# EXERCISES FOR PART 1/2 OF SCATTERING THEORY (SNAP 2019)

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**Exercise 1.** (Free resolvent in one dimension, physical space approach.)

(a) Let $\lambda \in \mathbb{C}$, $\operatorname{Im} \lambda > 0$. Find a distribution $u_\lambda(x) \in \mathscr{D}'(\mathbb{R}_x)$ such that

$$(-\partial_x^2 - \lambda^2) u_\lambda(x) = \delta(x), \tag{1}$$

and so that $|u_\lambda(x)| \to 0$ as $|x| \to \infty$.

(b) For $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$, set

$$R_0(\lambda)\varphi(x) := \int_{\mathbb{R}} u_\lambda(x-y)\varphi(y)\, dy. \tag{2}$$

Show that $(-\partial_x^2 - \lambda^2) R_0(\lambda)\varphi = \varphi$. We call $R_0(\lambda)$ the *free resolvent* of $-\partial_x^2$.

(c) Show that $R_0(\lambda)\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ for $\operatorname{Im} \lambda > 0$.

(d) Prove that $R_0(\lambda)$ extends from $\operatorname{Im} \lambda > 0$ to a meromorphic family of operators

$$R_0(\lambda)\colon \mathcal{C}_c^\infty(\mathbb{R}) \to \mathscr{D}'(\mathbb{R}), \quad \lambda \in \mathbb{C}. \tag{3}$$

(This means: for all $\varphi, \psi \in \mathbb{R}$, the complex-valued function $\lambda \mapsto \int_{\mathbb{R}} R_0(\lambda)\varphi(x) \cdot \psi(x)\, dx$ is meromorphic in $\lambda$.) Find its poles.

(e) For $\lambda \neq 0$, show that $R_0(\lambda)$ extends by continuity to a continuous map

$$R_0(\lambda)\colon L^2_c(\mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R}). \tag{4}$$

(This means: for any smooth cutoff function $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$, the *cutoff resolvent* $\rho R_0(\lambda)\rho\colon \mathcal{C}_c^\infty(\mathbb{R}) \to \mathscr{D}'(\mathbb{R})$ extends to a bounded linear map $\rho R_0(\lambda)\rho\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$.)

(f*) Show the following improvement of (4):

$$R_0(\lambda)\colon L^2_c(\mathbb{R}) \to H^2_{\mathrm{loc}}(\mathbb{R}), \quad \lambda \neq 0. \tag{5}$$

**Exercise 2.** (Waves in one dimension.) By solving the one-dimensional free wave equation explicitly, we will justify the phenomenon seen in the lecture: the solution becomes constant in any fixed compact set for late enough times.

Consider a function $u \in \mathcal{C}^\infty(\mathbb{R}^2)$ in two variables $(t, x)$ satisfying the free wave equation in $(1+1)$ dimensions:

$$(\partial_t^2 - \partial_x^2) u(t, x) = 0. \tag{6}$$

(a) By changing coordinates in equation (6) to $(w, z) = (t + x, t - x)$, show that there exist smooth functions $u_L, u_R \in \mathcal{C}^\infty(\mathbb{R})$ of one variable such that

$$u(t, x) = u_L(t + x) + u_R(t - x). \tag{7}$$

Conversely, show that every function of this form satisfies the wave equation (6).

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(b) Produce an explicit formula of the solution of the initial value problem

$$\begin{cases} (\partial_t^2 - \partial_x^2)u(t,x) = 0, & t \geq 0,\ x \in \mathbb{R}, \\ u(0,x) = 0, & x \in \mathbb{R}, \\ \partial_t u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{8}$$

for $u_0 \in \mathcal{C}^\infty(\mathbb{R})$.

(c) Let $R, R' > 0$. Suppose $u_0 \in \mathcal{C}_\mathrm{c}^\infty(B(0,R))$. Show that there exists $T = T(R,R') > 0$ such that $u(t,x)$ is constant for $t \geq T$, $|x| \leq R'$.

