QUASILINEAR WAVES AND TRAPPING: KERR-DE SITTER SPACE

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1. The setup and the results

Kerr-de Sitter space (M°, g_0) is a Lorentzian space-time of $1+3$ dimensions, which solves Einstein's equation with a cosmological constant. It models a rotating (with angular momentum a) black hole ('Kerr') of mass $M_\bullet$ in a space-time with cosmological constant Λ ('De Sitter'). In this talk, we will describe how to solve globally, and describe the asymptotic behavior of, certain quasilinear equations on M° of the form

$$\Box_{g(u,du)} = f + q(u, du),$$

where $g(0,0) = g_0$, for *small data* f. The key advance is overcoming the *normally hyperbolic trapping* by combining microlocal analysis and a Nash-Moser iteration.

To our knowledge, this is the first global result for the forward problem for a quasilinear wave equation on either a Kerr or a Kerr-de Sitter background. We remark, however, that Dafermos, Holzegel and Rodnianski [10] have constructed backward solutions for Einstein's equations on the Kerr background; for backward constructions the trapping does not cause difficulties. For concreteness, we state our results in the special case of Kerr-de Sitter space, but it is important to keep in mind that the setting is more general, for details see [30].

To proceed we need to describe Kerr-de Sitter space more precisely. To get started, we bordify M° to a smooth manifold with boundary, M, with M° as its interior, and $\partial M = X$ its boundary; see Figure 1, and also Figure 4 for a more complete picture. Here one can take $x = e^{-t_*}$ as a defining function of X, where t_* is a Kerr-star coordinate, see e.g. [16, 19, 46]. We work in a compact region Ω in M of the form

$$\Omega = \mathfrak{t}_1^{-1}([0,\infty)) \cap \mathfrak{t}_2^{-1}([0,\infty)),\ H_j = \mathfrak{t}_j^{-1}(\{0\}),$$

- with $\mathfrak{t}_j$ having forward, resp. backward, time-like differentials,
- with $\mathfrak{t}_j$ having linearly independent differentials at the common zero set, and
- with H_1 disjoint from X.

Here we want to impose vanishing Cauchy data at H_1; general Cauchy data can (essentially) be converted into this.

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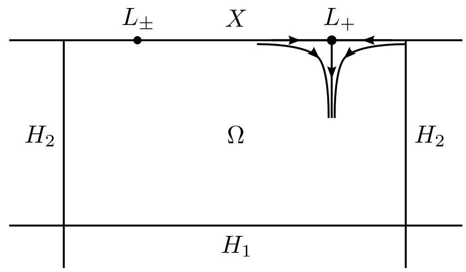

FIGURE 1. The region Ω in the bordification M of Kerr-de Sitter space.

The framework we need on M involves totally characteristic vector fields, i.e. vector fields $V \in \mathcal{V}_b(M)$ tangent to M. In local coordinates, with $n = 4$,

$$x, y_1, \dots, y_{n-1},\ x \geq 0,$$

these are linear combinations of

$$x\partial_x, \partial_{y_1}, \dots, \partial_{y_{n-1}},$$

with C^∞ coefficients. The dual metric $g^{-1} = G$ is then a smooth linear combination of symmetric products of these vector fields

$$x\partial_x \otimes_s x\partial_x,\ x\partial_x \otimes_s \partial_{y_j},\ \partial_{y_i} \otimes_s \partial_{y_j},$$

so the actual metric is a smooth linear combination of

$$\frac{dx}{x} \otimes_s \frac{dx}{x},\ \frac{dx}{x} \otimes_s dy_j,\ dy_i \otimes_s dy_j.$$

In particular, the Kerr-de Sitter metric g_0 is of such a form. Also write

$${}^b du = (x\partial_x u)\frac{dx}{x} + \sum_j (\partial_{y_j} u)\, dy_j;$$

thus, $a(u, {}^b du)$ is a short and invariant notation for

$$a(u, x\partial_x u, \partial_{y_1} u, \dots, \partial_{y_{n-1}} u).$$

We note here that analysis based on $\mathcal{V}_b(M)$ is sometimes called b-analysis, and in the elliptic setting it was extensively studied by Melrose [37], though in fact it originated in Melrose's study of hyperbolic boundary problems [36].

