GLOBAL ANALYSIS OF LINEAR AND NONLINEAR WAVE EQUATIONS ON COSMOLOGICAL SPACETIMES

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Abstract

We develop a general framework for the global analysis of linear and nonlinear wave equations on geometric classes of Lorentzian manifolds, based on microlocal analysis on compactified spaces. The main examples of manifolds that fit into this framework are cosmological spacetimes such as de Sitter and Kerr-de Sitter spaces, as well as Minkowski space, and perturbations of these spacetimes. In particular, we establish the global solvability of quasilinear wave equations on cosmological black hole spacetimes and obtain the asymptotic behavior of solutions using a novel approach to the global study of nonlinear hyperbolic equations. The framework directly applies to nonscalar problems as well, and we present linear and nonlinear results both for scalar equations and for equations on natural vector bundles.

To a large extent, our work was motivated by the black hole stability problem for cosmological spacetimes, and we expect the resolution of this problem to be within reach with the methods presented here.

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Chapter 1

Introduction

In this thesis, we give a detailed analysis of the *long-time behavior of linear and nonlinear waves on cosmological black hole spacetimes*, in particular establishing the *global-in-time existence and the asymptotic structure of nonlinear waves with small amplitudes.*

From a physical perspective, cosmological spacetimes are solutions to Einstein's (vacuum) field equations

$$\mathrm{Ric}(g) + \Lambda g = 0, \tag{1.0.1}$$

where $\Lambda > 0$ is the cosmological constant and g is a Lorentzian metric with signature $(+, -, \ldots, -)$ on a manifold M°. A special family of solutions (M°, g) of (1.0.1) describing spacetimes containing rotating black holes is the *Kerr-de Sitter family*, which is parameterized by the cosmological constant, the mass of the black hole and its angular momentum. A major open problem in the theory of general relativity is the black hole stability conjecture; in the present context, this conjecture asserts that spacetimes solving (1.0.1) which closely resemble a Kerr-de Sitter spacetime initially settle down to another Kerr-de Sitter spacetime for large times. Putting in an extra structure, namely choosing a gauge to eliminate the diffeomorphism invariance, Einstein's field equations can be recast as a quasilinear wave equation for the metric tensor g. In recent years, there has been a substantial amount of work on aspects of the black hole stability problem; the primary focus (mostly in the case $\Lambda = 0$) has been on obtaining a robust understanding of the decay of linear scalar waves on black hole spacetimes, modelled by the Cauchy problem for the equation

$$\Box_g u = 0 \tag{1.0.2}$$

for an unknown function u on M°, as well as of electromagnetic waves, described by Maxwell's equations; on the Minkowski spacetime, with $\Lambda = 0$ and without black holes, similar developments led to a proof of the stability of Minkowski space [20]. We refer to §§5.1.1 and 6.1.1 for a review of the literature.

We follow this tradition, but present a new perspective for the global study of waves that enables us to give the first proof of global existence, asymptotics and decay for *scalar and non-scalar quasilinear wave equations*, of a very general form, on black hole spacetimes. In fact, we believe that the global nonlinear stability of the Kerr-de Sitter family is within reach with the methods described here. We expect the general scheme of our approach, which originates from the work of Vasy [114], and many of the methods to be applicable and useful in other contexts as well.

Concretely, we adopt Melrose's philosophy [82, 83, 84] of studying natural differential operators P, e.g. the Laplace (or wave) operator, on a non-compact space M° by (partially) compactifying M° to a manifold M with boundary (or corners) by adding 'ideal boundaries,' the concrete choice of compactification being tied to the geometric structure of M° near infinity. The operator P then degenerates in a controlled manner at the added boundaries, and there are well-established tools with which one can analyze the resulting degeneracies. For instance, we compactify cosmological spacetimes equipped with a stationary metric (i.e. invariant under translations in a time coordinate t) by adding 'future infinity,' given by the vanishing of $\tau := e^{-t}$, to the spacetime. The wave operator $\Box_g$ on the compactified space then is a so-called b-differential operator, the 'b' standing for 'boundary' and indicating the precise nature of the degeneracy. This approach has proven to be very powerful for elliptic problems [82, 81], and we show here how to study hyperbolic equations like (1.0.2) from this point of view. The key observation is that the main qualitative properties of the cosmological spacetime can be read off from the structure of the null-geodesic flow, i.e. the paths of light rays, at the ideal boundary at future infinity: Horizons, like the event horizon which separates the black hole interior from the exterior region and the cosmological horizon defining the boundary of the observable universe, appear in the form of saddle points for the flow, see Figure 1.1, and trapped trajectories, i.e. light rays that neither escape through the cosmological horizon nor fall through the event horizon, are correctly encoded at future infinity as well – for any finite amount of time, there is no trapping from the global, spacetime point of view.

Note that thus encoding the causal structure of the spacetime is very convenient also

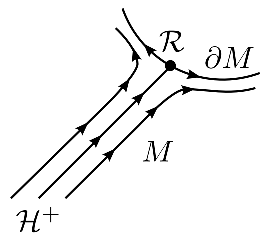

Figure 1.1: Null-geodesic flow near a horizon $\mathcal{H}^+$ in the compactified picture: Null-geodesics that do not exactly follow the horizon drift away from it exponentially; this is related to the red shift effect. The flow extends smoothly to future infinity $\partial M = \{\tau = 0\}$, and at the set $\mathcal{R}$ where the horizon intersects ∂M, the extended flow has a saddle point.

from the point of view of nonlinear problems, since even for spacetimes which are merely asymptotically stationary, i.e. settle down to stationary spacetimes exponentially fast, only the structure at future infinity matters for the study of regularity, asymptotics and decay. On stationary spacetimes, one can of course simply study spatial slices $\{t = \mathit{const}\}$ instead, but without modifications, this approach immediately breaks down for non-stationary spacetimes, whereas there are no additional complications whatsoever in the compactified b-picture.

The b-picture furthermore allows to *precisely* capture the structures responsible for asymptotic expansions, energy decay and global regularity of waves. (While we only consider the wave equation here, we stress that we only do this for the sake of simplicity: The concepts outlined here apply to much more general operators as well, including perturbations of wave operators by lower order terms, and in fact including suitable so-called b-pseudodifferential operators of arbitrary order.) Namely, they are encoded in the so-called normal operator family $\widehat{\Box_g}(\sigma)$, obtained by freezing the coefficients of the wave operator $\Box_g$ at future infinity (which produces a t-independent operator that agrees with $\Box_g$ up to exponentially decaying terms) and taking the Fourier transform in $-t$, with σ denoting the frequency (dual) variable. Just like in time-independent scattering theory, solutions of the equation (1.0.2) then have an asymptotic expansion

$$u(t,x) = \sum_j e^{-it\sigma_j} u_j a_j(x) + u'(t,x), \tag{1.0.3}$$

with x denoting points on $t = \mathit{const}$ slices, where the σ_j are poles of the meromorphic continuation of the inverse normal operator family $\widehat{\Box_g}(\sigma)^{-1}$, called *resonances* or *quasinormal* *modes*, and the a_j are *resonant states* and only depend on the operator $\Box_g$; the coefficients u_j are complex numbers, and the function u' is a remainder term whose regularity and decay we can quantify precisely. (At poles of order larger than 1, additional powers of t appear in the expansion (1.0.3).)

The global analysis of regularity and decay allows one to formulate the Cauchy or forward problem for linear wave-type equations as a *(non-elliptic) Fredholm problem* (or in fact an invertible one) on suitable global function spaces, which are weighted L^2-based Sobolev spaces on the compactified spacetime M and take into account the asymptotic expansion (1.0.3). Moreover, such a Fredholm statement, which is rather qualitative in nature, relies solely on qualitative properties of (M, g) and its causal structure, that is, on the global dynamics of the null-geodesic flow (structure of the horizons, the trapped set, the non-trapping or mildly trapping nature of the flow), and is very robust under perturbations of the metric. The non-elliptic Fredholm framework for the stationary operators $\widehat{\Box_g}(\sigma)$ developed by Vasy [114] is very closely related to the spacetime framework.

For applications to *nonlinear wave equations*, one needs more detailed quantitative information, most prominently boundedness and mode stability, which means excluding resonances σ with $\operatorname{Im}\sigma > 0$, which in the resonance expansion (1.0.3) would cause exponential growth of u in time, and excluding resonances on the real line as well, apart from allowing a simple resonance at $\sigma = 0$; then the expansion (1.0.3) shows that $u(t, x)$ solving (1.0.2) decays exponentially in t to a stationary state, which is a resonant state corresponding to the resonance at 0, or to 0 in the absence of a zero resonance. Such results, which amount to showing the triviality of the kernel of the (non-elliptic) partial differential operator $\widehat{\Box_g}(\sigma)$ for certain σ, rely on the exact form of the operator. (Thus, the non-elliptic Fredholm framework makes the global study of waves analogous to the study of elliptic equations, say on closed manifolds: Qualitative results, like the index or the smoothness of eigenfunctions, are robust and rely only on qualitative properties of the operator, whereas quantitative results, e.g. the triviality of the kernel, are rather sensitive to the precise form of the operator; but one already has a considerable amount of information from the Fredholm analysis when commencing the quantitative analysis.)

To demonstrate how one can then solve nonlinear wave equations, we consider as a simple example the scalar semilinear equation

$$\Box_g u = |\nabla u|^2 + f \tag{1.0.4}$$

on a Kerr-de Sitter spacetime (with small angular momentum a, or $a = 0$, which is Schwarzschild-de Sitter space), with vanishing initial data, and the small forcing f generating the nonlinear wave. In this case, one does have mode stability, and a crude form of the resonance expansion (1.0.3) for solutions of the linear equation $\Box_g u = 0$ reads $u = c + u'$, with c a constant and u' exponentially decaying in t. The global point of view suggests an iteration scheme for solving (1.0.4) in which one solves a linear equation globally at each step: Namely, with the initial guess $u_0 = 0$, we let

$$u_{k+1} = \Box_g^{-1}(|\nabla u_k|^2 + f), \quad k = 0, 1, \ldots. \tag{1.0.5}$$

(The actual iteration scheme is a bit more involved due to the loss of derivatives in the presence of trapping; on de Sitter space on the other hand, it works as written.) Then u_k converges to a solution $u = c + u'$ of the nonlinear equation (1.0.4), with c and u' as above. We point out that the stationary state c to which the nonlinear wave decays is automatically found by the global iteration scheme, which is made possible due to the fact that the linear analysis sees future infinity and thus the mechanism for asymptotics and decay at every step of the iteration; it is unclear and conceptually much less apparent how a solution scheme that proceeds via extending the solution for a finite time at each step would achieve this. The advantage of our global perspective becomes even more striking when we consider *quasilinear equations*, say

$$\Box_{g(u,\nabla u)} u = q(u, \nabla u) + f, \tag{1.0.6}$$

with $g(0,0)$ a Kerr-de Sitter metric, where now the metric, including its asymptotic form, can depend on the solution itself. Again, a global iteration scheme, solving a linear equation at each step, solves (1.0.6). We can consider this PDE not only in the scalar setting, but also in the setting of waves which are sections of a natural tensor bundle, e.g. the bundle of differential forms, for which the space of stationary states is no longer 1-dimensional. We therefore see that the problem of orbital stability for such equations can be resolved in a very natural way. (The black hole stability problem poses additional difficulties, specifically regarding the gauge freedom, and we do not discuss it further in this thesis.)

