

HORIZONS OF SOME ASYMPTOTICALLY STATIONARY SPACETIMES

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ABSTRACT. On a class of dynamical spacetimes which are asymptotic as $t \rightarrow \infty$ to a stationary spacetime containing a horizon \mathcal{H}_0 , we show the existence of a unique null hypersurface \mathcal{H} which is asymptotic to \mathcal{H}_0 . This is a special case of a general unstable manifold theorem for perturbations of flows which translate in time and have a normal sink at an invariant manifold in space. Examples of horizons \mathcal{H}_0 to which our result applies include event horizons of subextremal Kerr and Kerr–Newman black holes as well as event and cosmological horizons of subextremal Kerr–Newman–de Sitter black holes. In the Kerr(–Newman) case, we show that \mathcal{H} is equal to the boundary of the black hole region of the dynamical spacetime.

1. INTRODUCTION

In this note we give a short proof of (a generalization of) the recent result by Chen–Klainerman [CK24] on the regularity of the future event horizon of perturbations of subextremal Kerr black holes. Underlying our proof is a general unstable manifold theorem based on [Hin21]. A simple illustrative example, which is a toy model for the time 1 null-geodesic flow near the event horizon of Kerr in phase space, is as follows. Let $a \in \mathbb{R}$ and consider

$$f_0(t, \phi, z) = (t - 1, \phi + a - z, z/2), \quad t \in \mathbb{R}, \quad \phi \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}, \quad z \in (-1, 1). \quad (1.1)$$

This maps $\mathcal{H}_0 := \mathbb{R}_t \times \mathbb{S}_\phi^1 \times \{0\}$ into itself. Then we prove that for any perturbation $f(t, \phi, z) = f_0(t, \phi, z) + (0, \mathcal{O}(t^{-\alpha}), \mathcal{O}(t^{-\alpha}))$, $\alpha > 0$, there exist $t_0 \geq 1$ and a unique smooth submanifold $\mathcal{H} \subset [t_0, \infty) \times \mathbb{S}^1 \times (-1, 1)$ which is $\mathcal{O}(t^{-\alpha})$ -close to \mathcal{H}_0 and so that $f(\mathcal{H}) \cap \{t \geq t_0\} \subset \mathcal{H}$.

We set up and prove our general unstable manifold type theorem in §2. In §3, we put the construction of perturbed horizons into this setup. In essence, we construct a smooth submanifold Γ^u of phase space which asymptotes at late times to the conormal bundle Γ_0 (projected to the cosphere bundle) of the unperturbed event horizon. The key dynamical input is that the past geodesic flow on subextremal Kerr (and its generalizations), lifted to phase space, is a sink in directions normal to Γ_0 . We then show that Γ^u projects to the base as (and is the conormal bundle of) a smooth null hypersurface (Theorem 3.2). Under mild global assumptions on the perturbed Kerr spacetime, we show that this hypersurface coincides with the event horizon (Theorem 3.4).

2. A GENERAL UNSTABLE MANIFOLD THEOREM

Let \mathcal{X} be a closed manifold and $\Gamma \subset \mathcal{X}$ a smooth closed submanifold (in particular compact). Let $\bar{f}: \mathcal{X} \rightarrow \mathcal{X}$ be a smooth map with $\bar{f}(\Gamma) = \Gamma$ for which there exists an open neighborhood $\mathcal{U} \supset \Gamma$ so that $\bar{f}|_{\mathcal{U}}: \mathcal{U} \rightarrow \bar{f}(\mathcal{U})$ is a diffeomorphism. Since the differential $D\bar{f}: T_\Gamma \mathcal{X} \rightarrow T_\Gamma \mathcal{X}$ restricts to a vector bundle map

$$\Gamma_p \bar{f} := D_p \bar{f}|_{T_p \Gamma}: T_p \Gamma \rightarrow T_{\bar{f}(p)} \Gamma, \quad (2.1)$$

it also induces a map on the normal bundle $N\Gamma = T_\Gamma \mathcal{X}/T\Gamma$ which we denote

$$N_p \bar{f}: N_p \Gamma \rightarrow N_{\bar{f}(p)} \Gamma. \quad (2.2)$$

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Suppose that $\bar{f}|_\Gamma$ is an almost-isometry and Γ is a normal sink for \bar{f} in the following sense: there exist a fiber metric on $N\Gamma$ and, for all $\epsilon > 0$, a fiber metric on $T\Gamma$ so that

$$\|N_p \bar{f}\| < 1; \quad 1 - \epsilon < m(\Gamma_p \bar{f}) := \inf_{w \in T_p \Gamma, \|w\|=1} \|\Gamma_p \bar{f}(w)\| \leq \|\Gamma_p \bar{f}\| < 1 + \epsilon. \quad (2.3)$$

Define the map

$$f_0: \mathcal{M} := \mathbb{R} \times \mathcal{X} \rightarrow \mathcal{M}, \quad f_0(t, x) = (t - 1, \bar{f}(x)), \quad (2.4)$$

which maps $\Gamma_0 := \mathbb{R} \times \Gamma$ to itself. We measure the size of perturbations of f_0 using:

Definition 2.1 (Weight function). A weight function is a smooth function $0 < \rho \in \mathcal{C}^\infty([0, \infty))$ with

$$\rho' \leq 0, \quad \lim_{t \rightarrow \infty} \rho(t) = 0, \quad |\rho^{(k)}| \leq C_k \rho \quad \forall k \in \mathbb{N}_0. \quad (2.5)$$

Examples are $e^{-\alpha t}$ and $(1+t)^{-\alpha}$ for $\alpha > 0$. For $m \in \mathbb{N}_0 \cup \{\infty\}$, write $\mathcal{C}_b^m(\mathcal{M})$ for the space of $u \in \mathcal{C}^\infty(\mathcal{M})$ so that all of its up to m many derivatives along ∂_t and smooth vector fields on \mathcal{X} are bounded. For a bundle $E \rightarrow \mathcal{X}$, the space $\mathcal{C}_b^\infty(\mathcal{M}; E)$ is defined analogously using local trivializations of E . Furthermore $\rho \mathcal{C}_b^m(\mathcal{M}) = \{u: \rho^{-1}u \in \mathcal{C}_b^m(\mathcal{M})\}$.

Let $t_0 \geq 0$. Fix a Riemannian metric on \mathcal{X} and suppose $f: \mathcal{M} \cap \{t \geq t_0\} \rightarrow \mathcal{M}$, is a shift by -1 in t and a $\rho \mathcal{C}_b^\infty$ perturbation of \bar{f} in space. That is,

$$f(t, x) = (t - 1, \exp_{\bar{f}(x)} \tilde{V}(t, \bar{f}(x))), \quad \tilde{V}(t, \cdot) \in \mathcal{V}(\mathcal{X}) = \mathcal{C}^\infty(\mathcal{X}; T\mathcal{X}), \quad \tilde{V} \in \rho \mathcal{C}_b^\infty(\mathcal{M}; T\mathcal{X}). \quad (2.6)$$

Theorem 2.2 (Existence and uniqueness of the perturbed unstable manifold). *There exists a smooth submanifold $\Gamma^u \subset \mathcal{M} \cap \{t \geq t_0\}$ with the following properties:*

- (1) Γ^u is f -invariant in the sense that $f(\Gamma^u) \cap \{t \geq t_0\} = \Gamma^u$;
- (2) Γ^u approaches Γ_0 as $t \rightarrow \infty$ in a $\rho \mathcal{C}_b^\infty$ sense, in that Γ^u is the normal graph over Γ_0 of a function in $\rho \mathcal{C}_b^\infty(\Gamma_0; N\Gamma)$.