**Exercise 3.** (Free resolvent in three dimensions.) Let $\varphi \in \mathcal{C}_\mathrm{c}^\infty(\mathbb{R}^3)$. As in lecture, set

$$R_0(\lambda)\varphi(x) := \int_{\mathbb{R}^3} \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}\varphi(y)\,dy, \quad \lambda \in \mathbb{C}. \tag{9}$$

Prove by direct calculation that

$$(-\Delta - \lambda^2)R_0(\lambda)\varphi = \varphi. \tag{10}$$

**Exercise 4.** (Estimates in the upper half plane.)

(a) Show that there exists a constant $C > 0$ such that for all $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda > 0$, the following estimate holds ($d$ denoting Euclidean distance):

$$d(\lambda^2, [0,\infty)) \geq C|\lambda| \operatorname{Im}\lambda. \tag{11}$$

(b) Let $V \in L^\infty(\mathbb{R};\mathbb{C})$. Show that there exists $C' > 0$ such that the following holds: if $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda > C'$, and if $w \in H^2(\mathbb{R}^3)$ solves $(-\Delta + V - \lambda^2)w = 0$, then $w = 0$.

**Exercise 5.** (Analytic Fredholm theory for matrices.) Let $N \in \mathbb{N}$, and let $\Omega \subset \mathbb{C}$ denote a connected open set. Suppose $A(\lambda) \in \mathbb{C}^{N\times N}$ is an analytic matrix-valued function of $\lambda \in \Omega$. Prove that either $A(\lambda)$ is not invertible for any $\lambda \in \mathbb{C}$, or the inverse $A(\lambda)^{-1}$ is a meromorphic matrix-valued function on $\Omega$ (that is, its entries are meromorphic complex-valued functions).

**Exercise 6.** (An application of analytic Fredholm theory.) Let $K\colon X \to Y$ be a compact operator between two Banach spaces $X, Y$. By considering the analytic family $\mathbb{C} \ni z \mapsto I + zK$ of Fredholm operators, prove that the spectrum of $K$ is discrete and can only accumulate at 0.

**Exercise 7.** (Meromorphic continuation in one dimension.) Let $V \in L_\mathrm{c}^\infty(\mathbb{R};\mathbb{C})$. Following the arguments presented in lecture, show that the resolvent

$$R_V(\lambda) = (-\partial_x^2 + V - \lambda^2)^{-1}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \operatorname{Im}\lambda \gg 1, \tag{12}$$

admits a meromorphic continuation to a family of operators

$$R_V(\lambda)\colon L_\mathrm{c}^2(\mathbb{R}) \to L_\mathrm{loc}^2(\mathbb{R}), \quad \lambda \in \mathbb{C}. \tag{13}$$

(This means: $\rho R_V(\lambda)\rho$ is a meromorphic family of operators on $L^2(\mathbb{R})$ for any cutoff function $\rho \in \mathcal{C}_\mathrm{c}^\infty(\mathbb{R})$.) Make sure you carefully treat the pole of $R_0(\lambda)$ at $\lambda = 0$.)

**Exercise 8.** (Symmetry of resonances for real-valued potentials.) Let $V \in L_\mathrm{c}^\infty(\mathbb{R}^3;\mathbb{R})$ be *real-valued*. Show that if $\lambda \in \mathbb{C}$ is a resonance, then so is $-\bar{\lambda}$, the reflection of $\lambda$ across the imaginary axis. (Use complex conjugation.)

**Exercise 9.** (Meromorphic continuation for decaying potentials.) Let $T > 0$, and let $V \in L^\infty(\mathbb{R}^3)$ be a bounded potential satisfying $|V(x)| \leq Ce^{-T|x|}$ for some constant $C$. Show (by carefully following the proof in lecture for $V \in L^\infty_{\mathrm{c}}(\mathbb{R}^3)$) that the resolvent $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$ extends from $\operatorname{Im}\lambda \gg 1$ to a meromorphic family

$$R_V(\lambda)\colon L^2_{\mathrm{c}}(\mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R}), \quad \lambda \in \mathbb{C},\ \operatorname{Im}\lambda > -T. \tag{14}$$

**Exercise 10.** (Calculation of resonances in one dimension.) Fix $V_0 \in \mathbb{R}$ and $L > 0$, and define the potential $V \in L^\infty(\mathbb{R})$ by

$$V(x) := \begin{cases} V_0, & -L < x < L, \\ 0, & |x| \geq L. \end{cases} \tag{15}$$

(a) Derive a necessary and sufficient criterion for $\lambda \in \mathbb{C}$ to be a resonance of $-\partial_x^2 + V$. (Use the characterization of resonant states $(-\partial_x^2 + V - \lambda^2)u = 0$, $u(x) = u_\pm e^{i\lambda|x|}$, $\pm x \geq L$.) This will take the form of a transcendental equation.