To put the global iteration scheme (1.0.5) into context, we recall that the traditional way of solving nonlinear evolution equations proceeds by first establishing short time existence and then arguing that the solution persists as long as certain energies remain bounded, which, combined with a priori energy bounds or Strichartz estimates, gives global existence; this crucially relies on the finite speed of propagation for hyperbolic differential operators, which in particular guarantees that the forward solution to a (nonlinear) evolution equation at a given time is unaffected by any data that lie in its future. (There are natural problems even for wave equations which cannot be treated in this way, namely when considering the Feynman propagator instead of the advanced or retarded propagator [51].) Our approach on the other hand disregards the local evolution character of the wave equation to some extent and instead sets up a global framework which is more akin to elliptic problems. (Changing an elliptic equation in an arbitrary open set will in general affect its solutions *everywhere*, which renders a localized solution scheme for nonlinear elliptic equations impossible.) The type of problem one wants to study, for example the forward or the Feynman problem for the wave equation $\Box_g u = f$, merely dictates the *function spaces* on which one has invertibility or Fredholm properties of $\Box_g$. While we only study nonlinear small data problems here, it is an interesting question whether this global setup can also facilitate the analysis or provide a new perspective for *large data* evolution problems.

Our global approach to the study of linear and nonlinear wave equations at present only works on cosmological black hole spacetimes, i.e. when $\Lambda > 0$. (We can however study semilinear waves on spacetimes which are asymptotically Minkowskian.) There are two main complications arising in the study of spaces with $\Lambda = 0$, such as Schwarzschild and Kerr spacetimes: Firstly, due to the presence of an asymptotically flat end of a spatial slice $\{t = \mathit{const}\}$ (rather than an asymptotically hyperbolic end for cosmological spacetimes), the question of meromorphy of the inverse normal operator family becomes much more delicate, and in fact linear scalar waves only decay polynomially with a fixed rate [105]. Secondly, when studying quasilinear problems using a compactified perspective, the correct choice of compactification of, say, a perturbation of the Kerr spacetime depends on the long range part of the perturbation and thus would change at each step of our above iteration scheme, which therefore would not have a chance of converging. Regarding the first issue, it is reasonable to expect suitable bounds for the normal operator family to hold up to the real line, even though this has not been worked out yet, but it is unclear how to address the second issue. We point out that other, more traditional methods based on energy estimates and vector fields have been very successful in the treatment of linear scalar waves on exact Kerr spacetimes [27, 31]; some perturbative results are also available [105], as well as results for *semilinear* forward problems. See §5.1.1 for further references, including works also in the case $\Lambda < 0$, i.e. for anti-de Sitter universes.

Going back to the global Fredholm framework for linear wave equations, we need to understand global regularity and decay properties of waves, and as mentioned above, the normal operator analysis provides asymptotics and decay. In order to perform the regularity, i.e. high frequency, analysis on the other hand, we use *microlocal analysis* to convert information on the null-geodesic flow of the metric g *in phase space*, i.e. in the cotangent bundle of M° rather than on M° itself, into regularity properties of solutions to $\Box_g u = 0$. The connection between the two is the following: The null-geodesic flow is the flow of the Hamilton vector field of the Hamiltonian $G(x,\xi) := |\xi|^2_{g(x)}$ on T^*M°, while $\Box_g$ can be viewed as a *quantization* of G, thus replacing classical observables such as position x and momentum ξ by their quantum analogues, namely multiplication by x (for the position) and differentiation with respect to x (for the momentum), up to normalizations. The fundamental example illustrating this connection is the Duistermaat-Hörmander theorem on the propagation of singularities [38], which states that singularities of solutions of $\Box_g u = 0$ propagate along the corresponding classical trajectories, which in this case are null-geodesics; this statement indeed takes place in phase space and makes use of a refined notion of singularities. Since we work on compactified spaces M, which include a boundary at which the wave operator degenerates, we use Melrose's *b-calculus* to do the microlocal analysis on M, in particular analyzing the regularity of solutions near the boundary at future infinity by exploiting the aforementioned special structures present there. Thus, the microlocal approach is very clean conceptually and, as we show in this thesis, very powerful for global nonlinear problems, because it is very robust; as an important example, we mention the trapped set for Kerr (and Kerr-de Sitter) spacetimes: While the image of the set of trapped null-geodesics on the base M changes from a sphere for Schwarzschild (and Schwarzschild-de Sitter) to a set with non-empty interior (on a spatial slice/at future infinity) for rotating Kerr, the picture in phase space is essentially unchanged: The trapped set, located at future infinity, in the cotangent bundle is a smooth conic codimension 4 submanifold, with smooth stable and unstable manifolds. (This *normally hyperbolic nature* of the trapping was first noted by Wunsch and Zworski [124].)

In the remainder of the thesis, we will elaborate on the ideas outlined above. We refer to the introductions of the respective chapters for further background and more details, as well as for references to the literature. We begin in Chapter 2 by introducing the language of *b-analysis* which describes the geometry of our compactified spacetimes in a natural way, and show how de Sitter and Kerr-de Sitter spacetimes fit into this picture. The understanding of the qualitative properties of the global dynamics of the null-geodesic flow on these is of paramount importance, both conceptually and practically, for the analysis of wave equations in subsequent chapters. Next, in Chapter 3, we recall the 'classical' microlocal analysis on Euclidean space and on closed manifolds, including the propagation of singularities, before giving an account of *b-microlocal analysis*. The main new results proved there concern the b-microlocal regularity at horizons and trapped sets of the types that appear at future infinity for our cosmological spacetimes. The study of 'standard' local and global energy estimates for wave operators on b-geometries in Chapter 4 provides a means of obtaining the invertibility of the forward problems on weak function spaces; this is thus the first instance where *quantitative* assumptions come into play, in that such energy estimates and global invertibility statements only hold for hyperbolic *differential* operators, rather than more general b-pseudodifferential operators.

Chapter 5, see specifically §5.2.1, combines the results gathered thus far and provides the global Fredholm/invertibility framework for forward problems on (asymptotically) de Sitter, Kerr-de Sitter and Minkowski spacetimes. This is then used as in (1.0.5) to show the global solvability of semilinear wave equations on such spaces; since the resonances for scalar equations are known from previous works [13, 111, 40], we mostly consider scalar equations here, but the methods in principle apply to tensor-valued waves as well. We mention here that the study of asymptotically Minkowski spacetimes in §5.5 illustrates the relationship between qualitative and quantitative analysis in a different way: The function spaces on which we invert the wave operator, or in fact any pseudodifferential operator with a similar structure, are specified in terms of choices of (microlocal) regularity/decay conditions at the light cones at past and future infinity, while the identification of the resulting inverse *for the wave operator* (given the suitable choice of function spaces) with the forward propagator again requires the use of standard energy estimates.

We then proceed to demonstrate how to analyze linear, and thus nonlinear, tensor-valued waves on black hole spacetimes: In Chapter 6, we obtain the resonance expansion (1.0.3) with exponentially decaying remainder term for general tensor-valued waves on perturbations of Schwarzschild-de Sitter space. However, the wave operator acting on tensors of a certain rank may or may not satisfy mode stability, and there are in fact a number of natural wave operators on tensors which differ by 0-th order curvature terms, some of which do satisfy mode stability and some of which do not. For differential form-valued waves, we show in Chapter 7 that mode stability does hold for the Hodge wave operator $d\delta + \delta d$ on a large class of spacetimes, which, given the ensuing chapters, immediately implies the global solvability of quasilinear differential form-valued waves on cosmological black hole spacetimes, as indicated above.

The analysis of *quasilinear* waves necessitates the study of operators with non-smooth coefficients, since the nonlinearity can no longer be considered a lower order perturbation of the wave operator but must be treated directly. Thus, Chapter 8 develops the technical tools that enables us to do the global b-microlocal analysis for operators with non-smooth coefficients exactly as in the case of smooth coefficients. In the final chapter of this thesis, Chapter 9, we show how this leads to a general framework for the global study of *quasilinear forward problems* for wave-type equations. While the ideas of the arguments given there are essentially the same as in the semilinear or even linear setting of Chapter 5, the quasilinear nature of the problems considered requires the technical preliminaries of Chapter 8 and a slightly more sophisticated iteration scheme, namely Nash-Moser iteration, though we only need a very simple version thereof [99].

A large part of the material in this thesis is based on the following papers and preprints:

- Chapters 4 and 5 as well as parts of Chapter 3 are based on: Peter Hintz and András Vasy. Semilinear wave equations on asymptotically de Sitter, Kerr-de Sitter and Minkowski spacetimes. *Preprint, 2013.* Accepted for publication in *Analysis & PDE.*

- Parts of Chapter 3 are based on: Peter Hintz and András Vasy. Non-trapping estimates near normally hyperbolic trapping. *Math. Res. Lett.*, 21(6):1277–1304, 2014.

- Chapter 6 is based on: Peter Hintz. Resonance expansions for tensor-valued waves on Kerr-de Sitter space. *Preprint, 2015.*

- Chapter 7 is based on: Peter Hintz and András Vasy. Asymptotics for the wave equation on differential forms on Kerr-de Sitter space. *Preprint, 2015.*

- Chapters 8, 9 as well as parts of Chapter 2 are based on: Peter Hintz. Global well-posedness of quasilinear wave equations on asymptotically de Sitter spaces. *Preprint, 2013*, and: Peter Hintz and András Vasy. Global analysis of quasilinear wave equations on asymptotically Kerr-de Sitter spaces. *Preprint, 2014.*

Chapter 2

Structure of de Sitter and Kerr-de Sitter spacetimes

We discuss the two prime examples of spacetimes considered in this thesis in some detail, namely de Sitter space in §2.2, which is the analogue of flat Minkowski space in the context of a positive cosmological constant Λ, and Kerr-de Sitter space in §§2.3 and 2.4, which is a rotating black hole in a de Sitter universe and thus the analogue of the Kerr spacetime, which is a rotating black hole in an asymptotically flat (Minkowskian) universe. They will serve as motivations for the geometric settings in which we will work later on, and conversely as very explicit models to which our general theorems apply.

Both spacetimes are *stationary* in the sense that they are invariant under translations in a time variable t_*. This suggests that one should study asymptotic properties of geometric features like the (null-)geodesic flow and analytic ones like asymptotics of waves in terms of the quantity $\tau := e^{-t_*}$. Adding 'future infinity,' $\tau = 0$, to the spacetime, thus (partially) compactifying it, asymptotic features then appear in a concise way at $\tau = 0$, as we will detail in the subsequent sections, in particular §2.1.3. Note that the natural vector fields on such compactified spaces are $\partial_{t_*} = -\tau\partial_\tau$ and 'spatial' derivatives; these are precisely the vector fields tangent to $\tau = 0$. We give a brief overview of geometric aspects of such *b-spaces* ('b' for 'boundary') in §2.1; analytic aspects will be discussed in §3.3.

We shall also have the occasion to study waves on Minkowski space and more general asymptotically Minkowski spacetimes, but their study will be of a somewhat different flavor than that of cosmological spacetimes (with $\Lambda > 0$); we refer to §5.5 for details.

2.1 b-geometry

Throughout this section, M denotes a manifold with boundary $X = \partial M$. This will be the main case in our applications, but we point out that the study of b-geometry on manifolds with corners only requires minor, mostly notational modifications.

