18.950/9501 (S20): FINAL ASSIGNMENT

Due: Tuesday, May 12, by 11:59 PM EST, on Gradescope. **No late submissions accepted.** Collaboration is allowed, but you must write up your own solutions and cite all your collaborators. When writing down solutions, please state any formulas from class or from the book that you use. There are 10 problems in total; each problem is worth 10 points.

Problem 1. Let $\alpha\colon I \to \mathbb{R}^3$ be a regular parameterized curve with nonzero curvature k and nonzero torsion τ. We say that α is a *helix* if there exists a nonzero vector $v_0 \in \mathbb{R}^3$ so that the angle between α' and v_0 is constant. Prove:

(a) (7 points.) α is a helix if and only if k/τ is constant.
(b) (3 points.) α is a helix if and only if the lines $\mathbb{R} \ni t \mapsto \alpha(s) + tn(s)$ are parallel to a fixed (that is, independent of s) plane. Here, $n(s)$ is the normal vector of α.

Problem 2. Prove or disprove the existence of a smooth map ϕ from some[1] nonempty open subset $U \subset \mathbb{S}^2 = \{(x,y,z)\colon x^2+y^2+z^2 = 1\}$ to a nonempty open subset $V \subset \{(x,y,0)\}$ such that

(a) (2 points) ϕ preserves angles (i.e. for all $p \in U$, the angle between any two nonzero tangent vectors $v, w \in T_p\mathbb{S}^2$ is the same as that between $\mathrm{d}\phi_p(v)$ and $\mathrm{d}\phi_p(w)$).
(b) (5 points) ϕ preserves areas (i.e. the area of $\phi(R)$ is equal to the area of R for every bounded region $R \subset U$).
(c) (3 points) ϕ preserves lengths of curves (i.e. the length of $\phi(C)$ is equal to the length of C for every regular curve $C \subset U$).

Problem 3. Let $h\colon [0,1) \to \mathbb{R}$ be a smooth function, and define $\alpha(v) = (v, 0, h(v)) \in \mathbb{R}^3$ for $0 \le v < 1$. Define the set

$$S := \{(v\cos u, v\sin u, h(v))\colon v \in [0,1),\ u \in \mathbb{R}\}.$$

Give necessary and sufficient conditions on $h(v)$ which guarantee that S is a regular surface. (Note that this does not fit our definition of a surface of revolution, since the 'generating curve' α intersects the axis of rotation.) Check your result in the case $h(v) = \sqrt{1-v^2}$ (what surface is S then?).

Problem 4. Let $S = \{(x,y,z)\colon z^2 = x^2+y^2,\ z > 0\}$, which is a cone with its vertex removed. Note that this is the surface of revolution with rotation axis equal to the z-axis, and generating curve $C = \{(z,0,z)\colon z > 0\}$.

(a) (3 points.) Show that S is a regular surface.
(b) (7 points.) Let $p \in S$, and let $\gamma\colon (-\epsilon,\epsilon) \to S$ denote a parameterized geodesic on S passing through p. Show that γ can be extended to a parameterized geodesic $\mathbb{R} \to S$ if and only if γ is not a segment of a meridian of S.

Problem 5. Consider the surfaces S_1, S_2 parameterized by

$$x_1(u,v) = (u\cos v, u\sin v, \log u),$$

Date: May 1, 2020.

[1]That is, you are free to choose the sets U and V as you please (as long as they are nonempty and open, of course).

$$x_2(u,v) = (u\cos v, u\sin v, v),$$

with $u > 0$, $v \in \mathbb{R}$.

(a) (8 points.) Compute the coefficients of the first and second fundamental forms of S_1 and S_2.
(b) (1 point.) Show that the Gauss curvature of S_1 at $x_1(u,v)$ is equal to the Gauss curvature of S_2 at $x_2(u,v)$.
(c) (1 point.) Show that the mapping $x_2 \circ x_1^{-1} \colon S_1 \to S_2$ is not an isometry.

Problem 6. Construct, explicitly, a regular surface $S \subset \mathbb{R}^3$ with constant Gauss curvature -1. Show that your surface S is not complete.

Problem 7. Let S be a compact oriented surface with smooth field of unit normal vectors $N \colon S \to \mathbb{S}^2$. For $\lambda \in \mathbb{R}$ with $|\lambda|$ sufficiently small, let

$$S^\lambda = \{p + \lambda N(p) \colon p \in S\}.$$

Let us work in local coordinates $x(u,v)$ on S, and assume that $x^\lambda(u,v) := x(u,v) + \lambda N(u,v)$ is a local parameterization of S^λ.

(a) (5 points.) Compute the normal vector of S^λ and the differential of the Gauss map of S^λ.
(b) (5 points.) If κ_1, κ_2 are the principal curvatures of S, show that the principal curvatures $\kappa_1^\lambda, \kappa_2^\lambda$ of S_λ are given by

$$\kappa_1^\lambda = \frac{\kappa_1}{1-\lambda\kappa_1}, \quad \kappa_2^\lambda = \frac{\kappa_2}{1-\lambda\kappa_2}.$$

Problem 8. Let S be a compact regular surface in $\mathbb{R}^3$.

(a) (6 points.) Show that the Gauss map $N \colon S \to \mathbb{S}^2$ is surjective.
(b) (2 points.) Let $K_+(p) = \max(0, K(p))$, where $K(p)$ is the Gauss curvature of S at $p \in S$. Show that $\iint_S K_+ \,\mathrm{d}\sigma \geq 4\pi$.
(c) (2 points.) If S has nonzero genus (i.e. is a 'donut with at least 1 hole'), show that also $\iint_S K_- \,\mathrm{d}\sigma \geq 4\pi$, where $K_- = \max(0, -K(p))$.

Problem 9. Let (ρ, θ) be a system of geodesic polar coordinates on a regular surface.

(a) (5 points.) Let $\gamma(s)$ be a parameterized geodesic, given in local coordinates by $(\rho(s), \theta(s))$. Denote by $\phi(s)$ the angle (measured counterclockwise) from the curve $\theta = $ const (oriented in the direction of increasing ρ) to $\gamma'(s)$. Show that

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} + \left(\sqrt{G}\right)_\rho \frac{\mathrm{d}\theta}{\mathrm{d}s} = 0.$$

(b) (5 points.) Consider a geodesic triangle Δ, two of whose edges are radial geodesics. (Its third edge is thus a geodesic of the type studied in the first half of the problem.) Show *by direct computation* (i.e. without using the Gauss–Bonnet theorem) that

$$\sum_{i=1}^{3} \alpha_i = \pi + \iint_\Delta K \,\mathrm{d}\sigma,$$

where $0 < \alpha_i < \pi$, $i = 1,2,3$, are the internal angles of Δ.

Problem 10. Let S be a complete and *noncompact* regular surface. Let $p \in S$. Show that there exists an arc length parameterized geodesic $\gamma \colon [0,\infty) \to S$ with $\gamma(0) = p$ so that $d(\gamma(0), \gamma(s)) = s$ for all $s \in [0,\infty)$. (Here, d denotes the intrinsic distance on S.)