THE DIRICHLET-TO-NEUMANN MAP FOR A SEMILINEAR WAVE EQUATION ON LORENTZIAN MANIFOLDS

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Abstract. We consider the semilinear wave equation $\Box_g u + au^4 = 0$, $a \neq 0$, on a Lorentzian manifold (M, g) with timelike boundary. We show that from the knowledge of the Dirichlet-to-Neumann map one can recover the metric g and the coefficient a up to natural obstructions. Our proof rests on the analysis of the interaction of distorted plane waves together with a scattering control argument, as well as Gaussian beam solutions.

1. Introduction

Let (M, g) be a $(1+3)$-dimensional Lorentzian manifold with boundary ∂M, where the metric g is of signature $(-,+,+,+)$. We assume that $M = \mathbb{R} \times N$ where N is a connected manifold with boundary ∂N, and write the metric g as

$$g = -\alpha(\mathfrak{t}, x')\mathrm{d}\mathfrak{t}^2 + \kappa(\mathfrak{t}, x'), \tag{1.1}$$

where $x = (\mathfrak{t}, x') = (x^0, x^1, x^2, x^3)$ are local coordinates on M; here, $\alpha : \mathbb{R} \times N \to (0, \infty)$ is a smooth function and $\kappa(\mathfrak{t}, \cdot\,)$ is a Riemannian metric on N depending smoothly on $\mathfrak{t} \in \mathbb{R}$. The boundary $\partial M = \mathbb{R} \times \partial N$ of M is then timelike. Let $\nu = \nu_g$ denote the unit outer normal vector field to ∂M with respect to the metric g. Assume that ∂M is null-convex, which means that $\mathrm{II}(V, V) = g(\nabla_V \nu, V) \geq 0$ for all null vectors $V \in T(\partial M)$; see [22] for a discussion of this condition. We consider the semilinear wave equation

$$\begin{aligned} \Box_g u(x) + a(x)u(x)^4 &= 0, && \text{on } M, \\ u(x) &= f(x), && \text{on } \partial M, \\ u(\mathfrak{t}, x') &= 0, && \mathfrak{t} < 0, \end{aligned} \tag{1.2}$$

where $\Box_g = (-\det g)^{-1/2}\partial_j((-\det g)^{1/2} g^{jk}\partial_k)$ is the wave operator (d'Alembertian) on (M, g), and $a \in \mathcal{C}^\infty(M)$ is a nowhere vanishing function (thus either everywhere positive or everywhere negative).

We only consider boundary sources f with $\operatorname{supp} f \subset (0, T) \times \partial N$ and introduce the Dirichlet-to-Neumann (DN) map $\Lambda_{g,a}$ (measured in $(0, T) \times \partial N$) defined as

$$\Lambda_{g,a} f = \partial_\nu u|_{(0,T)\times\partial N} = \nu^j \partial_j u|_{(0,T)\times\partial N},$$

where u is the solution of (1.2). The well-posedness of the initial boundary value problem (1.2) with small Dirichlet data $f \in \mathcal{C}^m$, $m \geq 6$, can be established as in [23]. Thus $\Lambda_{g,a} f$ is well-defined for such f. We will study the inverse problem or recovering the Lorentzian metric g and the nonlinear coefficient a from $\Lambda_{g,a}$. We choose to consider the semilinear equation with a quartic nonlinear term because it requires the least amount of technicalities in the analysis of nonlinear interactions

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of distorted plane waves or Gaussian beams. Much as in the setting of manifolds without boundary (see the references below), more general nonlinearities can be dealt with in a similar way.

Since the work [31], rapid progress has been made on the study of inverse problems for nonlinear equations. See [37, 30, 36, 14, 47, 13, 44, 11, 20, 23, 12, 4, 35, 19, 32] for results on hyperbolic equations and [3, 10, 8, 34, 33, 21, 29, 27, 2, 28, 26, 9] for elliptic equations. For hyperbolic equations, the recovery of time-dependent coefficients is possible for some nonlinear equations, whereas the corresponding problems for linear equations are still largely open. See [46] for an overview of the recent progress on inverse problems for nonlinear hyperbolic equations.

The recovery of a Lorentzian metric from the *source-to-solution map* associated with a semilinear equation is considered in [31], where the authors use the nonlinear interactions of distorted plane waves to recover the light observation sets, and use them to reconstruct the Lorentzian geometry. The approach was then further generalized to deal with other different types of nonlinear equations in [37, 30, 36, 47, 44, 4]; recently in [43] it was shown that recovery of a Riemannian metric is possible when one measures the solution of a forced semilinear wave equation only at a single point for some time. Distorted plane waves can also be used to recover the coefficients of (linear) lower order terms [11, 12] and the nonlinear terms [37, 13, 23]. In the works [20, 45, 23, 19], Gaussian beams, instead of distorted planes waves, are used to study inverse problems for nonlinear wave equations. We will use both distorted plane waves and Gaussian beams in this paper.

First, notice that there a natural geometric obstruction to the reconstruction of g and a:

Lemma 1.1. *If $\Psi : M \to M$ is a diffeomorphism and $\Psi|_{\partial M} = \mathrm{Id}$, then*

$$\Lambda_{g,a} = \Lambda_{\Psi^* g, \Psi^* a}.$$

Proof. If $\Box_g u + au^4 = 0$, then

$$0 = \Psi^*(\Box_g u + au^4) = \Box_{\Psi^* g}(\Psi^* u) + (\Psi^* a)(\Psi^* u)^4.$$

Now, if $u|_{\partial M} = f$, then $(\Psi^* u)|_{\partial M} = f$ also; moreover, the outer unit normals at a point $x \in \partial M$ with respect to g and $\Psi^* g$ are equal to ν and $\Psi^*\nu = (\Psi^{-1})_*\nu$, respectively. Therefore, writing $\partial_{\Psi^*\nu}$ for differentiation along $\Psi^*\nu$, we compute at $x \in \partial M$

$$(\Lambda_{\Psi^* g, \Psi^* a} f)(x) = \big(\partial_{\Psi^*\nu}(\Psi^* u)\big)(x) = \big(\Psi^*(\partial_\nu u)\big)(x) = \partial_\nu u(x) = (\Lambda_{g,a} f)(x);$$

the penultimate equality uses that $\Psi(x) = x$. □

There is another invariance of the DN map $\Lambda_{g,a}$ related to a conformal change of the metric g.

Lemma 1.2. *Let β be a smooth function on M, such that*

$$\beta|_{\partial M} = 0, \quad \partial_\nu \beta|_{\partial M} = 0, \quad \Box_g e^{-\beta} = 0, \tag{1.3}$$

then we have

$$\Lambda_{e^{-2\beta} g, e^{-\beta} a} = \Lambda_{g,a}.$$

Proof. The key formula is the well-known

$$e^{3\beta} \Box_g (e^{-\beta} v) = \Box_{e^{-2\beta} g} v + \tilde{q} v, \quad \tilde{q} = e^{3\beta} \Box_g (e^{-\beta}), \tag{1.4}$$

valid for any Lorentzian metric g on a 4-dimensional spacetime. Indeed, the operators $e^{3\beta}\Box_g e^{-\beta}$ and $\Box_{e^{-2\beta} g} + \tilde{q}$ are both symmetric with respect to the volume density $|\mathrm{d}(e^{-2\beta} g)| = e^{-4\beta}|\mathrm{d}g|$ and have the same principal symbol, hence their difference, being a symmetric first order operator with real coefficients, is in fact an operator of order 0, and since both operators give the same result when acting on the constant function 1, the formula (1.4) follows.

Denote now $\widetilde{g} = e^{-2\beta} g$, $\widetilde{a} = e^{-\beta} a$ with β satisfying (1.3). Let $v = e^{\beta} u$, then we have

$$\Box_{\widetilde{g}} v + \widetilde{a} v^4 = \Box_{e^{-2\beta} g}(e^{\beta} u) + \widetilde{a} e^{4\beta} u^4 = e^{3\beta}(\Box_g u + qu + au^4) = e^{3\beta}(\Box_g u + au^4),$$

where $q = -e^{\beta} \Box_g e^{-\beta} = 0$ in M. Notice that, by the assumption on β, $\nu_{e^{-2\beta} g} = \nu_g$, and if $u|_{\partial M} = f$, then $v|_{\partial M} = f$ also. Moreover, we have

$$\Lambda_{e^{-2\beta} g, e^{-\beta} a} f = \partial_\nu v = \partial_\nu u = \Lambda_{g,a} f. \qquad \Box$$

Recall that a smooth curve $\mu : (a, b) \to M$ is called causal if $g(\dot{\mu}(s), \dot{\mu}(s)) \leq 0$ and $\dot{\mu}(s) \neq 0$ for all $s \in (a, b)$. Given $p, q \in M$, we denote $p \leq q$ if $p = q$ or p can be joined to q by a future-pointing causal curve. We say $p < q$ if $p \leq q$ and $p \neq q$. We denote the causal future of $p \in M$ by $J_g^+(p) = \{q \in M : p \leq q\}$ and the causal past of $q \in M$ by $J_g^-(q) = \{p \in M : p \leq q\}$. The curve μ is called time-like if $g(\dot{\mu}(s), \dot{\mu}(s)) < 0$ for all $s \in (a, b)$. We denote $p \ll q$ if $p \neq q$ and there is a future-pointing time-like path from p to q. Then the chronological future of $p \in M$ is the set $I_g^+(p) = \{q \in M \colon p \ll q\}$ and the chronological past of $q \in M$ is $I_g^-(q) = \{p \in M \colon p \ll q\}$. We also denote $J_g(p, q) := J_g^+(p) \cap J_g^-(q)$ and $I_g(p, q) := I_g^+(p) \cap I_g^-(q)$. We will reconstruct the metric g and the coefficient a in the set

$$\mathbb{U}_g = \bigcup_{p,q \in (0,T) \times \partial N} J_g(p, q). \tag{1.5}$$

We shall make the simplifying assumption that no null-geodesic in $\mathbb{U}_g$ has conjugate points (including on the boundary ∂M). The main result of this paper is as follows.

Theorem 1.3. *Suppose we are given two manifolds $M, \widetilde{M}$ smooth Lorentzian metrics $g, \widetilde{g}$ on $M, \widetilde{M}$ of the form (1.1) with respect to product decompositions $M = \mathbb{R} \times N$, $\widetilde{M} = \mathbb{R} \times \widetilde{N}$. Suppose $\partial N \cong \partial \widetilde{N}$, and identify $\mathbb{R} \times \partial N$ with $\mathbb{R} \times \partial \widetilde{N}$. Assume that null-geodesics in $(\mathbb{U}_g, g)$, resp. $(\mathbb{U}_{\widetilde{g}}, \widetilde{g})$ have no conjugate points. Suppose moreover we are given smooth nonvanishing functions $a, \widetilde{a}$ on $M, \widetilde{M}$. Assume that the Dirichlet-to-Neumann maps agree, that is, $\Lambda_{g,a} = \Lambda_{\widetilde{g},\widetilde{a}}$. Then there exists a diffeomorphism $\Psi \colon \mathbb{U}_g \to \mathbb{U}_{\widetilde{g}}$ with $\Psi|_{(0,T)\times\partial N} = \mathrm{Id}$ and a smooth function $\beta \in \mathcal{C}^\infty(M)$, $\beta|_{(0,T)\times\partial N} = \partial_\nu \beta|_{(0,T)\times\partial N} = 0$, so that, in $\mathbb{U}_g$,*

$$\Psi^* \widetilde{g} = e^{-2\beta} g, \quad \Psi^* \widetilde{a} = e^{-\beta} a, \quad \Box_g e^{-\beta} = 0.$$

In particular, if $a = \widetilde{a} = c \neq 0$ are equal to the same nonzero constant, then $\beta = 0$ and $\Psi^* \widetilde{g} = g$ in $\mathbb{U}_g$, that is, Ψ is an isometry.

Remark 1.4. We can not conclude from the result of Theorem 1.3 that $\beta = 0$ in $\mathbb{U}_g$, if the *Unique Continuation Principle* (UCP) fails for the operator $\Box_g$. If there exists a function $\beta \in \mathcal{C}^\infty(M)$ as in the statement of the Theorem, then $\Lambda_{e^{-2\beta} g, e^{-\beta} a} = \Lambda_{g,a}$ by Lemma 1.2, giving a counter-example for the unique determination of g and a (up to diffeomorphism) in $\mathbb{U}_g$ from $\Lambda_{g,a}$. In this sense, our result is sharp. We refer to [1] for the failure of UCP for an operator of the form $\Box_g + b$. It is an open problem to prove UCP or to construct a counterexample to UCP for $\Box_g$.

The structure of this paper is as follows. In Section 2, we show the determination of the jet of the metric g at the boundary ∂M. In Section 3, we present the proof of the determination of the conformal class of the metric g in the interior of the manifold. The nonlinear interaction of distorted plane waves is used for the proof. Due to the complications caused by possible reflections of waves at the boundary, we propose a "scattering control" scheme to suppress the reflections. In Section 4, we use Gaussian beam solutions to prove the uniqueness of the coefficient a and the conformal factor up to natural obstructions.

2. Boundary determination

In this section, we consider the determination of the jet of the metric on the boundary ∂M. This only uses the Dirichlet-to-Neumann map Λ_g^{lin} for the linearized equation

$$\begin{aligned} \Box_g u(x) &= 0, && \text{on } M, \\ u(x) &= f(x), && \text{on } \partial M, \\ u(\mathfrak{t}, x') &= 0, && \mathfrak{t} < 0. \end{aligned} \tag{2.1}$$

Note that we can recover Λ_g^{lin} from $\Lambda_{g,a}$ (without any restrictions on a except for smoothness) via

$$\Lambda_g^{\mathrm{lin}} f = \frac{\partial}{\partial \epsilon}\big(\Lambda_{g,a}(\epsilon f)\big)\Big|_{\epsilon=0}.$$

In particular, the assumptions of Theorem 1.3 imply that $\Lambda_g^{\mathrm{lin}} = \Lambda_{\widetilde{g}}^{\mathrm{lin}}$.

The discussion in the present section works in $(1+n)$ dimensions for any $n \geq 1$, and we shall proceed in this generality. We will recover g in Taylor series at $(0,T) \times \partial N$ from Λ_g^{lin} in boundary normal coordinates. Here, we recall:

Lemma 2.1. ([40]) *Let $U \subset \partial M$ be an open precompact subset. Then there exist $\epsilon > 0$, a neighborhood $V \subset M$ of U in M, and a diffeomorphism $\Psi\colon (\partial M \cap V) \times [0,\epsilon) \to V$ such that*

1. *$\Psi(x', 0) = x'$ for all $x' \in \partial M \cap V$;*
2. *$\Psi(x', x^n) = \gamma_{x'}(x^n)$, where $\gamma_{x'}(x^n)$ is the unit speed geodesic (with respect to g) issued from x' normal to ∂M.*

The pullback metric $\Psi^* g$ takes the form

$$\Psi^* g(x', x^n) = k(x^n, x', \mathrm{d}x') + \mathrm{d}x^n \otimes \mathrm{d}x^n, \tag{2.2}$$

where $k(x^n, x', \mathrm{d}x')$ is a Lorentzian metric on $\partial M \cap V$ depending smoothly on $x^n \in [0,\epsilon)$. In particular, if $(x^0, \dots, x^{n-1})$ are local boundary coordinates on ∂M, then in the coordinate system $(x^0, \dots, x^n)$, the metric tensor g takes the form

$$g = \sum_{0 \leq \alpha, \beta \leq n-1} g_{\alpha\beta} \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta + \mathrm{d}x^n \otimes \mathrm{d}x^n. \tag{2.3}$$

Indeed, we have $U \subset [a,b] \times \partial N$ for some $-\infty < a < b < \infty$; the existence of the unit speed geodesics $\gamma_{x'}(x^n)$ for $x^n \in [0,\epsilon')$ for some small $\epsilon' > 0$ (depending on a, b) then follows from basic ODE theory, as does the fact that the map $\Psi(x', x^n) = \gamma_{x'}(x^n)$ is a diffeomorphism for $x' \in U$ and $x^n \in [0,\epsilon)$ for sufficiently small $\epsilon \in (0,\epsilon']$. See also [42, Lemma 2.3].

We are interested in the special case $U = (0,T) \times \partial N$. We refer to the partial coordinate system (x', x^n) near $(0,T) \times \partial N$, or the full local coordinate system $(x^0, \dots, x^n)$, as *boundary normal coordinates*. Assume we have two metrics $g, \widetilde{g}$ such that $\Lambda_g^{\mathrm{lin}} = \Lambda_{\widetilde{g}}^{\mathrm{lin}}$. Let Ψ and $\widetilde{\Psi}$ be the diffeomorphisms as in Lemma 2.1 near $(-1, T+1) \times \partial N$ with respect to g and $\widetilde{g}$. (Recall that we are identifying ∂N and $\partial \widetilde{N}$.) Then $\widetilde{\Psi} \circ \Psi^{-1}$ is a diffeomorphism near $(0,T) \times \partial N$ fixing the boundary. Thus, g and $(\widetilde{\Psi} \circ \Psi^{-1})^* \widetilde{g}$ have a common boundary normal coordinate system near $\partial M = \partial \widetilde{M}$. Without loss of generality, we shall thus assume that g and $\widetilde{g}$ have the same boundary normal coordinates, i.e. they take the form (2.3) (with a priori possibly different coefficients $g_{\alpha\beta}$, $\widetilde{g}_{\alpha\beta}$) in the *same* (partial) coordinate system near $x_0 \in \partial M = \partial \widetilde{M}$.

Theorem 2.2. *If $\Lambda_g^{\mathrm{lin}} = \Lambda_{\widetilde{g}}^{\mathrm{lin}}$, then g and $\widetilde{g}$ in boundary normal coordinates are equal in Taylor series at $(0,T)\times\partial N$.*

Proof. The proof is adapted from the proof of Theorem 3.2 in [42]. We work in a fixed boundary normal coordinate system $x=(x',x^n)$ in a neighborhood of a point $x_0\in(0,T)\times\partial N\subset\partial M$; denote the corresponding coordinates on T^*M by $\xi=(\xi',\xi_n)$. Let $\xi^{0\prime}\in\mathbb{R}^n$ be such that $(\xi^{0\prime},0)\in T^*_{x_0}M$ is a future-pointing timelike covector (which in our special coordinates is thus dual to a tangent vector to ∂M), and denote by $\mathcal{U}\subset T^*(\partial M)$ a small neighborhood of $(x_0,\xi^{0\prime})$ so that for all $(x',\xi')\in\mathcal{U}$, the covector ξ' is future timelike. Let $\chi(x',\xi')$ be a smooth cutoff function with support in $\mathcal{U}$ and $\chi=1$ near $(x_0,\xi^{0\prime})$.

For any fixed $\xi'\in\mathbb{R}^n$, consider then the Dirichlet data

$$f(x')=e^{\mathrm{i}\lambda x'\cdot\xi'}\chi(x',\xi'),$$

with λ a large parameter. Then we construct (outgoing) geometric optics solutions near x_0 to the equation (2.1) of the form

$$u(x)\sim e^{\mathrm{i}\lambda\phi(x,\xi')}\sum_{j=0}^{\infty}\lambda^{-j}a_j(x,\xi'),$$

where the notation '$\sim$' means that the difference of u and the truncated sum over $j=0,\dots,J$ is bounded in L^∞ by λ^{-J-1}.