Moreover, Γ^u is the unique f -invariant submanifold within the class of manifolds approaching Γ_0 in a $\rho \mathcal{C}_b^1$ sense. Finally, given any $\tilde{\Gamma}^u$ which approaches Γ_0 in a $\tilde{\rho} \mathcal{C}_b^1$ sense for some weight function $\tilde{\rho}$, one obtains Γ^u as the \mathcal{C}_b^0 limit of iterates $\tilde{\Gamma}_0^u = \tilde{\Gamma}^u \cap \{t \geq t_0\}$, $\tilde{\Gamma}_{n+1}^u = f(\tilde{\Gamma}_n^u) \cap \{t \geq t_0\}$.

Proof. Fix any complementary subbundle $S \rightarrow \Gamma$ of $T_\Gamma \mathcal{X}$ to $T\Gamma$; thus $T_\Gamma \mathcal{X} \rightarrow N\Gamma$ induces

$$S \cong N\Gamma. \quad (2.7)$$

We fix a fiber inner product on S by pulling back the fiber inner product on $N\Gamma$.

If S is invariant under $D\bar{f}$, i.e. $D\bar{f}: S_p \rightarrow S_{\bar{f}(p)}$, then (2.3) implies the r -normal hyperbolicity (for every r) of \bar{f} at Γ , with trivial unstable normal bundle: in the notation of [Hin21, §2.3], one can take $\bar{N}^u = 0$ (trivial bundle over Γ) and $\bar{N}^s = S$. Since then all assumptions of [Hin21, Theorem 2.3] are satisfied, we obtain the existence and uniqueness of Γ^u as claimed. The final statement follows immediately from the proof of [Hin21, Theorem 2.3] which, following [HPS77], is based on a graph transform argument à la Hadamard. See in particular [Hin21, equation (2.22)] where the constant denoted C_Σ there can be taken arbitrary large but fixed (cf. [Hin21, equation (2.36)]).

To complete the proof, it thus suffices to show that the assumptions (2.1)–(2.3) imply the existence of a smooth Df -invariant complementary subbundle $\bar{N}^s \subset T_\Gamma \mathcal{X}$. We work in the splitting

$$T_\Gamma \mathcal{X} = T\Gamma \oplus S. \quad (2.8)$$

Write $\pi_S: T_\Gamma \mathcal{X} \rightarrow S$ for the projection, and $S_p \bar{f} = \pi_S \circ D_p \bar{f}|_{S_p}: S_p \rightarrow S_{\bar{f}(p)}$ for $N_p \bar{f}$ under the identification (2.7). The fiber over a point $p \in \Gamma$ of any complementary bundle can be written as

$$\bar{N}_p^s = \{(A_p(w), w): w \in S_p\} \quad (2.9)$$

for some $A_p \in \text{Hom}(S_p, T_p \Gamma)$. We thus need to find

$$A \in \text{Hom}(S, T\Gamma) \quad \text{s.t.} \quad D_p \bar{f}(A_p(w), w) = (A_{\bar{f}(p)}(w'), w') \quad \forall p \in \Gamma, \quad w \in S_p.$$

The desired A is thus an invariant section for the map $\ell: A \mapsto A'$ defined by

$$D_{\bar{f}^{-1}(p)}\bar{f}(A'_{\bar{f}^{-1}(p)}(w), w) = (A_p(w'), w'). \quad (2.10)$$

Define $\Gamma S\bar{f}$ by writing

$$D_p\bar{f} = \begin{pmatrix} \Gamma_p\bar{f} & \Gamma S_p\bar{f} \\ 0 & S_p\bar{f} \end{pmatrix}$$

in the splitting (2.8). In (2.10), we must then have $w' = S_{\bar{f}^{-1}(p)}\bar{f}(w)$, and therefore

$$A'_{\bar{f}^{-1}(p)} = (\Gamma_{\bar{f}^{-1}(p)}\bar{f})^{-1} \circ (A_p \circ S_{\bar{f}^{-1}(p)}\bar{f} - \Gamma S_{\bar{f}^{-1}(p)}\bar{f}). \quad (2.11)$$

Note that $\ell: A \mapsto A'$ is a fiber bundle map (in fact, affine linear) on the bundle $\text{Hom}(S, T\Gamma) \rightarrow \Gamma$ covering \bar{f}^{-1} . We estimate the operator norm k_p of the linearization of $\ell_p: A_p \mapsto A'_{\bar{f}^{-1}(p)}$ (which maps $B \mapsto (\Gamma_{\bar{f}^{-1}(p)}\bar{f})^{-1} \circ B \circ S_{\bar{f}^{-1}(p)}\bar{f}$) by

$$k_p \leq \|(\Gamma_{\bar{f}^{-1}(p)}\bar{f})^{-1}\| \|S_{\bar{f}^{-1}(p)}\bar{f}\| = m(\Gamma_{\bar{f}^{-1}(p)}\bar{f})^{-1} \|S_{\bar{f}^{-1}(p)}\bar{f}\|.$$

For every $r \in \mathbb{N}$ we can choose the fiber metric on $T\Gamma$ so that $\alpha_p := \|(\Gamma_{\bar{f}^{-1}(p)}\bar{f})^{-1}\| < 1 + \epsilon$ for any fixed $\epsilon > 0$. By (2.3), this implies

$$k_p \alpha_p^{r'} \leq \left(\sup_{p \in \Gamma} \|S_{\bar{f}^{-1}(p)}\bar{f}\| \right) (1 - \epsilon)^{-1} (1 + \epsilon)^{r'} < 1 \quad \forall r' = 0, \dots, r$$

if ϵ is small enough. [HPS77, Theorem 3.5] now applies (with $r \in \mathbb{N}$ there arbitrary) and produces the desired smooth section $A \in \text{Hom}(S, T\Gamma)$. \square

Remark 2.3 (Introductory example). In the context of example (1.1), we take $\mathcal{X} = \mathbb{S}_\phi^1 \times \mathbb{R}_z$, $\Gamma = \mathbb{S}_\phi^1 \times \{0\}$, and $\bar{f}(\phi, z) = (\phi + a - z, z/2)$; the non-compactness of \mathcal{X} in z is irrelevant since we are working near $z = 0$ anyway. Identifying $N_{(\phi, z)}\Gamma \cong S_{(\phi, z)} := \text{span}\{\partial_z\}$ and the norms $\|v\partial_\phi\| = |v|$, $\|w\partial_z\| = |w|$, we indeed have (2.3). The desired A_p is a scalar, and (2.11) reads $A'_{\bar{f}^{-1}(\phi, 0)} = 1^{-1} \cdot (A_{(\phi, 0)} \cdot \frac{1}{2} + 1)$ with fixed point $A \equiv 2$; and indeed $\bar{N}^s = \text{span}\{2\partial_\phi + \partial_z\}$ is invariant under $D\bar{f}$.