2.1.1 b-vector fields, operators and metrics; natural bundles

Denote by $\mathcal{V}_{\mathrm{b}}(M)$ the Lie algebra of vector fields on M which are tangent to the boundary; we call elements of $\mathcal{V}_{\mathrm{b}}(M)$ *b-vector fields.* Then $\mathcal{V}_{\mathrm{b}}(M)$ is the space of sections of a natural vector bundle ${}^{\mathrm{b}}TM$ over M, the *b-tangent bundle* [82, §2], which over the interior of M is naturally identified with TM. In local coordinates $(x, y) \in [0, \infty) \times \mathbb{R}^{n-1}$ near a point in X, with X locally given by $x = 0$, the bundle ${}^{\mathrm{b}}TM$ is spanned by $x\partial_x, \partial_{y_1}, \ldots, \partial_{y_{n-1}}$, and $\mathcal{V}_{\mathrm{b}}(M)$ is spanned over $\mathcal{C}^\infty(M)$ by these vector fields. Note that $x\partial_x$ is non-trivial as a b-vector field even at $x = 0$. The universal enveloping algebra $\mathrm{Diff}_{\mathrm{b}}(M)$ of linear combinations of products of b-vector fields is called the algebra of *b-differential operators*; elements of $\mathrm{Diff}_{\mathrm{b}}(M)$ act on $\mathcal{C}^\infty(M)$ as well as on $\dot{\mathcal{C}}^\infty(M)$, the space of smooth functions on M which vanish to infinite order at the boundary.

The bundle dual to ${}^{\mathrm{b}}TM$, denoted ${}^{\mathrm{b}}T^*M$ and called *b-cotangent bundle*, is spanned locally near the boundary by $\frac{dx}{x}, dy_1, \ldots, dy_{n-1}$; in particular, dx/x is non-singular (and non-trivial) as a b-covector at $x = 0$. We can then form the b-form bundle ${}^{\mathrm{b}}\Lambda M$, which is the exterior algebra generated by ${}^{\mathrm{b}}T^*M$; thus, b-differential forms are linear combinations of wedge products of $\frac{dx}{x}, dy_1, \ldots, dy_{n-1}$. The exterior derivative d on differential forms on M induces an *exterior b-differential* ${}^{\mathrm{b}}d \colon \mathcal{C}^\infty(M, {}^{\mathrm{b}}\Lambda M) \to \mathcal{C}^\infty(M, {}^{\mathrm{b}}\Lambda M)$, as follows from the observation

$$df = (x\partial_x f)\frac{dx}{x} + \sum_{j=1}^{n-1}(\partial_{y_j} f)dy_j =: {}^{\mathrm{b}}df.$$

In fact, this shows that ${}^{\mathrm{b}}d$ is a first order *b-differential operator*, ${}^{\mathrm{b}}d \in \mathrm{Diff}^1_{\mathrm{b}}(M, {}^{\mathrm{b}}\Lambda M)$. This is thus a re-interpretation of the differential df of f in terms of the 1-forms $\frac{dx}{x}$ and dy_j dual to the vector fields $x\partial_x$ and ∂_{y_j}, hence it is invariantly defined.

Over the boundary, ${}^{\mathrm{b}}TM$ and ${}^{\mathrm{b}}T^*M$ have natural subspaces: Indeed, the kernel of the natural map ${}^{\mathrm{b}}T_X M \to T_X M$ gives the *b-normal bundle of the boundary*, denoted ${}^{\mathrm{b}}NX$, and the image of the adjoint map $T^*_X M \to {}^{\mathrm{b}}T^*_X M$ gives the *b-cotangent bundle of the boundary*, denoted ${}^{\mathrm{b}}T^*X$, which is canonically isomorphic to the cotangent bundle T^*X;

we merely keep the notation ${}^{\mathrm{b}}T^*X$ to emphasize its nature as a subbundle of ${}^{\mathrm{b}}T^*_X M$. Thus, ${}^{\mathrm{b}}NX = \mathrm{span}\{x\partial_x\}$ if x is a boundary defining function of M, while ${}^{\mathrm{b}}T^*X$ is the annihilator of ${}^{\mathrm{b}}NX$ in ${}^{\mathrm{b}}T^*_X M$. Note that the natural b-bundles ${}^{\mathrm{b}}NX$ and ${}^{\mathrm{b}}T^*X$ are reversed compared to the natural bundles TX and N^*X in the smooth setting. In local coordinates $(x, y; \xi, \eta)$ of ${}^{\mathrm{b}}T^*M$, i.e. writing b-covectors as $\xi\,\frac{dx}{x} + \eta\,dy$, we have ${}^{\mathrm{b}}T^*X = \{(x = 0, y; \xi = 0, \eta)\}$.

Next, if $Z \subset X$ is a submanifold inside the boundary, we can naturally define its b-tangent bundle ${}^{\mathrm{b}}TZ$ within M, which is the preimage of TZ under the map ${}^{\mathrm{b}}T_Z M \to T_Z M$, equivalently, the bundle of all b-tangent vectors on M which are tangent to Z. The b-conormal bundle ${}^{\mathrm{b}}N^*Z$ of Z is the annihilator of ${}^{\mathrm{b}}TZ$ in ${}^{\mathrm{b}}T^*_Z M$. Splitting the boundary coordinates $y = (y', y'')$, with $Z = \{y' = 0\}$, and correspondingly splitting the dual variables $\eta = (\eta', \eta'')$, then ${}^{\mathrm{b}}N^*Z = \{(x = 0, y' = 0, y''; \xi = 0, \eta', \eta'' = 0)\}$; note here that $\frac{dx}{x}$ does *not* annihilate $x\partial_x \in {}^{\mathrm{b}}TZ$. Further, ${}^{\mathrm{b}}N^*Z$ is canonically isomorphic to the conormal bundle of Z *within* X. As a trivial example, if $Z = X$, then ${}^{\mathrm{b}}TX = {}^{\mathrm{b}}T_X M$, and ${}^{\mathrm{b}}N^*X$ is the 0-section of ${}^{\mathrm{b}}T^*_X M$.

A smooth *b-metric* g on M is a symmetric, non-degenerate section of the second symmetric tensor power $S^2\,{}^{\mathrm{b}}T^*M$, i.e. a linear combination

$$g = g_{00}\,\frac{dx^2}{x^2} + \sum_k \Big(g_{0k}\,\frac{dx}{x} \otimes dy_k + g_{k0}\,dy_k \otimes \frac{dx}{x}\Big) + \sum_{k\ell} g_{k\ell}\,dy_k \otimes dy_\ell \tag{2.1.1}$$

with $g_{ij} = g_{ji} \in \mathcal{C}^\infty(M)$ and (g_{ij}) non-degenerate. If the signature of g is Riemannian (resp. Lorentzian), we call g a *Riemannian* (resp. *Lorentzian*) *b-metric.* The volume density $|dg|$ of a b-metric is a non-vanishing b-density (more precisely, a b-1-density), and we can more consider b-density bundles ${}^{\mathrm{b}}\Omega^\alpha(M)$ $(\alpha \in \mathbb{R})$ in general, which are locally spanned by $|x^{-1}dx\,dy|^\alpha$. We thus have $|dg| = a(x, y)|x^{-1}dx\,dy|$ with $a > 0$ smooth.

We consider the geodesic flow of a b-metric g: In the interior of M, the flow, lifted to the cotangent bundle T^*M°, is generated by the Hamilton vector field H_G of the dual metric function G (up to a factor of 2 in the fiber direction); in local coordinates $z_1, \ldots, z_n$ on M° and the corresponding dual variables $\zeta_1, \ldots, \zeta_n$ (that is, writing covectors as $\sum_i \zeta_i\,dz_i$), the latter is defined by $G(z, \zeta) = |\zeta|^2_{G(z)} = \sum_{k\ell} G^{k\ell}\zeta_k\zeta_\ell$, with $G^{k\ell}$ the coefficients of the dual metric on T^*M, thus $(G^{k\ell}) = (g_{ij})^{-1}$, and the Hamilton vector field is given by

$$H_G = \sum_{j=1}^n (\partial_{\zeta_j} G)\partial_{z_j} - (\partial_{z_j} G)\partial_{\zeta_j} \tag{2.1.2}$$

Notice that $G \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*M)$, and G is homogeneous of degree 2 with respect to dilations in the fibers of ${}^{\mathrm{b}}T^*M$. Now suppose the coordinate system $z_1, \ldots, z_n$ is the restriction to M° of a coordinate system $x = z_1$, $y_i = z_i$ $(i = 2, \ldots, n)$. The natural coordinates ξ, η on the fibers of ${}^{\mathrm{b}}T^*M$ are defined by writing b-covectors as $\xi\,\frac{dx}{x} + \sum_k \eta_k\,dy_k$, thus $\xi = z_1\zeta_1$ and $\eta_k = \zeta_k$. Correspondingly, $\partial_{z_1} = \partial_x + \xi x^{-1}\partial_\xi$, $\partial_{\zeta_1} = x\partial_\xi$, and therefore

$$H_G = (\partial_\xi G)x\partial_x - (x\partial_x G)\partial_\xi + \sum_{i=2}^n (\partial_{\eta_i} G)\partial_{y_i} - (\partial_{y_i} G)\partial_{\eta_i}. \tag{2.1.3}$$

This directly shows that $H_G \in \mathcal{V}_{\mathrm{b}}({}^{\mathrm{b}}T^*M)$ is a b-vector field. In particular, its integral curves are either disjoint from $\partial\,{}^{\mathrm{b}}T^*M = {}^{\mathrm{b}}T^*_X M$ or entirely contained in it, and in this sense, the boundary X is 'at infinity.' Since we need to understand the causal structure of M, we will be interested in the structure of the null-geodesic flow, i.e. the flow of H_G within the *characteristic set* $\Sigma := \{G = 0\} \subset {}^{\mathrm{b}}T^*M \setminus o$, which is a conic subset.

We already remark here that the restriction

$$H_G|_{{}^{\mathrm{b}}T^*_X M} = \sum_{i=2}^n (\partial_{\eta_i} G)\partial_{y_i} - (\partial_{y_i} G)\partial_{\eta_i} \in \mathcal{V}({}^{\mathrm{b}}T^*_X M)$$

encodes some of the asymptotic behavior of the Hamilton flow in a natural way, as indicated in the introduction to this chapter, since it loses the dependence on x and its dual variable ξ; it will often be important to retain information on the behavior of the flow in the direction transverse to the boundary, and this is naturally accomplished by restricting H_G to ${}^{\mathrm{b}}T^*_X M$ *as a b-vector field*, thus keeping the first term in (2.1.3). We resume this discussion in §2.1.3.

2.1.2 Smoothness and conormality

We briefly digress to clarify the relevance of the notion of smoothness of b-objects, specifically b-metrics. For simplicity, assume that $M = [0, \infty)_x \times \mathbb{R}^{n-1}_y$. Then the smoothness of a b-metric g means that it be of the form (2.1.1) with coefficients $g_{ij}(x, y)$ which are smooth functions in the local coordinate chart; thus they are infinitely differentiable with respect to the vector fields $\partial_x, \partial_{y_1}, \ldots, \partial_{y_{n-1}}$. If g arises from a metric, not assumed to be stationary, on $\mathbb{R}_t \times \mathbb{R}^{n-1}_y$ by letting $x = e^{-t}$ (motivated by the introductory remarks at the beginning of this chapter), then the smoothness of g as a b-metric requires the coefficients of g in the (t, y)-coordinates to be smooth functions of e^{-t} and y, which is very restrictive and rather unnatural: For instance, rescaling the time variable and then compactifying, thus using e.g. $e^{-\alpha(y)t} = x^{\alpha(y)}$, $\alpha \in \mathcal{C}^\infty(\mathbb{R}^{n-1}_y)$ positive, as a boundary defining function, gives a different smooth structure on the compactified space. Notice however that if the metric g was 'stationary,' i.e. t-independent, it is a smooth b-metric on the compactified space independently of the choice of the boundary defining function $e^{-\alpha(y)t}$.