By the discussion in [42], we have the eikonal equation for the phase function

$$g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi+(\partial_n\phi)^2=0,\quad \phi|_{x^n=0}=x'\cdot\xi', \tag{2.4}$$

and the transport equations for the amplitudes

$$Ta_0=0,\qquad a_0|_{x^n=0}=\chi \tag{2.5a}$$

$$\mathrm{i}Ta_j=-\Box_g a_{j-1},\quad a_j|_{x^n=0}=0,\quad j\geq 1, \tag{2.5b}$$

where T is the transport operator defined by

$$Ta:=2g^{jk}\partial_j\phi\partial_k a+(\Box_g\phi)a=2\partial_n\phi\partial_n a+2g^{\alpha\beta}\partial_\alpha\phi\partial_\beta a+(\Box_g\phi)a. \tag{2.6}$$

Note that ∂_n is the *inward* pointing unit normal, so $\partial_n=-\partial_\nu$. Now, with the outgoing condition $\partial_n\phi|_{\partial M}<0$, the equation (2.4) implies that

$$\partial_n\phi((x',0),\xi')=\xi_n(x',\xi'):=-\sqrt{-g^{\alpha\beta}(x',0)\xi_\alpha\xi_\beta}<0. \tag{2.7}$$

In our boundary normal coordinates, we have

$$(\Lambda_g^{\mathrm{lin}}f)(x')\sim -e^{\mathrm{i}\lambda x'\cdot\xi'}\Big(\mathrm{i}\lambda\partial_n\phi(x',0,\xi')+\sum_{j=0}^{\infty}\lambda^{-j}\partial_n a_j(x',0,\xi')\Big). \tag{2.8}$$

The expression for$\Lambda_{\widetilde{g}}^{\mathrm{lin}}f$ is similar, with ϕ and a_j replaced by $\widetilde{\phi}$ and $\widetilde{a}_j$. Thus, under the assumption that $\Lambda_g^{\mathrm{lin}}f=\Lambda_{\widetilde{g}}^{\mathrm{lin}}f$, we have

$$\partial_n\phi((x',0),\xi')=\partial_n\widetilde{\phi}((x',0),\xi') \tag{2.9a}$$

$$\partial_n a_j((x',0),\xi')=\partial_n\widetilde{a}_j((x',0),\xi'),\quad j=0,1,\dots. \tag{2.9b}$$

By (2.9a) and (2.4), we have (with $\xi_\alpha=\xi'_\alpha$ for $\alpha=0,\dots,n-1$)

$$g^{\alpha\beta}\xi_\alpha\xi_\beta=\widetilde{g}^{\alpha\beta}\xi_\alpha\xi_\beta,$$

for ξ' in an open subset of $\mathbb{R}^n$. This implies $g|_{\partial M} = \widetilde{g}|_{\partial M}$.

Next, the transport equation (2.5a) for a_0 gives the identity

$$\begin{aligned} 0 &= 2\partial_n\phi\partial_n a_0 + \frac{1}{\sqrt{-\det g}}\partial_n\big(\sqrt{-\det g}\partial_n\phi\big) + Q(g) \\ &= 2\partial_n\phi\partial_n a_0 + \frac{1}{2\det g}(\partial_n \det g)\partial_n\phi + \partial_n^2\phi + Q(g) \end{aligned} \tag{2.10}$$

on $\{x'\colon \chi(x',\xi') = 1\}$ (which contains an open neighborhood of x_0), since $\partial_\beta a_0 = \partial_\beta\chi = 0$ there; here, Q is given by

$$Q(g) = \frac{1}{\sqrt{-\det g}}\partial_\alpha\big(\sqrt{-\det g}g^{\alpha\beta}\big)\xi_\beta$$

and thus depends only on $g|_{\partial M}$, which is already determined. Differentiating (2.4) along ∂_n, we obtain

$$2\xi_n\partial_n^2\phi = -\partial_n g^{\alpha\beta}\xi_\alpha\xi_\beta + R(g,\phi,\partial_n\phi), \quad \text{on } \partial M, \tag{2.11}$$

where $R(g,\phi,\partial_n\phi)$ depends on $g|_\partial, \phi|_{\partial M}, \partial_n\phi|_{\partial M}$ only. Using this to eliminate $\partial_n^2\phi$ in the identity (2.10) for $j=0$, multiplying by $2\xi_n = 2\partial_n\phi((x',0),\xi') = 2\partial_n\widetilde{\phi}((x',0),\xi')$, and using that $\partial_n\phi\partial_n a_0 = \partial_n\widetilde{\phi}\partial_n\widetilde{a}_0$, we thus obtain

$$\frac{\xi_n^2}{\det g}\partial_n\det g - \partial_n g^{\alpha\beta}\xi_\alpha\xi_\beta + \mathcal{R}(g,\phi,\partial_n\phi) = \frac{\xi_n^2}{\det\widetilde{g}}\partial_n\det\widetilde{g} - \partial_n\widetilde{g}^{\alpha\beta}\xi_\alpha\xi_\beta + \mathcal{R}(\widetilde{g},\widetilde{\phi},\partial_n\widetilde{\phi}),$$

where $\mathcal{R}(g,\phi,\partial_n\phi)$ depends on $g|_{\partial M}$, $\phi|_{\partial M}$, $\partial_n\phi|_{\partial M}$ only. Since $g|_{\partial M} = \widetilde{g}|_{\partial M}$, $\phi|_{\partial M} = \widetilde{\phi}|_{\partial M}$, $\partial_n\phi|_{\partial M} = \partial_n\widetilde{\phi}|_{\partial M}$, we have $\mathcal{R}(g,\phi,\partial_n\phi) = \mathcal{R}(\widetilde{g},\widetilde{\phi},\partial_n\widetilde{\phi})$. Using (2.7), we thus get

$$-\frac{1}{\det g}(\partial_n\det g)g^{\alpha\beta}\xi_\alpha\xi_\beta - \partial_n g^{\alpha\beta}\xi_\alpha\xi_\beta = -\frac{1}{\det\widetilde{g}}(\partial_n\det\widetilde{g})\widetilde{g}^{\alpha\beta}\xi_\alpha\xi_\beta - \partial_n\widetilde{g}^{\alpha\beta}\xi_\alpha\xi_\beta,$$

or equivalently

$$\frac{1}{\det g}\partial_n((\det g)g^{\alpha\beta}) = \frac{1}{\det\widetilde{g}}\partial_n((\det\widetilde{g})\widetilde{g}^{\alpha\beta}) \quad \text{on } \partial M. \tag{2.12}$$

Now, $\det g = \det\widetilde{g}$ on ∂M, thus for $h^{\alpha\beta} := (\det g)g^{\alpha\beta} = \widetilde{h}^{\alpha\beta}$ we have $\partial_n h^{\alpha\beta} = \partial_n\widetilde{h}^{\alpha\beta}$ on ∂M. Since $h_{\alpha\beta} = (\det g)^{-1}g_{\alpha\beta}$ with $\det h = (\det g)^{1-n}$, we thus have

$$\partial_n g^{\alpha\beta} = \partial_n((\det h)^{\frac{1}{1-n}}h^{\alpha\beta}) = \partial_n((\det\widetilde{h})^{\frac{1}{1-n}}\widetilde{h}^{\alpha\beta}) = \partial_n\widetilde{g}^{\alpha\beta}.$$

From the identity (2.11), we can then also conclude that $\partial_n^2\phi = \partial_n^2\widetilde{\phi}$ on ∂M.

Next, let us consider the transport equation (2.5b) for $j=1$. Since $a_1 = 0$ on ∂M, we have

$$2\mathrm{i}\partial_n a_1 = -\Box_g a_0 = -\frac{1}{\sqrt{-\det g}}\partial_n\big(\sqrt{-\det g}\partial_n a_0\big) = -\partial_n^2 a_0 + Q(g,\partial_n g).$$

By differentiating (2.10) once and (2.4) twice, we have

$$\begin{aligned} 2\mathrm{i}\partial_n a_1 &= \frac{1}{4\det g}(\partial_n^2\det g) + \frac{1}{2\partial_n\phi}\partial_n^3\phi + \mathcal{R}(g,\partial_n g,\phi,\partial_n\phi,\partial_n^2\phi,\partial_n a_0) \\ &= \frac{1}{4\det g}(\partial_n^2\det g) - \frac{1}{4(\partial_n\phi)^2}\partial_n^2 g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \mathcal{R}(g,\partial_n g,\phi,\partial_n\phi,\partial_n^2\phi,\partial_n a_0) \\ &= -\frac{1}{4}\frac{1}{\det g}\partial_n^2((\det g)g^{\alpha\beta})\xi_\alpha\xi_\beta + \mathcal{R}(g,\partial_n g,\phi,\partial_n\phi,\partial_n^2\phi,\partial_n a_0); \end{aligned}$$

here, the function $\mathcal{R}$ may vary from line to line, but it always only depends on g, $\partial_n g$, ϕ, $\partial_n\phi$, $\partial_n^2\phi$, $\partial_n a_0$ restricted on ∂M. Therefore, the identity (2.9b) for $j=1$ implies that

$$\frac{1}{\det g}\partial_n^2((\det g)g^{\alpha\beta}) = \frac{1}{\det \widetilde{g}}\partial_n^2((\det \widetilde{g})\widetilde{g}^{\alpha\beta}) \quad \text{on } \partial M.$$

and consequently $\partial_n^2 g = \partial_n^2\widetilde{g}$ on ∂M. Repeating this process, we can similarly establish $\partial_n^m g = \partial_n^m\widetilde{g}$ on ∂M for $m \geq 3, 4, \ldots$. The proof is complete. □

In terms of the form (2.2) of the metric in partial boundary normal coordinates, we have thus recovered the jet of $k \in \mathcal{C}^\infty\big([0,\epsilon)_{x^n}; \mathcal{C}^\infty((0,T)\times\partial N; S^2T^*((0,T)\times\partial N))\big)$ at $x^n=0$. Extending k in an arbitrary smooth fashion to $x^n \in (-2\epsilon,\epsilon)$ (and reducing ϵ further if necessary) thus furnishes us with an extension of g to a metric on a slightly larger open manifold

$$M_1 := (0,T)\times N_1, \qquad N \subset N_1^{\mathrm{int}}; \tag{2.13}$$

we denote the extended metric by the same letter g, and we may assume that the first coordinate $\mathfrak{t}$ is a time function on $M_1 \setminus M^{\mathrm{int}}$, i.e. its differential is timelike. In the notation used before the statement of Theorem 2.2, the a priori discontinuous Lorentzian metric defined as $(\widetilde{\Psi}\circ\Psi^{-1})^*\widetilde{g}$ on M and g on $M_1\setminus M$ is, in fact, smooth as a consequence of Theorem 2.2. We define

$$\mathcal{U}_1 := (0,T)\times(N_1\setminus N), \tag{2.14}$$

which thus is an open set which completely surrounds $(0,T)\times\partial N$. See Figure 2.1.

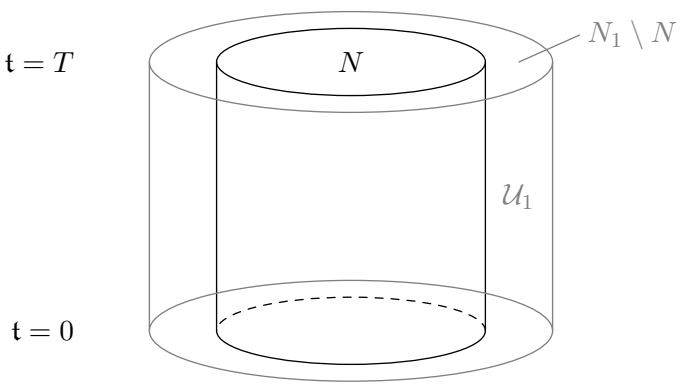

FIGURE 2.1. The enlarged manifold $(0,T)\times N_1$, the original manifold $(0,T)\times N$, and the 'sleeve' $\mathcal{U}_1$ surrounding $(0,T)\times N$.

3. Interior determination

In this section, we prove Theorem 1.3, except for the characterization of conformal factors β, in a sequence of steps.

(1) In §3.1, we describe null-geometric aspects of the extended manifold, and in §3.2 we construct distorted plane waves.
(2) In §3.3, we use a scattering control method to suppress reflections of distorted plane waves at ∂M.
(3) In §3.4, we give a description of the strongest singularity produced by the nonlinear interaction of four distorted plane waves.

(4) In §3.5, we use thus produced artificial point sources in M^{int} to reconstruct the conformal structure of the set $\mathbb{U}_g$ defined in (1.5).
(5) In §3.6, we use principal symbol arguments to determine the conformal class of (g, a).

We work with M_1 and $\mathcal{U}_1$ defined in (2.13)–(2.14). For $p \in M_1$, denote the set of light-like tangent vectors at p by

$$L_p M_1 = \{\zeta \in T_p M_1 \setminus \{0\} \colon g(\zeta, \zeta) = 0\}.$$

The set of light-like covectors at p is denoted by $L_p^* M_1$. The sets of future and past light-like vectors (covectors) are denoted by $L_p^+ M_1$ and $L_p^- M_1$ ($L_p^{*,+} M_1$ and $L_p^{*,-} M_1$). Define the future directed light-cone emanating from p by

$$\mathcal{L}^+(p) = \{\gamma_{p,\zeta}(t) \in M_1 \colon \zeta \in L_p^+ M_1,\ t \geq 0\} \subset M_1,$$

where $\gamma_{p,\zeta}(t)$ is the null-geodesic with $\gamma_{p,\zeta}(0) = p$, $\dot\gamma_{p,\zeta}(0) = \zeta$; the parameter t lies in the largest connected open interval $(t^-_{p,\zeta}, t^+_{p,\zeta})$ containing 0 for which $\gamma_{p,\zeta}$ is defined.

3.1. **Geometry of the extended manifold.** We will construct distorted plane waves whose singular support is concentrated near a suitably cut off geodesic inside of M_1. Since we are only assuming control on smooth geodesics inside M, we cannot expect to have good geometric properties of null-geodesics once they leave M and enter M_1; for instance, they could turn back and reenter M, which could lead to conjugate points and other complications. Thus, we introduce a number of notions designed to capture those null-geodesics which enter and exit (and then possibly again enter) M, and the times at which they do so. We begin by defining the set

$$\widetilde{E}_{\mathrm{i}} := \{(z_0, \zeta_0) \colon z_0 \in M_1 \setminus M^{\mathrm{int}},\ \zeta_0 \in L^+_{z_0} M_1,\ \exists\, t \in (0, t^+_{z_0,\zeta_0}) \text{ s.t. } \gamma_{z_0,\zeta_0}(t) \in (0,T) \times \partial N\}$$

containing all initial data of null-geodesics γ_{z_0,ζ_0} which intersect the boundary $(0,T) \times \partial N$ for *positive* affine parameter values. For $(z_0, \zeta_0) \in \widetilde{E}_{\mathrm{i}}$, we record with

$$\mathbf{t}_0(z_0, \zeta_0) := \inf\{t \in (0, t^+_{z_0,\zeta_0}) \colon \gamma_{z_0,\zeta_0}(t) \in (0,T) \times \partial N\} \in (0, t^+_{z_0,\zeta_0}) \tag{3.1}$$

the first positive time of contact with $(0,T) \times \partial N$. (Note that for $(z_0, \zeta_0) \in \widetilde{E}_{\mathrm{i}}$ with $z_0 \in (0,T) \times \partial N$ and ζ_0 inward pointing, $\mathbf{t}_0(z_0, \zeta_0)$ measures the time at which γ_{z_0,ζ_0} first *exits* $(0,T) \times N$.) We next discard those (z_0, ζ_0) for which γ_{z_0,ζ_0} has a tangential first contact with $(0,T) \times \partial N$; this leaves us with the set

$$E_{\mathrm{i}} := \{(z_0, \zeta_0) \in \widetilde{E}_{\mathrm{i}} \colon \dot\gamma_{z_0,\zeta_0}(\mathbf{t}_0(z_0, \zeta_0)) \notin T(\partial M)\},$$

the subscript 'i' standing for 'in'. We remark that the set $E_{\mathrm{i}}^{\mathrm{int}}$ containing initial data *outside* M,

$$E_{\mathrm{i}}^{\mathrm{int}} := E_{\mathrm{i}} \cap L^+(M_1 \setminus M), \tag{3.2}$$

is determined entirely by the restriction of g to $M_1 \setminus M^{\mathrm{int}}$. Note then that for $(z_0, \zeta_0) \in E_{\mathrm{i}}^{\mathrm{int}}$, the tangent vector $\dot\gamma_{z_0,\zeta_0}(\mathbf{t}_0(z_0, \zeta_0))$ is necessarily pointing into M^{int}. Thus, one can inquire whether γ_{z_0,ζ_0} leaves M^{int} again, which is equivalent to membership in E_{i} of the position and velocity when striking $(0,T) \times \partial N$; we thus set

$$E_{\mathrm{io}}^{\mathrm{int}} := \{(z_0, \zeta_0) \in E_{\mathrm{i}}^{\mathrm{int}} \colon (\gamma_{z_0,\zeta_0}(t), \dot\gamma_{z_0,\zeta_0}(t))|_{t=\mathbf{t}_0(z_0,\zeta_0)} \in E_{\mathrm{i}}\}, \tag{3.3}$$

with 'io' standing for 'in, out'; and for $(z_0, \zeta_0) \in E_{\mathrm{io}}^{\mathrm{int}}$, we record the exit time after the first entry as

$$\mathbf{t}_1(z_0, \zeta_0) := \mathbf{t}_0(z_0, \zeta_0) + \mathbf{t}_0(\gamma_{z_0,\zeta_0}(t), \dot\gamma_{z_0,\zeta_0}(t))|_{t=\mathbf{t}_0(z_0,\zeta_0)} \in (\mathbf{t_0}(z_0, \zeta_0), t^+_{z_0,\zeta_0}). \tag{3.4}$$

Continuing, the null-geodesic $\gamma_{z_0,\zeta_0}(t)$ may, after exiting, reenter $(0,T) \times N$ yet again; since we do not want to keep track of effects *inside* of M which are due to poorly controlled geometry *outside*

of M, we will in our constructions below stop singularities of waves prior to reentering. For now, we merely define

$$\begin{aligned} E_{\mathrm{ioi}}^{\mathrm{int}} &:= \{(z_0,\zeta_0) \in E_{\mathrm{io}}^{\mathrm{int}} \colon (\gamma_{z_0,\zeta_0}(t), \dot\gamma_{z_0,\zeta_0}(t))|_{t=\mathbf{t}_1(z_0,\zeta_0)} \in E_{\mathrm{i}}\}, \\ \mathbf{t}_2(z_0,\zeta_0) &:= \mathbf{t}_1(z_0,\zeta_0) + \mathbf{t}_0(\gamma_{z_0,\zeta_0}(t), \dot\gamma_{z_0,\zeta_0}(t))|_{t=\mathbf{t}_1(z_0,\zeta_0)} \in (\mathbf{t_0}(z_0,\zeta_0), t^+_{z_0,\zeta_0}), \quad (z_0,\zeta_0) \in E_{\mathrm{ioi}}^{\mathrm{int}}. \end{aligned} \tag{3.5}$$

In summary, we record the first entrance and exit time of γ_{z_0,ζ_0} into and out of $(0,T)\times N$, if they exist, and the first reentry time if applicable. See Figure 3.1.