For us, \bar{f} will arise as the time 1 flow of a vector field. We proceed to discuss the first condition in (2.3) in this context in some generality. Let M be a smooth manifold, and let $X \subset M$ a smooth compact submanifold. Let $V \in \mathcal{V}(M)$ be tangent to X . Write ϕ_V^t for the time t flow of V , and $N_x\phi_V^t: N_x X \rightarrow N_{\phi_V^t(x)} X$ for the map induced by $D_x\phi_V^t: T_x M \rightarrow T_{\phi_V^t(x)} M$ on the normal bundle. Write $\mathcal{I} \subset \mathcal{C}^\infty(M)$ for the ideal of functions vanishing on X . Define the *linearization* of V at X by

$$L_V \in \text{Diff}^1(X; N^*X), \quad L_V: df \mapsto d(Vf) \quad (f \in \mathcal{I}). \quad (2.12)$$

Since $\mathcal{C}^\infty(X; N^*X) \cong \mathcal{I}/\mathcal{I}^2$ via $df \mapsto f + \mathcal{I}^2$, $f \in \mathcal{I}$, and since $V: \mathcal{I}^j \rightarrow \mathcal{I}^j$, $j = 1, 2$, this is well-defined. (Equivalently, $L_V: \mathcal{C}^\infty(X; N^*X) \ni \xi \mapsto d\langle \xi, V \rangle \in \mathcal{C}^\infty(X; N^*X)$ where $\langle \cdot, \cdot \rangle$ is the dual pairing.) Furthermore, the Leibniz rule $L_V(g df) = g L_V(df) + (Vg)df$ holds. Given a local frame df_1, \dots, df_k of N^*X over $U \subset X$, we thus have (summation convention)

$$L_V(g^i df_i) = (Vg^j + \ell^j g^i) df_j, \quad g_i \in \mathcal{C}^\infty(U), \quad L_V(df_i) =: \ell^j_i df_j, \quad \ell^j_i \in \mathcal{C}^\infty(U). \quad (2.13)$$

Therefore, L_V induces a partial connection on N^*X along integral curves of V in X , and therefore a notion of parallel transport $P_V^t: N_x^*X \rightarrow N_{\phi_V^t(x)}^*X$. (Explicitly, setting $\gamma(s) = \phi_V^s(x)$, we have $P_V^t(t)w = w(t)$ where $w(0) = w = w^i df_i$ and $(w^j)' + \ell^j_i w^i = 0$.)

Lemma 2.4 (Parallel transport and normal linearization). $N_x\phi_V^t = (P_V^t(t)^*)^{-1}: N_x X \rightarrow N_{\phi_V^t(x)} X$.

Proof. We begin with a preliminary computation at x . For $f \in \mathcal{I}$, we compute

$$\left. \frac{d}{dt} (df \circ N_x\phi_V^t) \right|_{t=0} = \left. \frac{d}{dt} (df \circ D_x\phi_V^t) \right|_{t=0} = \left. \frac{d}{dt} ((\phi_V^t)^*(df)) \right|_{t=0} = \mathcal{L}_V(df) = d(Vf) = L_V(f).$$

More generally, if $f \in \mathcal{I}$, $g \in \mathcal{C}^\infty(M)$, then $(Vf)|_X = 0$ implies

$$\left. \frac{d}{dt}(g \, df \circ N_x \phi_V^t) \right|_{t=0} = d(i_V(g \, df)) + i_V d(g \, df) = g L_V(f) + (Vg)df = L_V(g \, df)$$

by the Leibniz rule for L_V . Thus $\left. \frac{d}{dt}(\xi \circ N_x \phi_V^t) \right|_{t=0} = L_V \xi$ for all $\xi \in \mathcal{C}^\infty(X; N^*X)$. We apply this to $(\phi_V^{t_0})^* \xi$ where $t_0 \in \mathbb{R}$ and observe $L_V((\phi_V^{t_0})^* \xi) = d((\phi_V^{t_0})^* \xi, V) = d((\phi_V^{t_0})^* \langle \xi, D_x \phi_V^t(V) \rangle) = (\phi_V^{t_0})^* d\langle \xi, V \rangle = (\phi_V^{t_0})^* L_V \xi$ to get

$$\frac{d}{dt}(\xi \circ N_x \phi_V^t) = (\phi_V^t)^* L_V \xi. \quad (2.14)$$

Let now $n_0 \in N_x X$ and $\xi_0 \in N_x^* X$. For $\xi := P_V^t(\xi_0) \in N_{\phi_V^t(x)} X$, we then have

$$\frac{d}{dt} \langle P_V^t(\xi_0), N_x \phi_V^t(n_0) \rangle = \frac{d}{dt} (\xi \circ N_x \phi_V^t)(n_0) = 0$$

by (2.14). Therefore, $\langle \xi_0, n_0 \rangle = \langle P_V^t(\xi_0), N_x \phi_V^t(n_0) \rangle$ for all t , as was to be shown. \square

Corollary 2.5 (Normal contraction rate). *Let $\theta \in \mathbb{R}$. Then the following are equivalent:*

- (1) \exists fiber inner product on NX with $\|N_x \phi_V^t\| \leq e^{-\theta t}$ for all $t \geq 0$, $x \in X$;
- (2) \exists fiber inner product on N^*X with $\left. \frac{d}{dt} \|P_V^t w\| \right|_{t=0} \geq \theta \|w\|$ for all $x \in X$, $w \in N_x^* X$.

Proof. Take dual inner products on NX and N^*X . \square

3. CONSTRUCTION OF PERTURBED HORIZONS

We consider Kerr–Newman(–de Sitter or –anti de Sitter) spacetimes with mass $\mathbf{m} \geq 0$, angular momentum a , charge Q , and cosmological constant $\Lambda \in \mathbb{R}$. We consider a value r_+ of the standard coordinate r at which there is a non-degenerate horizon. Special cases include subextremal Kerr ($\Lambda = 0$, and $r_+ = \mathbf{m} + \sqrt{\mathbf{m}^2 - a^2}$ is the radius of the event horizon), de Sitter space ($\mathbf{m} = 0$, $\Lambda > 0$, and $r_+ = \sqrt{3/\Lambda}$ is the radius of the cosmological horizon), and subextremal Kerr–(Newman–)de Sitter ($\Lambda > 0$, and r_+ is the radius of either the event or the cosmological horizon). In Boyer–Lindquist coordinates, the metric (see e.g. [PG06, §A]) is

$$g_0 = -\frac{\mu}{b^2 \varrho^2} (dt - a \sin^2 \theta \, d\phi)^2 + \varrho^2 \left(\frac{dr^2}{\mu} + \frac{d\theta^2}{\varkappa} \right) + \frac{\varkappa \sin^2 \theta}{b^2 \varrho^2} (a \, dt - (r^2 + a^2) \, d\phi)^2,$$

$$b = 1 + \frac{\Lambda a^2}{3}, \quad \varkappa = 1 + \frac{\Lambda a^2}{3} \cos^2 \theta, \quad \varrho^2 = r^2 + a^2 \cos^2 \theta, \quad \mu = (r^2 + a^2) \left(1 - \frac{\Lambda r^2}{3} \right) - 2mr + b^2 Q^2.$$