More precisely then, we consider equations of the form

$$\Box_{g(u, {}^b du)} = q(u, {}^b du) + f,$$

and we want a forward solution, i.e. for f supported away from H_1 in Ω, the solution should also be such. Here

$$q(u, {}^b du) = \sum_{j=1}^{N'} a_j u^{e_j} \prod_{k=1}^{N_j} X_{jk} u,$$

with $X_{jk} \in \mathcal{V}_b(M)$, $a_j \in C^\infty(M)$.

Theorem 1 (H.-V. [30]). *For $\alpha > 0$, $|a| \ll M_\bullet$, with $N_j \geq 1$ for all j, and with $f \in C_c^\infty(M)$ having sufficiently small H^{14} norm, the wave equation has a unique forward, smooth in M°, solution of the form $u = u_0 + \tilde{u}$, with $x^{-\alpha}\tilde{u}$ bounded, $u_0 = c\chi$, $\chi \equiv 1$ near ∂M.*

- For the Klein-Gordon equation
$$(\Box_{g(u,{}^b du)} - m^2)u = f + q(u, {}^b du),$$
$m > 0$ small, the analogous conclusion holds, without the u_0 term, and without the requirement $N_j \geq 1$. This is due to the absence of a '0 resonance'.
- The only reason $|a| \ll M_\bullet$ is assumed is to exclude possible resonances in $\operatorname{Im} \sigma \geq 0$, apart from the 0 resonance for the wave equation.
- The main constraint on solvability of the non-linear problem is thus resonances, discussed below.
- The setup works equally well for vector bundles.

For a more precise version, and also for the proofs, we need appropriate Sobolev spaces.

- Let L^2_{b} be the L^2 space relative to the density of any Riemannian or Lorentzian b-metric, which is thus of the form
$$\frac{|dx\, dy_1 \dots dy_{n-1}|}{x}.$$
- For $s \geq 0$ integer, H^s_b consists of elements of L^2_{b} with $V_1 \dots V_j u \in L^2_{\mathrm{b}}$ for $V_1, \dots V_j \in \mathcal{V}_b(M)$, $j \leq s$.
- The weighted Sobolev spaces are $H^{s,r}_b = x^r H^s_b$.
- We relax the requirements on the coefficients to
$$a_j \in C^\infty(M) + H^\infty_b(M),\ X_{jk} \in (C^\infty(M) + H^\infty_b(M))\mathcal{V}_b(M),$$
and the forcing to
$$f \in H^{\infty,\alpha}_b,\ \alpha > 0.$$

Theorem 2 (H.-V., [30]). *For $|a| \ll M_\bullet$, $\alpha > 0$ sufficiently small, and for $f \in H^{\infty,\alpha}_b$ of sufficiently small $H^{14,\alpha}_b$ norm, the wave equation (with $N_j \geq 1$ for all j) has a unique forward, smooth in M°, solution of the form $u = u_0 + \tilde{u}$, with $\tilde{u} \in H^{\infty,\alpha}_b$, $u_0 = c\chi$, $\chi \equiv 1$ near ∂M.*

The analogous conclusion holds for the Klein-Gordon equation, without the presence of the u_0 term, without the requirement $N_j \geq 1$.

As usual, $H^{\infty,\alpha}_b$ can be replaced by $H^{s,\alpha}_b$ for s sufficiently large: for suitably large C, s_0, and for $s \geq s_0$, $f \in H^{Cs,\alpha}_b$ gives rise to a solution $u \in H^{s,\alpha}_b$.