(b*) By approximately solving this equation, find an approximate formula for resonances $\lambda$ with large real part.

**Exercise 11.** (Potentials with a prescribed resonance.) The goal of this exercise is to show that resonances can appear anywhere in the complex plane.

(a) Construct a potential $V \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3;\mathbb{R})$ such that 0 is a resonance of $-\Delta + V$.

(b) Let $\lambda \in \mathbb{C}$. Construct a potential $V \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3;\mathbb{C})$ such that $\lambda$ is a resonance of $-\Delta + V$.

(c*) Let $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda < 0$. Construct a real-valued potential $V \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3;\mathbb{R})$ such that $\lambda$ is a resonance of $-\Delta + V$.

**Exercise 12.** (Waves and resolvents in three dimensions.) Let $U(t) := \sin(t\sqrt{-\Delta})/\sqrt{-\Delta}$. (This is defined using the Fourier transform $\mathcal{F}$ by $\mathcal{F}(U(t)f)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}\mathcal{F}f(\xi)$, $f \in \mathscr{S}(\mathbb{R}^3)$.)

(a*) Show that

$$U(t)f(x) = \frac{1}{4\pi t}\int_{\partial B(x,t)} f(y)\,dS(y), \quad t > 0. \tag{16}$$

(b) (Strong Huygens principle.) Suppose $f \in \mathcal{C}^\infty_{\mathrm{c}}(B(0,R))$. Prove that $B(0,R) \cap \operatorname{supp} U(t)f = \emptyset$ for $t > 2R$.

(c) Let $\operatorname{Im}\lambda > 0$. Show that $R_0(\lambda)\colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ can be expressed as

$$R_0(\lambda) = \int_0^\infty e^{i\lambda t}U(t)\,dt, \tag{17}$$

with convergence in operator norm.

(d) Let $\rho \in \mathcal{C}^\infty_{\mathrm{c}}(B(0,R))$. Show (using the strong Huygens principle and analytic continuation) that for all $\lambda \in \mathbb{C}$,

$$\rho R_0(\lambda)\rho = \int_0^{2R} e^{i\lambda t}\rho U(t)\rho\,dt. \tag{18}$$

(e) Show that $\|U(t)\|_{L^2\to H^1} = (1+t^2)^{1/2}$. Deduce that

$$\|\rho R_0(\lambda)\rho\|_{L^2\to H^1} \leq Ce^{2R(\operatorname{Im}\lambda)_-}. \tag{19}$$

(f) Make sure you understand the arguments given in lecture proving that for $j = 0, 1, 2$,

$$\|\rho R_0(\lambda)\rho\|_{L^2\to H^j} \leq Ce^{2R(\operatorname{Im}\lambda)_-}\langle\lambda\rangle^{j-1}. \tag{20}$$

**Exercise 13.** (Solving the wave equation using the resolvent.) Let $V \in L^\infty_c(\mathbb{R}^3;\mathbb{C})$. Let $u_0 \in H^1_c(\mathbb{R}^3)$, $u_1 \in L^2_c(\mathbb{R}^3)$, and consider the wave equation

$$\begin{cases} (\Box + V)u = (\partial_t^2 - \Delta_{\mathbb{R}^3} + V)u = 0, \\ u(0,x) = u_0(x), \\ \partial_t u(0,x) = u_1(x). \end{cases} \tag{21}$$

This has a unique solution $u \in \mathcal{C}^0(\mathbb{R}; H^1_c(\mathbb{R}^3)) \cap \mathcal{C}^1(\mathbb{R}; L^2_c(\mathbb{R}^3))$. Let $H(t)$ denote the Heaviside function ($H(t) = 1$ for $t \geq 0$ and $H(t) = 0$ for $t < 0$), and put $\tilde{u}(t,x) := H(t)u(t,x)$.

