18.950/9501 (S20): HOMEWORK 6

The book references are to do Carmo, *Differential Geometry of Curves and Surfaces*. (The numbers for the assigned problems are the same in both editions of the book.)

Due: Friday, Apr 10, on Gradescope.[1]

Exercise 1. Let V, W be two smooth vector fields on a regular surface S. For a smooth function $f\colon S \to \mathbb{R}$, recall that $V(f)\colon S \to \mathbb{R}$ is the smooth function defined by $V(f)(p) = df_p(V(p))$.

(1) Show that there exists a vector field X on S such that for all smooth functions $f\colon S \to \mathbb{R}$, we have $V(W(f)) - W(V(f)) = X(f)$. One calls X the *Lie bracket* of V and W, and writes $X = [V, W]$.

(2) Let $\vec{x}\colon U \subset \mathbb{R}^2 \to S$ be a local parameterization of S; write points in U as (ξ, η). Define vector fields V, W on $\vec{x}(U)$ to be the coordinate vector fields

$$V(\vec{x}(\xi,\eta)) := d\vec{x}_{(\xi,\eta)}(\partial_\xi) = \partial_\xi \vec{x}(\xi,\eta), \quad W(\vec{x}(\xi,\eta)) := d\vec{x}_{(\xi,\eta)}(\partial_\eta) = \partial_\eta \vec{x}(\xi,\eta).$$

Show that $[V, W] = 0$.[2]

(3) Give an example of vector fields V, W on $\mathbb{R}^2$ whose Lie bracket is nonzero, i.e. $[V, W] \neq 0$.

Exercise 2. Let S be a minimal surface (that is, its mean curvature vanishes). Show that the Gauss map $N\colon S \to \mathbb{S}^2$ is locally conformal, that is, $\langle dN_p(v_1), dN_p(v_2)\rangle = \lambda(p)^2 \langle v_1, v_2\rangle$ for all $p \in S$ and $v_1, v_2 \in T_pS$. Determine the conformal factor $\lambda(p)^2$.

Exercise 3. Prove that there does not exist a compact (i.e. closed and bounded) regular minimal surface $S \subset \mathbb{R}^3$.

Exercise 4. Chapter 4–2, Problem 1.

Exercise 5. Chapter 4–2, Problem 20.

Exercise 6. Chapter 4–3, Problem 1.

Date: April 2, 2020.

[1] See the course website, `https://math.mit.edu/~phintz/18.950-S20/`, for homework policies.

[2] Thus, $[V, W] = 0$ is a *necessary* condition for the vector fields V, W to be the coordinate vector fields of some parameterization. Cf. the first theorem of Lecture 11.