We require $b > 0$. We consider a value r_+ where $\mu(r_+) = 0$ and $\mu'(r_+) > 0$. These conditions are valid at the event horizon.¹ We pass to Kerr star type coordinates t_*, r, θ, ϕ_* regular near $r = r_+$,

$$dt = dt_* - \frac{b(r^2 + a^2)}{\mu} dr, \quad d\phi = d\phi_* - \frac{ba}{\mu} dr.$$

Writing covectors as $\zeta = \sigma dt_* + \xi dr + \eta_\theta d\theta + \eta_\phi d\phi_*$, we then compute for $G_0(\zeta) = g_0^{-1}(\zeta, \zeta)$:

$$G_0 := \varrho^2 G_0 = \mu \xi^2 + 2b((r^2 + a^2)\sigma + a\eta_\phi)\xi + \mathcal{C}_0, \quad \mathcal{C}_0 := \varkappa \eta_\theta^2 + \frac{b^2}{\varkappa \sin^2 \theta} (\eta_\phi + a \sin^2 \theta \, \sigma)^2. \quad (3.1)$$

Since $\varrho^2 g_0^{-1}(dr, dr) = \mu = 0$ at $r = r_+$, the hypersurface

$$\mathcal{H}_0 := \mathbb{R}_{t_*} \times \mathbb{S}_{\theta, \phi_*}^2 \times \{r_+\} \subset M = \mathbb{R}_{t_*} \times \mathbb{S}_{\theta, \phi_*}^2 \times (r_+ - 2\delta, r_+ + 2\delta) \quad (3.2)$$

is a null hypersurface (the *event horizon* for g_0), and G_0 vanishes on $N^* \mathcal{H}_0$, in particular on

$$N_-^* \mathcal{H}_0 = \{(t_*, r, \theta, \phi_*, \sigma, \xi, \eta_\theta, \eta_\phi) : r = r_+, \sigma = \eta_\theta = \eta_\phi = 0, \xi < 0\}.$$

¹One can equally well consider the case $\mu'(r_+) < 0$ which occurs at the cosmological horizon of a de Sitter type spacetime; we leave it to the reader to track the sign changes in the arguments below.

We consider here only the half $\xi < 0$ of $N^*\mathcal{H}_0 \setminus o$ since $H_{G_0}t_* = \partial_\sigma G_0 = 2b(r_+^2 + a^2)\xi < 0$ there. It will be convenient to work on the spherical cotangent bundle $S^*M = (T^*M \setminus o)/\mathbb{R}_+$ near $SN^*\mathcal{H}_0$ (the image of $N^*\mathcal{H}_0$ in S^*M) by introducing the fiber coordinates

$$\hat{\sigma} = \frac{\sigma}{\xi}, \quad \hat{\eta}_\theta = \frac{\eta_\theta}{\xi}, \quad \hat{\eta}_\phi = \frac{\eta_\phi}{\xi}.$$

For now, we keep track also of the inverse fiber-radial coordinate $\rho_\infty = -\frac{1}{\xi} > 0$. Then

$$\begin{aligned} \rho_\infty H_{G_0} &= -2b((r^2 + a^2)\partial_{t_*} + a\partial_{\phi_*}) - 2(\mu + b((r^2 + a^2)\hat{\sigma} + a\hat{\eta}_\phi))\partial_r \\ &\quad - (\mu' + 4br\hat{\sigma})(\rho_\infty\partial_{\rho_\infty} + \hat{\sigma}\partial_{\hat{\sigma}} + \hat{\eta}_\theta\partial_{\hat{\eta}_\theta} + \hat{\eta}_\phi\partial_{\hat{\eta}_\phi}) + \rho_\infty H_{\mathcal{G}_0}; \\ H_{\mathcal{G}_0} := \rho_\infty H_{\mathcal{G}_0} &= -2\mathcal{K}\hat{\eta}_\theta\partial_\theta - \frac{2b^2}{\mathcal{K}\sin^2\theta}(\hat{\eta}_\phi + a\sin^2\theta\hat{\sigma})(\partial_{\phi_*} + a\sin^2\theta\partial_{t_*}) \\ &\quad + \left(\mathcal{K}'\hat{\eta}_\theta^2 + \partial_\theta\left(\frac{b^2}{\mathcal{K}\sin^2\theta}(\hat{\eta}_\phi + a\sin^2\theta\hat{\sigma})^2\right)\right)\partial_{\hat{\eta}_\theta} \end{aligned} \quad (3.3)$$

By homogeneity, these induce vector fields on S^*M : this removes the term involving $\rho_\infty\partial_{\rho_\infty}$. We now restrict to the characteristic set $G_0^{-1}(0)$: note that $\partial_r G_0 = \mu'\xi^2 > 0$ at $N^*\mathcal{H}_0$, so r can be written as a function of $\sigma, \xi, \eta_\theta, \eta_\phi, \theta, \phi_*$ in a conic neighborhood, and thus (by homogeneity) as a function $r = r_0(\hat{\sigma}, \hat{\eta}_\theta, \hat{\eta}_\phi, \theta, \phi_*)$. Restricting to $\{\rho_\infty G_0 = 0\}$ thus allows us to further drop the ∂_r component. Finally normalizing the coefficient of ∂_{t_*} to be -1 , we thus define

$$\begin{aligned} V_0 &:= \frac{1}{2b(r^2 + a^2)}(\rho_\infty H_{G_0})|_{(\rho_\infty G_0)^{-1}(0)} = -\partial_{t_*} + \bar{V}, \\ \bar{V} &= -\frac{\mu'(r) + 4br\hat{\sigma}}{2b(r^2 + a^2)}(\hat{\sigma}\partial_{\hat{\sigma}} + \hat{\eta}_\theta\partial_{\hat{\eta}_\theta} + \hat{\eta}_\phi\partial_{\hat{\eta}_\phi}) - \frac{a}{r^2 + a^2}\partial_{\phi_*} + \frac{1}{2b(r^2 + a^2)}H_{\mathcal{G}_0}, \end{aligned} \quad (3.4)$$

where \bar{V} is tangent to

$$\Gamma := \{\hat{\sigma} = \hat{\eta}_\theta = \hat{\eta}_\phi = 0\} \subset \mathcal{X} := \mathbb{R}_{\hat{\sigma}} \times (T^*\mathbb{S}^2)_{\theta, \phi_*, \hat{\eta}_\theta, \hat{\eta}_\phi}. \quad (3.5)$$