One obtains a more natural notion of 'smoothness' for general, non-stationary b-metrics by observing that the vector fields ∂_t and $\partial_{y_1}, \ldots, \partial_{y_{n-1}}$ lift to a basis of the space of b-vector fields on the compactification, regardless of the choice $e^{-\alpha(y)t}$ of the boundary defining function. Therefore, we can naturally and invariantly define a class of b-metrics g on M by requiring that $g = g_0 + g'$, where g_0 is stationary, while the coefficients g'_{ij} of g' satisfy

$$V_1 \cdots V_k g'_{ij} \in x^\gamma L^\infty(M), \quad k \geq 0, V_1, \ldots, V_k \in \mathcal{V}_{\mathrm{b}}(M), \tag{2.1.4}$$

with $\gamma > 0$ uniformly over compact sets; in our applications, we will use L^2 instead of L^∞, because L^2-based spaces are more convenient for Fourier-based analysis. (Functions with iterated regularity under the application of vector fields tangent to a hypersurface, X in the case of (2.1.4), are called *conormal* to the hypersurface; see §3.3 for more on this.) In applications, the smooth b-metric g_0 will then be a 'stationary' metric, i.e. independent of the boundary defining function after fixing a collar neighborhood of the boundary, and the non-stationary, dynamical part of the metric is encoded in g'. We will refer to such metrics $g = g_0 + g'$ as *asymptotically stationary*; we will give a more concise description in §3.3, and discuss examples in Chapters 5, 6 and 9, see in particular §5.2.2.

2.1.3 Flow near infinity; radial compactifications

We continue to denote by g a b-metric on M, with dual metric function G. Since G is homogeneous of degree 2, H_G is homogeneous of degree 1 with respect to dilations in the fibers of ${}^{\mathrm{b}}T^*M$. It is often convenient to rescale homogeneous functions and vector fields so as to obtain objects on the quotient ${}^{\mathrm{b}}S^*M := ({}^{\mathrm{b}}T^*M \setminus o)/\mathbb{R}_+$. For vector fields however, one loses information about their behavior in the radial direction, i.e. along orbits of the $\mathbb{R}_+$-action. Following Vasy [114, §3], we therefore instead view ${}^{\mathrm{b}}S^*M$ as the boundary of the *radial compactification* ${}^{\mathrm{b}}\overline{T}^*M$ of ${}^{\mathrm{b}}T^*M$, a concept that we briefly review; see also [84, §1.8]. We start by defining the radial compactification of $\mathbb{R}^n$, which proceeds by adding a sphere at infinity: Concretely,

$$\overline{\mathbb{R}^n} := \big(\mathbb{R}^n \sqcup ([0,\infty) \times \mathbb{S}^{n-1})\big)/\sim, \quad (0,\infty)\times\mathbb{S}^{n-1} \ni (r,\omega) \sim r^{-1}\omega \in \mathbb{R}^n,$$

and $\overline{\mathbb{R}^n}$, equipped with the natural smooth structure, thus becomes a manifold with boundary. Then, we radially compactify each fiber of ${}^{\mathrm{b}}T^*M$ in this manner; the compactified fibers can be arranged to depend smoothly on the base point by first fixing a smooth function $\rho \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*M \setminus o)$, homogeneous of degree -1, which plays the role of the inverse distance r^{-1} to the origin near infinity in ${}^{\mathrm{b}}T^*M$, and ρ then extends smoothly to ${}^{\mathrm{b}}\overline{T}^*M$ and provides a defining function for ${}^{\mathrm{b}}S^*M$. (One of course needs to smooth out ρ near the zero section to obtain a globally smooth boundary defining function.) Thus, gluing the fiber-wise compactifications together, we obtain the fiber-radial compactification ${}^{\mathrm{b}}\overline{T}^*M$, a fiber bundle over M with typical fiber a closed ball in $\mathbb{R}^n$. This is a manifold with corners; in the case that M is a manifold with boundary, its boundary hypersurfaces are ${}^{\mathrm{b}}\overline{T}^*_X M$ and ${}^{\mathrm{b}}S^*M$, and the corner is ${}^{\mathrm{b}}S^*_X M$. See Figure 2.1.

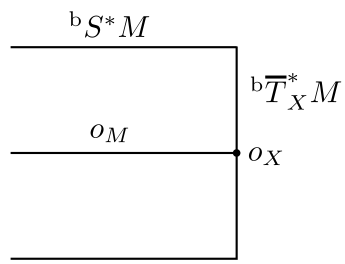

Figure 2.1: The radially compactified b-cotangent bundle ${}^{\mathrm{b}}\overline{T}^*M$ near ${}^{\mathrm{b}}\overline{T}^*_X M$; the cosphere bundle ${}^{\mathrm{b}}S^*M$, viewed as the boundary at fiber infinity of ${}^{\mathrm{b}}\overline{T}^*M$, is also shown, as well as the zero section $o_M \subset {}^{\mathrm{b}}\overline{T}^*M$ and the zero section over the boundary $o_X \subset {}^{\mathrm{b}}\overline{T}^*_X M$.

Now, for a function $f \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*M \setminus o)$ which is homogeneous of degree $m \in \mathbb{R}$, we can restrict $\rho^m f \in \mathcal{C}^\infty({}^{\mathrm{b}}\overline{T}^*M)$ to fiber infinity, thus $\rho^m f \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$. This provides an identification of homogeneous functions of fixed degree with elements of $\mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$. (This identification depends on the choice of ρ, hence we are really identifying homogeneous functions with sections of a line bundle over ${}^{\mathrm{b}}S^*M$; this bundle is canonically trivial only for $m = 0$.) The Hamilton vector field H_f is homogeneous of degree $(m-1)$; therefore, we can view

$$\mathsf{H}_f := \rho^{m-1} H_f \in \mathcal{V}({}^{\mathrm{b}}\overline{T}^*M),$$

which is tangent to ${}^{\mathrm{b}}S^*M$ and ${}^{\mathrm{b}}\overline{T}^*_X M$, as an element of $\mathcal{V}_{\mathrm{b}}({}^{\mathrm{b}}S^*M)$ by restriction. However, if $\mathsf{H}_f = 0$ at a point $(z,\zeta) \in {}^{\mathrm{b}}S^*M$, then H_f is *radial*, i.e. a multiple of the generator of dilations in the fibers of ${}^{\mathrm{b}}T^*M$. If it is a non-zero multiple, then H_f, while vanishing at (z,ζ) as a vector field, is non-trivial as a b-vector field there, and as such encodes whether H_f is radially pointing towards infinity in the fiber, or inwards towards 0. Thus, in these degenerate situations, we shall view

$$\mathsf{H}_f \in \mathcal{V}_{\mathrm{b}}({}^{\mathrm{b}}\overline{T}^*M),$$

with the relevant information encoded in $\mathsf{H}_f|_{{}^{\mathrm{b}}S^*M}$, the restriction to fiber infinity *as a b-vector field.*

Returning to the case of smooth b-metrics g, we conclude the discussion of b-geometry by showing how one can subdivide the task of studying the dynamics of the null-geodesic flow of H_G near the boundary into a study of the flow within the boundary and its behavior transverse to it. A concrete example to keep in mind is that of de Sitter space, detailed in the next section. Now, in local coordinates $(x,y;\xi,\eta)$ of ${}^{\mathrm{b}}T^*M$ near the boundary as before, the b-cotangent bundle of the boundary, ${}^{\mathrm{b}}T^*X$, is equal to $\{(0,y;0,\eta)\} \subset {}^{\mathrm{b}}T^*M$. (From the perspective of doing analysis on stationary spacetimes by viewing them as being foliated by isometric spacelike hypersurfaces X, ${}^{\mathrm{b}}T^*X$ is 'the same as' T^*X.) With x denoting a boundary defining function of M, we can then consider the subspace

$$T_\pm = \pm\frac{dx}{x} + {}^{\mathrm{b}}T^*X, \tag{2.1.5}$$

so in local coordinates as before, $T_\pm = \{(0,y;\pm 1,\eta)\} \subset {}^{\mathrm{b}}T^*_X M$. In fact, $T_\pm$ is well-defined independently of the choice of x, since $T_\pm = \{\varpi \in {}^{\mathrm{b}}T^*_X M \colon \varpi(x\partial_x) = \pm 1\}$, and the vector field $x\partial_x \in {}^{\mathrm{b}}T_X M$ does not depend on the choice of x. The point is that the characteristic set $\Sigma = G^{-1}(0)$, which is conic, can be identified with its intersection with $T_\pm$ away from the places where it intersects ${}^{\mathrm{b}}T^*X \setminus o$; and moreover, the Hamilton vector field H_G is tangent to $T_\pm$. Thus, the study of the Hamilton flow over X can be reduced to the study over $T_\pm$. Since G, being a polynomial, is fully homogeneous, not merely positively homogeneous, the restriction to T_+ of course suffices: Passing from T_+ to T_- merely requires changing the sign of all fiber coordinates and the direction of the Hamilton vector field. In order to encode the intersection $\Sigma \cap {}^{\mathrm{b}}T^*X$, which is a conic set, in this picture, we observe that we can radially compactify the affine spaces $T_\pm$, using the restriction to $T_\pm$ of the boundary defining function for fiber infinity used to compactify ${}^{\mathrm{b}}T^*M$. In other words, we can consider the closure of $T_\pm$ in ${}^{\mathrm{b}}\overline{T}^*M$; denote it by $\overline{T}_\pm$. Observe that the intersection of $\overline{T}_\pm$ with fiber infinity ${}^{\mathrm{b}}S^*M$ is the set ${}^{\mathrm{b}}S^*X = \partial({}^{\mathrm{b}}\overline{T}^*X)$. The characteristic set within ${}^{\mathrm{b}}T^*X$ then appears at fiber infinity of $\overline{T}_+$ and $\overline{T}_-$. See Figure 2.2. The rescaled Hamilton vector field $\mathsf{H}_G \in \mathcal{V}_{\mathrm{b}}({}^{\mathrm{b}}\overline{T}^*M)$ restricts to an element of $\mathcal{V}_{\mathrm{b}}(\overline{T}_\pm)$. Therefore, the flow generated by H_G *within the b-cotangent bundle of M over X* can be completely understood in terms of $\mathsf{H}_G|_{\overline{T}_\pm} \in \mathcal{V}_{\mathrm{b}}(\overline{T}_\pm)$.

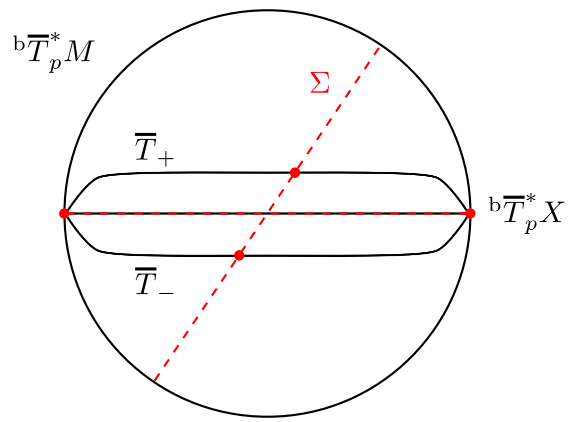

Figure 2.2: The radial compactification ${}^{\mathrm{b}}\overline{T}^*_pM$ of the fiber of ${}^{\mathrm{b}}T^*M$ over a point $p \in X = \partial M$, together with the (radial) compactifications ${}^{\mathrm{b}}\overline{T}^*_pX, \overline{T}_\pm$ of the natural affine subspaces ${}^{\mathrm{b}}T^*_pX, T_\pm \subset {}^{\mathrm{b}}T^*_pM$. In red: A conic subset Σ of ${}^{\mathrm{b}}T^*_pM$ and its identification with a subset of $\overline{T}_+ \cup \overline{T}_-$. (This is precisely the picture of the characteristic set for the static de Sitter metric in 2 spacetime dimensions over a point on the cosmological horizon; see §2.2.1.)