2. Previous results

In these expanded version of the lecture notes, we briefly discuss previous results on Kerr-de Sitter space and its perturbations, roughly following the introduction of [30]. The only paper the authors are aware of on non-linear problems in the Kerr-de Sitter setting is their earlier paper [29] in which the semilinear Klein-Gordon equation was studied. There is more work on the linear equation on perturbations of de Sitter-Schwarzschild and Kerr-de Sitter spaces: a rather complete analysis of the asymptotic behavior of solutions of the linear wave equation was given in [46], upon which the linear analysis of [30], described here, is ultimately based. Previously in exact Kerr-de Sitter space and for small angular momentum, Dyatlov [20, 19] has shown exponential decay to constants, even across the event horizon; see also the more recent work of Dyatlov [21]. Further, in de Sitter-Schwarzschild space (non-rotating black holes) Bachelot [3] set up the functional analytic scattering theory in the early 1990s, while later Sá Barreto and Zworski [4] and Bony and Häfner [7] studied resonances and decay away from the event horizon, Dafermos and Rodnianski in [12] showed polynomial decay to constants in this setting, and Melrose, Sá Barreto and Vasy [39] improved this result to exponential decay to constants. There is also physics literature on the subject, starting with Carter's discovery of this space-time [9, 8], either using explicit solutions in special cases, or numerical calculations, see in particular [49], and references therein. We also refer to the paper of Dyatlov and Zworski [24] connecting recent mathematical advances with the physics literature.

Wave equations on Kerr space (which has vanishing cosmological constant) have received more attention; on the other hand, they do not fit directly into our setting; see the introduction of [46] for an explanation and for further references. (See also [16] for more background and additional references.) For instance, polynomial decay on Kerr space was shown recently by Tataru and Tohaneanu [43, 42] and Dafermos, Rodnianski and Shlapentokh-Rothman [15, 14, 17], while electromagnetic waves were studied by Andersson and Blue [1] (see also Bachelot [2] in the Schwarzschild case), after pioneering work of Kay and Wald in [33] and [47] in the Schwarzschild setting. Normal hyperbolicity of the trapping, corresponding to null-geodesics that do not escape through the event horizons, in Kerr space was realized and proved by Wunsch and Zworski [48]; later Dyatlov extended and refined the result [22, 23]. Note that a stronger version of normal hyperbolicity is a notion that is stable under perturbations.

On the non-linear side, Luk [34] established global existence for forward problems for semilinear wave equations on Kerr space under a null condition, and Dafermos, Holzegel and Rodnianski [10] constructed *backward* solutions for Einstein's equations on Kerr space as already mentioned. Other recent works include [35, 44, 18, 13, 11, 6, 25, 26].

3. Non-linearities

As usual, the main part of solving small data problems for a non-linear PDE is solving linear PDE. However, the kind of linear PDE one needs to be able to handle depends on the non-linearity, and how it interacts with properties of the linear PDE. Here we run the solution scheme *globally*, on modifications as needed for the spaces $H_b^{s,r}$. The modifications add support conditions, as well as allow for terms corresponding to *resonances* of a linear equation, such as constants for the actual wave equation:

$$\mathcal{X}^{s,r} = H_b^{s,r}(\Omega)^{\bullet,-} \oplus \mathbb{C}.$$

Here $\bullet$ denotes the distributions supported in $\mathfrak{t}_1 \geq 0$ (the 'correct side' of H_1), while $-$ denotes the restriction of distributions to $\mathfrak{t}_2 > 0$ (again, the correct side of H_2), following Hörmander's notation [32, Volume 3, Appendix B]. (Thus, these distributions are 'supported' at H_1 and 'extendible' at H_2.) Here $\mathbb{C}$ is identified with $\mathbb{C}\chi$, $\chi \in C_c^\infty(M)$ supported in $\mathfrak{t}_1 > 0$, identically 1 near $\Omega \cap X$.

In order for $\mathcal{X}^{s,r}$ to be closed under multiplication, one needs $r \geq 0$ and $s > n/2$. In terms of derivatives, the best case scenario for $\Box_g^{-1}$ is the loss of one derivative relative to elliptic estimates (which happens even locally). The other main linear obstacle is *trapping* for a linear equation, to be discussed later, which causes further losses of derivatives.

We now give some examples:

- For de Sitter space, due to the 0 resonance, the best estimate one can get is
$$\Box_g^{-1} : H_b^{s-1,r}(\Omega)^{\bullet,-} \to \mathcal{X}^{s,r}$$
for suitable $r > 0$ small.
- For the Klein-Gordon equation on de Sitter space there is no resonance in the closed upper half plane:
$$(\Box_g - m^2)^{-1} : H_b^{s-1,r}(\Omega)^{\bullet,-} \to H_b^{s,r}(\Omega)^{\bullet,-}$$
for suitable $r > 0$ small.
- For Kerr-de Sitter space, due to trapping and resonances, the best estimate one can get is
$$\Box_g^{-1} : H_b^{s-1+\epsilon,r}(\Omega)^{\bullet,-} \to \mathcal{X}^{s,r}$$
for suitable $r > 0$, $\epsilon > 0$ small. (For K-G, ϵ remains, but the summand $\mathbb{C}$ in $\mathcal{X}$ can be dropped.)