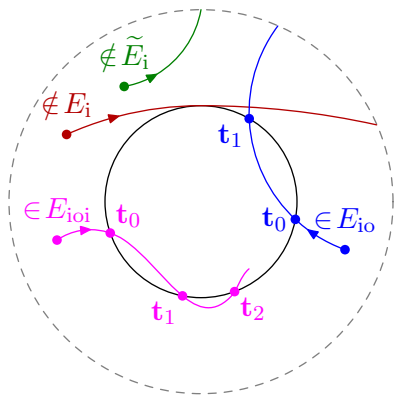

FIGURE 3.1. Illustration of the definitions (3.1)–(3.5), projected to the spatial manifold N_1. The boundary of N is the inner, solid black circle; the boundary of N_1 is the gray dashed outer circle.

We shall see in §3.3 how to constructively determine the sets $E_{\mathrm{io}}^{\mathrm{int}}$ and $E_{\mathrm{ioi}}^{\mathrm{int}}$ from the data of the inverse problem.

3.2. **Construction of distorted plane waves.** Distorted plane waves have singularities conormal to a submanifold of M_1 and can be viewed as Lagrangian distributions. We review the concept of Lagrangian distributions briefly, closely following the notation used in [37]. Recall that T^*M_1 is a symplectic manifold with canonical 2-form, given in local coordinates by $\omega = \sum_{j=1}^4 \mathrm{d}\xi_j \wedge \mathrm{d}x^j$. A submanifold $\Lambda \subset T^*M_1$ is called Lagrangian if $n := \dim \Lambda = 4$ and ω vanishes on Λ. For K a smooth submanifold of M_1, its conormal bundle

$$N^*K = \{(x,\zeta) \in T^*M_1 : x \in K,\ \langle \zeta, \theta\rangle = 0,\ \theta \in T_xK\}$$

is a Lagrangian submanifold of T^*M_1. Let Λ be a smooth conic Lagrangian submanifold of $T^*\widetilde{M}_1 \setminus 0$. We denote by $\mathcal{I}^\mu(\Lambda)$ the space of Lagrangian distributions of order μ associated with Λ. If $\Lambda = N^*K$ for some submanifold $K \subset M_1$, then $\mathcal{I}^\mu(K) := \mathcal{I}^\mu(N^*K)$ denotes the space of conormal distributions to K. For $u \in \mathcal{I}^\mu(\Lambda)$, one can define the principal symbol $\sigma_p(u)$ of u with

$$\sigma_p(u) \in S^{\mu+\frac{n}{4}}(\Lambda, \Omega^{1/2}\otimes L)/S^{\mu+\frac{n}{4}-1}(\Lambda, \Omega^{1/2}\otimes L),$$

where $\Omega^{1/2}$ is the half-density on M_1 and L is the Keller–Maslov line bundle of Λ. We refer to [16, Chapter 4] for the precise definition and more discussions.

Fix a Riemannian metric g^+ on M_1. For $z_0 \in M_1 \setminus M$, $\zeta_0 \in L^+_{z_0}M_1$, and $s_0 > 0$, denote by

$$\mathcal{W}(z_0,\zeta_0,a_0) = \{\eta \in L^+_{z_0}M_1 \colon \|\eta - \zeta_0\|_{g^+} < a_0\}$$

a neighborhood of ζ_0 inside the future light cone. Let

$$\begin{aligned} Y(z_0,\zeta_0,a_0,s_0) &= \{\gamma_{z_0,\eta}(s_0)\in M_1\colon \eta\in\mathcal{W}(z_0,\zeta_0,a_0)\},\\ K(z_0,\zeta_0,a_0,s_0,\bar t) &= \{\gamma_{z_0,\eta}(t)\in M_1\colon \eta\in\mathcal{W}(z_0,\zeta_0,a_0),\ t\in(s_0,\bar t)\},\\ \Lambda(z_0,\zeta_0,a_0,s_0,\bar t) &= \{(\gamma_{z_0,\eta}(t), r\dot\gamma_{z_0,\eta}(t)^\flat)\in T^*M_1\colon \eta\in\mathcal{W}(z_0,\zeta_0,a_0),\ t\in(s_0,\bar t),\ r\in\mathbb{R}\}, \end{aligned}$$

and

$$K(z_0,\zeta_0,a_0,s_0)=K(z_0,\zeta_0,a_0,s_0,t^+_{z_0,\eta}),\qquad \Lambda(z_0,\zeta_0,a_0,s_0)=\Lambda(z_0,\zeta_0,a_0,s_0,t^+_{z_0,\eta}).$$

Here, we use the parameter $a_0>0$ (which will always be chosen to be very small) to control the aperture of the congruence of null-geodesics emanating from z_0, while $s_0>0$ (also small) fixes the offset $t=s_0$ at which we start recording the null-geodesic. The set $\Lambda(z_0,\zeta_0,a_0,s_0)$ is the Lagrangian manifold that is the flowout from $L^{*,+}M_1\cap N^*Y(z_0,\zeta_0,a_0,s_0)$ under the Hamiltonian vector field H_p associated with $p(x,\xi)=g^{jk}(x)\xi_j\xi_k$. Moreover, $K(z_0,\zeta_0,a_0,s_0)$ is the projection $\pi\Lambda(z_0,\zeta_0,a_0,s_0)$ to the base; until the first conjugate point of $\gamma_{z_0,\eta}$, it is a smooth null hypersurface with conormal bundle equal to $\Lambda(z_0,\zeta_0,a_0,s_0)$.

Let now $(z_0,\zeta_0)\in E_{\mathrm{i}}^{\mathrm{int}}$, and consider a source

$$h_0\in\mathcal{I}^{\mu+3/2}(M_1;N^*Y(z_0,\zeta_0,a_0,s_0)) \tag{3.6}$$

conormal to Y. We wish to define a forward solution of the equation $\Box_g u_0=h_0$ adapted to our needs. Roughly speaking, we want u_0 to be an approximate solution, $\Box_g u_0-h_0\in\mathcal{C}^\infty$, until shortly before the time when γ_{z_0,ζ_0} reenters $(0,T)\times N$ after exiting for the first time; the plan is to manufacture u_0 to be singular in a small neighborhood of a *single* null-geodesic segment.

To accomplish this, note that the first intersection of γ_{z_0,ζ_0} with $(0,T)\times\partial N$ is transversal; this is an open condition in the initial conditions of the null-geodesics, hence we conclude that $(z_0,\eta)\in E_{\mathrm{i}}^{\mathrm{int}}$ for all $\eta\in\mathcal{W}(z_0,\zeta_0,2a_0)$ provided a_0 is sufficiently small. There are now three cases to consider:

(1) there exists $a_0>0$ so that $\mathcal{W}(z_0,\zeta_0,2a_0)\cap E_{\mathrm{io}}^{\mathrm{int}}=\emptyset$; that is, the curves $\gamma_{z_0,\eta}$ issuing from z_0 enter $(0,T)\times N$ but do not exit before time $\mathfrak{t}=T$. Or:
(2) one has $(z_0,\zeta_0)\in E_{\mathrm{io}}^{\mathrm{int}}$. The same then holds true for all $\eta\in\mathcal{W}(z_0,\zeta_0,2a_0)$ when $a_0>0$ is sufficiently small; that is, the curves $\gamma_{z_0,\eta}$ enter and exit $(0,T)\times N$ (and might return, to be discussed momentarily). Or:
(3) $(z_0,\zeta_0)\notin E_{\mathrm{io}}^{\mathrm{int}}$, but for all $a_0>0$ we have $\mathcal{W}(z_0,\zeta_0,2a_0)\cap E_{\mathrm{io}}^{\mathrm{int}}\neq\emptyset$. This happens if γ_{z_0,ζ_0} exits $(0,T)\times N$ precisely at $\mathfrak{t}=T$. The null convexity assumption on ∂M implies that γ_{z_0,ζ_0} exits transversally and thus does not return for a positive amount of time; but this means that we can choose $a_0>0$ so small that $\mathcal{W}(z_0,\zeta_0,2a_0)\cap E_{\mathrm{ioi}}^{\mathrm{int}}=\emptyset$.

In cases (1) and (3), we simply take $u_0:=\mathcal{Q}_g h_0$, where $\mathcal{Q}_g$ is a parametrix for $\Box_g$ microlocally near $\Lambda(z_0,\zeta_0,2a_0,s_0)\times N^*Y(z_0,\zeta_0,2a_0,s_0)$; more precisely, $\mathcal{Q}_g$ is a distinguished parametrix in the terminology of [17] corresponding to propagation of singularities in the forward direction in time.[1] By [38] we have $\mathcal{Q}_g\in\mathcal{I}^{-3/2}(\Lambda_g\setminus N^*\mathrm{Diag})$, where $\mathrm{Diag}=\{(x,y)\in M_1\times M_1,\ x=y\}$, and $\Lambda_g\subset T^*M_1\times T^*M_1$ is given by

$$\Lambda_g=\{(x,\xi,y,-\eta);\ g^{jk}(x)\xi_j\xi_k=0,\ (y,\eta)\in\Theta_{x,\xi}\},$$

[1]Alternatively, one can use a doubling construction to create a closed extension N_1 of N so that $((0,T)\times N_1,g)$ is globally hyperbolic, and take $\mathcal{Q}_g$ to be the forward fundamental solution.

where $\Theta_{x,\xi} \subset T^*M_1$ is the bicharacteristic of $\Box_g$ containing (x,ξ). Thus, in this case,

$$u_0 = \mathcal{Q}_g(h_0), \quad u_0|_{M_1\setminus Y} \in \mathcal{I}^\mu(M_1 \setminus Y;\, \Lambda(z_0,\zeta_0,a_0,s_0)) \tag{3.7}$$

is a conormal distribution whose singularities propagate along $\Lambda(z_0,\zeta_0,a_0,s_0)$ (cf. [31, Lemma 3.1]). We note that u_0 restricted to $(0,T)\times N$ satisfies $\Box_g u_0 \in \mathcal{C}^\infty((0,T)\times N)$.

It remains to consider case (2). Here we have $\mathbf{t}_1(z_0,\eta) < t^+_{z_0,\eta}$ for all $\eta \in \mathcal{W}(z_0,\zeta_0,2a_0)$. Due to the lower semicontinuity of the maximal existence time for ODEs, we may choose $\delta > 0$ and $a_0 > 0$ so that

$$\mathbf{t}_1(z_0,\eta) < \mathbf{t}_1(z_0,\zeta_0) + \delta, \qquad \eta \in \mathcal{W}(z_0,\zeta_0,2a_0).$$

In addition, in the case that $(z_0,\zeta_0) \in E^{\mathrm{int}}_{\mathrm{ioi}}$, i.e. the reentry time $\mathbf{t}_2(z_0,\zeta_0)$ (which is lower semicontinuous) is well-defined, we may shrink $a_0 > 0$ and choose $\delta > 0$ so small that

$$\mathbf{t}_1(z_0,\eta) < \mathbf{t}_1(z_0,\zeta_0) + \delta < \mathbf{t}_1(z_0,\zeta_0) + 3\delta < \mathbf{t}_2(z_0,\eta), \qquad \eta \in \mathcal{W}(z_0,\zeta_0,2a_0).$$

Let $\mathcal{Q}'_g$ denote a microlocal parametrix for the forward problem for $\Box_g$ near

$$\Lambda(z_0,\zeta_0,2a_0,s_0,\mathbf{t}_1(z_0,\zeta_0)+3\delta) \times N^*Y(z_0,\zeta_0,2s_0)$$

so that $\Box_g(\mathcal{Q}'_g h_0) - h_0$ is smooth near $\pi(\Lambda(z_0,\zeta_0,2a_0,s_0,\mathbf{t}_1(z_0,\zeta_0)+3\delta))$, where $\pi\colon T^*M_1 \to M_1$ is the projection map. In order to avoid the reentry of singularities, we now choose a cutoff function $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(M_1)$ with

$$\chi = 1 \text{ near } \pi\big(\Lambda(z_0,\zeta_0,a_0,s_0,\mathbf{t}_1(z_0,\zeta_0)+2\delta) \setminus \Lambda(z_0,\zeta_0,a_0,s_0/2,s_0)\big),$$

and put

$$\mathcal{Q}_g(h_0) := \chi\mathcal{Q}'_g(h_0). \tag{3.8}$$

This solves $\Box_g(\mathcal{Q}_g(h_0)) - h_0 - [\Box_g,\chi]\mathcal{Q}'_g(h_0) \in \mathcal{C}^\infty(M_1)$, which is smooth in $(0,T)\times N$ provided we take χ to have support in a sufficiently small neighborhood of $\pi\big(\Lambda(z_0,\zeta_0,2a_0,s_0/2,\mathbf{t}_1(z_0,\zeta_0)+3\delta)\big)$. Moreover, $\mathcal{Q}_g(h_0)$ itself is conormal of order μ along $K(z_0,\zeta_0,s_0)\cap M$ with principal symbol over M equal to the principal symbol of $\mathcal{Q}'_g$ times that of h_0 (see §3.4 for details).

Remark 3.1. The operator $\mathcal{Q}_g$ in (3.7) and (3.8) depends on various choices which are not recorded in the notation. However, in our application, only simple microlocal properties of $\mathcal{Q}_g(h_0)$ matter, and these are independent of choices. Alternatively, one can partition phase space T^*M_1 into different regions which roughly correspond to the separation into the cases (1)–(3), and define a single operator $\mathcal{Q}_g$ using a partition of unity; we leave the details to the reader.

3.3. **Scattering control.** Fix now $(z_0,\zeta_0) \in E^{\mathrm{int}}_{\mathrm{i}}$, $s_0 > 0$; let h_0 be as in (3.6) and pick it to be a classical conormal distribution with connected wave front set, and define $u_0 = \mathcal{Q}_g h_0$ by (3.7) or (3.8), as appropriate. Let us assume that either $(z_0,\zeta_0) \notin E^{\mathrm{int}}_{\mathrm{io}}$, or otherwise that $\gamma_{z_0,\zeta_0}(t)$ has no conjugate point for $t \in [0,\mathbf{t}_1(z_0,\zeta_0)+\delta]$ for some $\delta > 0$; we discuss this condition, and how to arrange it, after the proof of Proposition 3.2 below.

Define now

$$\begin{aligned}\mathcal{U}(z_0,\zeta_0,a_0) &= \{\gamma_{z_0,\eta}(\mathbf{t}_0(z_0,\eta)), \dot\gamma_{z_0,\eta}(\mathbf{t}_0(z_0,\eta))\colon \eta \in \mathcal{W}(z_0,\zeta_0,a_0)\},\\ U(z_0,\zeta_0,a_0) &= \pi(\mathcal{U}(z_0,\zeta_0,a_0)) \subset \partial M,\end{aligned}$$

and denote by

$$\widetilde{U}(z_0,\zeta_0,a_0) \subset \partial M$$

the open a_0-neighborhood of $U(z_0,\zeta_0,a_0)$ with respect to the Riemannian metric g^+. Note that the singularities of the restriction $u_0|_{\widetilde{U}(z_0,\zeta_0,2a_0)}$ to the slightly larger set $\widetilde{U}(z_0,\zeta_0,2a_0)$ are in fact

contained in the smaller set $\widetilde{U}(z_0,\zeta_0,a_0)$; thus, we can choose $\widehat{f}_0$ which has compact support in $\widetilde{U}(z_0,\zeta_0,2a_0)$ and so that $\widehat{f}_0-u_0|_{\widetilde{U}}$ is smooth in $\widetilde{U}(z_0,\zeta_0,2a_0)$. Extend $\widehat{f}_0$ by 0 to $\partial M\setminus\widetilde{U}(z_0,\zeta_0,2a_0)$. Importantly, since the metric g on $M_1\setminus M$ is already known, we can use the value of u_0 on $M_1\setminus M$ to determine the symbol of $\widehat{f}_0$ on $(0,T)\times\partial N$ by continuity. Denote by $\widehat{v}_0$ to be the solution to the initial-boundary value problem

$$\begin{aligned} \Box_g\widehat{v}_0(x) &= 0 && \text{on } M,\\ \widehat{v}_0(x) &= \widehat{f}_0(x) && \text{on } \partial M,\\ \widehat{v}_0(\mathfrak{t},x') &= 0, && \mathfrak{t}<0. \end{aligned} \tag{3.9}$$

We note that the singularities of the solution $\widehat{v}_0$ might encounter several reflections at ∂M. However, before the first reflection we have $\widehat{v}_0 = u_0 \bmod C^\infty$. The goal of this section is the construction of additional boundary sources that remove multiple reflections. In order to be useful for our inverse problem, this needs to be accomplished based on the knowledge of Λ_g^{lin}, but without knowing the metric g itself inside M. The procedure is close in spirit to the scattering control introduced in [7].

Proposition 3.2. *Let u_0 and $\widehat{f}_0$, depending on the choice of the aperture parameter $a_0>0$, be as above. There exists a constructive procedure, which only uses the Dirichlet-to-Neumann map Λ_g^{lin} and the knowledge of g on $M_1\setminus M^{\mathrm{int}}$, for the determination of a value $a_0>0$ and the construction of a boundary source $\widehat{f}_1$ on $(0,T)\times\partial N$ with the property that the solution of the equation*

$$\begin{aligned} \Box_g v_0(x) &= 0 && \text{on } M,\\ v_0(x) &= f_0(x) := \widehat{f}_0(x)+\widehat{f}_1(x) && \text{on } \partial M,\\ v_0(\mathfrak{t},x') &= 0, && \mathfrak{t}<0, \end{aligned} \tag{3.10}$$

satisfies $v_0-u_0\in\mathcal{C}^\infty((0,T)\times N)$. That is, v_0 does not undergo any reflection at ∂M.

Proof. If $(z_0,\eta)\notin E_{\mathrm{io}}^{\mathrm{int}}$ for all $\eta\in\mathcal{W}(z_0,\zeta_0,a_0)$, there is nothing to do: we automatically have $\widehat{v}_0=u_0$ modulo $\mathcal{C}^\infty((0,T)\times N)$ and can thus take $\widehat{f}_1\equiv 0$. We discuss how to determine whether or not $(z_0,\eta)\notin E_{\mathrm{io}}^{\mathrm{int}}$ holds at the end of the proof.

Otherwise, i.e. if there exists $\eta\in\mathcal{W}(z_0,\zeta_0,a_0)$ so that $(z_0,\eta)\in E_{\mathrm{io}}^{\mathrm{int}}$, let

$$\begin{aligned} &\mathcal{V}(z_0,\zeta_0,a_0)=\{\gamma(\mathbf{t}_1(z_0,\eta)),\dot\gamma(\mathbf{t}_1(z_0,\eta))\colon \eta\in\mathcal{W}(z_0,\zeta_0,a_0),\ (z_0,\eta)\in E_{\mathrm{io}}^{\mathrm{int}}\},\\ &V(z_0,\zeta_0,a_0):=\pi(\mathcal{V}(z_0,\zeta_0,a_0)). \end{aligned} \tag{3.11}$$

Since $\mathbf{t}_1(z_0,\zeta_0)>\mathbf{t}_0(z_0,\zeta_0)$, we can take a_0 small enough such that $\overline{\widetilde{U}(z_0,\zeta_0,2a_0)}\cap V(z_0,\zeta_0,2a_0)=\emptyset$. We return to the determination of the set $V(z_0,\zeta_0,a_0)$ of *first* exit points of $\gamma_{z_0,\eta}$ at the end of the proof.