Now, if $p \in \overline{T}_\pm$ is a point with $\mathsf{H}_G|_p \neq 0$, the $x\partial_x$-component $\mathsf{H}_G(x)$ of $\mathsf{H}_G \in \mathcal{V}_{\mathrm{b}}({}^{\mathrm{b}}\overline{T}^*M)$, which disappears when considering $\mathsf{H}_G|_{\overline{T}_\pm}$, has a single order of vanishing at $X \ni p$ as a vector field, hence does not affect the qualitative behavior of the H_G-flow near p. However, if $\mathsf{H}_G|_p = 0$ (as an element of ${}^{\mathrm{b}}T_p(\overline{T}_\pm)$), then in order to understand the nature of the critical point p of the Hamiltonian flow, one does need information on $x^{-1}\mathsf{H}_G(x)$: In the situations considered below, critical points p of $\mathsf{H}_G|_{\overline{T}_\pm}$ will be sources/sinks for the flow within $\overline{T}_\pm$, and whether they are sources/sinks or saddle points with a single stable/unstable direction transverse to the boundary for the 'full' flow in ${}^{\mathrm{b}}\overline{T}^*M$ depends precisely on whether $x^{-1}\mathsf{H}_G(x)$ is positive or negative at p (provided we are in the non-degenerate situation that the latter quantity is non-vanishing).

In §3.3.4, we will complete the above discussion by showing that the Hamilton flow of G on $T_\pm$ is equal (under certain natural identifications) to the (semiclassical) Hamilton flow of the so-called normal operator family (which is a family of operators on X depending on a parameter $\sigma \in \mathbb{C}$, obtained by Mellin transforming in x) of the Laplace-Beltrami operator Δ_g, and the transverse component $x^{-1}\mathsf{H}_G(x)$ of the flow can also be understood in terms of this family. This then shows that properties of the normal operator family of Δ_g directly translate to properties of the Hamilton flow of the b-metric g, and vice versa.

2.2 de Sitter space

We consider $(n+1)$-dimensional Minkowski space $\mathbb{R}^{n+1}_z$ with metric $g_0 := dz_{n+1}^2 - dz_1^2 - \cdots - dz_n^2$. Then n-dimensional de Sitter space is the one-sheeted hyperboloid

$$M_0^\circ = \{z_{n+1}^2 - z_1^2 - \cdots - z_n^2 = -1\}$$

with metric g induced by g_0; thus, g has signature $(+,-,\ldots,-)$. Moreover, M_0° inherits the usual time orientation from the ambient Minkowski space, in which $\partial_{z_{n+1}}$ is future timelike. We can introduce global coordinates using the map $\mathbb{R}_{z_{n+1}} \times \mathbb{S}^{n-1}_\theta, \quad (z_{n+1}, \theta) \mapsto ((1+z_{n+1}^2)^{1/2}\theta, z_{n+1}) \in \mathbb{R}^{n+1}$, and the metric becomes

$$g = \frac{dz_{n+1}^2}{1+z_{n+1}^2} - (1+z_{n+1}^2)d\theta^2$$

We compactify M_0°, first at future infinity by introducing $x = z_{n+1}^{-1}$ in $z_{n+1} \geq 1$, say, so the metric becomes

$$g = x^{-2}\left(\frac{dx^2}{1+x^2} - (1+x^2)d\theta^2\right) =: x^{-2}\bar{g}, \tag{2.2.1}$$

where $\bar{g}$ is a smooth Lorentzian metric down to $x = 0$, and likewise at past infinity; thus, we have compactified M_0° to a cylinder

$$M_0 \cong [-1,1]_T \times \mathbb{S}^{n-1},$$

say with $T = 1 - x$ near $x = 0$, and $T = 0$ at $z_{n+1} = 0$. The metric g is a so-called *0-metric*, see [81]. Null-geodesics of g are merely reparametrizations of null-geodesics of the conformally related metric $\bar{g}$.

2.2.1 Static model of de Sitter space

From the point of view of causality, one can localize the study of de Sitter space M_0 by picking a point q, say $T = 1$, $\theta = e_1 \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, at future infinity and considering only the interior M of the backward light cone from q, intersected with $\{T \geq 0\}$ for convenience; we call M_S° the *static model of de Sitter space*.[1] For an illustration, see Figure 2.3.

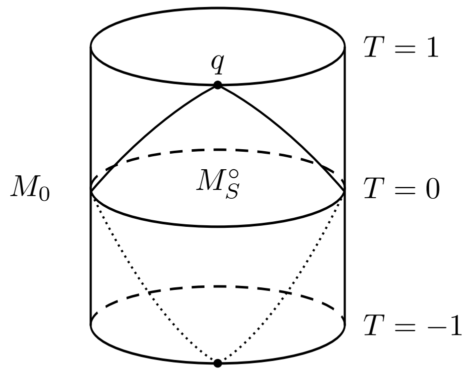

Figure 2.3: The 'future half' of the static model M_S° of de Sitter space, a submanifold of (compactified) de Sitter space M_0, is the backward light cone from the point q at future infinity, intersected with $T \geq 0$. The full static model is the intersection of the interiors of the backward light cone from q and the corresponding point at past infinity.

We make this explicit in the coordinates $z_1, \ldots, z_{n+1}$ of the ambient Minkowski space: Namely, for each fixed $\omega \in \mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}_{z_2,\ldots,z_n}$, the affine curve

$$\gamma_\omega(z_{n+1}) = (z_{n+1}, \omega; z_{n+1}) \in M_0^\circ \subset \mathbb{R}^{1+(n-1)+1}$$

is a geodesic on de Sitter space M_0°, and written in the coordinates $x = z_{n+1}^{-1}$, $\theta = (z_1^2 + \cdots + z_n^2)^{-1/2}(z_1, \ldots, z_n) \in \mathbb{S}^{n-1}$ introduced in the previous section, it is equal to

$$\gamma_\omega(z_{n+1}) = (x, \theta(x)), \quad \theta(x) = (1 + x^2)^{-1/2}(e_1 + x\omega).$$

Thus, we see that the family $\{\gamma_\omega : \omega \in \mathbb{S}^{n-2}\}$ exactly sweeps out the backward light cone from the point $x = 0$, $\theta = e_1$, thus is the boundary of the static model M_S°. In other words,

[1] Strictly speaking, this is only the future half of the static model; the full static model is the intersection of the interior of the backward light cone from $(T = 1, \theta = e_1)$ with the forward light cone from $(T = -1, \theta = e_1)$, see [114, §4] for details.

in the Minkowskian coordinates,

$$M_S^\circ = \{(z_1, \ldots, z_{n+1}) \in M_0^\circ \colon z_{n+1} \geq 0, z_2^2 + \cdots + z_n^2 < 1\}.$$

The backward light cone is a *cosmological horizon* for M_S°: Any causal (timelike or null) future-oriented curve in M_0°, starting at a point in M_S°, which crosses the cosmological horizon, can never return to M_S°.

On M_S°, one can choose coordinates $t \in \mathbb{R}, Y \in \mathbb{R}^{n-1}$, $|Y| < 1$, with respect to which the metric g is t-independent, and writing $Y = r\omega$, $r \in (0,1)$, $\omega \in \mathbb{S}^{n-2}$, away from $Y = 0$, one has

$$g = (1 - r^2)\, dt^2 - (1 - r^2)^{-1}\, dr^2 - r^2\, d\omega^2;$$

this is the special case of the Schwarzschild-de Sitter metric (2.3.1) with vanishing black hole mass $M_\bullet$ and cosmological constant $\Lambda = 3$, see §2.3. One can compactify this at future infinity using $\widetilde{x} = e^{-t}$ as a boundary defining function; the coordinate singularity of the metric at $r = 1$ can then be resolved by means of a suitable blow up of the corner $\widetilde{x} = 0, r = 1$, with g extending smoothly and non-degenerately past the front face of the blow-up near the side face $\widetilde{x} = 0$; see [114, §4] for details. In practice, this procedure amounts to performing a singular change of coordinates at $r = 1$, and we will give details for Schwarzschild-de Sitter and Kerr-de Sitter black holes in §§2.3 and 2.4.

We describe a different way of arriving at such a smooth extension of the static metric past the cosmological horizon. First, we recall the relation of hyperbolic space $\mathbb{H}^n = \{z_{n+1}^2 - z_1^2 - \cdots - z_n^2 = 1, z_{n+1} > 0\}$ as a subset of Minkowski space to the upper half plane model: Define the global coordinate chart

$$\Phi \colon \mathbb{H}^n \ni (z_1, z_2, \ldots, z_n, z_{n+1}) \mapsto (x, y),$$
$$x = \frac{2}{z_1 + z_{n+1}} \in (0, \infty), \quad y = \frac{2(z_2, \ldots, z_n)}{z_1 + z_{n+1}} \in \mathbb{R}^{n-2},$$

then the induced metric on $\mathbb{H}^n$ takes the simple form $x^{-2}(dx^2 + dy^2)$. The above map Φ is in fact well-defined on $\{z_1 + z_{n+1} > 0\}$, and restricting Φ to $M_0^\circ \cap \{z_1 + z_{n+1} > 0\}$, the metric on M_0° has the form

$$g = x^{-2}(dx^2 - dy^2).$$

Moreover, the point q at future infinity singled out above has coordinates $x = 0, y = 0$, where we extended Φ by continuity to $M_0 \cap \{z_1 + z_{n+1} > 0\}$, and the backward light cone from q is simply the set $\{|y| = x\}$; the static model, compactified at future infinity, therefore is

$$M_S = \{|y| < x, x \geq 0\}.$$

Blowing up $(0, 0)$ spherically, we can introduce coordinates $\tau = x \in [0, 1), Y = y/x \in \mathbb{R}^{n-1}$, $|Y| < 1$ near the interior of the front face, with respect to which

$$g = (1 - |Y|^2)\,\frac{d\tau^2}{\tau^2} - 2Y\frac{d\tau}{\tau}\,dY - dY^2 = \frac{d\tau^2}{\tau^2} - \Big(Y\frac{d\tau}{\tau} + dY\Big)^2 \tag{2.2.2}$$

which extends non-degenerately as a Lorentzian b-metric past the cosmological horizon $|Y| = 1$. Moreover, τ and $1/z_{n+1}$ are comparable (i.e. bounded by constant multiples of each other) near q, and in terms of the static time coordinate t, we have $\tau \sim e^{-t}$ over compact subsets of M_S, i.e. away from the cosmological horizon. In fact, we can define a new time coordinate t_* by

$$\tau = e^{-t_*},$$

which is thus smooth on M_S° *up to* (and beyond) the cosmological horizons. We point this out here since the extension of the metric across horizons in the Schwarzschild-de Sitter setting in §2.3 will involve a rescaled time coordinate t_* in exactly the same fashion.

