18.950/9501 (S20): HOMEWORK 5

The book references are to do Carmo, *Differential Geometry of Curves and Surfaces*. (The numbers for the assigned problems are the same in both editions of the book.)

Due: Thursday, Apr 2, on Gradescope.[1]

Exercise 1. Show that at a hyperbolic point of a regular surface, the principal directions bisect the asymptotic directions.

Exercise 2. Let S be a regular surface, $p \in S$.

(1) Show that the sum of the normal curvatures for any pair of orthogonal directions at p is constant.
(2) Show that if the mean curvature at p is zero, and p is not a planar point (that is, $dN_p \neq 0$), then p has two orthogonal asymptotic directions.

Exercise 3. A curve C is called a *line of curvature* of a regular surface S if each tangent vector of C is a principal direction of S. Suppose two regular surfaces S_1, S_2 intersect in a regular curve C, and the angle between the normal vectors of S_1 and S_2 is $\theta(p)$, $p \in C$. Assume that C is a line of curvature of S_1. Show that C is a line of curvature of S_2 if and only if $\theta(p)$ is constant.

Exercise 4.
(1) Let $R > 0$. Suppose $\alpha\colon I \to \mathbb{R}^3$ is a regular parameterized curve in $\mathbb{R}^3$ with the property that $\|\alpha(s)\| \leq R$ and $\|\alpha(s_0)\| = R$. Show that the curvature of α at s_0 satisfies the inequality $k(s_0) \geq 1/R$.
(2) Let S be a compact (that is, closed and bounded) regular surface. Show that there exists a point $p \in S$ with positive Gauss curvature.

Exercise 5. Let $I \subset \mathbb{R}$ be an open interval, $\alpha\colon I \to \mathbb{R}^3$ a regular parameterized curve, and $\beta\colon I \to \mathbb{R}^3$ a smooth function with $\beta \neq 0$. We define a parameterized surface by

$$x(u, v) = \alpha(u) + v\beta(u), \qquad (u, v) \in I \times \mathbb{R}.$$

This is called a *ruled surface*, with *rulings* β and *directrix* α. (An example is a cylinder, with α a circle and β a constant vector.) Show that a regular ruled surface has Gauss curvature $K \leq 0$.

Exercise 6. Chapter 3–3, Problem 13.

Date: March 5, 2020. Updated: March 16, 2020.

[1]See the course website, `https://math.mit.edu/~phintz/18.950-S20/`, for homework policies.