ extends from $\operatorname{Im}\lambda \gg 1$ to a meromorphic family of operators*

$$R_V(\lambda) \colon L^2_{\mathrm{c}}(\mathbb{R}^3) \to H^2_{\mathrm{loc}}(\mathbb{R}^3). \tag{3.11}$$

Proof. To obtain the meromorphic continuation of $R_V(\lambda)$, it suffices to meromorphically continue the inverse of $I + VR_0(\lambda)$. Since $R_0(\lambda)$ does not act on $L^2(\mathbb{R}^3)$ when $\operatorname{Im}\lambda < 0$,

[4] Recall that a Fredholm operator is a bounded linear operator $A \colon X \to Y$, where X, Y are Banach spaces, such that $\dim\ker A < \infty$, $\operatorname{ran} A$ is closed, and $\dim\operatorname{coker} A < \infty$, where $\operatorname{coker} A = Y/\operatorname{ran} A$. The *index* of a Fredholm operator is $\operatorname{ind} A = \dim\ker A - \dim\operatorname{coker} A$. The simplest example is $A = I$, or more generally A invertible. If A is Fredholm and $K \colon X \to Y$ is compact, then $A + K$ is Fredholm as well. If $t \mapsto A(t)$ is a continuous family of operators and t varies over a connected topological space, and each $A(t)$ is Fredholm, then the index $\operatorname{ind} A(t)$ is constant, i.e. independent of t. Applying this to the family of Fredholm operators $A(t) = I + tK$, $t \in \mathbb{C}$, where $K \colon X \to Y$ is compact, we see that $\operatorname{ind} A(t) = \operatorname{ind} I = 0$; in particular, $A(1) = I + K$ has index 0.

we insert compactly supported cutoffs to relate this to the inversion of an analytic family of operators on $L^2(\mathbb{R}^3)$. Thus, fix $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ with $\rho \equiv 1$ near $\operatorname{supp} V$. We then write

$$\begin{aligned} I + VR_0(\lambda) &= I + VR_0(\lambda)(1-\rho) + VR_0(\lambda)\rho \\ &= \bigl(I + VR_0(\lambda)(1-\rho)\bigr)\bigl(I + VR_0(\lambda)\rho\bigr) \end{aligned} \tag{3.12}$$

using $(1-\rho)V = 0$. The first factor is invertible as an operator on $L^2_c(\mathbb{R}^3)$:

$$\bigl(I + VR_0(\lambda)(1-\rho)\bigr)^{-1} = I - VR_0(\lambda)(1-\rho). \tag{3.13}$$

The second factor can be written as

$$I + VR_0(\lambda)\rho = I + K(\lambda), \quad K(\lambda) := V\rho R_0(\lambda)\rho. \tag{3.14}$$

The crucial point is now that

$$K(\lambda)\colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \quad \text{is a compact operator.} \tag{3.15}$$

Indeed, if $|x| < R$ near $\operatorname{supp}\rho$, then by Theorem 2.3, $K(\lambda)$ equals the composition

$$L^2(\mathbb{R}^3) \to L^2(B(0,R)) \xrightarrow{\rho} L^2(B(0,R)) \xrightarrow{\rho R_0(\lambda)} H^2(B(0,R)) \hookrightarrow L^2(B(0,R)) \hookrightarrow L^2(\mathbb{R}^3), \tag{3.16}$$

which is a compact operator by Rellich's compactness theorem (applied to the second-to-last arrow).

By (the proof of) Proposition 3.1, $I + K(\lambda) \in \mathcal{L}(L^2(\mathbb{R}^3))$ is invertible for $\operatorname{Im}\lambda \gg 1$. Since $I + K(\lambda)$, $\lambda \in \mathbb{C}$, is an analytic family of Fredholm operators, the *analytic Fredholm theorem*, see Theorem 3.5 below, then implies that

$$(I + K(\lambda))^{-1} \in \mathcal{L}(L^2(\mathbb{R}^3)) \tag{3.17}$$

is meromorphic.

For $\operatorname{Im}\lambda > 0$, we can thus expand the formula (3.7) into

$$R_V(\lambda) = R_0(\lambda)(I + K(\lambda))^{-1}\bigl(I - VR_0(\lambda)(1-\rho)\bigr)\colon L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3), \tag{3.18}$$

giving the meromorphic continuation to $\operatorname{Im}\lambda > 0$.

To continue this to all $\lambda \in \mathbb{C}$, we restrict the domain to $L^2_c(\mathbb{R}^3)$: the third factor in (3.18) maps this into $L^2_c(\mathbb{R}^3)$, and the first factor $R_0(\lambda)$ maps $L^2_c(\mathbb{R}^3) \to H^2_{\mathrm{loc}}(\mathbb{R}^3)$. It thus remains to show that the middle factor $(I + K(\lambda))^{-1}$ maps

$$(I + K(\lambda))^{-1}\colon L^2_c(\mathbb{R}^3) \to L^2_c(\mathbb{R}^3). \tag{3.19}$$

But if $(I + K(\lambda))u = f \in L^2_c(\mathbb{R}^3)$, then $u = f - K(\lambda)u \in L^2_c(\mathbb{R}^3)$ (due to the factor of V in the definition of $K(\lambda)$). The proof is complete. $\square$

For the convenience of the reader, we recall the analytic Fredholm theorem (in a slightly sharper form than used above). First:

Definition 3.4. Let $\Omega \subset \mathbb{C}$ be a connected open set. Let X, Y denote Banach spaces. We then say that $\Omega \ni z \mapsto B(z) \in \mathcal{L}(X,Y)$ is a meromorphic family of operators if for any $z_0 \in \Omega$ there exist $J \in \mathbb{N}_0$ and operators B_j, $j = 1, \ldots, J$, of finite rank as well as a family of operators $z \mapsto B_0(z) \in \mathcal{L}(X,Y)$, holomorphic near z, such that

$$B(z) = B_0(z) + \sum_{j=1}^{J} (z - z_0)^{-j} B_j \quad \text{near } z_0. \tag{3.20}$$

Theorem 3.5. *(Analytic Fredholm theorem.) Let $\Omega \subset \mathbb{C}$ be an open connected set, and let X, Y denote two Banach spaces. Suppose $\Omega \ni z \mapsto B(z) \in \mathcal{L}(X, Y)$ is an analytic family of Fredholm operators. Then:*

(1) *either $B(z)$ is not invertible for any $z \in \Omega$,*
(2) *or $z \mapsto B(z)^{-1}$ is a meromorphic family of operators.*

We leave the proof of this in the case $X = \mathbb{C}^N$, $Y = \mathbb{C}^M$ as an exercise. (The second possibility can only occur when $N = M$.)