Thus, Γ is a cross section of $SN^*\mathcal{H}_0$ projected off the r -coordinate; and $\bar{V} \in \mathcal{V}(\mathcal{X})$. Let $\mathcal{U} \subset \mathcal{X}$ denote a precompact open neighborhood of Γ so that the time 1 flow $e^{\bar{V}}$ of \bar{V} remains in \mathcal{X} . Define²

$$\bar{f} := e^{\bar{V}} : \mathcal{U} \rightarrow \mathcal{X}.$$

Lemma 3.1 (Normal sink nature of Γ). *Fix the standard metric on \mathbb{S}^2 on $\Gamma = \mathbb{S}_{\theta, \phi_*}^2$. Then $\bar{f}|_\Gamma$ is an isometry. There exists a fiber metric on $N\Gamma$ so that $\|N_p \bar{f}\| < 1$, $p \in \Gamma$, and thus (2.3) holds.*

Proof. The first part follows from $\bar{f}|_\Gamma : (\theta, \phi_*) \mapsto (\theta, \phi_* - \frac{a}{r_+^2 + a^2})$ being a rotation. For the second part, we write $\kappa := \frac{\mu'(r_+)}{2b(r_+^2 + a^2)} > 0$ and compute the linearization $L_{\bar{V}}$ of \bar{V} at Γ (see (2.12)) to map $\xi \mapsto -\kappa\xi$ for $\xi = d\hat{\sigma}, d\hat{\eta}_\theta, d\hat{\eta}_\phi$. We thus read off from $L_{\bar{V}}$ the bundle endomorphism

$$\text{End}(N^*\Gamma) = \text{End}(\mathbb{R}d\hat{\sigma} \oplus T_o^*(T^*\mathbb{S}^2)) \ni \ell = -\kappa.$$

This is the map encoded by ℓ^i_j in (2.13) upon using as a local frame $d\hat{\sigma}$ and differentials of fiber-linear functions on $T^*\mathbb{S}^2$ (for a cover of \mathbb{S}^2 by coordinate charts).

Fix the fiber inner product $\langle \cdot, \cdot \rangle := I \oplus \mathfrak{g}$ where \mathfrak{g} is the standard inner product on $T_o^*(T^*\mathbb{S}^2)$ coming from $T^*\mathbb{S}^2$; then $\langle w, \ell w \rangle = -\kappa\langle w, w \rangle$. Let $\|u\| := \langle u, u \rangle^{1/2}$. Given $w \in N_x^*\Gamma$, we then have for $w' := -\ell w$ the inequality $\frac{d}{dt}\|w + tw'\| \Big|_{t=0} \geq \kappa\|w\|$ (in fact, equality holds). Corollary 2.5 gives $\|N_p \bar{f}\| = \|N_p \phi_V^1\| \leq e^{-\kappa} < 1$, as desired. (This is a somewhat refined version of arguments appearing in Vasy [Vas13, §6.3] and, in greater generality, Gannot [Gan19].) \square

²If we smoothly cut off \bar{V} to 0 outside of an open neighborhood of Γ so that the resulting vector field \bar{V}' vanishes outside of \mathcal{U} , then we can also replace \mathcal{X} by a compact manifold, and $e^{\bar{V}'} : \mathcal{X} \rightarrow \mathcal{X}$ diffeomorphically. This puts us into the setting of §2. Since all considerations for now are local near Γ , we permit ourselves the minor imprecision of working with (3.5).

Theorem 3.2 (Construction of the perturbed horizon). *Let ρ be a weight function. Let g be a Lorentzian metric on $M = \mathbb{R}_{t_*} \times X$, $X = \mathbb{S}^2 \times (r_+ - 2\delta, r_+ + 2\delta)$, so that $g - g_0 \in \rho\mathcal{C}_b^\infty$ on $\{t_* \geq 0\}$, i.e. the metric coefficients in stationary frames consisting of ∂_{t_*} and local frames of TX are of class $\rho\mathcal{C}_b^\infty$. Then there exists t_0 so that there exists a unique smooth null hypersurface $\mathcal{H} \subset [t_0, \infty) \times X$ which approaches \mathcal{H}_0 in a $\rho\mathcal{C}_b^\infty$ sense. One can take $t_0 = 0$ when $\|g - g_0\|_{\rho\mathcal{C}_b^2}$ is sufficiently small.*

Proof. • Construction of \mathcal{H} . For a subset $A \subset S^*M$, denote by $C(A) \subset T^*M \setminus o$ its conic extension. Let $G(\zeta) = g^{-1}(\zeta, \zeta)$, $\mathbf{G} = \varrho^2 G$. For a t_* -independent neighborhood \mathcal{U}_0 of $\Gamma_0 = \mathbb{R}_{t_*} \times \Gamma$, we have $\rho_\infty H_{\mathbf{G}} t_* < 0$ on \mathcal{U}_0 and $\partial_r \mathbf{G} > 0$ on $C(\mathcal{U}_0)$. The latter allows us to write $\{\rho_\infty^2 \mathbf{G} = 0\} \cap \mathcal{U}_0$ as the graph of a smooth function $r: \mathcal{X} \rightarrow (r_+ - 2\delta, r_+ + 2\delta)$ approaching the analogous function r_+ for $\rho_\infty^2 \mathbf{G}_0$ in a $\rho\mathcal{C}_b^\infty$ sense. (See [Hin21, Lemma 4.6] for a similar argument.) Write $F: \mathcal{X} \rightarrow \mathcal{X} \times (r_+ - 2\delta, r_+ + 2\delta)$, $x \mapsto (x, r(x))$, and define F_0 analogously using r_0 in place of r . Then

$$V := F^* \left(\frac{1}{-\rho_\infty H_{\mathbf{G}} t_*} (\rho_\infty H_{\mathbf{G}})|_{S^*M} \right) \in \mathcal{V}(\mathbb{R}_{t_*} \times \mathcal{X})$$

is a $\rho\mathcal{C}_b^\infty$ -perturbation of V_0 in (3.4) and satisfies $V t_* = -1$. By Lemma 3.1, we can apply Theorem 2.2 to its time 1 flow $f = \phi_V^1$ to obtain a smooth f -invariant submanifold

$$\Gamma^u \subset [t_0, \infty) \times \mathcal{X}$$

approaching Γ_0 in a $\rho\mathcal{C}_b^\infty$ sense. Applying the same result, for Nt_* in place of t_* , to the time $1/N$ flow of V shows that Γ^u is also $\phi_V^{1/N}$ -invariant; taking $N \rightarrow \infty$ implies that V is tangent to Γ^u .