The simplest setting for an equation like
$$\Box_{g(u,{}^b du)} u = q(u, {}^b du) + f$$
is if g is actually independent of u, i.e. $\Box = \Box_g$ is fixed, so the equation is semilinear, for then
$$u = \Box_g^{-1}(q(u, {}^b du) + f).$$
Then the contraction mapping principle/Picard iteration can be used provided $\Box_g^{-1}$ and q are well-behaved:
$$u_{k+1} = \Box_g^{-1}(q(u_k, {}^b du_k) + f).$$
As $\Box_g^{-1}$ loses a derivative relative to elliptic estimates even in the best case scenario, one cannot simply replace $\Box_{g(u,{}^b du)}$ by its linearization and put the difference on the right hand side. If the trapping causes further losses of derivatives, one would need $q = q(u)$! We refer to [29] for more detail.

For quasilinear equations,
$$\Box_{g(u)} u = q(u, {}^b du) + f,$$
without trapping losses and g depending on u only, one can instead run a modified, Newton-type at the second order level, solution scheme
$$u_{k+1} = \Box_{g(u_k)}^{-1}(q(u_k, {}^b du_k) + f).$$
This still gives well posedness in the sense that (ignoring modifications due to resonances) for small $f \in H_b^{s-1,r}$ the solution u of small $H_b^{s,r}$-norm is unique, and in $H_b^{s-1,r}$ it depends continuously on f in the $H_b^{s-1,r}$ norm. This approach requires providing a (global) linear theory for operators with $H_b^{s,r}$-type coefficients, with estimates that are uniform in the $H_b^{s,r}$ coefficients when they are bounded by appropriate (small) constants; this was achieved by Hintz [27] building in part on earlier work of Beals and Reed [5]. At the level of a multiplication operator (multiplication by u here), this corresponds to
$$\|uv\|_{H_b^{s,r}} \leq C\|u\|_{H_b^{s,r}}\|v\|_{H_b^{s,r}},$$
which is valid for $s > n/2$, $r \geq 0$.

For quasilinear equations on Kerr-de Sitter space, due to the trapping losses, we use a Nash-Moser iterative scheme. Here for simplicity we use X. Saint Raymond's version [41]: one solves

$$\phi(u;f)=0,\ \phi(u;f)=\Box_{g(u,{}^b du)}u-q(u,{}^b du)-f,$$

by using the solution operator $\psi(u;f)$ for the linearization $\phi'(u;f)$ of ϕ in u:

$$\psi(u;f)\phi'(u;f)w=w,$$

and letting $u_0=0$,

$$u_{k+1}=u_k-S_{\theta_k}\psi(u_k;f)\phi(u_k;f),$$

where $\theta_k\to\infty$, S_{θ_k} is a smoothing operator $\mathcal{X}^{s,r}\to\mathcal{X}^{s,r}$.

Again, this needs the linear theory for $H_b^{s,r}$-coefficients. Further, one needs *tame* estimates. At the level of a multiplication operator, this corresponds to

$$\|uv\|_{H_b^{s,r}}\le C(\|u\|_{H_b^{s_0,r}}\|v\|_{H_b^{s,r}}+\|u\|_{H_b^{s,r}}\|v\|_{H_b^{s_0,r}}),$$

which is valid for $s\ge s_0>n/2$, $r\ge 0$. Here s is a 'high' (regularity), s_0 a 'low' norm; what one does not want is the product of high norms, i.e. one wants an estimate with a linear bound in high norms. For further details, including more sophisticated tame bounds, we refer to [30].