Since the intersection of $K(z_0,\zeta_0,a_0)$ with ∂M at $V(z_0,\zeta_0,a_0)$ is transversal, the solution $\widehat{v}_0$ can be written as the sum of two *conormal* distributions associated to the incoming null hypersurface $K(z_0,\zeta_0,a_0)$ (up until $V(z_0,\zeta_0,a_0)$) on the one hand, and the reflected null hypersurface on the other hand. Thus, we can write $\widehat{v}_0=\widehat{v}_0^{\mathrm{inc}}+\widehat{v}_0^{\mathrm{ref}}$, where

$$\begin{aligned} \widehat{v}_0^{\mathrm{inc}}(x) &= (2\pi)^{-1}\int_{\mathbb{R}} e^{\mathrm{i}\phi^{\mathrm{inc}}(x)\theta}a^{\mathrm{inc}}(x,\theta)\mathrm{d}\theta,\\ \widehat{v}_0^{\mathrm{ref}}(x) &= (2\pi)^{-1}\int_{\mathbb{R}} e^{\mathrm{i}\phi^{\mathrm{ref}}(x)\theta}a^{\mathrm{ref}}(x,\theta)\mathrm{d}\theta, \end{aligned} \tag{3.12}$$

for x in a neighborhood of V inside of M, and $\theta \in \mathbb{R}$; here, since our source h_0 is a classical (one-step polyhomogeneous) conormal distribution, the amplitudes a^{inc} and a^{ref} are such as well, with asymptotic expansions as $|\theta| \to \infty$

$$a^{\bullet}(x,\theta) \sim \sum_{j\geq 0} a_j^{\bullet}(x,\theta), \qquad \bullet = \mathrm{inc},\ \mathrm{ref}, \qquad a_j \in S^{\mu+\frac{1}{2}-j}. \tag{3.13}$$

We can assume that $\phi^{\mathrm{inc}}|_V = \phi^{\mathrm{ref}}|_V$. We remark that $\mathrm{d}(\phi^{\bullet}|_{\partial M})$ is timelike, since V is necessarily spacelike, cf. [22, Lemma 2.5]; by switching the sign of θ, we may assume that $\mathrm{d}(\phi^{\bullet}|_{\partial M})$ is *future* timelike. Moreover, the phase functions ϕ^{inc} and ϕ^{ref} both satisfy the eikonal equations (2.4), but their normal derivatives differ by a sign, that is,

$$\partial_\nu \phi^{\mathrm{inc}} = -\partial_\nu \phi^{\mathrm{ref}} \text{ on } V.$$

Invoking the zero Dirichlet boundary condition $\widehat{v}_0^{\mathrm{inc}} + \widehat{v}_0^{\mathrm{ref}} = 0$ on V, the amplitudes satisfy the matching conditions

$$a_j^{\mathrm{ref}}|_V = -a_j^{\mathrm{ref}}|_V, \quad j = 0, 1, \ldots. \tag{3.14}$$

The Neumann data of $\widehat{v}_0$ are thus

$$\begin{aligned}\Lambda_g^{\mathrm{lin}}(\widehat{f}_0)|_V &= \partial_\nu \widehat{v}_0|_V \\ &= (2\pi)^{-1} \int e^{\mathrm{i}\phi^{\mathrm{inc}}(x)\theta} \left(2\mathrm{i}\theta(\partial_\nu \phi^{\mathrm{inc}}) a_0^{\mathrm{inc}} + \partial_\nu(a_0^{\mathrm{inc}} + a_0^{\mathrm{ref}}) + 2\mathrm{i}\theta(\partial_\nu \phi^{\mathrm{inc}}) a_1^{\mathrm{inc}} + \cdots \right) \mathrm{d}\theta.\end{aligned} \tag{3.15}$$

Note that $\phi^{\bullet}|_{\partial M}$ in a neighborhood of V can be assumed to be known (with certain freedom of choice as long as $\{(x, \theta \mathrm{d}(\phi^{\bullet}|_{\partial M})) : \theta \in \mathbb{R}, \phi^{\bullet}(x) = 0\}$ is a parametrization of the Lagrangian submanifold associated with $\Lambda_g^{\mathrm{lin}}(\widehat{f}_0)$ near V). Since the metric g on $M_1 \setminus M$ is known, the value of $\partial_\nu \phi^{\mathrm{inc}}|_V < 0$ is known. Thus, from the principal symbol of $\Lambda_g^{\mathrm{lin}}(\widehat{f}_0)$ on V,

$$\sigma_p(\Lambda_g^{\mathrm{lin}}(\widehat{f}_0)) = 2\mathrm{i}\theta(\partial_\nu \phi^{\mathrm{inc}}) a_0^{\mathrm{inc}}, \tag{3.16}$$

we can determine the value of a_0^{inc} on V.

The crucial insight is then that from the knowledge of a_0^{inc} thus obtained from the Dirichlet-to-Neumann map, and from the full symbol of $\Lambda_g^{\mathrm{lin}}(\widehat{f}_0)|_V$, we can reconstruct the full jet of all $a_j^{\bullet}$ with $(j, \bullet) \neq (0, \mathrm{inc})$. Indeed, recall the matching conditions (3.14), and note that the amplitudes satisfy the transport equations

$$\begin{aligned}\mathrm{i}T^{\mathrm{inc}} a_j^{\mathrm{inc}} &= -\Box_g a_{j-1}^{\mathrm{inc}}, \\ \mathrm{i}T^{\mathrm{ref}} a_j^{\mathrm{ref}} &= -\Box_g a_{j-1}^{\mathrm{ref}},\end{aligned} \tag{3.17}$$

for $j \geq 1$, where the transport operators T^{inc} and T^{ref} are defined in (2.6) with $\phi = \phi^{\mathrm{inc}}$ and $\phi = \phi^{\mathrm{ref}}$. By solving these transport equations in the exterior domain $M_1 \setminus M$, the amplitudes a_0^{inc} and a_0^{ref} in can be determined in $M_1 \setminus M$. Consequently, $\partial_\nu(a_0^{\mathrm{inc}} + a_0^{\mathrm{ref}})|_V$ can be determined. Then from the subprincipal symbol of $\Lambda_g^{\mathrm{lin}}(\widehat{f}_0)$, we can determine $a_1^{\mathrm{inc}}|_V$. Continuing with this process, we can determine $a_j|_V$ for all $j = 0, 1, 2, \ldots$. Therefore, the boundary trace $\widehat{v}_0^{\mathrm{ref}} = -\widehat{v}_0^{\mathrm{inc}}|_V = -u_0|_V$ of the reflected distorted plane wave can be determined, modulo smooth functions, from $\Lambda_g^{\mathrm{lin}}(\widehat{f}_0)$. But this means that we can construct the boundary source

$$f_0 = \widehat{f}_0 + \widehat{f}_1, \qquad \widehat{f}_1 := -\widehat{v}_0^{\mathrm{ref}},$$

which has the desired property. Carefully note that this uses that u_0 itself does not have artificial singularities introduced at later times; this is precisely due to our cutoff construction in (3.8) for singularities which enter, then leave, and *do not reenter* $(0,T)\times N$ since we cut them off in time.

It remains to tie up two loose ends from the beginning of the proof. Firstly, whether or not the condition $(z_0,\eta)\notin E_{\mathrm{io}}^{\mathrm{int}}$ holds for all $\eta\in\mathcal{W}(z_0,\zeta_0,a_0)$ can be checked as follows. First, for sufficiently small $a_0>0$, the set of singularities of the Neumann data $\Lambda_g^{\mathrm{lin}}(\widehat{f}_0)$ inside the set $\widetilde{U}(z_0,\zeta_0,a_0)$ is contained in the singular set of $\widehat{f}_0$; this follows from the transversality of $\gamma_{z_0,\eta}$ at the first point of intersection with $(0,T)\times\partial N$. For such a_0, we thus rule out that the first reflection happens inside $\widetilde{U}(z_0,\zeta_0,a_0)$. Next, if $\Lambda_g^{\mathrm{lin}}(\widehat{f}_0)$ has no singularities outside of $\widetilde{U}(z_0,\zeta_0,a_0)$, we can conclude that $(z_0,\eta)\notin E_{\mathrm{io}}^{\mathrm{int}}$; indeed, otherwise (as we are assuming that $\gamma_{z_0,\eta}(t)$ has no conjugate points near $[0,\mathbf{t}_1(z_0,\eta)]$) the Neumann trace would have a singularity by (3.16).

Secondly, the determination of the first exit points, in the case that $(z_0,\eta)\in E_{\mathrm{io}}^{\mathrm{int}}$ for some $\eta\in\mathcal{W}(z_0,\zeta_0,a_0)$ no matter how small $a_0>0$ is, can be accomplished as follows. For sufficiently small $a_0>0$, the singular set of $\Lambda_g^{\mathrm{lin}}(\widehat{f}_0)$ will have at least one connected component disjoint from the component contained in $\widetilde{U}(z_0,\zeta_0,a_0)$; one of them is the place of first reflection. (This is the place where we take advantage of our choice of the source as having connected wave front set, thus singular set.) The key point is that if we perform the above scattering control construction near a component of the singular set of $\Lambda_g^{\mathrm{lin}}(\widehat{f}_0)$ which is disjoint from $V(z_0,\zeta_0,a_0)$, the solution v_0 of the modified equation (3.10) will remain singular at $V(z_0,\zeta_0,a_0)$. That is, we can determine whether we performed scattering control at the correct, i.e. first, reflection point simply by checking if, after scattering control, the singular set of the Neumann data of the solution v_0 of (3.10) has a single connected component. □

Regarding the assumption on the absence of conjugate points along γ_{z_0,ζ_0}, note that for $(z_1,\zeta_1):=\gamma_{z_0,\zeta_0}(\mathbf{t}_0(z_0,\zeta_0))\in(0,T)\times\partial N$, the null-geodesic γ_{z_1,ζ_1} has no conjugate points prior to hitting ∂M at the point $\gamma_{z_0,\zeta_0}(\mathbf{t}_1(z_0,\zeta_0))$. Thus, for some small $\delta>0$, $\gamma_{z_0,\zeta_0}(t)$ has no conjugate points for $t\in[\mathbf{t}_0(z_0,\zeta_0)-\delta,\mathbf{t}_1(z_0,\zeta_0)+\delta]$. Correspondingly, distorted plane waves in M_1 generated by conormal sources near $\gamma_{z_0,\zeta_0}(\mathbf{t}_0(z_0,\zeta_0)-\delta)$ are conormal until a bit beyond the exit out of $(0,T)\times N$, and in particular the Neumann data have a conormal singularity near $\gamma_{z_0,\zeta_0}(\mathbf{t}_1(z_0,\zeta_0))$; the singularity in turn is the zero level set of the restriction to ∂M of a phase function ϕ^{inc} which extends, as a solution of the eikonal equation, *smoothly* to a neighborhood of $\gamma_{z_0,\zeta_0}(\mathbf{t}_1(z_0,\zeta_0))$ inside $M_1\setminus M^{\mathrm{int}}$.

Thus, given (z_0,ζ_0), we can determine $\delta_0>0$ so that for all $\delta\in(0,\delta_0)$, the phase function ϕ^{inc} is smooth in a δ_0-neighborhood of $\gamma_{z_0,\zeta_0}(\mathbf{t}_1(z_0,\zeta_0))$. Moreover, δ_0 can be chosen to depend continuously on (z_0,ζ_0), $\|\zeta\|_{g^+}=1$. Thus, we can (constructively) take the extension $N_1\setminus N$ in (2.14) sufficiently small so that the absence of conjugate points holds for all starting points $z_0\in M_1\setminus M^{\mathrm{int}}$.

3.4. **Nonlinear interactions of distorted plane waves.** For $j=1,2,3,4$, take $z_j\in M_1\setminus M$ and $\zeta_j\in L^+_{z_j}M_1$ so that $(z_j,\zeta_j)\in E_{\mathrm{i}}^{\mathrm{int}}$; we recall that this condition can be checked only using the knowledge of $g|_{M_1\setminus M^{\mathrm{int}}}$, which we have already constructed. Choosing $a_0,s_0>0$ sufficiently small, and taking sources

$$h_j\in\mathcal{I}^{\mu+3/2}(M_1;N^*Y(z_j,\zeta_j,a_0,s_0)) \tag{3.18}$$

which we take to be classical conormal with connected wave front set, we can then use Proposition 3.2 to produce Dirichlet data f_j such that the solution v_j to the equation

$$\begin{aligned} \Box_g v_j(x) &= 0, && \text{on } M, \\ v_j(x) &= f_j(x), && \text{on } \partial M, \\ v_j(\mathfrak{t}, x') &= 0, && \mathfrak{t} < 0, \end{aligned} \tag{3.19}$$

is the restriction to M of a conormal distribution in $\mathcal{I}^\mu(M_1; \Lambda(z_j, \zeta_j, a_0, s_0))$ which agrees with $u_j := \mathcal{Q}_g(h_j)$ on $(0,T) \times N$.

As Dirichlet data for four interacting distorted plane waves, we then take

$$f = \sum_{j=1}^4 \epsilon_j f_j; \tag{3.20}$$

denote by $v = \sum_{j=1}^4 \epsilon_j v_j$ the solution of the linear initial-boundary value problem $\Box_g v = f$, $v|_{\partial M} = f$, $v|_{\mathfrak{t}<0} = 0$. Let u denote the solution of the *nonlinear* equation (1.2) with $f = \sum_{j=1}^4 \epsilon_j f_j$. Writing $w = Q_g(F)$ if w solves the linear wave equation

$$\begin{aligned} \Box_g w(x) &= F, && \text{on } M, \\ w(x) &= 0, && \text{on } \partial M, \\ w &= 0, && \mathfrak{t} < 0, \end{aligned}$$

the wave produced by the four-fold intersection is then

$$\mathcal{U}^{(4)} := \partial_{\epsilon_1}\partial_{\epsilon_2}\partial_{\epsilon_3}\partial_{\epsilon_3} u|_{\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_4=0} = Q_g(a v_1 v_2 v_3 v_4). \tag{3.21}$$

We proceed to describe the microlocal structure of $\mathcal{U}^{(4)}$ under certain nondegeneracy assumptions. Let us abbreviate

$$K_j = K(z_j, \zeta_j, a_0, s_0, \bar{t}_j), \qquad \Lambda_j = \Lambda(z_j, \zeta_j, a_0, s_0, \bar{t}_j),$$

where $\bar{t}_j$ is fixed to be larger than $\mathbf{t}_1(z_j, \zeta_j)$ if the latter is defined and equal to $t^+_{z_j,\zeta_j}$ otherwise, and $\bar{t}_j < \mathbf{t}_2(z_j, \zeta_j)$ if defined; and $a_0, s_0 > 0$ are chosen small. Assume that

(1) K_i, K_j, $i \neq j$, intersect transversally at a co-dimension 2 submanifold $K_{ij} \subset M^{\text{int}}$;
(2) K_i, K_j, K_k, i, j, k distinct, intersect transversally at a co-dimension 3 submanifold $K_{ijk} \subset M^{\text{int}}$.

Under the assumption that $z_j \notin \gamma_{z_i,\zeta_i}([0, t^+_{z_i,\zeta_i}))$ for $i \neq j$, these two conditions are automatically satisfied if we choose the aperture a_0 of the sources h_j, see (3.18), sufficiently small. Indeed, in the limit $a_0 \to 0$, the sets K_j tend to the null-geodesics γ_{z_j,ζ_j}, the tangent vectors of which are equal to the normal vector of K_j at points on γ_{z_j,ζ_j}; but any two γ_{z_j,ζ_j} necessarily intersect transversally (unless one is contained in the other, which we are assuming is not the case), and a third null-geodesic γ_{z_k,ζ_k} has, at a triple intersection point $\gamma_{z_i,\zeta_i}(t_i) = \gamma_{z_j,\zeta_j}(t_j) = \gamma_{z_k,\zeta_k}(t_k)$, a tangent vector not collinear to either of the other two. Indeed, the only lightlike tangent vectors in the span of $\dot\gamma_{z_i,\zeta_i}(t_i)$ and $\dot\gamma_{z_j,\zeta_j}(t_j)$ (which is a 2-dimensional linear Lorentzian vector space) are scalar multiples of $\dot\gamma_{z_i,\zeta_i}(t_i)$ and $\dot\gamma_{z_j,\zeta_j}(t_j)$; this implies the transversality of K_i, K_j, K_k for sufficiently small $s_0 > 0$.

In order to proceed, let us make the additional assumption that

$$\begin{gathered} \text{for all } s_0 > 0,\ K_1,\ K_2,\ K_3,\ K_4 \text{ intersect transversally at a point } q_0 \in M^{\text{int}}, \\ \text{prior to exiting } (0,T) \times N \text{ for the first time (if they exit at all).} \end{gathered} \tag{3.22}$$

We discuss the verification of this assumption using the already reconstructed data at the end of this section; here, we note that due to our assumption on the absence of conjugate points in M, there is at most a single such intersection point q_0 when all (z_j, ζ_j) are sufficiently close to each another, which will indeed be the setup in the next section. We then put

$$\Lambda_{ij} = N^* K_{ij}, \quad \Lambda_{ijk} = N^* K_{ijk}, \qquad \Lambda_{q_0} = T^*_{q_0} M \setminus 0;$$

these are all Lagrangian submanifolds in T^*M_1. For any $\Gamma \subset T^*M_1$, we denote by Γ^g the flow-out of $\Gamma \cap L^{*,+}M_1$ under the geodesic flow of g lifted to T^*M_1. Finally, set

$$\Lambda^{(1)} = \bigcup_{i=1}^4 \Lambda_i; \quad \Lambda^{(2)} = \bigcup_{i,j=1}^4 \Lambda_{ij}; \quad \Lambda^{(3)} = \bigcup_{i,j,k=1}^4 \Lambda_{ijk};$$
$$K^{(1)} = \bigcup_{i=1}^4 K_i; \quad K^{(2)} = \bigcup_{i,j=1}^4 K_{ij}; \quad K^{(3)} = \bigcup_{i,j,k=1}^4 K_{ijk};$$
$$\Xi = \Lambda^{(1)} \cup \Lambda^{(3),g} \cup \Lambda_{q_0}.$$

We then can write

$$\mathcal{U}^{(4)} = \mathcal{U}^{\mathrm{inc}} + \mathcal{U}^{\mathrm{ref}}, \tag{3.23}$$

in $(0,T) \times N$, where (cf. [37, Proposition 3.11])

$$\mathcal{U}^{\mathrm{inc}} = \mathcal{Q}_g(a v_1 v_2 v_3 v_4) \in \mathcal{I}^{4\mu+3/2}(\Lambda^g_{q_0} \setminus \Xi);$$

here $\mathcal{Q}_g$ denotes a forward parametrix for $\Box_g^{-1}$ on M_1 microlocally near $(\Lambda^g_{q_0} \setminus \Xi) \times \Lambda_{q_0}$ which is truncated after the passage of null-bicharacteristics out of M into $M_1 \setminus M$, analogously to the construction of (3.8); and $\mathcal{U}^{\mathrm{ref}}$ is the wave generated by the reflection of $\mathcal{U}^{\mathrm{inc}}$ at the boundary ∂M.