Proof of Theorem 3.5. The index of $B(z)$ is constant; if it is non-zero, the first alternative holds. Assume thus that $\operatorname{ind} B(z) = 0$. Denote by $\Omega' \subset \Omega$ the subset on which $B(z)$ is invertible; thus Ω' is open, and $B(z)^{-1}$ is analytic on Ω'.

If $\Omega' \neq \Omega$, pick $z_0 \in \Omega \cap \partial\Omega'$. Let $X_1 = \ker B(z_0)$, $Y_0 = \operatorname{ran} B(z_0)$, and pick complementary closed subspaces $X_0 \subset X$, $Y_1 \subset Y$, such that

$$X = X_0 \oplus X_1, \quad Y = Y_0 \oplus Y_1. \tag{3.21}$$

We have $\operatorname{ind} B(z_0) = \dim X_1 - \dim Y_1 = 0$, so X_1, Y_1 have the same finite dimension. In the splitting (3.21), $B(z)$ takes the form

$$B(z) = \begin{pmatrix} B_{00}(z) & B_{01}(z) \\ B_{10}(z) & B_{11}(z) \end{pmatrix}, \tag{3.22}$$

where $B_{01}(z_0) = 0$, $B_{10}(z_0) = 0$, $B_{11}(z_0) = 0$, and $B_{00}(z_0)\colon X_0 \to Y_0$ is invertible. Since the set of invertible operators in $\mathcal{L}(X_0, Y_0)$ is open, $B_{00}(z)$ is invertible for z near z_0.

Now $B(z)$ is invertible if and only if $B(z)(x_0, x_1) = (0, 0)$ only has the trivial solution; for z near z_0, this gives $x_0 = -B_{00}(z)^{-1}B_{01}(z)x_1$ and then

$$B^\sharp(z)x_1 \equiv \big(B_{11}(z) - B_{10}(z)B_{00}(z)^{-1}B_{01}(z)\big)x_1 = 0. \tag{3.23}$$

Thus, the invertibility of $B(z)$ is equivalent to that of $B^\sharp(z) \in \mathcal{L}(X_1, Y_1)$; and if $B^\sharp(z)$ is invertible, then we have

$$B(z)^{-1} = \begin{pmatrix} B_{00}^{-1} + B_{00}^{-1}B_{01}(B^\sharp)^{-1}B_{10}B_{00}^{-1} & -B_{00}^{-1}B_{01}(B^\sharp)^{-1} \\ -(B^\sharp)^{-1}B_{10}B_{00}^{-1} & (B^\sharp)^{-1} \end{pmatrix}. \tag{3.24}$$

By assumption, $B^\sharp(z_0) = 0$, but $B^\sharp(z)$ is invertible along a sequence of $z \in \Omega'$ tending to z_0. Fixing bases of X_1, Y_1, the determinant $\det B^\sharp(z)$ is thus an analytic function near z_0 which does not vanish identically; its inverse its therefore meromorphic, implying that $B^\sharp(z)$ is invertible in a punctured neighborhood of z_0 with meromorphic inverse (cf. the finite-dimensional exercise). The explicit formula (3.24) then implies that $B(z)^{-1}$ is meromorphic near z_0 as well, finishing the proof. □

Having the meromorphic continuation of $R_V(\lambda)$ in our hands, we now study its poles in more detail.

Definition 3.6. Let $V \in L^\infty_{\mathrm{c}}(\mathbb{R}^3; \mathbb{C})$.

(1) The poles of the meromorphic continuation of $R_V(\lambda) = (P_V - \lambda^2)^{-1}$ are called *(scattering) resonances.*

(2) If λ_0 is a resonance and $R_V(\lambda) = B_0(\lambda) + \sum_{j=1}^{J} (\lambda - \lambda_0)^{-j} B_j$, with $B_0(\lambda)$ holomorphic near λ_0 and $B_J \neq 0$, we call J the *order* of the resonance. For $J = 1$, we call

$$m_V(\lambda_0) := \dim B_1(L^2_{\mathrm{c}}(\mathbb{R}^3)) \tag{3.25}$$

the *multiplicity* of λ_0. In this case, any element of $B_1(L^2_{\mathrm{c}}(\mathbb{R}^3))$ is called a *resonant state*. If $m_V(\lambda_0) = 1$, we say that λ_0 is a *simple* resonance.

We have shown that the set of resonances is a discrete subset of $\mathbb{C}$ and contained in a half space $\{\operatorname{Im}\lambda < C\}$ for some constant C. The multiplicity of a resonance is defined in general as the dimension of the space spanned by $B_j(L^2_{\mathrm{c}}(\mathbb{R}^3))$, $j = 1, \ldots, J$. For simplicity, we will only concern ourselves here with simple resonances (which is the generic situation, see [DZ19, Theorem 4.39]).

Next, we give a characterization of non-zero resonances as those complex numbers λ_0 for which there exists an *outgoing* solution u of $(P_V - \lambda_0^2)u = 0$, where 'outgoing' means being in the range of the free resolvent:

Theorem 3.7. *A complex number $\lambda_0 \neq 0$ is a resonance if and only if there exists $0 \neq \varphi \in L^2_{\mathrm{c}}(\mathbb{R}^3)$ such that for $u = R_0(\lambda_0)\varphi$, we have $(P_V - \lambda_0^2)u = 0$.*

Proof. Let $\rho \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$ be identically 1 near $\operatorname{supp} V$. Given φ as in the statement of the theorem, we have

$$0 = (-\Delta + V - \lambda_0^2)R_0(\lambda_0)\varphi = (I + VR_0(\lambda_0))\varphi = \big(I + VR_0(\lambda_0)(1-\rho)\big)(I + K(\lambda_0))\varphi, \tag{3.26}$$

where $K(\lambda) = V\rho R_0(\lambda)\rho$ (see (3.12)). Since the second to last operator is invertible (see (3.13)), this implies $I + K(\lambda)$ has non-trivial kernel at λ_0, hence its inverse has a pole there. Using the formula (3.18) for $R_V(\lambda)$ and the fact that the first factor $R_0(\lambda)$ there is injective on $L^2_{\mathrm{c}}(\mathbb{R}^3)$, while the third factor is an isomorphism on $L^2_{\mathrm{c}}(\mathbb{R}^3)$, we conclude that $R_V(\lambda)$ has a pole at λ_0.