The dual metric of g is

$$G = (Y\partial_Y - \tau\partial_\tau)^2 - \partial_Y^2. \tag{2.2.3}$$

Concretely, with $r = |Y|$ and $\omega = r^{-1}Y$, we introduce $\mu = 1 - r^2$ as a defining function of $r = 1$, and compute

$$G = -4\mu r^2\partial_\mu^2 + 4r^2\tau\partial_\tau\,\partial_\mu + (\tau\partial_\tau)^2 - r^{-2}\partial_\omega^2, \tag{2.2.4}$$

valid away from $r = 0$, which extends non-degenerately to $\mu < 0$. The same blow-up procedure can be applied to more general, asymptotically de-Sitter like spaces, see §2.2.2. Since at the horizon $\mu = 1$, we expect the null-geodesic flow to be somewhat degenerate, we study the flow in a slightly enlarged domain

$$\Omega = \{\mathfrak{t}_1 \geq 0, \mathfrak{t}_2 \geq 0\}, \quad \mathfrak{t}_1 = \tau_0 - \tau, \quad \mathfrak{t}_2 = \mu + \delta \tag{2.2.5}$$

with $\tau_0 > 0$ fixed, $\delta > 0$ small. Thus, $\mathfrak{t}_1$ defines a Cauchy hypersurface H_1, while $\mathfrak{t}_2$ defines a hypersurface H_2; see Figure 2.4 below. The domain Ω will be the model for the types of domains on which we shall later study linear and nonlinear wave equations. We view

$$\Omega \subset M := \{\mu > -2\delta, 0 \leq \tau < 2\tau_0\},$$

with M given the (dual) metric (2.2.4) in $\mu \leq 0$. We introduce M here merely to have an ambient manifold to work in; the point is that Ω is a domain with boundary in M, with the boundaries H_1 and H_2 being 'artificial,' namely contained in M° if disjoint from ∂M or intersecting ∂M transversally (see below), while the only boundary of Ω that deserves this name from the b-perspective is

$$Y := \Omega \cap \partial M = \{\mu \geq -\delta, \tau = 0\},$$

the boundary of Ω at future infinity.

We proceed to analyze the null-geodesic flow, lifted to the cotangent bundle, and the global causal structure of Ω. Concretely, we first check:

Proposition 2.2.1. *The domain Ω enjoys the following properties:*

(1) Ω is compact,

(2) the differentials of $\mathfrak{t}_1$ and $\mathfrak{t}_2$ have the opposite timelike character near their respective zero sets within Ω, more specifically, $\mathfrak{t}_1$ is future timelike, $\mathfrak{t}_2$ past timelike,

(3) the artificial boundary hypersurfaces $H_j := \mathfrak{t}_j^{-1}(0)$, $j = 1, 2$, intersect the boundary ∂M transversally, and H_1 and H_2 intersect only in the interior of M, and they do so transversally,

(4) the defining function τ of future infinity of M has $d\tau/\tau$ timelike on $\Omega \cap \partial M$, with timelike character opposite to the one of $\mathfrak{t}_1$, i.e. $d\tau/\tau$ is past oriented.

Proof. (1) is clear. With $\mathfrak{t}_j$, $j = 1, 2$, defined in (2.2.5), we compute

$$\begin{aligned} G({}^{\mathrm{b}}d\mathfrak{t}_1, {}^{\mathrm{b}}d\mathfrak{t}_1)|_{\mathfrak{t}_1=0} &= G\Big(-\tau\frac{d\tau}{\tau}, -\tau\frac{d\tau}{\tau}\Big)|_{\tau=\tau_0} = \tau_0^2 > 0, \\ G({}^{\mathrm{b}}d\mathfrak{t}_2, {}^{\mathrm{b}}d\mathfrak{t}_2)|_{\mathfrak{t}_2=0} &= G(d\mu, d\mu)|_{\mu=-\delta} = 4\delta(1+\delta) > 0, \\ G({}^{\mathrm{b}}d\mathfrak{t}_1, {}^{\mathrm{b}}d\mathfrak{t}_2)|_{\mathfrak{t}_1=\mathfrak{t}_2=0} &= -4(1+\delta)\tau_0 < 0, \end{aligned}$$

hence $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are timelike with opposite timelike character; indeed, $\mathfrak{t}_1$ is future oriented and $\mathfrak{t}_2$ is past oriented, as is $d\tau/\tau$. Moreover, $d\mathfrak{t}_2$ and $d\tau$ are clearly linearly independent at $Y \cap H_2$, as are $d\mathfrak{t}_1$ and $d\mathfrak{t}_2$ at $H_1 \cap H_2$. This establishes (2) and (3). Finally, (4) follows from $G\big(\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\big) = 1 > 0$. □

Next, we establish properties of the null-geodesic flow. Denote by $\Sigma = G^{-1}(0)$ the characteristic set, i.e. the (dual) light cones for the metric g, by

$$\Sigma_\Omega := \Sigma \cap {}^{\mathrm{b}}S^*_\Omega M$$

the characteristic set over the domain Ω, and by $\mathcal{R} \subset \Sigma_\Omega$ the radial set: This is the set of all points in Σ_Ω, identified with half-lines in ${}^{\mathrm{b}}T^*M$, at which the Hamilton vector field H_G is radial; equivalently, $\mathcal{R}$ is the set of critical points of the rescaled Hamilton vector field $\mathsf{H}_G \in \mathcal{V}({}^{\mathrm{b}}S^*_\Omega M)$ within Σ_Ω. We will view Σ_Ω and $\mathcal{R}$ as conic subsets of ${}^{\mathrm{b}}T^*_\Omega M \setminus o$ whenever convenient. We will show in Proposition 2.2.3 below that the characteristic set Σ has two components, $\Sigma = \Sigma_+ \cup \Sigma_-$, corresponding to the backward (+) and forward (−) light cones. In order to capture the behavior of the H_G-flow near the radial set, we then make the following general definition:

Definition 2.2.2. A smooth submanifold $L \subset {}^{\mathrm{b}}S^*_Y M \subset {}^{\mathrm{b}}\overline{T}^*_Y M$ is called a *generalized b-radial set* if the following holds for one choice of signs:

(1) $L \subset \Sigma$ is given by $\mathcal{L} \cap {}^{\mathrm{b}}S^*_Y M$, where $\mathcal{L}$ is a smooth submanifold of Σ transversal to ${}^{\mathrm{b}}S^*_Y M$, with H_G tangent to $\mathcal{L}$,

(2) for a defining function $\widehat{\rho}$ of fiber infinity ${}^{\mathrm{b}}S^*M$ within ${}^{\mathrm{b}}\overline{T}^*M$, and a defining function τ of ∂M within M, we have

$$\widehat{\rho}^{-1}\mathsf{H}_G\widehat{\rho} = \mp\beta_0, \quad -\tau^{-1}\mathsf{H}_G\tau = \mp\widetilde{\beta}\beta_0 \tag{2.2.6}$$

at L, with $\beta_0, \widetilde{\beta} \in \mathcal{C}^\infty(L)$ positive,

(3) there exists a quadratic defining function ρ_0 of $\mathcal{L}$ within Σ, see below, such that

$$\mp \mathsf{H}_G\rho_0 - \beta_1\rho_0 \geq 0 \tag{2.2.7}$$

holds, with $\beta_1 > 0$ near L, modulo terms that vanish cubically at L.

The function ρ_0 being a quadratic defining function means that it vanishes quadratically at $\mathcal{L}$ (and vanishes only at $\mathcal{L}$), with the vanishing non-degenerate, in the sense that the Hessian is positive definite on the normal bundle of $\mathcal{L}$ within Σ, corresponding to ρ_0 being a sum of squares of linear defining functions whose differentials span the conormal bundle of $\mathcal{L}$ within Σ.

Then $L \subset \Sigma$ is a sink (top signs)/source (bottom signs) within ${}^{\mathrm{b}}S^*_Y M$ in the sense that nearby bicharacteristics *within* ${}^{\mathrm{b}}S^*_Y M$ all tend to L as the parameter along them goes to $\pm\infty$; in fact, the behavior of the rescaled flow on ${}^{\mathrm{b}}\overline{T}^*_Y M$ is sink/source even in the fiber-radial direction. At L however, there is also a unstable/stable manifold, namely $\mathcal{L}$: Indeed, bicharacteristics in $\mathcal{L}$ remain there by the tangency of H_G to $\mathcal{L}$; further $\tau \to 0$ along them as the parameter goes to $\mp\infty$ by (2.2.6), at least sufficiently close to $\tau = 0$, since L is defined in $\mathcal{L}$ by $\tau = 0$. Notice that we do *not* assume that the Hamilton vector field be radial at L: While there might be non-trivial dynamics within L, the above definition is designed to only capture the saddle point dynamics in the directions normal to L. For the static de Sitter spacetime, L indeed consists of radial points, and the manifold $\mathcal{L}$ is (one half of) the conormal bundle of the cosmological horizon. For rotating black holes with non-zero angular momentum, discussed in §2.4, there are non-trivial dynamics within L, but the qualitative behavior in the normal directions is the same as in the present, static de Sitter context.

Proposition 2.2.3. *The null-geodesic flow on Ω has the following properties:*

(5) The characteristic set $\Sigma = G^{-1}(0)$ is a smooth codimension 1 submanifold transversal to ${}^{\mathrm{b}}S^*_Y M$ and has the form $\Sigma = \Sigma_+ \cup \Sigma_-$ with $\Sigma_\pm$ a union of connected components of Σ,

(6) the radial set is the union $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$ with $\mathcal{R}_\pm \subset \Sigma_\pm$; put $L_\pm = \partial\mathcal{R}_\pm \subset {}^{\mathrm{b}}S^*_\Omega M$, then $L_\pm$ is a generalized b-radial set in the sense of Definition 2.2.2; concretely, L_+ (resp. L_-) is a sink (resp. source) for the rescaled Hamilton flow within ${}^{\mathrm{b}}\overline{T}^*_Y M \setminus o$, with an unstable (resp. stable) direction transversal to ${}^{\mathrm{b}}\overline{T}^*_Y M$,

(7) the metric g is non-trapping *in the following sense: All bicharacteristics in Σ_Ω from any point in $\Sigma_\Omega \cap (\Sigma_+ \setminus L_+)$ flow (within Σ_Ω) to ${}^{\mathrm{b}}S^*_{H_1} M \cup L_+$ in the forward direction (i.e. either enter ${}^{\mathrm{b}}S^*_{H_1} M$ in finite time or tend to L_+) and to ${}^{\mathrm{b}}S^*_{H_2} M \cup L_+$ in the backward direction, and from any point in $\Sigma_\Omega \cap (\Sigma_- \setminus L_-)$ to ${}^{\mathrm{b}}S^*_{H_2} M \cup L_-$ in the forward direction and to ${}^{\mathrm{b}}S^*_{H_1} M \cup L_-$ in the backward direction.*

See Figure 1.1 for the flow near the radial set $\mathcal{R}$, and also Figures 2.4 and 5.1.