In order to describe the symbol of $\mathcal{U}^{\mathrm{inc}}$, we introduce some further notation: define the set of initial data for null-geodesics inside $(0,T) \times N^{\mathrm{int}}$ which strike $(0,T) \times \partial N$ by

$$\mathcal{S} := \{(z_0, \zeta_0) \colon z_0 \in (0,T) \times N^{\mathrm{int}},\ \zeta_0 \in L^+_{z_0} M,\ \exists\, t \in (0, t^+_{z_0,\zeta_0}) \text{ s.t. } \gamma_{z_0,\zeta_0}(t) \in (0,T) \times \partial N\}; \tag{3.24}$$

importantly, the null convexity of ∂M implies that γ_{z_0,ζ_0} in fact strikes ∂M *transversally* and thus exits M for a positive amount of time. For $(z_0, \zeta_0) \in \mathcal{S}$, we then define the exit time

$$\mathbf{t}_0(z_0, \zeta_0) := \inf\{t \in (0, t^+_{z_0,\zeta_0}) \colon \gamma_{z_0,\zeta_0}(t) \in (0,T) \times \partial N\}.$$

If $(q_0, \zeta) \in \mathcal{S}$ and $(y, \eta) = (\gamma_{q_0,\zeta}(\mathbf{t}_0(q_0,\zeta)), \dot\gamma_{q_0,\zeta}(\mathbf{t}_0(q_0,\zeta)^\flat)$, the wave $\mathcal{U}^{\mathrm{inc}}$ incident to $(0,T) \times \partial N$ then has principal symbol given by (cf. [37, Proposition 3.12])

$$\sigma_p(\mathcal{U}^{\mathrm{inc}})(y,\eta) = -24(2\pi)^{-3} a(q_0) \sigma_p(\mathcal{Q}_g)(y, \eta, q_0, \zeta^\flat) \prod_{j=1}^4 \sigma_p(v_j)(q_0, \theta_j). \tag{3.25}$$

Here $\theta_j \in N^*_{q_0} K_j$ denote the unique covectors for which $\zeta^\flat = \sum_{j=1}^4 \theta_j$, and $\sigma_p(\mathcal{Q}_g)$ is the principal symbol of the FIO $\mathcal{Q}_g$ on $\Lambda^g_{q_0}$ away from $N^*\mathrm{Diag}$.

We can say more: in terms of (3.23), we can write, near the point $y \in (0,T) \times \partial N$,

$$\begin{aligned} \mathcal{U}^{\mathrm{inc}}(x) &= (2\pi)^{-1} \int e^{\mathrm{i}\phi^{\mathrm{inc}}(x)\theta} a^{\mathrm{inc}}(x,\theta) \mathrm{d}\theta \\ \mathcal{U}^{\mathrm{ref}}(x) &= (2\pi)^{-1} \int e^{\mathrm{i}\phi^{\mathrm{ref}}(x)\theta} a^{\mathrm{ref}}(x,\theta) \mathrm{d}\theta, \end{aligned} \tag{3.26}$$

where the phase functions $\phi^{\rm inc}$ and $\phi^{\rm ref}$ satisfy the eikonal equation (2.4) and the boundary conditions $\phi^{\rm inc}=\phi^{\rm ref}$ and $\partial_\nu\phi^{\rm inc}=-\partial_\nu\phi^{\rm ref}$ on ∂M, while the symbols $a^\bullet$ are as in (3.13). The Dirichlet boundary condition implies that $a^{\rm ref}(x,\theta)=-a^{\rm inc}(x,\theta)$ for $x\in\partial M$. Then

$$\partial_\nu\mathcal{U}^{(4)}|_{\partial M}=(2\pi)^{-3}\int e^{{\rm i}\phi^{\rm inc}(x)\theta}\Big(2{\rm i}\theta(\partial_\nu\phi^{\rm inc})a^{\rm inc}+\partial_\nu(a^{\rm inc}+a^{\rm ref})\Big)\,{\rm d}\theta. \tag{3.27}$$

Therefore, the contribution of the four-fold interaction to the Dirichlet-to-Neumann map is

$$\begin{aligned}&\sigma_p\big(\partial_{\epsilon_1}\partial_{\epsilon_2}\partial_{\epsilon_3}\partial_{\epsilon_4}\Lambda_{g,a}\left(\epsilon_1f_1+\epsilon_2f_2+\epsilon_3f_3+\epsilon_4f_4\right)|_{\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_4=0}\big)(x,\theta)\\&\quad=\sigma_p(\partial_\nu\mathcal{U}^{(4)}|_{\partial M})(x,\theta)\\&\quad=-2{\rm i}\theta\xi_na_0^{\rm inc}(x,\theta),\qquad \xi_n:=-\sqrt{-g^{\alpha\beta}(x)\xi'_\alpha\xi'_\beta},\quad \xi':={\rm d}(\phi^{\rm inc}|_{\partial M}).\end{aligned}$$

Thus, we can recover $a_0^{\rm inc}(x,\theta)$ for $x\in\partial M$ (and then the complete jet of $a_j^\bullet$, $j=0,1,2,\ldots$, and $\phi^\bullet$ at ∂M, where $\bullet={\rm inc},{\rm ref}$; and therefore we can recover the (full) symbol of $\mathcal{U}^{\rm inc}|_{\partial M}$ and then the symbol of $\mathcal{U}^{\rm inc}$ in $M_1\setminus M^{int}$.

Thus, if the interaction point $q_0\in M^{\rm int}$ is such that the set of $\zeta\in L^+_{q_0}M$ with $(q_0,\zeta)\in\mathcal{S}$ is nonempty, then this set is automatically open, and thus $\mathcal{U}^{\rm inc}$ is singular on a codimension 1 (i.e. 2-dimensional) submanifold of $(0,T)\times\partial N$. If there does not exist $\zeta\in L^+_{q_0}M$ with $(q_0,\zeta)\in\mathcal{S}$, then $\mathcal{U}^{\rm inc}$ is smooth on $(0,T)\times\partial N$.

We argue that we can verify if assumption (3.22) is satisfied: indeed, if for sufficiently small $a_0>0$ the null hypersurfaces K_1,K_2,K_3,K_4 do not intersect all at once, the four-fold interaction $\mathcal{U}^{(4)}$ is smooth away from $\Xi=\Lambda^{(1)}\cup\Lambda^{(3),g}$. But the intersection of the projection of Ξ to the base with ∂M tends to a set of codimension 2 (meaning: of Hausdorff dimension $3-2=1$); note here that the 1-dimensional Hausdorff measure of any triple intersection K_{ijk} tends to 0 as $a_0>0$, and for any $p\in K_{ijk}$, the dimension of the set of lightlike directions in $N^*_pK_{ijk}$ (i.e. the set of lightlike covectors which have unit length with respect to g^+) is equal to 2 (compared to the dimension of the set of lightlike directions in $N^*_{q_0}M$, which is equal to 3). See also [31, §3.3.2].

Similarly, if the intersection $\bigcap_{j=1}^4K_j$ is nonempty but not transversal (but, due to the absence of conjugate points, still only consists of a single point), the wave front set of the product $v_1v_2v_3v_4$ is contained in $\bigcup N^*K_{ijk}$, thus $\mathcal{U}^{(4)}$ is singular only on a small set in the sense just explained.

3.5. **Determination of the conformal class of the metric.** We continue using the notation of the previous section; thus, we take $z_j\in M_1\setminus M$ and $\zeta_j\in L^+_{z_j}M_1$ for $j=1,2,3,4$ with $(z_j,\zeta_j)\in E^{\rm int}_{\rm i}$, and consider sources h_j and solutions of the initial-boundary value problem v_j with Dirichlet data f_j; we moreover take $f=\sum_{i=1}^4\epsilon_if_i$ as in (3.20) and u as in (1.2), and put $v_j=\frac{\partial}{\partial\epsilon_j}u|_{\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_4=0}$, which solves the linear initial-boundary value problem with source f_j. Thus,

$$\partial_\nu v_j|_{(0,T)\times\partial N}=\frac{\partial}{\partial\epsilon_j}\Lambda_{g,a}(f)|_{\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_4=0}=\frac{\partial}{\partial\epsilon_j}\Lambda_{g,a}\left(\sum_{i=1}^4\epsilon_if_i\right)\Big|_{\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_4=0}.$$

From now on, whenever $(z_j,\zeta_j)\in E^{\rm int}_{\rm ioi}$, we shall write $\widetilde{\gamma}_{z_j,\zeta_j}$ for the restriction of γ_{z_j,ζ_j} to parameters $t\in[0,\mathbf{t}_2(z_j,\zeta_j))$, i.e. we stop the null-geodesic right before it would reenter $(0,T)\times\partial N$; otherwise, we let $\widetilde{\gamma}_{z_j,\zeta_j}=\gamma_{z_j,\zeta_j}$ with domain $[0,t^+_{z_j,\zeta_j})$. First, we test whether the intersection point q_0 of γ_{z_j,ζ_j} lies on the boundary; but $\bigcap_{j=1}^4\gamma_{z_j,\zeta_j}=q_0\in(0,T)\times\partial N$ if and only if $q_0\in\lim_{s_0\to0}\bigcap_{j=1}^4{\rm singsupp}\,(\partial_\nu v_j|_{(0,T)\times\partial N})$.

As discussed in the previous section, we can determine whether the four-fold intersection of the null hypersurfaces K_j is nonempty and transversal. Assuming thus that it is, then away from

$$\mathcal{K}^{(3)} = \pi(\Lambda^{(3),g})$$

which bounds the set of spacetime points to which singularities arising from triple intersections could propagate, $\mathcal{U}^{\mathrm{inc}}$ is the restriction to M of an element of $\mathcal{I}^{4\mu+3/2}(\Lambda^g_{q_0} \setminus (\Xi \cup T^*_{\mathcal{K}^{(3)}} M_1))$; and as discussed after (3.27), we can recover the full symbol of $\mathcal{U}^{\mathrm{inc}}$ restricted to $(0,T) \times \partial N$. Upon taking the aperture a_0 of our distorted plane waves to 0, the set

$$\{(\mathfrak{t}, x) \in (0,T) \times \partial N \colon \mathcal{U}^{\mathrm{inc}} \text{ is smooth near } (0,\mathfrak{t}) \times \{x\} \text{ but not at } (\mathfrak{t}, x)\}$$

tends to an open dense subset of the boundary light observation set

$$\mathcal{E}_M(q_0) := ((0,T) \times \partial N) \cap \{\gamma_{q_0,\zeta_0}(\mathfrak{t}_0(q_0,\zeta_0)) \colon \zeta_0 \in L^+_{q_0} M\}$$

(which, due to the 'no conjugate points' hypothesis, is a smooth spacelike hypersurface), and thus we can recover $\mathcal{E}_M(q_0)$.

Now, we claim that every point $q_0 \in \mathbb{U}_g$ lying in $(0,T) \times N^{\mathrm{int}}$ is the transversal intersection of four distorted plane waves originating outside $(0,T) \times N$. To see this, we first prove the following result:

Lemma 3.3. *Given $q_0 \in \mathbb{U}_g \cap ((0,T) \times N^{\mathrm{int}})$, there exists a null-geodesic $\mu \colon [0,1] \to M$ with $\mu(0) \in (0,T) \times \partial N$ and $\mu(1) = q_0$, and which moreover has no cut points.*

Proof. By definition of $\mathbb{U}_g$, there exists $(\mathfrak{t}_0, x_0) \in (0,T) \times \partial N$ so that q_0 lies in the causal future of $(\mathfrak{t}_0, x_0)$; on the other hand, q_0 does *not* lie in the causal future of (T, x_0). Thus, we have

$$\bar{\mathfrak{t}} := \sup\{t \in (0,T) \colon q_0 \in J^+_g((t,x_0))\} \in [\mathfrak{t}_0, T).$$

We claim that there exists a null-geodesic in M joining $(\bar{\mathfrak{t}}, x_0)$ to q_0. We can select $t_j \in (0,T)$ and causal curves $\gamma_j \colon [t_j, \mathfrak{t}(q_0)] \to M$ so that $t_1 \le t_2 \le \ldots \nearrow \bar{\mathfrak{t}}$ and $\gamma_j(t_j) = (t_j, x_0)$, $\gamma_j(\mathfrak{t}(q_0)) = q_0$; here, we parameterize the γ_j to that $\mathfrak{t} \circ \gamma_j(t) = t$. Since N is compact, the Arzelà–Ascoli theorem implies that upon passing to a subsequence of the γ_j, we may assume the existence of a limit $\lim_{j\to\infty} \gamma_j(s) = \gamma(s)$ for all $s \in [\bar{\mathfrak{t}}, \mathfrak{t}(q_0)]$; note here that $\gamma(\bar{\mathfrak{t}}) = (\bar{\mathfrak{t}}, x_0)$ and $\gamma(\mathfrak{t}(q_0)) = q_0$, and $s \mapsto \gamma(s)$ is Lipschitz since each γ_j is Lipschitz (with uniform Lipschitz constant). Let $s_0 \in [\bar{\mathfrak{t}}, \mathfrak{t}(q_0)]$ be the supremum of all $s \in [\bar{\mathfrak{t}}, \mathfrak{t}(q_0))$ so that $\gamma(s) \in \partial M$; in particular, $s_0 < \mathfrak{t}(q_0)$.

We will use $\gamma|_{[s_0,\mathfrak{t}(q_0)]}$ to construct the desired null-geodesic μ. Consider any $s \in (s_0, \mathfrak{t}(q_0)]$, and suppose that we have already re-defined γ to be a smooth reparameterized null-geodesic on $[s, \mathfrak{t}(q_0)]$. (This is initially satisfied for $s = \mathfrak{t}(q_0)$.) Then $\gamma(s) \in (0,T) \times N^{\mathrm{int}}$. Denote by $\mathcal{U} \subset M^{\mathrm{int}}$ a geodesically convex neighborhood of $q = \gamma(s)$ with closure $\bar{\mathcal{U}} \subset M^{\mathrm{int}}$, and let $\epsilon > 0$ be such that $\gamma_{q,\zeta}(1) \in \mathcal{U}$ for all $\zeta \in T_q M$ with $|\zeta|_{g^+} \le 3\epsilon$. Since $\gamma(s_0) \notin \mathcal{U}$, there exist $s' \in (s_0, s)$ and $\zeta \in T_q M$ with $|\zeta|_{g^+} = \epsilon$ so that

$$q' := \gamma(s') = \gamma_{q,\zeta}(1).$$

By construction of γ, the tangent vector ζ is necessarily past causal.

Suppose that ζ was past *timelike*; we lead this to a contradiction to the definition of $\bar{\mathfrak{t}}$ by perturbing γ_j on $[t_j, \mathfrak{t}(q')]$ to a causal curve which starts at a later time on $(0,T) \times \partial N$. The argument given here is close to that in [39, Proposition 46], but since we need to take particular care that our curves remain inside the domain $(0,T) \times N$ with boundary, we present the details.

For $\digamma > 1$ (chosen momentarily), $j \in \mathbb{N}$, and $\delta > 0$ small, define the curve[2]

$$\tilde{\gamma}_{j,\delta}(t) := \gamma_j(t) + \big(\delta e^{\digamma(t-\mathfrak{t}(q'))}, 0\big), \quad t \in [t_j, \mathfrak{t}(q')];$$

we claim that this curve is timelike. Indeed, using the form (1.1) of the metric, and writing $\gamma_j(t) = (t, x_j'(t))$, the causal nature of γ_j implies that

$$|\dot{\gamma}_j(t)|^2_{g(t,x_j'(t))} = -\alpha(t, x_j'(t)) + |\dot{x}_j'(t)|^2_{\kappa(t,x_j'(t))} \leq 0.$$

Writing $w(t) = \delta e^{\digamma(t-\mathfrak{t}(q'))}$, we have

$$\begin{aligned}|\dot{\tilde{\gamma}}_{j,\delta}(t)|^2_{g(t+w(t),x_j'(t))} &= -(1+w'(t))^2\alpha(t+w(t), x_j'(t)) + |\dot{x}_j'(t)|^2_{\kappa(t+w(t),x_j'(t))} \\ &= |\dot{\gamma}_j(t)|^2_{g(t,x_j'(t))} - \Big((1+w'(t))^2\alpha(t+w(t), x_j'(t)) - \alpha(t, x_j'(t)) \\ &\qquad + |\dot{x}_j'(t)|^2_{\kappa(t+w(t),x_j'(t))} - |\dot{x}_j'(t)|^2_{\kappa(t,x_j'(t))}\Big).\end{aligned}$$

We claim that the big parenthesis is nonnegative; indeed, denoting $\alpha_0 = \inf_M \alpha > 0$, and noting that $w' = \digamma w$, we can bound it from below for $t \in [t_j, \mathfrak{t}(q')]$ (so $0 < w(t) \leq \delta$) by

$$(2w'(t) + w'(t)^2)\big(\alpha(t, x_j'(t)) - Cw(t)\big) - Cw(t) \geq w\big((2\digamma + \digamma^2 w)(\alpha_0 - C\delta) - C\big) \geq \digamma\alpha_0 w > 0$$

for some constant C only depending on the metric g (using that α and κ are $\mathcal{C}^1$), provided we fix $\digamma$ sufficiently large and consider $\delta \in (0, \delta_0]$ where $\delta_0 > 0$ is sufficiently small (and can be selected independently of j). The idea behind this construction is that one can dominate error terms of size w—arising from computing norms at points differing by $w(t)$ in the first coordinate—by a small constant times w'—arising in the expression for the tangent vector of $\tilde{\gamma}_{j,\delta}$.

Furthermore, as $j \to \infty$ and $\delta > 0$, we have $\tilde{\gamma}_{j,\delta}(\mathfrak{t}(q')) = \gamma_j(\mathfrak{t}(q')) + (\delta, 0) \to q'$. Therefore, there exist $j_0 \in \mathbb{N}$ and $\delta_0 > 0$ so that for all $j \geq j_0$ and $\delta \in (0, \delta_0]$, one can write $\tilde{\gamma}_{j,\delta}(\mathfrak{t}(q')) = \gamma_{q,\zeta_{j,\delta}}(1)$ where $\zeta_{j,\delta} \in T_qM$, $|\zeta_{j,\delta}|_{g^+} < 2\epsilon$, is past timelike still. Fix such $\delta > 0$. Then for sufficiently large j, the concatenation of $\tilde{\gamma}_{j,\delta}|_{[t_j,\mathfrak{t}(q')]}$ and $\gamma_{q,\zeta_{j,\delta}}(1-r)$, $r \in [0,1]$, and $\gamma|_{[s,\mathfrak{t}(q_0)]}$ is a piecewise smooth timelike curve connecting $(t_j + \delta, x_0)$ to q_0; this can be smoothed out to produce a *smooth* timelike curve from $(t_j + \delta e^{-\digamma(\mathfrak{t}(q')-t_j)}, x_0)$ to q_0. But as $j \to \infty$, we have $(t_j + \delta e^{-\digamma(\mathfrak{t}(q')-t_j)}, x_0) \to (\bar{\mathfrak{t}} + \delta e^{-\digamma(\mathfrak{t}(q')-\bar{\mathfrak{t}})}, x_0)$; since $\bar{\mathfrak{t}} + \delta e^{-\digamma(\mathfrak{t}(q')-\bar{\mathfrak{t}})} > \bar{\mathfrak{t}}$, we obtain a contradiction to the definition of $\bar{\mathfrak{t}}$.

We conclude that ζ must be past *lightlike*, and we re-define γ by replacing the segment $\gamma|_{[s',s]}$ with the null-geodesic segment $\gamma_{q,\zeta}(1-r)$, $r \in [0,1]$, reparameterized by $\mathfrak{t}$. Repeat this construction for s' in place of s: we obtain $s'' \in (s_0, s')$ (so $s_0 < s'' < s' < s$), and a re-definition of γ so that $\gamma'' = \gamma|_{[s'',s']}$ and $\gamma' = \gamma|_{[s',s]}$ are null-geodesics. Consider the piecewise smooth curve $\gamma|_{[s'',s]}$; if it had a break point at $\gamma(s')$, i.e. if $\dot{\gamma}(s'+0)$ was not a scalar multiple of $\dot{\gamma}(s'-0)$, then by a standard short cut argument [39, §10], we could construct, for all sufficiently small $\delta > 0$, a strictly timelike curve from any point in a δ-neighborhood (with respect to g^+) of $\gamma(s'')$ to $\gamma(s) = q_0$, and then an argument as in the previous paragraph would produce a contradiction to the definition of $\bar{\mathfrak{t}}$. Therefore, $\gamma|_{[s'',s]}$ is, in fact, a smooth reparameterized null-geodesic.