We prove the converse only for simple resonances. Writing

$$R_V(\lambda) = (\lambda - \lambda_0)^{-1} B_1 + B_0(\lambda) \tag{3.27}$$

with $B_0(\lambda)$ holomorphic near λ_0, we have

$$I = (P_V - \lambda^2)R_V(\lambda) = (\lambda - \lambda_0)^{-1}(P_V - \lambda_0^2)B_1 + (\text{holomorphic near } \lambda_0), \tag{3.28}$$

hence $(P_V - \lambda_0^2)B_1 = 0$. Pick $\psi \in L^2_{\mathrm{c}}(\mathbb{R}^3)$ such that $B_1\psi \neq 0$.

Now, note that $(-\Delta - \lambda^2)R_V(\lambda) = I - VR_V(\lambda)$; applying $R_0(\lambda)$ from the left, we get

$$R_V(\lambda) = R_0(\lambda) - R_0(\lambda)VR_V(\lambda) \tag{3.29}$$

(first for $\operatorname{Im}\lambda \gg 1$, then for all $\lambda \in \mathbb{C}$ by analytic continuation). Integrating this along a small circle around λ_0, we find

$$B_1 = -R_0(\lambda_0)VB_1. \tag{3.30}$$

Applying this to ψ and putting $\varphi := -VB_1\psi \in L^2_{\mathrm{c}}(\mathbb{R}^3)$, we obtain $R_0(\lambda_0)\varphi = B_1\psi \in \ker(P_V - \lambda_0^2)$, as desired. ∎

4. Resonance expansion of waves

Our next goal is to make the resonance expansion of scattered waves mentioned (and observed) in §1 precise.

Theorem 4.1. *Let $V \in L^\infty_{\mathrm{c}}(\mathbb{R}^3;\mathbb{C})$, and suppose all resonances $\lambda_1, \lambda_2, \ldots$ of $P_V = -\Delta + V$ are simple.*[5] *Denote by $u(t,x)$ the solution of*

$$\begin{cases} (\partial_t^2 - \Delta + V)u(t,x) = 0, & t \geq 0,\ x \in \mathbb{R}^3, \\ u(0,x) = \psi(x) \in H^1_{\mathrm{c}}(\mathbb{R}^3), \\ \partial_t u(0,x) = \varphi(x) \in L^2_{\mathrm{c}}(\mathbb{R}^3). \end{cases} \tag{4.1}$$

Then, for any $A > 0$, we have

$$u(t,x) = \sum_{\operatorname{Im}\lambda_j > -A} e^{-i\lambda_j t} a_j(x) + E_A(t), \tag{4.2}$$

where the sum is finite, a_j is a resonant state (thus solving $(P_V - \lambda_j^2)a_j = 0$) given by

$$e^{-i\lambda_j t} a_j(x) = \operatorname{Res}_{\lambda=\lambda_j}\bigl(e^{-i\lambda t}(iR_V(\lambda)\varphi + \lambda R_V(\lambda)\psi)\bigr), \tag{4.3}$$

and for any $R > 0$, there exist constants $C_{R,A}$ and $T_{R,A}$ such that

$$\|E_A(t)\|_{H^2(B(0,R))} \leq C_{R,A} e^{-tA}(\|\psi\|_{H^1} + \|\varphi\|_{L^2}), \quad t \geq T_{R,A}. \tag{4.4}$$

We recall that the initial value problem (4.1) has a unique solution[6]

$$u \in \mathcal{C}^0(\mathbb{R}_t; H^1_{\mathrm{c}}(\mathbb{R}^3)) \cap \mathcal{C}^1(\mathbb{R}_t; L^2_{\mathrm{c}}(\mathbb{R}^3)). \tag{4.6}$$

The compact support in the x-variables is inherited from the initial data by finite speed of propagation. For simplicity, we shall here only consider the case that

$$\psi \equiv 0, \quad \varphi \in L^2_{\mathrm{c}}(\mathbb{R}^3). \tag{4.7}$$

(The exercises touch on the general case.) We rephrase (4.1) as a *forcing* problem: setting $\tilde{u}(t,x) = H(t)u(t,x)$, one computes that $(\partial_t^2 - \Delta + V)\tilde{u} = \delta(t)\varphi(x)$; moreover, $\tilde{u}$ is the unique solution of this equation which vanishes for $t < 0$. The advantage of this formulation is that it is directly amenable to taking the Fourier transform in t, as we shall see momentarily.

We henceforth study $\tilde{u}$; changing notation, we thus study, for $\varphi \in L^2_{\mathrm{c}}(\mathbb{R}^3)$, the equation

$$(\partial_t^2 - \Delta + V)u(t,x) = \delta(t)\varphi(x), \tag{4.8}$$

$$u(t,x) = 0,\ t < 0; \tag{4.9}$$

cf. (1.1). Formally taking the Fourier transform in t gives

$$(P_V - \lambda^2)\hat{u}(\lambda,x) = \varphi(x), \quad \hat{u}(\lambda,x) = \int_{\mathbb{R}} e^{i\lambda t} u(t,x)\,dt, \tag{4.10}$$

---

[5] In the general case, the expansion (4.2) may have additional algebraic factors, i.e. the expansion becomes $\sum e^{-i\lambda_j t} t^k a_{j,k}(x) + E_A(t)$, where for any j, k is bounded.