4. Linear problems

We now discuss the linear analysis in more detail. As already present in elliptic problems [37], there are two aspects of the linear analysis:

- *b-regularity analysis*: provides estimates for the PDE at high b-frequencies, i.e. estimates of the form

  $$\|u\|_{H_b^{s,r}}\le C(\|Lu\|_{H_b^{s',r}}+\|u\|_{H_b^{\tilde s,r}})$$

  with $\tilde s<s$ (in many cases arbitrary). This provides no additional decay, and is thus *not* sufficient for global Fredholm-type properties since $H_b^{s,r}\to H_b^{\tilde s,\tilde r}$ compact needs $s>\tilde s$ *and* $r>\tilde r$.
- *Normal operator analysis*: provides a framework for understanding decay and asymptotic properties of solutions.

The *normal operator* of $L=\sum_{|\alpha|\le 2}a_{j,\alpha}(x,y)(xD_x)^jD_y^\alpha$ is obtained by freezing coefficients at $x=0$:

$$N(L)=\sum_{|\alpha|\le 2}a_{j,\alpha}(0,y)(xD_x)^jD_y^\alpha,$$

so it is dilation invariant in x. Mellin transforming in x gives

$$\hat L(\sigma)=\sum_{|\alpha|\le 2}a_{j,\alpha}(0,y)\sigma^jD_y^\alpha.$$

Now, the b-regularity analysis gives uniform control of $\hat L(\sigma)$ in strips $|\operatorname{Im}\sigma|<C$ as $|\sigma|\to\infty$. However, $\hat L(\sigma)^{-1}$ may have finitely many poles σ_j in such a strip; these are the *resonances*. For Fredholm theory, we need weights r (for $H_b^{s,r}$) such that there are no resonances σ_j with $\operatorname{Im}\sigma_j=-r$. In an elliptic setting, with a global problem on $X=\partial M$ for simplicity,

$$\hat L(\sigma):H^s(X)\to H^{s-2}(X).$$

(Examples: cylindrical ends, asymptotically Euclidean spaces.) In our setting

$$\hat{L}(\sigma) : \{u \in H^s(\Omega \cap X)^- : \ \hat{L}(\sigma)u \in H^{s-1}(\Omega \cap X)^-\} \to H^{s-1}(\Omega \cap X)^-.$$

Here the principal symbol of $\hat{L}(\sigma)$ is independent of σ, and thus so is the space on the left hand side. It is a first order coisotropic space.

In an elliptic setting on M, with r as above,

$$L : H_b^{s,r} \to H_b^{s-2,r}$$

is Fredholm. Adding to this spaces of resonant states, such as in $\mathcal{X}^{s,r}$ above, maintains Fredholm properties, and if done correctly, can give invertibility.

In our non-elliptic settings one loses at least a derivative. The typical scenario is

$$L : \{u \in H_b^{s,r} : \ Lu \in H_b^{s-1,r}\} \to H_b^{s-1,r}$$

being Fredholm. This works for all r with no resonances with $\operatorname{Im} \sigma_j = -r$ if either there is no trapping, or even with normally hyperbolic trapping if $r < 0$.

- For $L = \Box_g$, with or without trapping, the forward solution satisfies

  $$L^{-1} : H_b^{s-1,r} \to H_b^{s,r}$$

  if $r < 0$.
- Adding resonant state spaces, one gets invertibility even for $r > 0$. For $r > 0$ small, no trapping,

  $$L^{-1} : H_b^{s-1,r} \to H_b^{s,r} \oplus \mathbb{C};$$

  in general all the resonant states with $\operatorname{Im} \sigma_j > -r$ should be added to the right hand side.
- The trapping losses are all as $|\sigma| \to \infty$, so

  $$\hat{L}(\sigma) : H^{s-1} \to H^s$$

  is still a meromorphic Fredholm family, but its high energy behavior of the inverse is lossy in $\operatorname{Im} \sigma \leq 0$.

In order to see where such statements come from we need to discuss microlocal analysis.

5. Microlocal analysis

In our discussion of microlocal analysis let's start with the boundaryless setting, such as X.

- The theory is microlocal, i.e. one works with $A \in \Psi^0(X)$ to microlocalize.
- Recall that the principal symbol $a = \sigma_0(A)$ is a function on $S^*X = (T^*X \setminus o)/\mathbb{R}^+$ (with $\mathbb{R}^+$ acting by dilations in the fibers of the cotangent bundle), and the wave front set $\mathrm{WF}'(A)$ is a subset of S^*X.
- The characteristic set $\mathrm{Char}(A)$ of A is the zero set of a; the elliptic set is its complement.
- For general order A the situation is similar, except the principal symbol is a homogeneous object on $T^*X \setminus o$, or the section of a line bundle on S^*X.