We can proceed in this fashion any finite number of times, starting at $s = \mathfrak{t}(q_0)$. (Note that we never reach a point on ∂M in finitely many steps, as all constructions take place inside of M^{int}.) Thus, we obtain $s_1 \in [s_0, s')$ and a re-definition of γ on $(s_1, \mathfrak{t}(q_0)]$ so that $\gamma|_{(s_1,\mathfrak{t}(q_0)]}$ is a

[2]We work here in the product splitting $(0,T) \times N$ of M, and write for a point $p = (t,x) \in M$ and a real number $a \in (-t, T-t)$: $p + (a,0) := (t+a, x)$. A more intrinsic (but notationally less convenient) perspective which avoids using the product splitting proceeds by instead using the exponential map based at p and evaluated at a times a fixed future timelike vector on M which is tangent to ∂M.

smooth reparameterized null-geodesic. In the case that $s_1 > s_0$, we can continue the construction at $\gamma(s_1)$. Therefore, we may in fact assume $s_1 = s_0$. In conclusion, the thus re-defined curve $\gamma|_{[s_0,\mathfrak{t}(q_0)]}$ is a null-geodesic joining $\gamma(s_0) \in (0,T) \times \partial N$ with $\gamma(\mathfrak{t}(q_0)) = q_0$; we define μ to be a smooth reparameterization of $\gamma|_{[s_0,\mathfrak{t}(q_0)]}$.

Finally, we argue that μ has no cut points. If this were false, we could replace a segment of μ by another null-geodesic segment so that the resulting curve was not smooth. But then we could again appeal to a short cut argument and obtain a contradiction to the definition of $\bar{t}$. This finishes the proof. □

Denote now by $\gamma\colon [0,1] \to M$ a null-geodesic joining a point $\gamma(0) \in (0,T) \times \partial N$ to q_0. Consider $v_1 := -\dot{\gamma}(1) \in L^-_{q_0}M$, and let $\mathbf{t}_1 > 0$ be such that $\gamma(0) = \gamma_{q_0,v_1}(\mathbf{t}_1)$. For small $\delta > 0$, we then have

$$z_1 := \gamma_{q_0,v_1}(\mathbf{t}_1 + \delta) \in (0,T) \times (N_1 \setminus N);$$

moreover, if we take $v_2, v_3, v_4 \in L^-_{q_0}M$ to be very close (with respect to g^+) to v_1 and so that v_1, v_2, v_3, v_4 are linearly independent, then

$$z_j := \gamma_{q_0,v_j}(\mathbf{t}_1 + \delta) \in (0,T) \times (N_1 \setminus N)$$

as well. As initial velocities of null-geodesics at z_j, we then take

$$\zeta_j := -\dot{\gamma}_{q_0,v_j}(\mathbf{t}_1 + \delta).$$

This construction guarantees that the γ_{z_j,ζ_j} intersect transversally at q_0, as desired; moreover, by taking (z_j, ζ_j) to be close enough to (z_1, ζ_1), they do not intersect prior to q_0, since otherwise γ_{z_1,ζ_1} would have to have a conjugate point on $[0, \mathbf{t}_1 + \delta]$, contradicting our assumption (for small enough δ).

As a particular consequence of our arguments thus far, we can determine the collection

$$\{\mathcal{E}_M(q_0)\colon q_0 \in \mathbb{U}_g \cap M^{\mathrm{int}}\}$$

of *earliest* boundary light observation sets. An application of [22, Theorem 1.2] then recovers $\mathbb{U}_g \cap M^{\mathrm{int}}$ as a smooth manifold, together with the conformal class of g. (In fact, the full strength of [22, Theorem 1.2] is not needed, as we are only keeping track of the earliest light observation sets, which suffice for the reconstruction procedure of the reference to work.)

We would like to improve this and recover the structure of $\mathbb{U}_g$ as a smooth manifold with boundary $\mathbb{U}_g \cap \partial M = (0,T) \times \partial N$. In order to accomplish this, we take advantage of the extended manifold $M_1 \supset (0,T) \times \partial N$ as follows. First of all, we can extend $\mathcal{E}_M(q_0)$ for $q_0 \in \mathbb{U}_g \cap M^{\mathrm{int}}$ to

$$\mathcal{E}_{M_1}(q_0) \subset M_1 \setminus M^{\mathrm{int}}$$

by taking the union of $\mathcal{E}_M(q_0)$ with $\gamma_{z,\zeta}([0,\delta))$ where $z \in \mathcal{E}_M(q_0)$, ζ is the unique (up to a positive scalar multiple) outward pointing null vector normal to $\mathcal{E}_M(q_0)$ at z (see [22, Lemma 2.5]), and $\delta > 0$ is arbitrary unless $\gamma_{z,\zeta}$ reenters $(0,T) \times N$, in which case we take $\delta = \mathbf{t}_1(z,\zeta)$. Thus, the intersection with the exterior of M, i.e. $\mathcal{E}_{M_1}(q_0) \cap M_1 \setminus M$, is a smooth null hypersurface in a sufficiently small neighborhood of any $z \in \mathcal{E}_M(q_0)$.

For $q_0 \in (0,T) \times \partial N$, we can similarly consider all tangent vectors $\zeta \in L^+_{q_0}M$ which point out of M, and take the union $\mathcal{E}_{M_1}(q_0)$ over all null-geodesic segments $\gamma_{q_0,\zeta}([0,\delta))$ with $\delta > 0$ defined in the same way as before.

In order to recover the smooth structure of $\mathbb{U}_g$ near $q_0 \in (0,T) \times \partial N$, we now follow the strategy of [22]. Namely, the earliest observation time along 4 generic timelike curves in $M_1 \setminus M$ defined in a neighborhood of q_0 provide smooth local coordinates in the intersection of a small neighborhood

$\mathcal{U} \subset M$ of q_0 with M^{int}; and they remain smooth down to $\mathcal{U} \cap \partial M$. This produces a smooth local coordinate system on $\mathcal{U}$, as desired. This completes the reconstruction of $\mathbb{U}_g$ as a smooth manifold and the conformal class of g on it.

Remark 3.4. An alternative procedure constructs the intersection with $M_1 \setminus M$ of the future light cones based at all points

$$q_0 \in \mathbb{U}_g \cup ((0,T) \times (N_1 \setminus N)); \tag{3.28}$$

the idea is that for a null-geodesic starting at q_0 which strikes $(0,T) \times \partial N$, we can determine (from the Dirichlet-to-Neumann map) the place and tangent vector on $(0,T) \times \partial N$ where it exits (if it exits at all), from where one can extend it using the metric g in $M_1 \setminus M$. Thus, we can construct the (earliest) light observation sets of all points (3.28) as measured in $M_1 \setminus M$; a straightforward application of the methods of [31] then reconstructs the set in (3.28) as a smooth manifold equipped with the conformal class of a Lorentzian metric.

3.6. **Determination of the conformal factor.** From now on, we just assume, without loss of generality, $g = e^{2\beta}\widetilde{g}$ with $\beta = 0$ on $M_1 \setminus M$; and we restrict to $\mathbb{U}_g = \mathbb{U}_{\widetilde{g}}$. Let Σ_g be the characteristic set, i.e.,

$$\Sigma_g = \{(x,\xi) \in T^*M_1 \colon |\xi|^2_{g(x)^{-1}} = 0\}.$$

Note that $\Sigma_g = \Sigma_{\widetilde{g}}$. Denote the Lagrangians Λ^g and $\Lambda^{\widetilde{g}}$ to be the flowouts of Σ_g under the Hamiltonians associated with g and $\widetilde{g}$. By [37, Proposition 4.5] we have $\Lambda^g = \Lambda^{\widetilde{g}}$ and the principal symbols of $\mathcal{Q}_g, \mathcal{Q}_{\widetilde{g}} \in \mathcal{I}^{-2}(N^*\mathrm{diag} \setminus \Lambda^g)$ satisfy

$$\sigma_p(\mathcal{Q}_g) = e^{2\beta}\sigma_p(\mathcal{Q}_{\widetilde{g}}).$$

The principal symbols in $\mathcal{I}^{-3/2}(\Lambda^g \setminus N^*\mathrm{diag})$ satisfy

$$\sigma_p(\mathcal{Q}_g)(x,\xi,y,\eta) = e^{-\beta(x)}\sigma_p(\mathcal{Q}_{\widetilde{g}})(x,\xi,y,\eta)e^{3\beta(y)},$$

where (y,η) is joined to (x,ξ) by a null-bicharacteristic on Λ^g. Therefore, the solutions v_i, resp. $\widetilde{v}_i$ of the linear initial-boundary value problems (3.19) with respect to g, resp. $\widetilde{g}$ with the same boundary source f_j satisfy

$$\sigma_p(v_i)(q_0,\theta_j) = e^{-\beta(q_0)}\sigma_p(\widetilde{v}_i)(q_0,\theta_j),$$

for $\theta_j \in N^*_{q_0}K_j$. By [37, Proposition 3.12], we have (cf. (3.25))

$$\sigma_p(\widetilde{\mathcal{U}}^{\mathrm{inc}})(y,\eta) = -\frac{1}{24}(2\pi)^{-3}\sigma_p(\mathcal{Q}_{\widetilde{g}})(y,\eta,q_0,\zeta^\flat)\widetilde{a}(q_0)\prod_{i=1}^4 \sigma_p(\widetilde{v}_i)(q_0,\theta_j)$$

and

$$\begin{aligned}
\sigma_p(\mathcal{U}^{\mathrm{inc}})(y,\eta) &= -\frac{1}{24}(2\pi)^{-3}\sigma_p(\mathcal{Q}_g)(y,\eta,q_0,\zeta^\flat)a(q_0)\prod_{i=1}^4 \sigma_p(v_i)(q_0,\theta_j) \\
&= -\frac{1}{24}(2\pi)^{-3}\sigma_p(\mathcal{Q}_{\widetilde{g}})(y,\eta,q_0,\zeta^\flat)e^{3\beta(q_0)}a(q_0)e^{-4\beta(q_0)}\prod_{i=1}^4 \sigma_p(\widetilde{v}_i)(q_0,\theta_j) \\
&= -\frac{1}{24}(2\pi)^{-3}\sigma_p(\mathcal{Q}_{\widetilde{g}})(y,\eta,q_0,\zeta^\flat)a(q_0)e^{-\beta(q_0)}\prod_{i=1}^4 \sigma_p(\widetilde{v}_i)(q_0,\theta_j).
\end{aligned}$$

Therefore $\sigma_p(\widetilde{\mathcal{U}}^{\mathrm{inc}})(y,\eta) = \sigma_p(\mathcal{U}^{\mathrm{inc}})(y,\eta)$ implies that $\widetilde{a}(q_0) = a(q_0)e^{-\beta(q_0)}$. If we assume a priori that $a(q_0) = \widetilde{a}(q_0) = c \neq 0$ for all q_0, then this implies $e^{-\beta(q_0)} = 1$, hence $\beta(q_0) = 0$ for all q_0.

This finishes the proof of a large part of Theorem 1.3; it only remains to prove that $\Box_g e^\beta = 0$, which we will do in the next section.

4. Reconstruction using Gaussian beams

In this section, we will prove:

Proposition 4.1. *Suppose we are given two smooth Lorentzian metrics $g, \widetilde{g}$ and two smooth functions $a, \widetilde{a}$ on M of the form (1.1), and assume that null-geodesics in $(\mathbb{U}_g, g)$, resp. $(\mathbb{U}_{\widetilde{g}}, \widetilde{g})$ have no conjugate points. Assume further that $\widetilde{g}$ and g are in the same conformal class, i.e., there exists a smooth function β on M such that $\widetilde{g} = e^{-2\beta} g$. If $\Lambda_{\widetilde{g},\widetilde{a}} = \Lambda_{g,a}$, then we have $\widetilde{a} = e^{-\beta} a$ and $\Box_g e^{-\beta} = 0$ in $\mathbb{U}_g$.*

By the boundary determination, we have $\beta|_{(0,T)\times\partial N} = \partial_\nu \beta|_{(0,T)\times\partial N} = 0$. Denote by $\Lambda_{g,q,a}$ the Dirichlet-to-Neumann map for the equation $\Box_g u + qu + au^4 = 0$. We start from the fact, shown in the proof of Lemma 1.2, that

$$\Lambda_{\widetilde{g},\widetilde{a}} = \Lambda_{g,q,e^\beta \widetilde{a}} = \Lambda_{g,0,a} = \Lambda_{g,a},$$

where $-q = e^\beta \Box_g e^{-\beta}$. (See also [42, Lemma 2].) Therefore, we only need to consider the equations

$$\Box_g \widetilde{v} + q\widetilde{v} + e^\beta \widetilde{a} \widetilde{v}^4 = 0,$$

and

$$\Box_g v + av^4 = 0,$$

on the same Lorentzian manifold (M, g). It is easy to see that Proposition 4.1 is then a direct consequence of the following lemma.

Lemma 4.2. *Assume that null-geodesics in $(\mathbb{U}_g, g)$ have no conjugate points. If $\Lambda_{g,q,e^\beta\widetilde{a}} = \Lambda_{g,0,a}$, then $q = 0$, $\widetilde{a} = e^{-\beta} a$ in $\mathbb{U}_g$.*

Instead of using distorted plane waves as in Section 3, we will use Gaussian beam solutions to prove Proposition 4.1. The Gaussian beams have also been used for various inverse problems for both elliptic an hyperbolic equations [24, 5, 6, 15, 18, 20, 21]. Note that the fact $\widetilde{a} = e^{-\beta} a$ has already been proved in previous section using distorted plane waves, and this section includes an alternative proof.

4.1. Construction of Gaussian beam solutions.

We will construct Gaussian beam solutions [41] for the *linear* equation

$$\Box_g \widetilde{u} + q\widetilde{u} = 0, \quad (\text{and} \quad \Box_g u = 0).$$

A Gaussian beam can be thought as a wave packet traveling along a null-geodesic γ. We first construct Gaussian beam solutions on the manifold M_1 without boundary. The construction can be done in Fermi coordinates in a neighborhood of γ. Assume that γ passes through a point $p \in M$ and joins two points $\gamma(\tau_-)$ and $\gamma(\tau_+)$ on the boundary $\mathbb{R} \times \partial N$. We will use Fermi coordinates Φ on the extended manifold M_1 in a neighborhood of $\gamma([\tau_-, \tau_+])$, denoted by $(z^0 := \tau, z^1, z^2, z^3)$, such that $\Phi(\gamma(\tau)) = (\tau, 0)$ (cf. [20, Lemma 1]).

The Gaussian beams are of the form

$$\widetilde{u}_\rho(x) = e^{\mathrm{i}\rho\varphi(x)} \widetilde{\mathfrak{a}}_\rho(x),$$

with

$$\varphi=\sum_{k=0}^{N}\varphi_k(\tau,z'),\quad \widetilde{\mathfrak{a}}_\rho(\tau,z')=\chi\left(\frac{|z'|}{\delta}\right)\sum_{k=0}^{N}\rho^{-k}\widetilde{a}_k(\tau,z'),\quad \widetilde{a}_k(\tau,z')=\sum_{j=0}^{N}\widetilde{a}_{k,j}(\tau,z') \tag{4.1}$$

in a neighborhood of γ,

$$\mathcal{V}=\big\{(\tau,z')\in M_1:\tau\in\big[\tau_- -\tfrac{\epsilon}{\sqrt{2}},\tau_+ +\tfrac{\epsilon}{\sqrt{2}}\big],\,|z'|<\delta\big\}. \tag{4.2}$$

Here for each j, φ_j and $\widetilde{a}_{k,j}$ are a complex valued homogeneous polynomials of degree j with respect to the variables z^i, $i=1,2,3$, and $\delta>0$ is a small parameter. The smooth function $\chi:\mathbb{R}\to[0,+\infty)$ satisfies $\chi(t)=1$ for $|t|\le\frac{1}{4}$ and $\chi(t)=0$ for $|t|\ge\frac{1}{2}$.

By calculation, one can verify that

$$\begin{aligned}(\Box_g+q)(\widetilde{\mathfrak{a}}_\rho e^{\mathrm{i}\rho\varphi})&=e^{\mathrm{i}\rho\varphi}(\rho^2(\mathcal{S}\varphi)\widetilde{\mathfrak{a}}_\rho-\mathrm{i}\rho\mathcal{T}\widetilde{\mathfrak{a}}_\rho+(\Box_g+q)\widetilde{\mathfrak{a}}_\rho),\\ \mathcal{S}\varphi&=\langle\mathrm{d}\varphi,\mathrm{d}\varphi\rangle_g,\\ \mathcal{T}\widetilde{a}&=2\langle\mathrm{d}\varphi,\mathrm{d}\widetilde{a}\rangle_g-(\Box_g\varphi)\widetilde{a}.\end{aligned} \tag{4.3}$$

We construct the phase φ and the amplitude $\widetilde{\mathfrak{a}}_\rho$ such that

$$\frac{\partial^\Theta}{\partial z^\Theta}(\mathcal{S}\varphi)(\tau,0)=0, \tag{4.4}$$

$$\frac{\partial^\Theta}{\partial z^\Theta}(\mathcal{T}\widetilde{a}_0)(\tau,0)=0, \tag{4.5}$$

$$\frac{\partial^\Theta}{\partial z^\Theta}(-\mathrm{i}\mathcal{T}\widetilde{a}_k+(\Box_g+q)\widetilde{a}_{k-1})(\tau,0)=0 \tag{4.6}$$

for $\Theta=(\Theta_0=0,\Theta_1,\Theta_2,\Theta_3)$ with $|\Theta|\le N$. For more details we refer to [20]. For the construction of the phase function φ, we can take

$$\varphi_0=0,\quad \varphi_1=z^1,\quad \varphi_2(\tau,z)=\sum_{1\le i,j\le 3}H_{ij}(\tau)z^iz^j.$$

Here H is a symmetric matrix with $\Im H(\tau)>0$; the matrix H satisfies a Riccati ODE,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}H+HCH+D=0,\quad \tau\in\big(\tau_- -\tfrac{\epsilon}{2},\tau_+ +\tfrac{\epsilon}{2}\big),\quad H(0)=H_0,\ \text{with } \Im H_0>0, \tag{4.7}$$

where C, D are matrices with $C_{11}=0$, $C_{ii}=2$, $i=2,3$, $C_{ij}=0$, $i\ne j$ and $D_{ij}=\frac{1}{4}(\partial^2_{ij}g^{11})$.

Lemma 4.3 ([18, Lemma 3.2])**.** *The Riccati equation (4.7) has a unique solution. Moreover the solution H is symmetric and $\Im(H(\tau))>0$ for all $\tau\in(\tau_- -\frac{\delta}{2},\tau_+ +\frac{\delta}{2})$. For solving the above Riccati equation, one has $H(\tau)=Z(\tau)Y(\tau)^{-1}$, where $Y(\tau)$ and $Z(\tau)$ solve the ODEs*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau)&=CZ(\tau), & Y(\tau_-)&=Y_0,\\ \frac{\mathrm{d}}{\mathrm{d}\tau}Z(\tau)&=-D(\tau)Y(\tau), & Z(\tau_-)&=Y_1=H_0Y_0.\end{aligned}$$

In addition, $Y(\tau)$ is nondegenerate.