[6] If V is real-valued, then P_V is self-adjoint on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$, and the solution of (4.1) is given using the functional calculus for P_V by

$$u(t,x) = \cos(t\sqrt{P_V})\psi(x) + \frac{\sin(t\sqrt{P_V})}{\sqrt{P_V}}\varphi(x). \tag{4.5}$$

Using Stone's formula for the spectral measure, the operators appearing here can be expressed in terms of the resolvent, and one can proceed with the proof of Theorem 4.1 from there. This is the approach taken in [DZ19, §§2.3,3.3].

suggesting we try to construct the solution of (4.8) using the inverse Fourier transform,

$$u(t,x) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=M} e^{-i\lambda t}R_V(\lambda)\varphi(x)\,d\lambda, \tag{4.11}$$

where the constant M will be chosen appropriately and so that $R_V(\lambda)$ does not have poles on the integration contour. Formally differentiating (4.11) shows that it indeed solves (4.8).

We now make sense of the integral (4.11):

Lemma 4.2. *For $M > 0$ sufficiently large, the formula (4.11) defines an element of $L^2_{\mathrm{loc}}(\mathbb{R}_t; L^2(\mathbb{R}^3_x))$ which vanishes for $t < 0$, and thus solves (4.8)–(4.9).*

Proof. We use the estimate (3.4) on $R_V(\lambda)$, which implies that

$$\|R_V(\lambda)\varphi\|_{L^2(\mathbb{R}^3)} \le \frac{C}{|\lambda|\operatorname{Im}\lambda}\|\varphi\|_{L^2(\mathbb{R}^3)} \tag{4.12}$$

for $\operatorname{Im}\lambda \ge M$, $M > 0$ large enough. (In particular, there are no resonances in this half-space.) Writing $\lambda = \sigma + iM$, this is square-integrable in σ. Plancherel's theorem therefore implies that

$$u(t,x) = \frac{1}{2\pi}e^{Mt}\int_{\mathbb{R}} e^{-i\sigma t}R_V(\sigma+iM)\varphi\,d\sigma \in L^2_{\mathrm{loc}}(\mathbb{R}_t; L^2(\mathbb{R}^3_x)). \tag{4.13}$$

Next, for $t < 0$, we have $|e^{-i\lambda t}| = e^{-(\operatorname{Im}\lambda)|t|} \to 0$ as $\operatorname{Im}\lambda \to +\infty$. Thus, using Cauchy's theorem, we can shift the contour of integration to $\operatorname{Im}\lambda = M'$ for any $M' \ge M$ (using that the integrand has no poles in $\operatorname{Im}\lambda \ge M$); this gives

$$\begin{aligned} u(t,x) &= \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=M'} e^{-i\lambda t}R_V(\lambda)\varphi(x)\,d\lambda \\ &= \frac{1}{2\pi}e^{-M'|t|}\int_{\mathbb{R}} e^{-i\sigma t}R_V(\sigma+iM')\varphi(x)\,d\sigma \end{aligned} \tag{4.14}$$

Let $T_- < T_+ < 0$. Using (4.12), we obtain

$$\|u\|_{L^2([T_-,T_+];L^2(\mathbb{R}^3_x))} \le \frac{e^{-M'T_-}}{2\pi}\|R_V((-)+iM')\varphi\|_{L^2(\mathbb{R}_\sigma;L^2(\mathbb{R}^3_x))} \le C\frac{e^{-M'T_-}}{2\pi}. \tag{4.15}$$

Letting $M' \to \infty$, this implies that $u(t,x) = 0$ for $t \in [T_-, T_+]$. Taking $T_- \to -\infty$ and $T_+ \nearrow 0$ proves the claim. $\square$

To obtain the resonance expansion (4.2), we want to shift the contour into the *lower* half plane so that we see the exponentially decaying contributions from resonances and resonant states. (Note that the above proof only captures u in spaces allowing for exponential growth as $t \to \infty$, cf. the exponential prefactor in (4.13).)

The estimate (4.12) is a *high energy estimate*, in that it gives quantitative control for fixed $\operatorname{Im}\lambda$ as $|\operatorname{Re}\lambda| \to \infty$. (This smells like semiclassical tools should be very useful for studying the resolvent in this limit, and indeed they are; cf. part 2 of the scattering theory course.) In order to justify the contour shifting into the lower half plane, we need corresponding estimates there.

We first prove such estimates for the free resolvent by relating the free resolvent $R_0(\lambda)$ to the wave propagator

$$U(t) := \sin(t\sqrt{-\Delta})/\sqrt{-\Delta}. \tag{4.16}$$

This is defined using the Fourier transform $\mathcal{F}$ by $\mathcal{F}(U(t)f)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}\mathcal{F}f(\xi)$, $f \in \mathscr{S}(\mathbb{R}^3)$.[7] We leave it as an exercise to prove the explicit representation formula

$$U(t)f(x) = \frac{1}{4\pi t}\int_{\partial B(x,t)} f(y)\,dS(y), \quad t > 0. \tag{4.17}$$

This implies the *strong Huygens principle*:

$$f \in \mathcal{C}_c^\infty(B(0,R)) \quad \Longrightarrow \quad B(0,R) \cap \operatorname{supp} U(t)f = \emptyset, \quad t > 2R. \tag{4.18}$$

Let now $\operatorname{Im}\lambda > 0$. Then $R_0(\lambda)\colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ can be expressed as

$$R_0(\lambda) = \int_0^\infty e^{i\lambda t} U(t)\,dt, \tag{4.19}$$

with convergence in operator norm. This follows, upon passing to the Fourier transform in space, from the complex analysis calculation (exercise)

$$\int_0^\infty e^{i\lambda t}\frac{\sin(t|\xi|)}{|\xi|}\,dt = \frac{1}{\xi^2 - \lambda^2}, \quad \operatorname{Im}\lambda > 0. \tag{4.20}$$

Proposition 4.3. *For $R > 0$ and $\rho \in \mathcal{C}_c^\infty(B(0,R))$, we have*

$$\|\rho R_0(\lambda)\rho\|_{L^2 \to H^j} \le C\langle\lambda\rangle^{j-1} e^{2R(\operatorname{Im}\lambda)_-}, \quad j = 0,1,2. \tag{4.21}$$

Proof. Let $\rho \in \mathcal{C}_c^\infty(B(0,R))$. Using the sharp Huygens principle, we conclude that $\rho U(t)\rho = 0$ for $T > 2R$, hence

$$\rho R_0(\lambda)\rho = \int_0^{2R} e^{i\lambda t}\rho U(t)\rho\,dt. \tag{4.22}$$

This holds for $\operatorname{Im}\lambda > 0$ in view of (4.19), and then for all λ by analytic continuation.