(a) Show that $(\Box + V)\tilde{u}(t,x) = f(t,x) := \delta'(t)u_0(x) + \delta(t)u_1(x)$.
(b) For $C > 0$ sufficiently large, show that

$$v(t,x) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = C} e^{-i\lambda t} R_V(\lambda)(u_1 - i\lambda u_0)\,d\lambda \tag{22}$$

is well-defined (as a distribution) and satisfies $(\Box + V)v = f$. (Pick $C$ so that $R_V(\lambda)$ satisfies good estimates for $\operatorname{Im}\lambda \geq C$ and has no resonances there.)
(c) Using the Paley–Wiener theorem, or arguing directly, show that $v(t,x) = 0$ for $t < 0$.
(d) Show that the difference $w(t,x) := \tilde{u}(t,x) - v(t,x)$ vanishes. (By construction, $w(t,x) = 0$ for $t < 0$; moreover $(\Box + V)w(t,x) = 0$. Use energy estimates to conclude.) Therefore, $\tilde{u}(t,x)$ is given by (22).

**Exercise 14.** (Regularity of solutions of wave equations.) Let $V \in \mathcal{C}^\infty_c(\mathbb{R}^3;\mathbb{C})$,[1] and suppose $u(t,x)$ is the unique solution of the wave equation

$$\begin{cases} (\partial_t^2 - \Delta + V)u = 0, & t \in \mathbb{R},\ x \in \mathbb{R}^3, \\ u(0,x) = u_0(x) \in H^1_c(\mathbb{R}^3), \\ \partial_t u(0,x) = u_1(x) \in L^2_c(\mathbb{R}^3); \end{cases} \tag{23}$$

we have $u \in \mathcal{C}^0(\mathbb{R}; H^1(\mathbb{R}^3)) \cap \mathcal{C}^1(\mathbb{R}; L^2(\mathbb{R}^3))$. If $R > 0$ is such that $\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset B(0,R)$, show that $u \in \mathcal{C}^\infty(\Omega)$ in the domain

$$\Omega = \{(t,x) \colon |x| < |t| - R\}. \tag{24}$$

In particular, $u(t,x)$ is smooth in any fixed compact subset of $\mathbb{R}^3_x$ when $t$ is large enough. (It suffices to prove this for $t > 0$. Using the previous exercise, reduce to a statement about solutions of $(\partial_t^2 - \Delta + V)u = f \in \mathscr{D}'(\mathbb{R}^4)$ where $t \geq 0$ on $\operatorname{supp} f$ and on $\operatorname{supp} u$. Then use the microlocal propagation of singularities.)

**Exercise 15.** (Resonances for real-valued potentials.) Let $V \in L^\infty_c(\mathbb{R};\mathbb{R})$, $P_V = -\partial_x^2 + V$.

(a) (Absence of embedded eigenvalues.) Show that $P_V$ has no non-zero real resonances. (Given a resonant state $u(x)$ corresponding to a resonance $0 \neq \lambda \in \mathbb{R}$, evaluate $\int_{-R}^R (P_V - \lambda^2)u \cdot \bar{u} - u \cdot (P_V - \lambda^2)\bar{u}\,dx$ for $R \gg 1$ in two different ways using that $u(x) = a_\pm e^{\pm i\lambda x}$ for $\pm x \gg 1$.)
(b) Suppose $V \geq 0$. Show that the resonances $\lambda$ of $P_V$ satisfy $\lambda = 0$ or $\operatorname{Im}\lambda < 0$. (You only need to study $\operatorname{Im}\lambda > 0$. Given a resonant state $u$, integrate by parts in $0 = \langle (P_V - \lambda^2)u, u\rangle_{L^2(\mathbb{R})}$ and consider real and imaginary parts.)
(c) If $V > 0$ on a set of positive measure, show that 0 is not a resonance of $P_V$.
(d) Prove the last two results in three spatial dimensions.

[1] The compact support assumption can be dropped easily here.