Lemma 4.4 ([18, Lemma 3.3])**.** *The following identity holds:*

$$\det(\Im(H(\tau))|\det(Y(\tau))|^2=c_0$$

with c_0 independent of τ.

We see that the matrix $Y(\tau)$ satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}Y + CDY = 0, \quad Y(\tau_-) = Y_0, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau_-) = CY_1. \tag{4.8}$$

As in [20], we have the following estimate by the construction of $\widetilde{u}_\rho$

$$(\|\Box_g + q)\widetilde{u}_\rho\|_{H^k(M)} \leq C\rho^{-K}, \qquad K = \frac{N+1-k}{2} - 1. \tag{4.9}$$

For the amplitude, we first notice that the principal term $\widetilde{a}_0$, which satisfies (4.5), is independent of q. Setting $|\Theta| = 0$ in (4.5), one can get the equation for $\widetilde{a}_{0,0}$

$$2\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{a}_{0,0} + \mathrm{Tr}(CH)\widetilde{a}_{0,0} = 0.$$

By Lemma 4.3 and Lemma 4.4, we have

$$\mathrm{Tr}(CH) = \frac{\mathrm{d}}{\mathrm{d}\tau}\log\det Y.$$

Therefore, we can take

$$\widetilde{a}_{0,0}(\tau, 0) = \det(Y(\tau))^{-\frac{1}{2}}.$$

For more details, we refer to [20]. The terms $\widetilde{a}_{0,k}$, $j = 1, 2, \cdots, N$ can be constructed successively by solving linear first order ODEs resulted from (4.5). This finishes the construction of $\widetilde{a}_0$.

For the construction of $\widetilde{a}_1$, we use equation (4.6). Set $k = 1$ and $|\Theta| = 0$, then

$$2\frac{\partial}{\partial\tau}\widetilde{a}_{1,0}(\tau, 0) + \mathrm{Tr}(CH)\widetilde{a}_{1,0}(\tau, 0) = -\mathrm{i}(\Box_g + q)\widetilde{a}_0(\tau, 0). \tag{4.10}$$

Therefore we can take

$$\widetilde{a}_{1,0}(\tau, 0) = -\frac{\mathrm{i}}{2}\det(Y(\tau))^{-\frac{1}{2}}\int_{\tau_-}^{\tau}\left[(\Box_g\widetilde{a}_0)(\tau', 0) + q(\tau', 0))\det(Y(\tau')^{\frac{1}{2}}\right]\mathrm{d}\tau'.$$

Notice that $\widetilde{a}_{1,0}(\tau_-, 0) = 0$, and $\frac{\partial}{\partial\tau}\widetilde{a}_{1,0}(\tau_-, 0)$ depends on q only via $q(\tau_-, 0)$. Inductively one can prove that $\frac{\partial^j}{\partial\tau^j}\widetilde{a}_{1,0}(\tau_-, 0)$ depends on q only via $\frac{\partial^k}{\partial\tau^k}q(\tau_-, 0)$ for $k = 0, 1, \ldots, j$ for each $j \in \mathbb{N}$ by (4.10). For any $|\Theta| = k$, $1 \leq k \leq N$, we have

$$2\frac{\partial}{\partial\tau}\frac{\partial^\Theta}{\partial z^\Theta}\widetilde{a}_{1,k}(\tau, 0) + \mathrm{Tr}(CH)\frac{\partial^\Theta}{\partial z^\Theta}\widetilde{a}_{1,k}(\tau, 0) = \mathcal{E}_k, \tag{4.11}$$

where $\mathcal{E}_k$ depends on $g, q, \varphi, \widetilde{a}_0$ along with their derivatives on γ, and $\frac{\partial^\alpha}{\partial z^\alpha}\widetilde{a}_1(\tau, 0), |\alpha| \leq k-1$. We can take

$$\frac{\partial^\Theta}{\partial z^\Theta}\widetilde{a}_{1,k}(\tau_-, 0) = 0,$$

and solve the equation (4.11) for $\frac{\partial^\Theta}{\partial z^\Theta}\widetilde{a}_{1,k}(\tau, 0)$. This determines $\widetilde{a}_1$. By a similar argument as above, we conclude that $\frac{\partial^j}{\partial\tau^j}\frac{\partial^\Theta}{\partial z^\Theta}\widetilde{a}_1(\tau_-, 0)$ depends on q only via the jets of q at $(\tau_-, 0)$, for any $j \in \mathbb{N}$. The functions $\widetilde{a}_2, \ldots, \widetilde{a}_N$ can be determined in a similar way. This completes the construction of the Gaussian beam solution $\widetilde{u}_\rho$.

Construction of Gaussian beams with reflections at the boundary. Next, we take into account the reflections of Gaussian beams at the boundary ∂M. Now assume $\gamma : [\mathbf{t}_0, T]$ is a broken null-geodesic with $\mathbf{t}_1 < \mathbf{t}_2 < \cdots < \mathbf{t}_N$, where $\mathbf{t}_j \in (\mathbf{t}_0, T)$ for each $j = 1, \cdots, N$, the times of reflections at the boundary. Also assume that $\gamma(\mathbf{t}_0) \in \partial M$ and $\gamma(T) \notin \partial M$. We will construct Gaussian beam solutions concentrating near γ. Gaussian beams with reflections on Riemannian

manifolds are constructed in [25]. We only give details about how to deal with the first reflection at time $\mathbf{t}_1$, and all the subsequent reflections can be dealt in the same way. Consider Gaussian beam solutions of the form

$$\widetilde{u}_\rho = \widetilde{u}_\rho^{\mathrm{inc}} + \widetilde{u}_\rho^{\mathrm{ref}},$$

where $\widetilde{u}_\rho^{\mathrm{inc}}$ is the Gaussian beam solution associated with the geodesic $\gamma|_{[\mathbf{t}_0,\mathbf{t}_1]}$, and $\widetilde{u}_\rho^{\mathrm{ref}}$ is the Gaussian beam solution associated with the geodesic $\gamma|_{[\mathbf{t}_1,\mathbf{t}_2]}$. We will construct $\widetilde{u}_\rho^{\mathrm{inc}}$ and $\widetilde{u}_\rho^{\mathrm{ref}}$ such that $(\widetilde{u}_\rho^{\mathrm{inc}} + \widetilde{u}_\rho^{\mathrm{ref}})|_{\partial M}$ is small near $\gamma(\mathbf{t}_1)$. Here we denote

$$\widetilde{u}_\rho^{\mathrm{inc}} = e^{\mathrm{i}\rho\varphi^{\mathrm{inc}}}\widetilde{\mathfrak{a}}_\rho^{\mathrm{inc}}, \qquad \widetilde{u}_\rho^{\mathrm{ref}} = e^{\mathrm{i}\rho\varphi^{\mathrm{ref}}}\widetilde{\mathfrak{a}}_\rho^{\mathrm{ref}},$$

both of the form (4.1). We can take $\varphi^{\mathrm{ref}}|_{\partial M} = \varphi^{\mathrm{inc}}|_{\partial M}$ up to N-th order at $\gamma(\mathbf{t}_1) \in \partial M$. In particular, we have

$$\mathrm{d}(\varphi^{\mathrm{inc}}|_{\partial M})|_{\gamma(\mathbf{t}_1)} = \dot{\gamma}(\mathbf{t}_1-)^\flat|_{T\partial M},$$
$$\mathrm{d}(\varphi^{\mathrm{ref}}|_{\partial M})|_{\gamma(\mathbf{t}_1)} = \dot{\gamma}(\mathbf{t}_1+)^\flat|_{T\partial M}.$$

We can also choose $\widetilde{\mathfrak{a}}^{\mathrm{ref}}|_{\partial M} = -\widetilde{\mathfrak{a}}^{\mathrm{inc}}|_{\partial M}$ up to N-th order at $\gamma(\mathbf{t}_1)$.

Let R_1 be a small neighborhood of $\gamma(\mathbf{t}_1)$ on ∂M such that $\widetilde{u}_\rho^{\mathrm{inc}}$ and $\widetilde{u}_\rho^{\mathrm{ref}}$ are compactly supported in R_1. Let (y, x_n) be the coordinates near $\gamma(\mathbf{t}_1)$ such that ∂M is parametrized by $y \mapsto (y, x_n)$. First notice that on R_1,

$$|e^{\mathrm{i}\rho\varphi^\bullet}| \le e^{-C'\rho|y|^2}.$$

By above considerations, we have

$$|\varphi^{\mathrm{ref}} - \varphi^{\mathrm{inc}}| \le C|y|^{N+1},$$

and

$$|\widetilde{\mathfrak{a}}_\rho^{\mathrm{ref}} + \widetilde{\mathfrak{a}}_\rho^{\mathrm{inc}}| \le C|y|^{N+1}$$

on R_1. We can write $\widetilde{u}_\rho$ as

$$\widetilde{u}_\rho = (e^{\mathrm{i}\rho\varphi^{\mathrm{inc}}} - e^{\mathrm{i}\rho\varphi^{\mathrm{ref}}})\widetilde{\mathfrak{a}}_\rho^{\mathrm{inc}} + e^{\mathrm{i}\rho\varphi^{\mathrm{ref}}}(\widetilde{\mathfrak{a}}_\rho^{\mathrm{ref}} + \widetilde{\mathfrak{a}}_\rho^{\mathrm{inc}}).$$

We have

$$e^{\mathrm{i}\rho\varphi^{\mathrm{inc}}} - e^{\mathrm{i}\rho\varphi^{\mathrm{ref}}} = \mathrm{i}\rho(\varphi^{\mathrm{inc}} - \varphi^{\mathrm{ref}}) \int_0^1 e^{\mathrm{i}\rho(s\varphi^{\mathrm{inc}}+(1-s)\varphi^{\mathrm{ref}})}\mathrm{d}s$$

and consequently

$$|e^{\mathrm{i}\rho\varphi^{\mathrm{inc}}} - e^{\mathrm{i}\rho\varphi^{\mathrm{ref}}}| \le C\rho|y|^{N+1}e^{-C'\rho|y|^2}.$$

Thus we obtain that

$$|\widetilde{u}_\rho|_{R_1}| \le C\rho|y|^{N+1}e^{-C'\rho|y|^2}.$$

Taking the first derivative of $\widetilde{u}_\rho$ in y, we have

$$D_y\widetilde{u}_\rho = \mathrm{i}\rho e^{\mathrm{i}\rho\varphi^{\mathrm{inc}}}D_y\varphi^{\mathrm{inc}}\widetilde{\mathfrak{a}}_\rho^{\mathrm{inc}} + e^{\mathrm{i}\rho\varphi^{\mathrm{inc}}}D_y\widetilde{\mathfrak{a}}_\rho^{\mathrm{inc}} + \mathrm{i}\rho e^{\mathrm{i}\rho\varphi^{\mathrm{ref}}}D_y\varphi^{\mathrm{ref}}\widetilde{\mathfrak{a}}_\rho^{\mathrm{ref}} + e^{\mathrm{i}\rho\varphi^{\mathrm{ref}}}D_y\widetilde{\mathfrak{a}}_\rho^{\mathrm{ref}}.$$

Notice that $|D_y\varphi^{\mathrm{inc}}| \le C|y|$ on R_1, we obtain the estimate

$$|D_y\widetilde{u}_\rho|_{R_1}| \le C\rho|y|^{N+2}e^{-C'\rho|y|^2} + C|y|^N e^{-C'\rho|y|^2}.$$

Inductively, one can show that

$$\left|\partial_y^\alpha\widetilde{u}_\rho|_{R_1}\right| \le C\sum_{k+j=|\alpha|}\rho^k|y|^k e^{-C'\rho|y|^2}|y|^{N+1-j}.$$

By change of variables $y = \rho^{-1/2} y'$, we obtain

$$\int_{R_1} |\partial_y^\alpha \widetilde{u}_\rho|^2 \mathrm{d}S\mathrm{d}t \leq C\rho^{-(N-|\alpha|+1)-\frac{3}{2}},$$

and therefore

$$\|\widetilde{u}_\rho\|_{H^k(R_1)} \leq C\rho^{-\frac{N-k+1}{2}-\frac{3}{4}}. \tag{4.12}$$

We can also construct the Gaussian beam solution associated with the same broken null-geodesic γ for the equation $\Box_g u = 0$ by just setting $q = 0$ in the above process. Considering only the reflection at $\gamma(\mathbf{t}_1)$, we denote the solution to be

$$u_\rho = u_\rho^{\mathrm{inc}} + u_\rho^{\mathrm{ref}}.$$

Since the incidence waves are what will be used explicitly, we shall denote

$$\widetilde{u}_\rho^{\mathrm{inc}}(x) = e^{\mathrm{i}\rho\varphi^{\mathrm{inc}}(x)}\widetilde{\mathfrak{a}}_\rho^{\mathrm{inc}}(x), \quad u_\rho^{\mathrm{inc}}(x) = e^{\mathrm{i}\rho\varphi^{\mathrm{inc}}(x)}\mathfrak{a}_\rho^{\mathrm{inc}}(x),$$

with

$$\widetilde{\mathfrak{a}}_\rho^{\mathrm{inc}}(\tau, z') = \chi\left(\frac{|z'|}{\delta}\right)\sum_{k=0}^N \rho^{-k}\widetilde{a}_k(\tau, z'), \quad \widetilde{a}_k(\tau, z') = \sum_{j=0}^N \widetilde{a}_{k,j}(\tau, z').$$

$$\mathfrak{a}_\rho^{\mathrm{inc}}(\tau, z') = \chi\left(\frac{|z'|}{\delta}\right)\sum_{k=0}^N \rho^{-k}a_k(\tau, z'), \quad a_k(\tau, z') = \sum_{j=0}^N a_{k,j}(\tau, z').$$

In above notations we have used the fact that the phase function φ^{inc}, since independent of q, is the same for $\widetilde{u}_\rho^{\mathrm{inc}}$ and u_ρ^{inc}. We remark here that the following estimate, analogous to (4.12), also holds for u_ρ,

$$\|u_\rho\|_{H^k(R_1)} \leq C\rho^{-\frac{N-k+1}{2}-\frac{3}{4}}. \tag{4.13}$$

Also we have $\widetilde{a}_0 = a_0$, as they are also independent of q. Now we have

$$a_1(\tau, 0) = a_{1,0}(\tau, 0) = -\frac{\mathrm{i}}{2}\det(Y(\tau))^{-\frac{1}{2}}\int_{\mathbf{t}_0}^\tau \left[(\Box_g a_0)(\tau', 0)\det(Y(\tau'))^{\frac{1}{2}}\right]\mathrm{d}\tau'.$$

Recall that the jet of q vanishes at $\gamma(\mathbf{t}_0) \in \partial M$. By above discussion, $(\widetilde{\mathfrak{a}}_\rho - \mathfrak{a}_\rho)|_{\partial M}$ vanishes at $\gamma(\mathbf{t}_0)$ up to N-th order. Then we have

$$\left|\partial_y^\alpha(\widetilde{u}_\rho - u_\rho)\right| \leq C\sum_{k+j=|\alpha|}\rho^k|y|^k e^{-C'\rho|y|^2}|y|^{N+1-j}.$$

Similar as (4.12) and (4.13), we have

$$\|\widetilde{u}_\rho - u_\rho\|_{H^k(R_0)} \leq C\rho^{-\frac{N-k+1}{2}-\frac{3}{4}}, \tag{4.14}$$

where R_0 is a small neighborhood of $\gamma(\mathbf{t}_0)$ on ∂M, and $\widetilde{u}_\rho|_{\partial M}, u_\rho|_{\partial M}$ are supported in R_0. Combining the estimates (4.12), (4.13), and (4.14), we finally get

$$\|\widetilde{u}_\rho - u_\rho\|_{H^k(\partial M)} \leq C\rho^{-\frac{N-k+1}{2}-\frac{3}{4}}.$$

In conclusion, for any $K \geq 0$, we can take N large enough such that

$$\|\widetilde{u}_\rho - u_\rho\|_{H^k(\partial M)} \leq C\rho^{-K},$$

which is valid with an arbitrary number of reflections of γ.

4.2. **Proof of Lemma 4.2.** For any point $x \in \partial M$, and an inward pointing null-vector $\xi \in L_x^+ M_1$, consider the broken null-geodesic $\gamma = \gamma_{x,\xi}$ starting from (x, ξ). We assume that $\gamma(T) \notin \partial M$. By the discussions above, we can construct boundary sources $f_{\rho,x,\xi}$ such that the solutions $\widetilde{u}$, u to the equations

$$\begin{aligned} (\Box_g + q)\widetilde{u} &= 0, && \text{on } M, \\ \widetilde{u} &= f_{\rho,x,\xi}(x), && \text{on } \partial M, \\ \widetilde{u}(\mathfrak{t}, x') &= 0, && \mathfrak{t} < 0, \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} \Box_g u &= 0, && \text{on } M, \\ u &= f_{\rho,x,\xi}(x), && \text{on } \partial M, \\ u(\mathfrak{t}, x') &= 0, && \mathfrak{t} < 0, \end{aligned} \tag{4.16}$$

satisfy

$$\widetilde{u} = \widetilde{u}_\rho + \widetilde{R}_\rho, \qquad u = u_\rho + R_\rho,$$

where $\widetilde{u}_\rho, u_\rho$ are the Gaussian beam solutions associated with the broken null-geodesic $\gamma_{x,\xi}$ constructed above, and

$$\|f_{\rho,x,\xi}(x) - \widetilde{u}_\rho\|_{H^k(\partial M)} = \mathcal{O}(\rho^{-K}), \quad \|f_{\rho,x,\xi}(x) - u_\rho\|_{H^k(\partial M)} = \mathcal{O}(\rho^{-K}). \tag{4.17}$$

For example, one can take $f_{\rho,x,\xi} = \widetilde{u}_\rho|_{\partial M}$ or $f_{\rho,x,\xi} = u_\rho|_{\partial M}$, and let the remainder terms be the solutions to the equations

$$\begin{aligned} (\Box_g + q)\widetilde{R}_\rho &= -(\Box_g + q)\widetilde{u}_\rho, && \text{on } M, \\ \widetilde{R}_\rho &= f_{\rho,x,\xi}(x) - \widetilde{u}_\rho, && \text{on } \partial M, \\ \widetilde{R}_\rho(\mathfrak{t}, x') &= 0, && \mathfrak{t} < 0, \end{aligned}$$

and

$$\begin{aligned} \Box_g R_\rho &= -(\Box_g + q)u_\rho, && \text{on } M, \\ R_\rho &= f_{\rho,x,\xi}(x) - u_\rho(x), && \text{on } \partial M, \\ R_\rho(\mathfrak{t}, x') &= 0, && \mathfrak{t} < 0, \end{aligned}$$

The parameter δ in (4.1) can be taken small enough to ensure that $\widetilde{u}_\rho = 0$, $u_\rho = 0$ near $\{t = 0\}$. By (4.9), we have

$$\|(\Box_g + q)\widetilde{u}_\rho\|_{H^k(M)} = \mathcal{O}(\rho^{-K}), \quad \|\Box_g u_\rho\|_{H^k(M)} = \mathcal{O}(\rho^{-K}). \tag{4.18}$$

By (4.17) and (4.18), we have

$$\|\widetilde{R}_\rho\|_{H^{k+1}(M)}, \ \|R_\rho\|_{H^{k+1}(M)} \le C\rho^{-K}.$$

Fix a point $q_0 \in \mathbb{U}_g$. There exist $\theta_0, \theta_1 \in L_{q_0}^{*,+} M$ and real numbers $s^-(q_0, \theta_1) < 0 < s^+(q_0, \theta_0)$ such that (cf. [23] or the arguments in §3.5)

$$x_1 = \gamma_{q_0,\theta_1^\sharp}(s^-(q_0, \theta_1)) \in (0, T) \times \partial N, \quad x_0 = \gamma_{q_0,\theta_0^\sharp}(s^+(q_0, \theta_0)) \in (0, T) \times \partial N, \tag{4.19}$$

Denote

$$\begin{aligned} \xi_1 &= \dot{\gamma}_{q_0,\theta_1^\sharp}(s^-(q_0, \theta_1)) \in L_{x_1}^+ M_1, \\ \xi_0 &= -\dot{\gamma}_{q_0,\theta_0^\sharp}(s^+(q_0, \theta_0)) \in L_{x_0}^- M_1. \end{aligned}$$

We can choose θ_0, θ_1 such that the two broken null geodesics γ_{x_1,ξ_1} (starting from x_1 going forward in $\mathfrak{t}$) and γ_{x_0,ξ_0} (starting from x_0 going backward in $\mathfrak{t}$) intersect only at q_0. Indeed, we claim that

may arrange this by choosing x_0, x_1 as in the proof of Lemma 3.3 and using a small perturbation argument: the lemma already shows that they cannot intersect once more along γ_{x_1,ξ_1} between x_1 and q_0 unless they are the same curves. This situation however can be avoided by perturbing θ_0 slightly. For q_0 in a dense subset of $\mathbb{U}_g$, we can guarantee that $\gamma_{x_1,\xi_1}(T), \gamma_{x_0,\xi_0}(0) \notin \partial M$.