Write now $\|U(t)\|_{L^2\to H^1}^2 = \|U(t)\|_{L^2\to L^2}^2 + \|\sqrt{-\Delta}U(t)\|_{L^2\to L^2}^2$, and use that

$$\begin{aligned} \|U(t)\|_{L^2\to L^2} &= \sup_{\xi\in\mathbb{R}^3}\left|\frac{\sin(t|\xi|)}{|\xi|}\right| = |t|, \\ \|\sqrt{-\Delta}U(t)\|_{L^2\to L^2} &= \sup_{\xi\in\mathbb{R}^3}|\cos(t|\xi|)| = 1; \end{aligned} \tag{4.23}$$

therefore $\|U(t)\|_{L^2\to H^1} = (1+t^2)^{1/2}$. Plugging this into (4.22) gives

$$\|\rho R_0(\lambda)\rho\|_{L^2\to H^1} \le Ce^{2R(\operatorname{Im}\lambda)_-} \tag{4.24}$$

(with C depending on ρ), thus proving (4.21) for $j = 1$.

For the case $j = 0$, write

$$\begin{aligned} \lambda\rho R_0(\lambda)\rho &= \int_0^{2R} -i\partial_t(e^{i\lambda t})\rho U(t)\rho\,dt \\ &= -i\rho(e^{2i\lambda R}U(2R) - U(0))\rho + i\int_0^{2R} e^{i\lambda t}\rho\partial_t U(t)\rho\,dt. \end{aligned} \tag{4.25}$$

Since $\partial_t U(t) = \sqrt{-\Delta}U(t)$, we conclude using (4.23) that

$$|\lambda|\|\rho R_0(\lambda)\rho\|_{L^2\to L^2} \le Ce^{2R(\operatorname{Im}\lambda)_-}, \tag{4.26}$$

[7]Given $\varphi \in L^2(\mathbb{R}^3)$, the function $u(t,x) = U(t)\varphi(x)$ solves the initial value problem $(\partial_t^2 - \Delta)u = 0$, $u(0,x) = 0$, $\partial_t u(0,x) = \varphi(x)$.

which gives the desired result upon division by $|\lambda|$.

Finally, for $j = 2$, fix $\widetilde{\rho} \in \mathcal{C}_c^\infty(B(0,R))$ with $\rho_1 = 1$ near $\operatorname{supp}\rho$. Since $-\Delta R_0(\lambda) = \lambda^2 R_0(\lambda)$, we can estimate

$$\begin{aligned}
&\|\rho R_0(\lambda)\rho\|_{L^2\to H^2} \\
&\qquad \le C\big(\|\Delta\rho R_0(\lambda)\rho\|_{L^2\to L^2} + \|\rho R_0(\lambda)\rho\|_{L^2\to L^2}\big) \\
&\qquad \le C\big(\|[\Delta,\rho](\widetilde{\rho}R_0(\lambda)\widetilde{\rho})\rho\|_{L^2\to L^2} + \|\rho\lambda^2 R_0(\lambda)\rho\|_{L^2\to L^2} + \|\rho R_0(\lambda)\rho\|_{L^2\to L^2}\big) \\
&\qquad \le C\big(\|\widetilde{\rho}R_0(\lambda)\widetilde{\rho}\|_{L^2\to H^1} + (|\lambda|^2+1)\|\rho R_0(\lambda)\rho\|_{L^2\to L^2}\big) \\
&\qquad \le Ce^{2R(\operatorname{Im}\lambda)_-}\langle\lambda\rangle.
\end{aligned} \tag{4.27}$$

This completes the proof of the proposition. ∎

The proof of the corresponding result for $R_V(\lambda)$ is now easy:

Proposition 4.4. *Let $V \in L_c^\infty(\mathbb{R}^3;\mathbb{C})$, and let $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^3)$. Then for all $A \in \mathbb{R}$ and $\delta < 1/\operatorname{diam}(\operatorname{supp} V)$, there exists constants C, C_1, C_2 such that*

$$\|\rho R_V(\lambda)\rho\|_{L^2\to H^j} \le C\langle\lambda\rangle^{j-1}e^{C_1(\operatorname{Im}\lambda)_-}, \quad j = 0, 1, 2, \tag{4.28}$$

for all λ with

$$|\lambda| \ge C_2, \quad \operatorname{Im}\lambda \ge -A - \delta\log(1+|\lambda|). \tag{4.29}$$

In particular, there are only finitely many resonances in the region $\{\lambda \in \mathbb{C}\colon \operatorname{Im}\lambda \ge -A - \delta\log(1+|\lambda|)\}$ for any $A > 0$.

Proof. Without loss, we may assume $\rho \equiv 1$ near $\operatorname{supp} V$. Let $R > \operatorname{diam}(\operatorname{supp} V)/2$. By translating V if necessary, we can pick $\chi \in \mathcal{C}_c^\infty(B(0,R))$ with $\chi \equiv 1$ near $\operatorname{supp} V$. We then recall the formula

$$\rho R_V(\lambda)\rho = \rho R_0(\lambda)\circ(I + V\chi R_0(\lambda)\chi)^{-1}\circ(I - VR_0(\lambda)(1-\chi))\rho \tag{4.30}$$

from (3.18) and (3.14). We thus conclude that (4.28) holds for those λ for which

$$\|V\chi R_0(\lambda)\chi\|_{L^2\to L^2} \le \frac{1}{2}, \tag{4.31}$$

since the middle factor in (4.30) is then given by a Neumann series with operator norm ≤ 2. But by Proposition 4.3, we have

$$\|V\chi R_0(\lambda)\chi\|_{L^2\to L^2} \le C\|V\|_{L^\infty}(1+|\lambda|)^{-1}e^{2R(\operatorname{Im}\lambda)_-}, \tag{4.32}$$

which for $\operatorname{Im}\lambda > -A - \delta\log(1+|\lambda|)$ with $\delta < 1/(2R)$ is bounded by

$$Ce^{2RA_+}\|V\|_{L^\infty}(1+|\lambda|)^{-1+2R\delta} \le \frac{1}{2} \tag{4.33}$$

for $|\lambda|$ sufficiently large. ∎

We are now ready to prove the resonance expansion of scattered waves:

Proof of Theorem 4.1. As discussed above, we only study the case of initial data $(\psi,\varphi) = (0,\varphi)$, $\varphi \in L_c^2(\mathbb{R}^3)$, with solution in $t > 0$ given in terms of the resolvent by the formula (4.11) (which, precisely speaking, is the solution of the forward forcing problem (4.8)–(4.9)). Choosing a cutoff $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^3)$, $\rho \equiv 1$ near $B(0,R) \cup \operatorname{supp}\varphi$, we have

$$\rho(x)u(t,x) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = M} e^{-i\lambda t}\rho R_V(\lambda)\rho\varphi(x)\,d\lambda, \quad M \gg 1,\ t > 0. \tag{4.34}$$

We now deform the contour. Let $A \in \mathbb{R}$ be as in the statement of the theorem (namely, the desired decay rate of the remainder term), and let

$$0 < \delta < \min\left(\frac{1}{C_1}, \frac{1}{\operatorname{diam}(\operatorname{supp} V)}\right), \tag{4.35}$$

with C_1 given by Proposition 4.4. Choose R large enough so that all resonances $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda > -A - \delta\log(1 + |\operatorname{Re}\lambda|)$ satisfy $|\lambda| < R$. We use the contours

$$\begin{aligned} \Gamma &:= \{\lambda - i(A + \delta\log(1 + |\operatorname{Re}\lambda|))\colon \lambda \in \mathbb{R}\}, \\ \Gamma_R &:= \Gamma \cap \{|\operatorname{Re}\lambda| \leq R\}, \\ \gamma_R^\pm &:= \{\pm R + is\colon\ -A - \delta\log(1 + R) \leq s \leq M\}, \\ \Gamma_R^{||} &:= \gamma_R^- \cup \gamma_R^+, \\ \Gamma_R^- &:= (-\infty + iM, -R + iM) \cup (R + iM, \infty + iM). \end{aligned} \tag{4.36}$$

The resonances above the contour Γ are thus all contained in the domain

$$\Omega_A := \{\lambda \in \mathbb{C}\colon\ \operatorname{Im}\lambda \geq -A - \delta\log(1 + |\operatorname{Re}\lambda|)\}. \tag{4.37}$$

See Figure 4.1. By (slightly) increasing A and R, we can make sure that there are no resonances on the contours themselves.

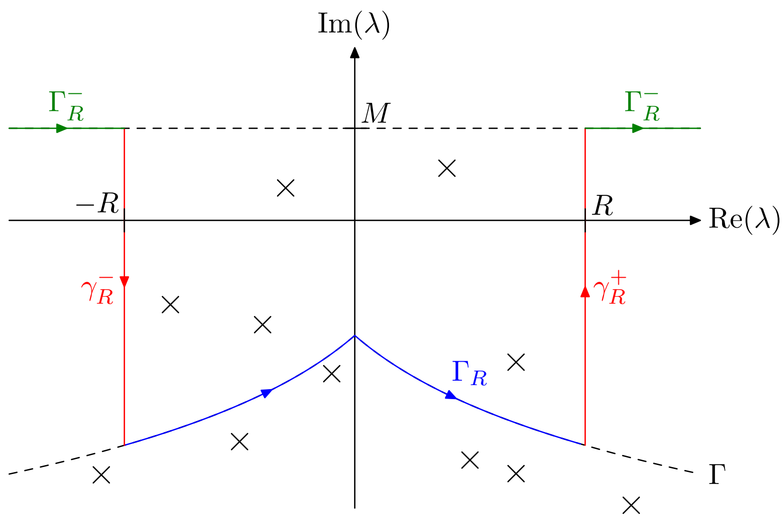


FIGURE 4.1. The integration contours (4.36), with orientations indicated by arrows. Also shown are a few examplary resonances: their imaginary parts are all $< M$, and the resonances above Γ have real parts less than R. The domain Ω_A from (4.37) is the subset of $\mathbb{C}$ lying above Γ.

By the residue theorem, the poles of $R_V(\lambda)$ in Ω_A will give a contribution

$$\Pi_A(t) := i\sum_{\lambda_j \in \Omega_A} \operatorname{Res}_{\lambda=\lambda_j}\big(e^{-i\lambda t}\rho R_V(\lambda)\rho\big) \tag{4.38}$$

upon shifting the contour in (4.34). Concretely, we have

$$\rho(x)u(t,x) = \Pi_A(t)\varphi + U_{\Gamma_R}(t)\varphi + U_{\Gamma_R^{||}}(t)\varphi + U_{\Gamma_R^-}(t)\varphi, \tag{4.39}$$

where $u_\gamma(t) = \frac{1}{2\pi}\int_\gamma e^{-i\lambda t}\rho R_V(\lambda)\rho\,d\lambda$ (with the orientations in Figure 4.1). We wish to take $R\to\infty$ and demonstrate that the final two terms (which, naively, seem to give exponentially growing contributions) do not contribute in the limit.