We now map Γ^u back into $\{\rho_\infty^2 \mathbf{G} = 0\}$ via F . Thus $C(F(\Gamma^u)) \subset \mathbf{G}^{-1}(0)$, and $H_{\mathbf{G}}$ is tangent to $C(F(\Gamma^u))$. Increasing t_0 if necessary, the base projection $\pi: C(F(\Gamma^u)) \rightarrow [t_0, \infty) \times X$ has rank 3 since this is true for $\pi: C(F_0(\Gamma_0)) = N^* \mathcal{H}_0 \setminus o \rightarrow M$; its image is thus a smooth hypersurface

$$\mathcal{H} = \pi(C(F(\Gamma^u))) \subset \{t_* \geq t_0\} \cap M.$$

By construction, for $z_0 \in \mathcal{H}$ and $0 \neq \zeta_0 \in T_{z_0}^* M$ with $(z_0, \zeta_0) \in C(F(\Gamma^u))$, the $H_{\mathbf{G}}$ -integral curve starting at (z_0, ζ_0) remains in $C(F(\Gamma^u))$ as long as it remains in $\{t_* \geq t_0\}$. Therefore, its projection along π , which is a null-geodesic, remains in \mathcal{H} . We draw two conclusions: first, $T_{z_0} \mathcal{H}$ is null or Lorentzian, as it contains a null vector ζ_0^\sharp ; second,

$$\mathcal{H} \text{ is ruled by null-geodesics.} \tag{3.6}$$

• \mathcal{H} is null: reduction. The main task is to show that \mathcal{H} (i.e. each $T_{z_0} \mathcal{H}$) is null. It suffices to prove this for $\mathcal{H} \cap \{t_* \geq t_1\}$ for any $t_1 \geq t_0$. We recall the classical argument. First, let $L \in \mathcal{V}(\mathcal{H})$ be a positive multiple of ζ_0^\sharp at each $z_0 \in \mathcal{H}$ with $(z_0, \zeta_0) \in C(F(\Gamma^u))$. Then the integral curves of L are reparameterized past null-geodesics, so $\nabla_L L = cL$, $c \in \mathcal{C}^\infty(\mathcal{H})$. For $X \in \mathcal{C}^\infty(\mathcal{H} \cap \{t_* = t_1\}; T\mathcal{H})$, define $\tilde{X} \in \mathcal{V}(\mathcal{H} \cap \{t_0 \leq t_* \leq t_1\})$, $\tilde{X}|_{t_* = t_1} = X$, via $\mathcal{L}_L \tilde{X} = 0$. Then $Lg(L, \tilde{X}) = g(\nabla_L L, \tilde{X}) + g(L, [L, \tilde{X}]) + \frac{1}{2} \tilde{X}g(L, L) = cg(L, \tilde{X})$. Since $g(L, \tilde{X}) = g(L, X) = 0$ at $t_* = t_1$, we obtain $g(L, \tilde{X}) \equiv 0$. Since for $z_0 \in \mathcal{H} \cap \{t_0 \leq t_* \leq t_1\}$, the space of $\tilde{X}|_{z_0} \in T_{z_0} \mathcal{H}$ for such X is all of $T_{z_0} \mathcal{H}$, this shows that $T_{z_0} \mathcal{H} = L^\perp$ is null indeed.

• Decomposition of M in $t_* \geq t_1$. Fix now $t_1 \geq t_0$ so that dr is timelike at $r = r_+ - \delta$ for $t_* \geq t_1$, and so that at each point $z_0 \in M \cap \{t_* \geq t_1\}$ with $r(z_0) = r_+ + \delta$ there exist future null vectors $W_+, W_0 \in T_{z_0} M$, with smooth dependence on z_0 , so that $dr(W_+) > 0$, $dr(W_0) = 0$. (Since such W_+, W_0 exist for the metric g_0 , a time t_1 with this property exists.) Define the sets

$$\begin{aligned} B &:= \{z \in M \cap \{t_* \geq t_1\} : \forall \text{ maximal future null-geodesics } \gamma: [0, \bar{s}) \rightarrow M, \gamma(0) = z, \\ &\quad \text{we have } \underline{r} := \liminf_{s \rightarrow \bar{s}} r(\gamma(s)) < r_+\}, \\ E &:= \{z \in M \cap \{t_* \geq t_1\} : \exists \text{ future causal curve } \gamma: [0, \bar{s}) \rightarrow M, \gamma(0) = z, \\ &\quad \text{so that } \lim_{s \rightarrow \bar{s}} t_*(\gamma(s)) = \infty, \bar{r} := \limsup_{s \rightarrow \bar{s}} r(\gamma(s)) > r_+\}. \end{aligned} \tag{3.7}$$

We shall prove that

$$M \cap \{t_* \geq t_1\} = B \sqcup (\mathcal{H} \cap \{t_* \geq t_1\}) \sqcup E. \quad (3.8)$$

Given $z \in B$ and a maximal future null-geodesic $\gamma: [0, \bar{s}) \rightarrow M$ starting at z , suppose that $t_* \circ \gamma$ is bounded; then γ must exit M , so $r = r_+ - 2\delta$. If $t_* \circ \gamma$ is unbounded, we use that dr is timelike for $r \leq r'$ for any fixed $r' < r_+$ when t_* is sufficiently large: this forces $r \circ \gamma$ to be monotonically decreasing for sufficiently large s , and γ exits M through $r = r_+ - 2\delta$ again. Thus, B is open and disjoint from E . Moreover, $B \supset \{r = r_+ - \delta, t_* \geq t_1\}$ is non-empty.

Similarly, when $r' > r_+$ is fixed, $r(\gamma(s)) \geq r'$, and $t_*(\gamma(s))$ is large enough depending on r' , then γ can be continued past $\gamma(s)$ as a future timelike curve which reaches $\{r = r_+ + \delta\}$ since the same is true on (M, g_0) . This implies that E is open. Moreover, $E \supset \{r = r_+ + \delta, t_* \geq t_1\}$ is non-empty.

Let now $z \in (M \cap \{t_* \geq t_1\}) \setminus (B \cup E)$. Since $z \notin B$, there exists a maximal future null-geodesic γ with $\gamma(0) = z$ and $r \geq r_+$. Since $z \notin E$, $t_* \circ \gamma$ is bounded or $\bar{r} \leq r_+$. If $\bar{r} > r_+$ and $t_* \circ \gamma$ is bounded, then $r \circ \gamma$ must leave every compact subset of $(r_+ - 2\delta, r_+ + 2\delta)$, which in view of $r \geq r_+$ forces $r(\gamma(s)) \geq r_+ + \delta$ for some $s \in [0, \bar{s})$. Continuing γ past $\gamma(s)$ as an integral curve of W_0 produces a future causal curve starting at z along which t_* is unbounded and $\bar{r} = r_+ + \delta$, in contradiction to $z \notin E$. Therefore, $\bar{r} \leq r_+$ and thus

$$\lim_{s \rightarrow \bar{s}} r(\gamma(s)) = r_+, \quad \lim_{s \rightarrow \bar{s}} t_*(\gamma(s)) = \infty.$$