Microlocal elliptic estimates for an operator $P \in \Psi^m(X)$ are of the form

$$\|B_1 u\|_{H^s} \leq C(\|B_3 P u\|_{H^{s-m}} + \|u\|_{H^{\tilde{s}}})$$

if $B_j \in \Psi^0(X)$, B_3 elliptic on $\mathrm{WF}'(B_1)$, $\mathrm{WF}'(B_1)$ disjoint from $\mathrm{Char}(P)$, $s, \tilde{s}$ arbitrary.

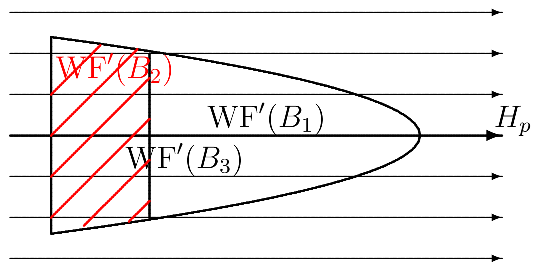

FIGURE 2. The wave front set of the operators B_j for real principal type estimates, i.e. propagation of singularities. Here B_3 has slightly larger wave front set than $\mathrm{WF}'(B_1) \cup \mathrm{WF}'(B_2)$.

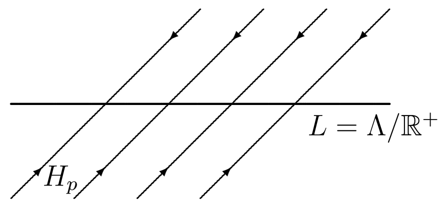

FIGURE 3. A submanifold of radial points L which is a sink in the normal directions. The figure should be understood as one in the cosphere bundle, $S^*X = (T^*X \setminus o)/\mathbb{R}^+$.

Real principal type estimates correspond to propagation of singularities: one can control u microlocally somewhere in terms of control on it at another point on the bicharacteristic through it, and of course of Pu:

$$\|B_1 u\|_{H^s} \leq C(\|B_2 u\|_{H^s} + \|B_3 Pu\|_{H^{s-m+1}} + \|u\|_{H^{\tilde{s}}});$$

here $s, \tilde{s}$ arbitrary, B_3 elliptic on $\mathrm{WF}'(B_1)$, and the bicharacteristic of P from every point in $\mathrm{WF}'(B_1) \cap \mathrm{Char}(P)$ reaches $\mathrm{Ell}(B_2)$ while remaining in $\mathrm{Ell}(B_3)$; see Figure 2. These are typically proved by *positive commutator estimates*, which are essentially microlocal energy estimates, see [31].

This real principal type estimate means that one has control of u if one controls it somewhere else – but one needs a starting point. One way this works is for Cauchy problems, where one works with spaces of *supported distributions*; one propagates estimates from outside the support. Another way this works is if the bicharacteristics approach submanifolds L which are normally sources or sinks for the bicharacteristic flow (see Figure 3), and at which one has estimates without the B_2 term. In this case there is a threshold regularity s_0, and the result depends on whether $s > s_0$ or $s < s_0$. Here s_0 depends on the principal symbol of $\frac{1}{2i}(P - P^*)$; if $P - P^* \in \Psi^{m-2}(X)$, then it is $s_0 = (m-1)/2$.

- For $s > \tilde{s} > s_0$, the estimates are of the form

  $$\|B_1 u\|_{H^s} \leq C(\|B_3 Pu\|_{H^{s-m+1}} + \|u\|_{H^{\tilde{s}}}),$$

  i.e. one has control without having to make assumptions on u elsewhere. Here B_1 is elliptic on L, B_3 elliptic on $\mathrm{WF}'(B_1)$, and all bicharacteristics from $\mathrm{WF}'(B_1) \cap \mathrm{Char}(P)$ tend to L in either the forward or backward direction (depending on sink/source) while remaining in the elliptic set of B_3.