To do this, we first assume $\varphi \in H^2_{\mathrm{c}}(B(0,R))$ (so that we get more decay of $R_V(\lambda)$ in a weak function space, cf. the bounds in Proposition 4.3). Since $R_V(\lambda)(-\Delta+V) = \lambda^2 R_V(\lambda) + I$ on $H^2_{\mathrm{c}}(\mathbb{R}^3)$, we have

$$\rho R_V(\lambda)\rho\varphi = \rho R_V(\lambda)\varphi = \lambda^{-2}\rho\big(R_V(\lambda)(-\Delta+V)\varphi - \varphi\big), \tag{4.40}$$

whose $H^1(\mathbb{R}^3)$-norm is bounded by

$$\|\rho R_V(\lambda)\rho\varphi\|_{H^1} \le Ce^{C_1(\operatorname{Im}\lambda)_-}\langle\lambda\rangle^{-2}\|\varphi\|_{H^2} \tag{4.41}$$

by Proposition 4.4. Therefore,

$$\begin{aligned} \|U_{\Gamma_R^-}(t)\varphi\|_{H^1} &\le C\int_R^\infty \langle\lambda+iM\rangle^{-2}\|\varphi\|_{H^2}\,d\lambda \le \frac{C}{R}\|\varphi\|_{H^2} \to 0, \\ \|U_{\Gamma_R^{||}}(t)\varphi\|_{H^1} &\le C\frac{M+\log(1+R)}{1+R^2}e^{C_1\delta\log(1+R)}\|\varphi\|_{H^2} \to 0, \end{aligned} \tag{4.42}$$

as $R\to\infty$. We have now proved

$$\rho u(t,x) = \Pi_A(t)\varphi + U_\Gamma(t)\varphi, \quad \varphi\in H^2_{\mathrm{c}}(\mathbb{R}^3),\ \operatorname{supp}\varphi\subset B(0,R). \tag{4.43}$$

Under these assumptions, we proceed to show

$$\|U_\Gamma(t)\varphi\|_{H^2} \le Ce^{-tA}\|\varphi\|_{L^2}. \tag{4.44}$$

Since the space of φ in (4.43) is dense in $L^2(B(0,R))$, this will imply the decomposition (4.43) for all $\varphi\in L^2(B(0,R))$ and thus finish the proof. In order to show (4.44), we estimate, using (4.28) with $j=2$,

$$\begin{aligned} \|U_\Gamma(t)\varphi\|_{H^2} &\le Ce^{-At}\int_{\mathbb{R}} e^{-t\delta\log(1+|\lambda|)}\cdot(1+|\lambda|)e^{C_1\delta\log(1+|\lambda|)}\|\varphi\|_{L^2}\,d\lambda \\ &\le Ce^{-At}\int_{\mathbb{R}}(1+|\lambda|)^{-(t-C_1)\delta+1}\|\varphi\|_{L^2}\,d\lambda, \end{aligned} \tag{4.45}$$

which for $t > C_1 + 3\delta^{-1}$ is bounded by $Ce^{-At}\int_{\mathbb{R}}(1+|\lambda|)^{-2}\|\varphi\|_{L^2}\,d\lambda$, proving (4.44). □

Remark 4.5. Note that the remainder $E_A(t)$ in Theorem 4.1 is more regular than one would naively expect given the regularity of the initial data. For $V=0$, i.e. the free wave equation, this follows from the sharp Huygens principle. For $V\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$, we leave it as an exercise to show that this follows from the propagation of singularities. What Theorem 4.1 entails is (a weak form of) propagation of H^2-regularity of u even for somewhat irregular potentials $V\in L^\infty_{\mathrm{c}}(\mathbb{R}^3)$.

5. More on resonances

We study the (typically most interesting) case of *real-valued* potentials in more detail. Thus, from now on, fix

$$V\in L^\infty_{\mathrm{c}}(\mathbb{R}^3;\mathbb{R}), \quad P_V = -\Delta + V. \tag{5.1}$$

We start with a simple result on resonances in the open upper half plane:

Proposition 5.1. *Any resonance λ with $\operatorname{Im}\lambda > 0$ is purely imaginary, $\lambda = i\mu$, $\mu \in (0,\infty)$. Moreover, $i\mu$ is a resonance if and only if $-\mu^2$ is an eigenvalue of P_V.*

Proof. A resonant state $u \neq 0$ corresponding to a resonance λ, $\operatorname{Im}\lambda > 0$, decays exponentially fast as $|x| \to \infty$, hence lies in $L^2(\mathbb{R}^3)$. Integrating by parts gives

$$0 = \int_{\mathbb{R}^3} (P_V - \lambda^2)u \cdot \bar{u}\,dx = \int_{\mathbb{R}^3} |\nabla u|^2 + (V - \lambda^2)|u|^2\,dx. \tag{5.2}$$

If $\lambda \notin i(0,\infty)$, then $\operatorname{Im}\lambda^2 \neq 0$, so taking the imaginary part of (5.2) gives $\int |u|^2\,dx = 0$, hence $u = 0$. This contradiction shows that $\lambda = i\mu$, and $P_V u = -\mu^2 u$, as claimed.

Conversely, if $u \in L^2(\mathbb{R}^3)$, $P_V u = -\mu^2 u$, then $u \in H^2(\mathbb{R}^3)$ by elliptic regularity. Therefore, $P_V + \mu^2 \colon H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is not injective, hence $i\mu$ is a resonance of P_V. $\square$

The real action starts on the real line. (This is typically also the last line of defense where one can prove general results which do not involve some kind of asymptotic, such as semiclassical, limit.) The following result is called *Rellich's uniqueness theorem*, see also Remark 5.3.

Theorem 5.2. *P_V has no non-zero real resonances.*

Proof. Let $0 \neq \lambda \in \mathbb{R}$. Fix $\rho \in \mathcal{C}^\infty_c(\mathbb{R}^3)$, $\rho \equiv 1$ near $\operatorname{supp} V$. If R_V has a pole at λ, then $I + VR_0(\lambda)\rho$ is not invertible on $L^2(\mathbb{R}^3)$ (see (3.18)), hence there exists $\varphi \in L^2(\mathbb{R}^3)$ such that

$$\varphi = -VR_0(\lambda)\rho\varphi; \tag{5.3}$$

multiplying by ρ, the right hand side remains unchanged, hence $\varphi = \rho\varphi \in L^2_c(\mathbb{R}^3)$. Defining $u = R_0(\lambda)\varphi$, we then have

$$(-\Delta - \lambda^2)u = \varphi = -VR_0(\lambda)\varphi = -Vu, \tag{5.4}$$

so upon applying $R_0(\lambda)$ to (5.3),

$$(P_V - \lambda^2)u = 0, \quad u = -R_0(\lambda)Vu. \tag{5.5}$$

By Theorem 2.5, we have

$$u(x) = \frac{e^{i\lambda r}}{4\pi r}\bigl(h(\omega) + \mathcal{O}(r^{-1})\bigr), \quad x = r\omega, \quad h(\omega) = -\widehat{Vu}(\lambda\omega). \tag{5.6}$$