Proof of Proposition 2.2.3. We introduce coordinates on the b-cotangent bundle by writing b-covectors as

$$\sigma\,\frac{d\tau}{\tau}+\zeta\,dY,\quad \text{resp.}\quad \sigma\,\frac{d\tau}{\tau}+\xi\,d\mu+\eta\,d\omega,$$

and the dual metric function is then given by

$$G=-4r^2\mu\xi^2+4r^2\sigma\xi+\sigma^2-r^{-2}|\eta|^2=(Y\cdot\zeta-\sigma)^2-|\zeta|^2, \tag{2.2.8}$$

see (2.2.4) and (2.2.3). Correspondingly, we compute the Hamilton vector field by formula (2.1.3) to be

$$\begin{aligned} H_G&=4r^2(-2\mu\xi+\sigma)\partial_\mu-(4\xi^2(1-2r^2)-4\sigma\xi-r^{-4}|\eta|^2)\partial_\xi\\ &\qquad+(4r^2\xi+2\sigma)\tau\partial_\tau-r^{-2}H_{|\eta|^2}\\ &=2(Y\cdot\zeta-\sigma)(Y\partial_Y-\zeta\partial_\zeta-\tau\partial_\tau)-2\zeta\cdot\partial_Y. \end{aligned} \tag{2.2.9}$$

We begin by proving (5), i.e. that $G^{-1}(0)$ is a smooth conic 1-codimensional submanifold of ${}^{\mathrm{b}}T^*M\setminus o$ transversal to ${}^{\mathrm{b}}T^*_YM$. We have to show that $dG\neq 0$ whenever $G=0$. We compute

$$\begin{aligned} dG&=(4\xi^2(1-2r^2)-4\sigma\xi-r^{-4}|\eta|^2)d\mu+4r^2(-2\mu\xi+\sigma)d\xi\\ &\qquad+(4(1-\mu)\xi+2\sigma)d\sigma-r^{-2}d|\eta|^2. \end{aligned}$$

Thus if $dG=0$, all coefficients have to vanish, thus $\sigma=2\mu\xi$ and $\sigma=2(\mu-1)\xi$, giving $\xi=0$ and thus $\sigma=0$, hence also $\eta=0$. Thus dG vanishes only at the zero section of ${}^{\mathrm{b}}T^*M$ in this coordinate system. In the coordinates valid near $r=0$, we compute

$$dG=2(Y\cdot\zeta-\sigma)\zeta\cdot dY+2\big((Y\cdot\zeta-\sigma)Y-2\zeta\big)\cdot d\zeta-2(Y\cdot\zeta-\sigma)\,d\sigma,$$

thus $dG=0$ implies $Y\cdot\zeta=\sigma$, hence $\zeta=0$ and then $\sigma=0$. The transversality statement is clear since dG and $d\tau$ are linearly independent at Σ by inspection. Moreover, from (2.2.8), we have

$$G=(\sigma+2r^2\xi)^2-4r^2\xi^2-r^{-2}|\eta|^2, \tag{2.2.10}$$

and thus $\Sigma = \Sigma_+ \cup \Sigma_-$, where

$$\Sigma_\pm = \{\pm(\sigma + 2r^2\xi) > 0\} \cap \Sigma = \{\pm(\sigma - Y \cdot \zeta) > 0\},$$

because $G = 0$, $\sigma + 2r^2\xi = 0$ implies $\xi = \eta = 0$, thus $\sigma = 0$, hence $\{\sigma + 2r^2\xi = 0\}$ does not intersect the characteristic set $G^{-1}(0)$, and similarly in the (Y, τ, ζ, σ) coordinates.

Next, we locate the radial set: Since g is a Lorentzian b-metric, the Hamilton vector field H_G cannot be radial except at the boundary $Y = \partial M$ at future infinity, where $\tau = 0$. In the coordinate system near $r = 0$, one easily checks using (2.2.9) that there are no radial points over $Y = 0$. At radial points, we then moreover have $H_G\mu = 4r^2(-2\mu\xi + \sigma) = 0$, thus $\sigma = 2\mu\xi$. Further, the vanishing of $H_{|\eta|^2}$ at a radial point requires $\eta = 0$. Now, if $\xi = 0$, then $\sigma = 0$, i.e. all fiber variables vanish and we are outside the characteristic set Σ; thus $\xi \neq 0$. At points where $\sigma = 2\mu\xi, \eta = 0, \tau = 0$, the expression for G simplifies to $G = 4r^2\mu\xi^2 + 4\mu^2\xi^2 = 4\mu\xi^2$, which does not vanish unless $\mu = 0$. Hence, $\mu = 0, \tau = 0, \eta = 0, \sigma = 0$, and we easily check that these conditions are also sufficient for a point in this coordinate patch to be a radial point. Therefore, the radial set is $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$ with

$$\begin{aligned} \mathcal{R}_\pm &= \{\mu = 0, \tau = 0, \eta = 0, \sigma = 0, \pm\xi > 0\} \\ &= \{\tau = 0, \sigma = 0, Y = \mp\zeta/|\zeta|\} \subset \Sigma. \end{aligned}$$

Clearly, we have $\mathcal{R}_\pm \subset \Sigma_\pm$. To analyze the flow near $L_\pm := \partial\mathcal{R}_\pm \subset {}^{\mathrm{b}}S^*M$, we introduce normalized coordinates

$$\widehat{\rho} = \frac{1}{\xi}, \widehat{\eta} = \frac{\eta}{\xi}, \widehat{\sigma} = \frac{\sigma}{\xi}$$

and consider the homogeneous degree 0 vector field $\mathsf{H}_G := |\widehat{\rho}|H_G$. We get a first qualitative understanding of the dynamics near $L_\pm$ by looking at the linearization W of $\pm\mathsf{H}_G = \widehat{\rho}H_G$, following the arguments of [8, §3]. Note that $\langle\xi\rangle^{-1}$ is a defining function of the boundary of ${}^{\mathrm{b}}\overline{T}^*M$ at fiber infinity near $L_\pm$. The coordinate vector fields in the new coordinate system are

$$\partial_\eta = \widehat{\rho}\partial_{\widehat{\eta}}, \quad \xi\partial_\xi = -\widehat{\rho}\partial_{\widehat{\rho}} - \widehat{\eta}\partial_{\widehat{\eta}} - \widehat{\sigma}\partial_{\widehat{\sigma}}.$$

Hence

$$\begin{aligned} \widehat{\rho}H_G = {} & 4r^2(-2\mu + \widehat{\sigma})\partial_\mu + (4(1 - 2r^2) - 4\widehat{\sigma} - r^{-4}|\widehat{\eta}|^2)(\widehat{\rho}\partial_{\widehat{\rho}} + \widehat{\eta}\partial_{\widehat{\eta}} + \widehat{\sigma}\partial_{\widehat{\sigma}}) \\ & + (4r^2 + 2\widehat{\sigma})\tau\partial_\tau - r^{-2}\widehat{\rho}H_{|\eta|^2}. \end{aligned}$$

We have $\widehat{\rho}H_G \in \mathcal{V}_{\mathrm{b}}({}^{\mathrm{b}}\overline{T}^*M)$ (where it is defined). Since $\widehat{\rho}H_G$ vanishes (as a vector field) at a radial point $q \in {}^{\mathrm{b}}\overline{T}^*M$, it maps the ideal $\mathcal{I}$ of functions in $C^\infty({}^{\mathrm{b}}\overline{T}^*M)$ vanishing at q into itself. The linearization of $\widehat{\rho}H_G$ at q then is the vector field $\widehat{\rho}H_G$ acting on $\mathcal{I}/\mathcal{I}^2 \cong T_q^*{}^{\mathrm{b}}\overline{T}^*M$, where the isomorphism is given by $f + \mathcal{I}^2 \mapsto df|_q$. Computing the linearization W of $\widehat{\rho}H_G$ at q now amounts to ignoring terms of $\widehat{\rho}H_G$ that vanish to at least second order at q, which gives

$$W = 4(-2\mu + \widehat{\sigma})\partial_\mu - 4(\widehat{\rho}\partial_{\widehat{\rho}} + \widehat{\eta}\partial_{\widehat{\eta}} + \widehat{\sigma}\partial_{\widehat{\sigma}}) + 4\tau\partial_\tau - 2K^{ij}(\omega)\widehat{\eta}_j\partial_{\omega_i},$$

where we introduced a local coordinate system on the sphere. We read off the eigenvectors and corresponding eigenvalues:

$$\begin{aligned} d\widehat{\rho}, d\widehat{\eta}, d\widehat{\sigma} &\text{ with eigenvalue } -4, \\ d\mu - d\widehat{\sigma} &\text{ with eigenvalue } -8, \\ d\tau &\text{ with eigenvalue } +4, \\ d\omega_i - \tfrac{1}{2}K^{ij}d\widehat{\eta}_j &\text{ with eigenvalue } 0. \end{aligned}$$

Thus, L_+ (L_-) is a sink (source) of the Hamilton flow within ${}^{\mathrm{b}}S^*_YM$, with an unstable (stable) direction normal to the boundary. More precisely, the τ-independence of the metric suggests the definition

$$\mathcal{L}_\pm = \partial\{\mu = 0, \sigma = 0, \eta = 0, \pm\xi > 0\} \subset {}^{\mathrm{b}}S^*M$$

of the unstable (stable) manifold, so that $L_\pm = {}^{\mathrm{b}}S^*_YM \cap \mathcal{L}_\pm$; moreover $\mathcal{L}_\pm \subset \Sigma$, and H_G is tangent to $\mathcal{L}_\pm$; indeed,

$$H_G = 4\xi^2\partial_\xi + 4\xi\tau\partial_\tau \text{ at } \mathcal{L}_\pm. \tag{2.2.11}$$

Now, going back to the full rescaled Hamilton vector field H_G, we have at $L_\pm$ (in fact, at $\mathcal{L}_\pm$):

$$|\widehat{\rho}|^{-1}\mathsf{H}_G|\widehat{\rho}| = \mp\beta_0, \quad -\tau^{-1}\mathsf{H}_G\tau = \mp\widetilde{\beta}\beta_0 \tag{2.2.12}$$

with $\beta_0 = 4$ and $\widetilde{\beta} = 1$, thus establishing condition (2.2.6) for generalized b-radial sets; furthermore, near $\mathcal{L}_\pm$,

$$\mp\mathsf{H}_G\widehat{\eta} = 4\widehat{\eta}, \quad \mp\mathsf{H}_G\widehat{\sigma} = 4\widehat{\sigma}, \quad \mp\mathsf{H}_G(\mu - \widehat{\sigma}) = 8(\mu - \widehat{\sigma})$$

modulo terms that vanish quadratically at $\mathcal{L}_\pm$, hence, putting $\beta_1 = 8$, the quadratic defining function $\rho_0 := \widehat{\eta}^2 + \widehat{\sigma}^2 + (\mu - \widehat{\sigma})^2$ of $\mathcal{L}_\pm$ within Σ satisfies $\mp \mathsf{H}_G\rho_0 - \beta_1\rho_0 \geq 0$ modulo terms that vanish cubically at $L_\pm$ (in fact at $\mathcal{L}_\pm$), thus (2.2.7) holds. This establishes (6).

It remains to check the non-trapping assumption (7). Let us first analyze the flow in ${}^{\mathrm{b}}T^*_\Omega M \setminus {}^{\mathrm{b}}T^*_Y M$; recall from §2.1.1 that bicharacteristics intersecting ${}^{\mathrm{b}}T^*_Y M$ are in fact contained in ${}^{\mathrm{b}}T^*_Y M$, and correspondingly bicharacteristics containing points in ${}^{\mathrm{b}}T^*_\Omega M \setminus {}^{\mathrm{b}}T^*_Y M$ stay in ${}^{\mathrm{b}}T^*_\Omega M \setminus {}^{\mathrm{b}}T^*_Y M$. There,

$$\pm H_G\tau = \pm 2(\sigma + 2r^2\xi)\tau > 0 \text{ on } \Sigma_\pm. \tag{2.2.13}$$

In particular, in $\Sigma_\pm \setminus {}^{\mathrm{b}}T^*_Y M$, bicharacteristics reach ${}^{\mathrm{b}}T^*_{H_1} M$ (i.e. $\tau = \tau_0$) in finite time in the forward $(+)$, resp. backward $(-)$, direction. We show that they stay within ${}^{\mathrm{b}}T^*_\Omega M$: For this, observe that $G = 0$ and $\mu < 0$, thus $r > 1$, imply

$$2|\xi| \leq 2r|\xi| \leq |\sigma + 2r^2\xi|$$

by equation (2.2.10). In fact, if $\xi \neq 0$, the first inequality is strict, and if $\xi = 0$, the second inequality is strict, and we conclude the strict inequality

$$2|\xi| < |\sigma + 2r^2\xi| \text{ if } G = 0, \mu < 0.$$

Hence, on $(\Sigma_\pm \setminus {}^{\mathrm{b}}T^*_Y M) \cap \Sigma_\Omega$, if $\mu < 0$, then

$$\pm H_G\mu = \pm 4r^2(\sigma + 2r^2\xi - 2\xi) > 0, \tag{2.2.14}$$

thus in the forward (on Σ_+), resp. backward (on Σ_-), direction, bicharacteristics cannot cross ${}^{\mathrm{b}}T^*_{H_2} M = \{\mu = -\delta\}$.