- For $s < s_0$,

$$\|B_1 u\|_{H^s} \leq C(\|B_2 u\|_{H^s} + \|B_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{\tilde{s}}});$$

where now $\mathrm{WF}'(B_2)$ is disjoint from L, B_1 elliptic on L, so one propagates estimates from outside L to L, B_3 elliptic on $\mathrm{WF}'(B_1)$, and all bicharacteristics from $(\mathrm{WF}'(B_1) \cap \mathrm{Char}(P)) \setminus L$ tend to $\mathrm{Ell}(B_2)$ in either the forward or backward direction (depending on source/sink) while remaining in the elliptic set of B_3.

These statements are again proved (under the appropriate assumptions) by positive commutator estimates; see [46, Section 2] and also [45].

This structure happens in $X = \partial M$ for Schwarzschild-de Sitter space ($P = \hat{L}(\sigma)$ is the Mellin transformed normal operator), via *radial sets*, where the Hamilton vector field H_p is tangent to the dilation orbits in $T^*X \setminus o$. Note that there is no dynamics within the radial set. (Asymptotically Euclidean scattering theory has similar phenomena, see especially Melrose's work [38].) In X for Kerr-de Sitter space there is non-trivial dynamics within the radial set (rotating black hole), but the normal dynamics is again source/sink. In both cases, $L = SN^*(Y \cap X)$, where Y is the *event horizon* of the black hole or the *cosmological horizon* of the de Sitter end; see [46] and Figure 4. Furthermore, $s_0 = (m-1)/2 + \beta r$, $r = -\operatorname{Im}\sigma$, where β arises as the negative of the ratio of the eigenvalues of the linearization of the Hamilton flow normally to L *within* M, namely the eigenvalue corresponding to the defining function of fiber infinity in the cotangent bundle and the eigenvalue corresponding to the boundary defining function of M (see [29, Section 2] for a discussion of this perspective).

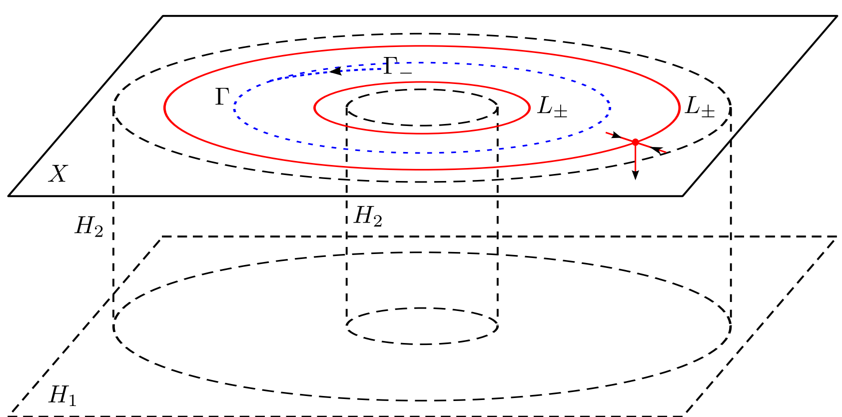

FIGURE 4. A more complete picture of Kerr-de Sitter space with $L_\pm$ the projection of the radial sets (from the cotangent bundle), and Γ the projection of the trapped set.

With

$$P : \{u \in H^s(\Omega \cap X)^- : \ Pu \in H^{s-m+1}(\Omega \cap X)^-\} \to H^{s-m+1}(\Omega \cap X)^-,$$

and

$$P^*: \{u \in H^{s'}(\Omega \cap X)^\bullet :\ P^* u \in H^{s'-m+1}(\Omega \cap X)^\bullet\} \to H^{s'-m+1}(\Omega \cap X)^\bullet,$$

$s > (m-1)/2 + \beta r$, $s' < (m-1)/2 - \beta r$, we get the required Fredholm estimates; here we want $s' = -s + m - 1$ for duality. (Here $r = -\operatorname{Im} \sigma$ as above.) The a priori control for P comes from L; for P^* it comes from the Cauchy surface $H_2 \cap X$.

Turning to $L \in \Psi_b^m(M)$, where $\Psi_b^m(M)$ is the b-pseudodifferential algebra (corresponding to $\mathcal{V}_b(M)$ and the Sobolev spaces $H_b^{s,r}(M)$), the situation is similar, except one has to use $\Psi_b(M)$ to microlocalize; see [46] and especially [29].