Lift γ to phase space as an integral curve of H_G and project it to S^*M . Call the resulting curve $\alpha: [0, \bar{s}) \rightarrow S^*M$; we reparameterize it so that $(t_* \circ \alpha)' = 1$. The projection $[\alpha]$ of α to the quotient S^*M/\mathbb{R} of S^*M by t_* -translations remains in a compact set. Denote by $\varpi \in S^*M/\mathbb{R}$ a point in the ω -limit set of $[\alpha]$; it satisfies $r(\varpi) = r_+$. We claim that

$$\varpi \in SN_+^* \mathcal{H}_0 / \mathbb{R} \quad (3.9)$$

where $SN_+^* \mathcal{H}_0$ is the *future* component, with $\xi > 0$, of the spherical conormal bundle of \mathcal{H}_0 . Denote by $L^+ \subset S^*M \cap \{r \leq r_+\}$ the projection to S^*M of the future component of $\mathbf{G}_0^{-1}(0) \cap (T^*M \setminus o)$ in $r \leq r_+$. To show (3.9), note that $\rho_\infty H_{G_0} r < 0$ has a strictly negative upper bound in a t_* -translation invariant neighborhood $\mathcal{U}_0 \subset S^*M$ of $\{r \leq r_+\} \cap (L^+ \setminus \mathcal{V})$, where \mathcal{V} is any fixed t_* -translation invariant neighborhood of $SN_+^* \mathcal{H}_0$. Indeed, in $r < r_+$ we have $H_{G_0} r < 0$ along future null-bicharacteristics since dr is timelike there due to $\mu(r) < 0$, cf. (3.1). By continuity, $H_{G_0} r \leq 0$ in $r = r_+$ in the future characteristic set; but $H_{G_0} r = 0$ and $r = r_+$ imply by (3.3) that $(r^2 + a^2)\sigma + a\eta_\phi = 0$, hence from $\mathbf{G}_0 = 0$ also $\mathcal{C}_0 = 0$ by (3.1). This implies $\sigma = \eta_\theta = \eta_\phi = 0$, so one is at $N^* \mathcal{H}_0$. — We then also have a strict negative upper bound for $\rho_\infty H_G r$ on $\mathcal{U}_0 \cap \mathbf{G}^{-1}(0) \cap \{t_* \geq t_2\}$ for sufficiently large $t_2 \geq t_1$. If (3.9) were false, we could choose \mathcal{V} to have closure disjoint from ϖ ; we would then obtain a contradiction to $r \geq r_+$.

Having established (3.9) for all points in the ω -limit set of $[\alpha]$, we infer that α in fact converges to $SN_+^* \mathcal{H}_0$, and thus $-\alpha$ (fiberwise minus sign) converges to $SN_-^* \mathcal{H}_0$ as $t_* \rightarrow \infty$; and its image is an integral curve of $\rho_\infty H_G$. Upon shifting the argument of $-\alpha$ by a constant, we thus have $F^{-1}((-\alpha)(s)) = (s, \hat{\sigma}(s), \omega(s), \hat{\eta}(s)) \in \mathbb{R}_{t_*} \times \mathbb{R}_{\hat{\sigma}} \times (T^*S^2)_{\omega, \hat{\eta}}$ where $|\hat{\sigma}(s)| + |\hat{\eta}(s)| \rightarrow 0$ as $s \rightarrow \infty$. Using Lemma 3.3 below, there exists a weight function $\tilde{\rho}$ so that $|\hat{\sigma}(s)| + |\hat{\eta}(s)| \leq \tilde{\rho}(s)$. In the application of Theorem 2.2 in the proof of Theorem 3.2, we may then start with $\tilde{\Gamma}^u$ which approaches Γ_0 in a $\tilde{\rho}_b^1$ sense and contains the image A of $F^{-1} \circ \alpha$ intersected with $\{t_* \geq t_1\}$. Since α is an integral curve of H_G , every iterate $\tilde{\Gamma}_n^u$ in the notation of Theorem 2.2 contains A , and thus so does the limit Γ^u . Therefore, $\gamma([0, \bar{s})) \subset \mathcal{H}$, and hence $z = \gamma(0) \in \mathcal{H}$.

Let now $\nu \in \mathcal{C}^\infty(M \cap \{t_* \geq t_1\})$ denote a defining function of \mathcal{H} , i.e. $\mathcal{H} = \{\nu = 0\}$, $d\nu \neq 0$ on \mathcal{H} , and $\pm\nu > 0$ at $\{r = r_+ \pm \delta\}$. Define the (open) sets $B_\pm := B \cap \{\pm\nu > 0\}$, $E_\pm := E \cap \{\pm\nu > 0\}$. We have $\{\nu < 0\} = B_- \sqcup E_-$; since $B_- \neq \emptyset$, we get $E_- = \emptyset$ and $B \supset B_- = \{\nu < 0\}$. This implies $E \cap \mathcal{H} = \emptyset$ since the neighborhood of a point in $E \cap \mathcal{H}$ would contain a point lying both in

$\{\nu < 0\} \subset B$ and in E . By (3.6), we have $B \cap \mathcal{H} = \emptyset$. Finally, $\{\nu > 0\} = B_+ \sqcup E_+$ with $E_+ \neq \emptyset$, and thus $B_+ = \emptyset$, $E \supset E_+ = \{\nu > 0\}$. This gives $B = B_-$, $E = E_+$, and proves (3.8).

• *\mathcal{H} is null: conclusion.* Let $z_0 \in \mathcal{H} \cap \{t_* \geq t_1\}$. If $T_{z_0}\mathcal{H} \subset T_{z_0}M$ were not a lightlike hyperplane, it would contain a future timelike vector. Since the set of future timelike vectors is open, there would exist a future timelike vector $W \in T_{z_0}M$ pointing into E (i.e. $d\nu(W) > 0$), and hence a causal curve γ with $\gamma(0) = z_0$ and $\gamma'(0) = W$ would enter E . This would imply $z_0 \in E$, contradicting (3.8).

• *Uniqueness of \mathcal{H} .* Let \mathcal{H} be a null hypersurface which approaches \mathcal{H}_0 in a $\rho\mathcal{C}_b^\infty$ sense. Note that $\xi < 0$ on one component of $\{t_* \geq t_0\} \cap (N^*\mathcal{H} \setminus o)$ when t_0 is sufficiently large. Thus $SN^*\mathcal{H}$ approaches $SN^*\mathcal{H}_0$ in a $\rho\mathcal{C}_b^\infty$ sense, and hence $F^{-1}(SN^*\mathcal{H})$ approaches Γ_0 in a $\rho\mathcal{C}_b^\infty$ sense. The uniqueness part of Theorem 2.2 now yields the claim. (This argument also gives $C(F(\Gamma^u)) = N^*\mathcal{H} \setminus o$.)

