18.950/9501 (S20): HOMEWORK 4

The book references are to do Carmo, *Differential Geometry of Curves and Surfaces*. (The numbers for the assigned problems are the same in both editions of the book.)

Due: Thursday, Mar 5, in class.[1]

Exercise 1. Chapter 2–4, Problem 15.

Exercise 2. Chapter 2–5, Problem 5.

Exercise 3. Suppose the regular connected curve C lies in the xz plane and is parameterized by

$$(a, b) \ni v \mapsto (f(v), 0, g(v)),$$

where f, g are smooth functions, and $f(v) > 0$ for $v \in (a, b)$. The *surface of revolution* S with *generating curve* C is the set

$$S = \{(f(v)\cos u, f(v)\sin u, g(v))\colon u \in \mathbb{R},\ v \in (a, b)\}.$$

(1) Prove that S is a regular surface.
(2) Prove that S is orientable.
(3) Suppose C has finite length $\ell > 0$. Parameterize C by arc length, so $C = \alpha((0, \ell))$, where $\alpha\colon (0, \ell) \to \mathbb{R}^3$ is smooth with $\|\alpha'(s)\| = 1$ for all $s \in (0, \ell)$. Show that the area of S is $2\pi \int_0^\ell \rho(s)\, ds$, where $\rho(s)$ is the distance of $\alpha(s)$ to the z-axis. (That is, $\rho(s)$ is the x-component of $\alpha(s)$.)

Exercise 4. Chapter 2–6, Problem 1.

Exercise 5. Let S_1, S_2 be regular surfaces, and suppose S_2 is orientable. Suppose the smooth map $\phi\colon S_1 \to S_2$ is a local diffeomorphism at every $p \in S_1$. (This means: every $p \in S_1$ has a neighborhood $V \subset S_1$ so that the restriction $\phi|_V\colon V \to \phi(V)$ is a diffeomorphism.) Prove that S_1 is orientable.

Exercise 6. Let S be a regular surface, and assume S is compact and orientable; denote by $N\colon S \to \mathbb{R}^3$ a smooth field of unit normal vectors on S.

(1) Let $p_0 \in S$. Show that there exists an open neighborhood $V_{p_0} \subset S$ of p_0 and a positive number $\epsilon_{p_0} > 0$ so that

$$V_{p_0,a} := \{p + aN(p)\colon p \in V_{p_0}\}$$

is a regular surface for all $a \in (-\epsilon_{p_0}, \epsilon_{p_0})$.
(2) Show that there exists a finite number $k \in \mathbb{N}$ and a collection of points $p_0, \ldots, p_k \in S$ so that the open neighborhoods V_{p_i} from part (i) cover S: that is, $S = \bigcup_{i=0}^k V_{p_i}$. Conclude that there exists $\epsilon > 0$ so that

$$S_a := \{p + aN(p)\colon p \in S\}$$

is a regular surface for all $a \in (-\epsilon, \epsilon)$.
(3) Given an example of a compact orientable surface S and a number $a \in \mathbb{R}$ such that S_a is *not* a regular surface.

Date: March 4, 2020.

[1]See the course website, `https://math.mit.edu/~phintz/18.950-S20/`, for homework policies.