- In particular, microlocal elliptic and real principal type estimates are unchanged.
- From the perspective of M, the normal sources/sinks L within X are actually saddle points, with the normal direction to X having the opposite stable/unstable nature relative to X. (This corresponds to the *red-shift effect*.)
- In this case, on $H_b^{s,r}$, one can propagate estimates through L from outside X into X (to L and beyond) if $s > (m-1)/2 + \beta r$, and from inside X (a punctured neighborhood of L) to L and to outside X if $s < (m-1)/2 + \beta r$: β being a scale relating s and r due to the linearization eigenvalues mentioned above.
- If all bicharacteristics, except those within components of the generalized radial set go to H_1, resp. H_2 in the two directions, as in de Sitter space, this gives estimates
$$\|u\|_{H_b^{s,r}(\Omega)^{\bullet,-}} \leq C(\|Lu\|_{H_b^{s-m+1,r}(\Omega)^{\bullet,-}} + \|u\|_{H^{\tilde{s},r}(\Omega)^{\bullet,-}}),$$
$$\|u\|_{H_b^{s',r'}(\Omega)^{-,\bullet}} \leq C(\|L^* u\|_{H_b^{s'-m+1,r'}(\Omega)^{-,\bullet}} + \|u\|_{H^{\tilde{s}',r'}(\Omega)^{-,\bullet}}),$$
$s' = -s + m - 1$, $r' = -r$.

If this *non-trapping assumption* is not satisfied, these estimates need not hold.

- In Kerr-de Sitter space, the trapped set Γ is in ${}^b S_X^* M$, and corresponds to the *photon sphere* of Schwarzschild-de Sitter space.
- It can be considered as a subset of T^*X, and then shows up in high energy, or semiclassical, estimates for $\hat{L}(\sigma)$.
- It is *normally hyperbolic*: there are smooth transversal stable/unstable submanifolds $\Gamma_\pm$ with intersection Γ.
- Further, normally hyperbolic trapping is the only trapping: outside $L \cup \Gamma$, in both the forward and the backward directions, all bicharacteristics need to tend to either H_j or to L or to Γ, with tending to Γ is only allowed in one of the two directions.
- In this case, the estimates above are valid for $r < 0$ only! (Growing spaces.) However, the estimates are valid for rough coefficients, and indeed they are tame estimates. (There are actually some estimates valid for $r = 0$; see [28].)

However, for $r > 0$ small (with a precise dynamical bound), Dyatlov [23] (earlier results are due to Wunsch and Zworski [48], and more general results are due to Nonnenmacher and Zworski [40]; for us Dyatlov's version is convenient) has shown for the Mellin transformed normal operator the lossy estimates (in terms of derivatives relative to non-trapping), which in turn give the lossy estimates for

$N(L)$. This is valid for $L = \Box_g$ or $\Box_g - m^2$, g the Kerr-de Sitter metric, or *smooth* perturbations.

Since our linearized operator L_u depends on (the rough!) u, this is not enough. But, as the coefficients of L_u are in $\mathbb{C} \oplus H_b^{s,r}$ with $r > 0$, one can treat the second term as a perturbation: one can combine Dyatlov's decaying estimates for L_c^{-1} ($c \in \mathbb{C}$) with the rough coefficient estimates on $H_b^{s,r'}$, $r' < 0$; once the coefficients of $L_u - L_c$ have sufficient decay, they map $H_b^{s,r'}$ to $H_b^{s-m,r}$. Notice that there are no tameness issues for L_c (c has only a 'low regularity' part).

Altogether this gives a tame estimate for the solution operator S for $s_0 > n/2 + 1/2$, $\alpha > 0$ small, $s > n/2 + 2$, $0 < r \leq \alpha$,

$$\|Sf\|_{\mathcal{X}^{s,r}} \leq C(s, \|u\|_{\mathcal{X}^{s_0,\alpha}})(\|f\|_{H_b^{s+3,r}(\Omega)^{\bullet,-}} + \|f\|_{H_b^{s_0,r}(\Omega)^{\bullet,-}} \|u\|_{\mathcal{X}^{s+4,r}}).$$

This then plugs into the Nash-Moser framework and gives the global solvability and asymptotics (decay to constants) result stated at the beginning of these notes.

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