Finally, the fact that one can take $t_0 = 0$ when $\|g - g_0\|_{\rho\mathcal{C}_b^2}$ is sufficiently small, and thus the vector fields $\rho_\infty H_{G_0}|_{S^*M}$ and $\rho_\infty H_G|_{S^*M}$ are close in $\rho\mathcal{C}_b^1$, follows from an inspection of the proof of [Hin21, Theorem 2.3]. \square

Lemma 3.3 (Weight function). *Let $d: [0, \infty) \rightarrow [0, \infty)$ with $\lim_{s \rightarrow \infty} d(s) = 0$. Then there exists a weight function ρ with $d(s) \leq \rho(s)$ for all s .*

Proof. Replace d by $s \mapsto \sup_{s' \geq s} d(s')$; then d is monotonically decreasing. Define the sequence

$$\bar{d}(n) := \max\{2^{-k}d(n-k) : k = 0, 1, \dots, n\}, \quad n \in \mathbb{N}_0.$$

Then $\bar{d}(n) \geq d(n)$ for all n , and $\bar{d}(n)$ does not fall off too quickly in that

$$\bar{d}(n+1) = \max(d(n+1), \max\{2^{-k-1}d(n-k) : k = 0, \dots, n\}) \geq \frac{1}{2}\bar{d}(n).$$

Fix $\phi \in \mathcal{C}^\infty([0, 1])$ so that $\phi = 1$ on $[0, \frac{1}{3}]$, $\phi' < 0$ on $(\frac{1}{3}, \frac{2}{3})$, and $\phi = 0$ on $[\frac{2}{3}, 1]$. Set

$$\rho(n+\eta) = 2(\phi(\eta)\bar{d}(n) + (1-\phi(\eta))\bar{d}(n+1)), \quad n \in \mathbb{N}_0, \eta \in [0, 1].$$

This is equal to $\bar{d}(n)$ on $[\max(n - \frac{1}{3}, 0), n + \frac{1}{3}]$ and thus smooth. Moreover, $\rho(n+\eta) \geq 2\phi(\eta)\bar{d}(n) + (1-\phi(\eta))\bar{d}(n) \geq \bar{d}(n) \geq d(n) \geq d(n+\eta)$, and

$$|\rho^{(k)}(n+\eta)| = 2|\phi^{(k)}(\eta)|(\bar{d}(n) - \bar{d}(n+1)) \leq 2|\phi^{(k)}(\eta)|\bar{d}(n) \leq 2\|\phi\|_{\mathcal{C}^k}\rho(n+\eta). \quad \square$$

While \mathcal{H} in Theorem 3.2 is constructed locally, the definition of the event horizon of an asymptotically flat spacetime with future affine complete null infinity \mathcal{S}^+ —namely, the boundary of the causal past $J^-(\mathcal{S}^+)$ of \mathcal{S}^+ —is global. We show that when (M, g_0) is subextremal Kerr–Newman, then \mathcal{H} is the event horizon of (M, g) under mild assumptions on the spacetime. (These assumptions are satisfied by the spacetimes arising in nonlinear stability proofs [DHRT21, KS23, GKS22, She23].)

Theorem 3.4 (Smoothness of the event horizon of asymptotically subextremal Kerr–Newman spacetimes). *Let g_0 denote a subextremal Kerr–Newman spacetime defined on $M = \mathbb{R}_{t_*} \times X$, $X = [r_+ - 2\delta, \infty) \times \mathbb{S}^2$, with the level sets of t_* spacelike and transversal to the future event horizon $\mathcal{H}_0 = \{r = r_+\}$. Let g be a Lorentzian metric on $[0, \infty) \times X$ so that $g - g_0 \in \rho\mathcal{C}_b^\infty$ (in the frame on $\mathbb{R}_{t_*} \times \mathbb{R}^3 \supset \mathbb{R}_{t_*} \times X$ given by coordinate vector fields). Assume moreover that there exists a radius R_0 so that every point in $\{t_* \geq 0, r \geq R_0\}$ can be connected to future null infinity \mathcal{S}^+ by a causal curve. Define $\mathcal{H} \subset \{t_* \geq t_0\}$ using Theorem 3.2, and denote by \mathcal{B} , resp. \mathcal{E} the component of $(M \cap \{t_* \geq t_0\}) \setminus \mathcal{H}$ on which r is bounded, resp. unbounded. Then:*

- (1) every future causal curve starting in \mathcal{B} remains there;
- (2) starting at every point in \mathcal{E} , there exists a future causal curve tending to a point at \mathcal{S}^+ ;
- (3) the only future causal curves remaining in \mathcal{H} are the null-generators of \mathcal{H} .

In particular, \mathcal{H} is the event horizon of (M, g) .

An analogous result holds (by the same proof), *mutatis mutandis*, on asymptotically subextremal Kerr–Newman–de Sitter and –anti de Sitter spacetimes, with the event horizon defined to be the boundary of the causal past of the conformal boundary.

Remark 3.5 (The radius R_0). The existence of R_0 is a condition only on the far field behavior of g . If g_0 has mass parameter $m \in \mathbb{R}$, then this condition is satisfied, for example, when there exists $\delta > 0$ so that, in Boyer–Lindquist coordinates and using $F(r) := 1 - \frac{2m}{r}$, $h := g - g_0$,

$$h(\partial_t + F\partial_r, \partial_t + F\partial_r) = \mathcal{O}(r^{-1-\delta}), \quad h(\partial_t, \partial_r) = o(1), \quad h(\partial_r, \partial_r) = o(1), \quad r \rightarrow \infty. \quad (3.10)$$

Taking g_0 to be the Schwarzschild metric (Kerr–Newman being an $\mathcal{O}(r^{-2})$ perturbation thereof) and using the tortoise coordinate r_* with $dr_* = \frac{dr}{F}$, the radial curve $\gamma(s) = (t(s), r_*(s)) := (t + s, R_0 + s - a(s))$ with $\dot{\gamma}(s) = \partial_t + (1 - \dot{a})\partial_{r_*} = \partial_t + (1 - \dot{a})F\partial_r$ is timelike if and only if $F^{-1}g_0(\dot{\gamma}(s), \dot{\gamma}(s)) = -1 + (1 - \dot{a})^2 = -2\dot{a} + \dot{a}^2 < 0$, i.e. $\dot{a} \in (0, 2)$; and it limits to a point at null infinity if $\lim_{s \rightarrow \infty} a(s) < \infty$. A possible choice is thus $\dot{a} = s^{-1-\delta/2}$; and then $g_0(\dot{\gamma}(s), \dot{\gamma}(s)) < -s^{-1-\delta/2}$ for large enough s . In view of (3.10), we then have $g(\dot{\gamma}(s), \dot{\gamma}(s)) < -\frac{1}{2}s^{-1-\delta/2}$ for large s , as desired.

Proof of Theorem 3.4. We define B, E by (3.7) with t_0 in place of t_1 . Given $r' > r_+$, note that on (M, g_0) we can connect any point $z' \in M$ with $r(z') = r'$ to a point with $r \geq R_0$ using a future timelike curve with compact interval of definition; the same thus remains true on (M, g) when, in addition, $t_*(z')$ is sufficiently large. Applying this for $z \in E$ and a future causal curve $\gamma: [0, \bar{s}) \rightarrow M$, with $\gamma(0) = z$ and $\bar{r} = \limsup_{s \rightarrow \bar{s}} r(\gamma(s)) > r_+$, to $z' = \gamma(s)$ where $t_*(\gamma(s))$ is large and $r(\gamma(s)) \geq r' := \frac{r_+ + \bar{r}}{2}$, we conclude that z can be connected to \mathcal{I}^+ by a causal curve. Therefore, $E \subset J^-(\mathcal{I}^+) \cap \{t_* \geq t_0\}$, whereas $(B \cup \mathcal{H}) \cap J^-(\mathcal{I}^+) = \emptyset$. This implies that \mathcal{H} is the boundary of $J^-(\mathcal{I}^+)$ in $\{t_* \geq t_0\}$, and $\mathcal{B} = B$, $\mathcal{E} = E$. The proof is complete. \square

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