In particular, we have

$$(\partial_r - i\lambda)u(x) = \mathcal{O}(r^{-2}), \quad r = |x|. \tag{5.7}$$

We now use a *boundary pairing argument*, relating the imaginary part of $\int u(P_V - \lambda^2)\bar{u}\,dx$ to the boundary value (encoded by h) of u via integration by parts. Concretely, using that

V is real-valued,

$$\begin{aligned} 0 &= \int_{B(0,R)} \big(u \cdot \overline{(P_V - \lambda^2)u} - (P_V - \lambda^2)u \cdot \bar{u}\big)\, dx \\ &= \int_{B(0,R)} \big(-u \cdot \Delta\bar{u} + \Delta u \cdot \bar{u}\big)\, dx \\ &= \int_{\partial B(0,R)} \big(-u \cdot \partial_r\bar{u} + \partial_r u \cdot \bar{u}\big)\, dS \\ &= 2i\lambda \int_{\partial B(0,R)} |u|^2\, dS + \int_{\partial B(0,R)} \mathcal{O}(R^{-3})\, dS, \end{aligned} \tag{5.8}$$

where we used (5.7) in the last step. Since $\lambda \neq 0$, we get $\int_{\partial B(0,R)} |u|^2\, dS = \mathcal{O}(R^{-1})$. Upon taking $R \to \infty$ and using (5.6), this gives $h \equiv 0$,[8] hence

$$\widehat{Vu}(\xi) = 0 \quad \forall\, \xi \in \mathbb{R}^n,\ |\xi|^2 = \lambda^2. \tag{5.9}$$

Now, $\Sigma = \{\xi \in \mathbb{C}^n : \xi \cdot \xi = \lambda^2\}$ is a connected complex hypersurface in $\mathbb{C}^n$, and the entire function[9] $\widehat{Vu}(\xi)$ vanishes on $\Sigma \cap \mathbb{R}^n$; it follows that $\widehat{Vu}(\xi) = 0$. (We leave the proof, which only requires single complex variable techniques, as an exercise.) Thus,

$$\frac{\widehat{Vu}(\xi)}{\xi \cdot \xi - \lambda^2} \tag{5.10}$$

is an entire function of $\xi \in \mathbb{C}^n$. Using (5.4), we conclude

$$(\xi \cdot \xi - \lambda^2)\hat{u}(\xi) = \widehat{Vu}(\xi). \tag{5.11}$$

By the Paley–Wiener–Schwartz theorem (which one can apply separately in each variable x_j and its dual momentum ξ_j), the fact that Vu has compact support thus implies that u has compact support:

$$u \in H^2_{\mathrm{c}}(\mathbb{R}^3). \tag{5.12}$$

We now appeal to a standard unique continuation principle which states that any solution of the equation $(P_V - \lambda^2)u = 0$ which vanishes on an open set must vanish identically. (See [DZ19, Lemma 3.34] and the subsequent discussion there for a proof.) The proof is complete. □

Remark 5.3. An equivalent formulation of Theorem 5.2 is: if u is an outgoing solution of $(P_V - \lambda^2)u = 0$ (in the sense that $u \in R_0(\lambda)(L^2_{\mathrm{c}}(\mathbb{R}^3))$), then $u = 0$.

Finally, we can rigorously justify the exponential decay observed numerically in the introduction (albeit here in three, not one dimension) for the positive (bump) potentials used there:

Theorem 5.4. *Suppose* $V \in L^\infty_{\mathrm{c}}(\mathbb{R}^3; \mathbb{R})$, $V \geq 0$.

(1) *All non-zero resonances* λ *of* P_V *satisfy* $\operatorname{Im}\lambda < 0$.

---

[8] One can show that this implies that $u \in \mathscr{S}(\mathbb{R}^3)$: if the leading order part of an outgoing function vanishes, then all terms in the Taylor series expansion at $r^{-1} = 0$ vanish. The next part of the proof shows that this rapid vanishing at infinity in fact implies the vanishing near infinity, i.e. for large r.

[9] Writing $\xi = (\xi_1, \ldots, \xi_n)$, this means: it is holomorphic in each of the ξ_j separately.

(2) *If* $V > 0$ *on a set of positive measure, then all resonances satisfy* $\operatorname{Im}\lambda < 0$*. In particular, solutions of the wave equation* $(\partial_t^2 - \Delta + V)u = 0$ *with compactly supported initial data as in Theorem 4.1 decay exponentially in time in any fixed compact subset of* $\mathbb{R}^3$*.*

The proof is left as an exercise.

References

[BH08] Jean-François Bony and Dietrich Häfner. Decay and non-decay of the local energy for the wave equation on the de Sitter–Schwarzschild metric. *Communications in Mathematical Physics*, 282(3):697–719, 2008.

[Dya12] Semyon Dyatlov. Asymptotic distribution of quasi-normal modes for Kerr–de Sitter black holes. *Annales Henri Poincaré*, 13(5):1101–1166, 2012.

[DZ19] Semyon Dyatlov and Maciej Zworski. *Mathematical theory of scattering resonances*, volume 200 of *Graduate Studies in Mathematics*. American Mathematical Society, 2019.

[HV18] Peter Hintz and András Vasy. The global non-linear stability of the Kerr–de Sitter family of black holes. *Acta mathematica*, 220:1–206, 2018.

[LIG16] LIGO Scientific Collaboration and Virgo Collaboration. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.*, 116:061102, 2016.

[Mel95] Richard B. Melrose. *Geometric scattering theory*. Stanford Lectures. Cambridge University Press, Cambridge, 1995.

[Zwo17] Maciej Zworski. Mathematical study of scattering resonances. *Bulletin of Mathematical Sciences*, 7(1):1–85, 2017.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139-4307, USA

Email address: phintz@mit.edu