Choose local coordinates so that g coincides with the Minkowski metric at q_0. Without loss of generality, one can then assume

$$\theta_0 = (-1, \pm\sqrt{1-r_0^2}, r_0, 0), \qquad \theta_1 = (-1, 1, 0, 0),$$

for some $r_0 \in [-1, 1]$ but with $\theta_0 \neq \theta_1$. Take a small parameter $\varsigma > 0$ and introduce two perturbations of $\xi^{(1)}$

$$\theta_2 = (-1, \sqrt{1-\varsigma^2}, \varsigma, 0), \qquad \theta_3 = (-1, \sqrt{1-\varsigma^2}, -\varsigma, 0).$$

Thus, $\theta_2, \theta_3 \in L^{*,+}_{q_0} M_1$. One can then write θ_0 as a linear combination of $\theta_1, \theta_2, \theta_3$,

$$\theta_0 = \alpha_1\theta_1 + \alpha_2\theta_2 + \alpha_3\theta_3,$$

with

$$\alpha_1 = \frac{-\sqrt{1-\varsigma^2} \pm \sqrt{1-r_0^2}}{1-\sqrt{1-\varsigma^2}}, \quad \alpha_2 = \frac{1 \mp \sqrt{1-r_0^2}}{2(1-\sqrt{1-\varsigma^2})} + \frac{r_0}{2\varsigma}, \quad \alpha_3 = \frac{1 \mp \sqrt{1-r_0^2}}{2(1-\sqrt{1-\varsigma^2})} - \frac{r_0}{2\varsigma}.$$

We refer to [23] for more details. Therefore, for $\kappa_0 = 1$ and $\kappa_j = -\alpha_j$, $j = 1, 2, 3$, we have

$$\kappa_0\theta_0 + \kappa_1\theta_1 + \kappa_2\theta_2 + \kappa_3\theta_3 = 0. \tag{4.20}$$

By possibly perturbing θ_0, we can assume that θ_0 and θ_1 are linearly independent. Then for $j = 1, 2, 3$, $\kappa_j \neq 0$ and

$$\lim_{\varsigma\to 0} |\kappa_j| = +\infty.$$

For $j = 2, 3$, denote

$$x_j = \gamma_{q_0,\theta_j}(s^-(q_0, \theta_j)),$$
$$\xi_j = \dot\gamma_{q_0,\theta_j}(s^-(q_0, \theta_j)) \in L^+_{x_j} M_1.$$

Note that we can choose ς small enough so that $x_j \in (0, T) \times \partial N$ for $j = 2, 3$.

Now denote $\gamma^{(j)} = \gamma_{x_j,\xi_j}$ for $j = 0, 1, 2, 3$, to be the broken null-geodesics with $\gamma^{(j)}(0) = x_j$, $\dot\gamma^{(j)}(0) = \xi_j$ and $\gamma^{(j)}(s_j) = q_0$ for some $s_j > 0$. For $j = 1, 2, 3$, we define $\widetilde{u}^{(j)}_\rho, u^{(j)}_\rho$ to be the Gaussian beam solutions which, before the first reflection, are of the form

$$\widetilde{u}^{(1)}_\rho = e^{\mathrm{i}\kappa_1\rho\varphi^{(1)}}\widetilde{\mathfrak{a}}^{(1)}_{\kappa_1\rho}, \quad \widetilde{u}^{(2)}_\rho = e^{\mathrm{i}\kappa_2\rho\varphi^{(2)}}\widetilde{\mathfrak{a}}^{(2)}_{\kappa_2\rho}, \quad \widetilde{u}^{(3)}_\rho = e^{\frac{\mathrm{i}}{2}\kappa_3\rho\varphi^{(3)}}\widetilde{\mathfrak{a}}^{(3)}_{\kappa_3\rho/2};$$
$$u^{(1)}_\rho = e^{\mathrm{i}\kappa_1\rho\varphi^{(1)}}\mathfrak{a}^{(1)}_{\kappa_1\rho}, \quad u^{(2)}_\rho = e^{\mathrm{i}\kappa_2\rho\varphi^{(2)}}\mathfrak{a}^{(2)}_{\kappa_2\rho}, \quad u^{(3)}_\rho = e^{\frac{\mathrm{i}}{2}\kappa_3\rho\varphi^{(3)}}\mathfrak{a}^{(3)}_{\kappa_3\rho/2}.$$

Notice the $\frac{1}{2}$-factor which we put into the definition of $\widetilde{u}^{(3)}_\rho$ and $u^{(3)}_\rho$. For $j = 1, 2, 3$, we can construct $\widetilde{u}^{(j)} = \widetilde{u}^{(j)}_\rho + \widetilde{R}^{(j)}_\rho$ and $u^{(j)} = u^{(j)}_\rho + R^{(j)}_\rho$ to be the solutions to (4.15) and (4.16) with $f_{\rho,x,\xi}$ taken to be $f^{(j)} := f_{\rho,x_j,\xi_j} = u^{(j)}_\rho|_{\partial M}$. We emphasize that $\widetilde{u}^{(j)}$ and $u^{(j)}$ satisfy the same boundary Dirichlet boundary conditions $\widetilde{u}^{(j)}|_{\partial M} = u^{(j)}|_{\partial M} = f^{(j)}$.

Similarly, we can construct Gaussian beams of the form

$$\widetilde{u}^{(0)}_\rho = e^{\mathrm{i}\kappa_0\rho\varphi^{(0)}}\widetilde{\mathfrak{a}}^{(0)}_{\kappa_0\rho}, \quad u^{(0)}_\rho = e^{\mathrm{i}\kappa_0\rho\varphi^{(0)}}\mathfrak{a}^{(0)}_{\kappa_0\rho},$$

before the first reflection, concentrating near $\gamma^{(0)}$. Then we can construct remainder terms $\widetilde{R}_\rho^{(0)}$, $R_\rho^{(0)}$ such that $\widetilde{u}^{(0)} = \widetilde{u}_\rho^{(0)} + \widetilde{R}_\rho^{(0)}$ and $u^{(0)} = u_\rho^{(0)} + R_\rho^{(0)}$ satisfy the backward wave equations

$$\begin{aligned} (\Box_g + q)\widetilde{u}^{(0)} &= 0, && \text{on } M, \\ \widetilde{u}^{(0)} &= f^{(0)}(x), && \text{on } \partial M, \\ \widetilde{u}^{(0)}(\mathfrak{t}, x') &= 0, && \mathfrak{t} > T, \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} \Box_g u^{(0)} &= 0, && \text{on } M, \\ u^{(0)} &= f^{(0)}(x), && \text{on } \partial M, \\ u^{(0)}(\mathfrak{t}, x') &= 0, && \mathfrak{t} > T. \end{aligned} \tag{4.22}$$

Here $f^{(0)} = u_\rho^{(0)}|_{\partial M}$, and δ is taken to be small enough such that $u_\rho^{(0)} = 0$ near $\{\mathfrak{t} = T\}$.

We are now in a position to prove Proposition 4.1. Notice that a fourth order linearization of the DN maps $\Lambda_{g,q,e^\beta\widetilde{a}}(f) = \Lambda_{g,0,a}(f)$, with $f = \epsilon_1 f^{(1)} + \epsilon_2 f^{(2)} + \epsilon_3 f^{(3)} + \epsilon_4 f^{(3)}$ (notice the repetition of $f^{(3)}$), leads to the integral identity

$$\begin{aligned} &\int_0^T \int_{\partial N} \frac{\partial^4}{\partial\epsilon_1\partial\epsilon_2\partial\epsilon_3\partial\epsilon_4} \Lambda_{g,q,e^\beta\widetilde{a}}(\epsilon_1 f^{(1)} + \epsilon_2 f^{(2)} + \epsilon_3 f^{(3)} + \epsilon_4 f^{(3)}) f^{(0)} \mathrm{d}S\mathrm{d}t \\ &= \int_0^T \int_{\partial N} \frac{\partial^4}{\partial\epsilon_1\partial\epsilon_2\partial\epsilon_3\partial\epsilon_4} \Lambda_{g,0,a}(\epsilon_1 f^{(1)} + \epsilon_2 f^{(2)} + \epsilon_3 f^{(3)} + \epsilon_4 f^{(3)}) f^{(0)} \mathrm{d}S\mathrm{d}t. \end{aligned}$$

and consequently

$$\widetilde{\mathcal{I}} := \int_M e^\beta \widetilde{a} \widetilde{u}^{(1)} \widetilde{u}^{(2)} (\widetilde{u}^{(3)})^2 \widetilde{u}^{(0)} \mathrm{d}V_g = \int_M a u^{(1)} u^{(2)} (u^{(3)})^2 u^{(0)} \mathrm{d}V_g =: \mathcal{I}, \tag{4.23}$$

via integration by parts. Observe that

$$\begin{aligned} \rho^2 \widetilde{\mathcal{I}} = \rho^2 \int_M & e^\beta \widetilde{a} e^{\mathrm{i}\rho S} \chi_1 \chi_2 \chi_3^2 \chi_0 (a_0^{(1)} + \rho^{-1}\kappa_1^{-1}\widetilde{a}_1^{(1)})(a_0^{(2)} + \rho^{-1}\kappa_2^{-1}\widetilde{a}_1^{(2)})(a_0^{(3)} + 2\rho^{-1}\kappa_3^{-1}\widetilde{a}_1^{(3)})^2 \\ & \times (a_0^{(0)} + \rho^{-1}\kappa_0^{-1}\widetilde{a}_1^{(0)}) \mathrm{d}V_g + \mathcal{O}(\rho^{-2}). \end{aligned} \tag{4.24}$$

Here χ_j is the cutoff function associated with $\widetilde{u}_\rho^{(j)}(u_\rho^{(j)})$ as in (4.1). The multiplication by ρ^2, resp. the $\mathcal{O}(\rho^{-2})$ remainder estimate are motivated by, resp. follow from the fact that $\int_M |e^{\mathrm{i}\rho S}| \mathrm{d}V_g = \mathcal{O}(\rho^{-2})$ and stationary phase, see (4.26) below. Here, we use that the phase function

$$S := \kappa_0 \varphi^{(0)} + \kappa_1 \varphi^{(1)} + \kappa_2 \varphi^{(2)} + \kappa_3 \varphi^{(3)},$$

which satisfies the following properties:

Lemma 4.5 ([20, Lemma 5])**.** *The function S is well-defined in a neighborhood of q_0 and*

(1) $S(q_0) = 0$;
(2) $\nabla S(q_0) = 0$;
(3) $\Im S(x) \geq c d(x, q_0)^2$ *for x in a neighborhood of q_0, where $c > 0$ is a constant.*

Similarly, we have

$$\begin{aligned} \rho^2 \mathcal{I} = \rho^2 \int_M & a e^{\mathrm{i}\rho S} \chi_1 \chi_2 \chi_3^2 \chi_0 (a_0^{(1)} + \rho^{-1}\kappa_1^{-1} a_1^{(1)})(a_0^{(2)} + \rho^{-1}\kappa_2^{-1} a_1^{(2)})(a_0^{(3)} + 2\rho^{-1}\kappa_3^{-1} a_1^{(3)})^2 \\ & \times (a_0^{(0)} + \rho^{-1}\kappa_0^{-1} a_1^{(0)}) \mathrm{d}V_g + \mathcal{O}(\rho^{-2}). \end{aligned} \tag{4.25}$$

By (4.23), (4.24), and (4.25), we obtain

$$\rho^2\int_M e^{\beta}\widetilde{a}\chi_1\chi_2\chi_3^2\chi_0 e^{\mathrm{i}\rho S}a_0^{(1)}a_0^{(2)}(a_0^{(3)})^2a_0^{(0)}\mathrm{d}V_g=\rho^2\int_M ae^{\mathrm{i}\rho S}\chi_1\chi_2\chi_3^2\chi_0 a_0^{(1)}a_0^{(2)}(a_0^{(3)})^2a_0^{(0)}\mathrm{d}V_g+\mathcal{O}(\rho^{-1}).$$

Since $\gamma^{(j)}$, $j=0,1,2,3$ only intersect at q_0, the function $\chi_1\chi_2\chi_3^2\chi_0$ is supported in a small neighborhood of q_0 and is equal to 1 at q_0. Using the method of stationary phase, we have

$$\rho^2\int_M e^{\beta}\widetilde{a}e^{\mathrm{i}\rho S}\chi_1\chi_2\chi_3^2\chi_0 a_0^{(1)}a_0^{(2)}(a_0^{(3)})^2a_0^{(0)}\mathrm{d}V_g=ce^{\beta(q_0)}\widetilde{a}(q_0)+\mathcal{O}(\rho^{-1}), \tag{4.26}$$

and

$$\rho^2\int_M ae^{\mathrm{i}\rho S}\chi_1\chi_2\chi_3^2\chi_0 a_0^{(1)}a_0^{(2)}(a_0^{(3)})^2a_0^{(0)}\mathrm{d}V_g=ca(q_0)+\mathcal{O}(\rho^{-1})$$

with some constant $c\neq 0$. Thus, by letting $\rho\to+\infty$ we conclude that

$$e^{\beta(q_0)}\widetilde{a}(q_0)=a(q_0).$$

Since q_0 can be any point in a dense subset of $\mathbb{U}_g$, we have $\widetilde{a}=e^{-\beta}a$ in $\mathbb{U}_g$.

Next, we consider the recovery of q. We will adapt the arguments of [20], where a similar problem is considered. We exclude the principal terms in (4.24) and (4.25) and get, upon multiplying by ρ,

$$\begin{aligned}
&\rho^2\int_M e^{\beta}\widetilde{a}e^{\mathrm{i}\rho S}\chi_1\chi_2\chi_3^2\chi_0\Big(\kappa_1^{-1}\widetilde{a}_1^{(1)}a_0^{(2)}(a_0^{(3)})^2a_0^{(0)}+\kappa_2^{-1}\widetilde{a}_1^{(2)}a_0^{(1)}(a_0^{(3)})^2a_0^{(0)}\\
&\qquad\qquad+4\kappa_3^{-1}\widetilde{a}_1^{(3)}a_0^{(1)}a_0^{(2)}a_0^{(3)}a_0^{(0)}+\kappa_0^{-1}\widetilde{a}_1^{(0)}a_0^{(1)}a_0^{(2)}(a_0^{(3)})^2\Big)\\
&=\rho^2\int_M ae^{\mathrm{i}\rho S}\chi_1\chi_2\chi_3^2\chi_0\Big(\kappa_1^{-1}a_1^{(1)}a_0^{(2)}(a_0^{(3)})^2a_0^{(0)}+\kappa_2^{-1}a_1^{(2)}a_0^{(1)}(a_0^{(3)})^2a_0^{(0)}\\
&\qquad\qquad+4\kappa_3^{-1}a_1^{(3)}a_0^{(1)}a_0^{(2)}a_0^{(3)}a_0^{(0)}+\kappa_0^{-1}a_1^{(0)}a_0^{(1)}a_0^{(2)}(a_0^{(3)})^2\Big)+\mathcal{O}(\rho^{-1}).
\end{aligned}$$

Letting $\rho\to+\infty$ and applying the method of stationary phase again, we obtain

$$\begin{aligned}
&e^{\beta(q_0)}\widetilde{a}(q_0)\left(\frac{c_{1,1}}{\kappa_1}\widetilde{a}_1^{(1)}(q_0)+\frac{c_{1,2}}{\kappa_2}\widetilde{a}_1^{(2)}(q_0)+\frac{c_{1,3}}{\kappa_3}\widetilde{a}_1^{(3)}(q_0)+\frac{c_{1,0}}{\kappa_0}\widetilde{a}_1^{(0)}(q_0)\right)\\
&=a(q_0)\left(\frac{c_{1,1}}{\kappa_1}a_1^{(1)}(q_0)+\frac{c_{1,2}}{\kappa_2}a_1^{(2)}(q_0)+\frac{c_{1,3}}{\kappa_3}a_1^{(3)}(q_0)+\frac{c_{1,0}}{\kappa_0}a_1^{(0)}(q_0)\right).
\end{aligned}$$

where $c_{1,j}$, $j=0,1,2,3$, are non-zero constants that do not depend on q (thus do not depend on β). For more details, we refer to [20]. Recall now that $e^{\beta(q_0)}\widetilde{a}(q_0)=a(q_0)$ and

$$\widetilde{a}_1^{(j)}(q_0)-a_1^{(j)}(q_0)=-\frac{\mathrm{i}}{2}\det(Y(s_j))^{-\frac{1}{2}}\int_0^{s_j}q(\gamma^{(j)}(s))\mathrm{d}s$$

for $j=0,1,2,3$. Arguing as in [20], one can then obtain

$$\sum_{j=0}^{3}\frac{c_{1,j}}{\kappa_j}\int_0^{s_j}q(\gamma^{(j)}(s))\mathrm{d}s=0.$$

Letting $\varsigma\to 0$, we have

$$\int_0^{s_0}q(\gamma^{(0)}(s))\mathrm{d}s=0,$$

since $\kappa_0 = 1$ and $\lim_{\varsigma\to 0} \kappa_j^{-1} = 0$ for $j = 1, 2, 3$ recalling the definition of κ_j around (4.20). Finally, by differentiating in s_0 we obtain $q(q_0) = 0$. Since $-e^{-\beta} q = \Box_g e^{-\beta}$, we finally conclude that

$$\Box_g e^{-\beta} = 0 \quad \text{in } \mathbb{U}_g.$$

This finishes the proof of Proposition 4.1.

For the proof of Theorem 1.3, there is one more remark we would like to make. Notice that in Proposition 4.1, we assumed $\widetilde{g} = e^{-2\beta} g$ in M, while in Section 3 we have only proved that $\widetilde{g} = e^{-2\beta} g$ in $\mathbb{U}_g = \mathbb{U}_{\widetilde{g}}$ (up to diffeomorphism). However, one can easily adapt the proof by using the property of finite speed of propagation. We shall leave Proposition 4.1 and Lemma 4.2 as they are, since they might be of independent interest.

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