Next, backward, resp. forward, bicharacteristics in $\mathcal{L}_\pm \setminus L_\pm$ tend to $L_\pm$ by (2.2.13), since H_G is tangent to $\mathcal{L}_\pm$, and $L_\pm = \mathcal{L}_\pm \cap \{\tau = 0\}$; in fact, by (2.2.6), more is true, namely these bicharacteristics, as curves in ${}^{\mathrm{b}}\overline{T}^* M \setminus o$, tend to $L_\pm$ if the latter is considered a subset of the boundary ${}^{\mathrm{b}}S^* M$ of ${}^{\mathrm{b}}\overline{T}^* M$ at fiber infinity. Now, consider backward, resp. forward, bicharacteristics γ in $(\Sigma_\pm \setminus \mathcal{L}_\pm) \cap {}^{\mathrm{b}}T^*_\Omega M$, including those within ${}^{\mathrm{b}}T^*_Y M$. By (2.2.13), τ is non-increasing along γ, and by (2.2.14), μ is strictly decreasing along γ once γ enters $\mu < 0$, hence it then reaches ${}^{\mathrm{b}}T^*_{H_2} M$ in finite time, staying within ${}^{\mathrm{b}}T^*_\Omega M$. We have to show that γ

necessarily enters $\mu < 0$ in finite time. Assume this is not the case. Then observe that

$$\mp H_G(\sigma - Y \cdot \zeta) = \mp 2|\zeta|^2 = \mp 2(\sigma - Y \cdot \zeta)^2 \text{ on } \Sigma_\pm, \tag{2.2.15}$$

thus $\sigma - Y \cdot \zeta$ converges to 0 along γ. Now on Σ, $|\zeta| = |\sigma - Y \cdot \zeta|$, thus, also ζ converges to 0, and moreover, on Σ, we have

$$|\sigma| \leq |Y \cdot \zeta| + |Y \cdot \zeta - \sigma| \leq (1 + |Y|)|\zeta|$$

since we are assuming $|Y| \leq 1$ on γ, hence σ converges to 0 along γ. But $H_G\sigma = 0$, i.e. σ is constant. Thus necessarily $\sigma = 0$, hence $G = 0$ gives $|Y \cdot \zeta| = |\zeta|$, and thus we must in fact have $|Y| = 1$ on γ, more precisely $Y = \mp\zeta/|\zeta|$; therefore γ lies in $\mathcal{L}_\pm$, which contradicts our assumption $\gamma \not\subset \mathcal{L}_\pm$. Hence, γ enters $|Y| > 1$ in finite time, and so, as we have already seen, reaches ${}^{\mathrm{b}}T^*_{H_2}M$ in finite time.

Finally, we show that forward, resp. backward, bicharacteristics γ in $(\Sigma_\pm \cap {}^{\mathrm{b}}T^*_Y M \setminus \mathcal{R}_\pm) \cap \Sigma_\Omega$ tend to $L_\pm$. By equation (2.2.15), $\pm(\sigma - Y \cdot \zeta) \to \infty$ (in finite time) along γ, hence $|\zeta| = |\sigma - Y \cdot \zeta|$ on $\gamma \subset \Sigma$ tends to ∞, and therefore

$$|Y| \geq \frac{|Y \cdot \zeta|}{|\zeta|} \geq \frac{|\sigma - Y \cdot \zeta|}{|\zeta|} - \frac{|\sigma|}{|\zeta|} \to 1$$

because σ is constant along γ. On the other hand, at points on γ where $|Y| > 1$, i.e. $\mu < 0$, we have $\pm H_G\mu > 0$ by (2.2.14). We conclude that γ tends to $|Y| = 1$, i.e. $\mu = 0$. Moreover,

$$\left(Y \cdot \frac{\zeta}{|\zeta|} - \frac{\sigma}{|\zeta|}\right)^2 = 1 \text{ on } \Sigma,$$

thus $\big|Y \cdot \zeta/|\zeta|\big| \to 1$ along γ; together with $|Y| \to 1$, this implies $Y \to \mp\zeta/|\zeta|$, and since σ is constant and $|\zeta| \to \infty$, we conclude that γ tends to $L_\pm$. This concludes the proof of the non-trapping nature of the flow (7). □

Remark 2.2.4. Conditions (6) and thus (7) are not stable under arbitrary perturbations of g as a b-metric, and it will in fact be crucial later that they can be relaxed. Namely, we do not need to require that null-bicharacteristics of a small perturbation of g tend to $L_\pm$, but only that they reach a fixed small neighborhood of $L_\pm$; this condition *is* stable under perturbations. See Remark 3.3.11. We moreover point out that changing g by an exponentially decaying (in the time coordinate $-\log\tau$) conormal metric perturbation, see §2.1.2, does not affect any of the properties established in Propositions 2.2.1 and 2.2.3 if the size of the perturbation is sufficiently small, as the Hamilton vector field at ∂M is unaffected by such a perturbation.

As indicated in §2.1.3, Proposition 2.2.3 can also be proved by working only over the sets $\overline{T}_\pm \subset {}^{\mathrm{b}}\overline{T}^*_Y M$ (see (2.1.5)), and the flow in $\overline{T}_\pm$ is precisely the Hamilton flow of the semiclassical principal symbol of the Mellin transformed normal operator family. Thus, the description of the semiclassical flow near the horizon in [115, Lemma 3.2] and in the static region in [114, §4.6] yields the same result, apart from the description of the unstable/stable direction at the radial set, which is transversal to the boundary; the latter can be recovered from the parameter-dependence of the normal operator family. See §3.3.4 for details.

2.2.2 Asymptotically de Sitter-like spaces

As a slight generalization of the construction of the static model of de Sitter space from the global space, we now consider an asymptotically de Sitter-like space $(\widetilde{M},\widetilde{g})$, which means [111] that $\widetilde{M}\cong[-1,1]_T\times\widetilde{X}$ is an n-dimensional manifold with two connected boundary components $\widetilde{X}_+$ (at future infinity, $T=1$) and $\widetilde{X}_-$ (past infinity, at $T=-1$); furthermore, for the boundary defining function $x=1\mp T$ near $T=\pm 1$, the metric $\widetilde{g}$ has the form

$$\widetilde{g}=\frac{dx^2-h}{x^2},$$

where h is a symmetric 2-tensor on $\widetilde{M}$, and $h|_{\widetilde{X}}$ is in fact a Riemannian metric *on* $\widetilde{X}$.

Remark 2.2.5. The geometric condition is that $\widetilde{g}=x^{-2}g_0$ for a smooth metric g_0 on $\widetilde{M}$, and $g_0(dx,dx)=1$ at $\widetilde{X}_\pm$, which makes the asymptotic curvature constant and the boundary at infinity $\widetilde{X}_\pm$ spacelike. Analogously to asymptotically hyperbolic spaces, where this was shown by Graham and Lee [55], on such a space one can always introduce a product decomposition $[0,\delta)_x\times(\partial\widetilde{M})_y$ near $\partial\widetilde{M}$, possibly changing x, such that the metric has a warped product structure $g_0=dx^2-h(x,y,dy)$, $\widetilde{g}=x^{-2}g_0$.

Thus, an asymptotically de Sitter-like space is the Lorentzian analogue of the Riemannian conformally compact spaces of Mazzeo and Melrose [81]. If h is *even* [57], i.e. only even powers of x appear in the Taylor expansion of h at $\widetilde{X}$, we say that $(\widetilde{M},\widetilde{g})$ is an *even asymptotically de Sitter-like space*. We now fix a point $q\in\widetilde{X}_+$ at future infinity and consider the homogeneous blow-up $[\widetilde{M}; q]$, and within it the lift of the interior of the backward light cone from q; we denote by g the lift of the metric $\widetilde{g}$ to $[\widetilde{M}; q]$. Introducing local geodesic coordinates y of $\widetilde{X}_+$ near q, which is then given by $x = y = 0$, this means that we use $Y = y/x \in \mathbb{R}^{n-1}$ and $\tau = x$ as coordinates near the front face of the blow-up, and then

$$g = (1 - |Y|^2)\,\frac{d\tau^2}{\tau^2} - 2Y\frac{d\tau}{\tau}\,dY - dY^2$$

plus a section of $S^{2\mathrm{b}}T^*\widetilde{M}$ that vanishes at $\tau = 0$. Thus, g is a b-metric near the front face (but away from the side face), and g agrees to first order at $\tau = 0$ with the static de Sitter metric, see (2.2.2). Furthermore, the intersection of the lift of the backward light cone from q with the front face $x = 0$ is equal to the set $\{x = 0, |Y| = 1\}$. Thus, the interior of the backward light cone from q, which we denote by M_S°, is a generalization of the static model of de Sitter space; if $\widetilde{M}$ is actual de Sitter space, then M_S° is the actual static model. We bordify M_S° at future infinity by adding $\tau = 0$, thus obtaining M_S. We can consider a neighborhood $M = [0, \infty)\tau \times X$ of M_S in $[\widetilde{M}; q]$, where X is given by

$$X = \{x = 0, |Y| < 1 + 2\delta\} \tag{2.2.16}$$

for $\delta > 0$ small, and then a domain $\Omega \subset M$, defined as in (2.2.5). Since g and the static de Sitter metric agree at $\partial\Omega$, Propositions 2.2.1 and 2.2.3 continue to hold in the present context as long as the initial surface H_1 is sufficiently close to $\tau = 0$, e.g. when $H_1 = \mathfrak{t}_1^{-1}(0)$ with $\mathfrak{t}_1 = \tau_0 - \tau$, $\tau_0 > 0$ small, and we take $H_2 = \mathfrak{t}_2^{-1}(0)$ with $\mathfrak{t}_2 = 1 + \delta - |Y|$ near $|Y| = 1 + \delta$. See Figure 2.4. We call (Ω, g) a *generalized static model*.

2.3 Schwarzschild-de Sitter space

The Schwarzschild-de Sitter black hole in $n \geq 4$ spacetime dimensions is the space $M_S^\circ = \mathbb{R}_t \times X_S$, $X_S = (r_-, r_+)_r \times \mathbb{S}^{n-2}_\omega$, with $r_\pm$ defined below, equipped with the stationary metric

$$g_0 = \mu\,dt^2 - (\mu^{-1}\,dr^2 + r^2\,d\omega^2), \tag{2.3.1}$$

where $d\omega^2$ is the round metric on the sphere $\mathbb{S}^{n-2}$, and

$$\mu = 1 - \frac{2M_\bullet}{r^{n-3}} - \lambda r^2, \quad \lambda = \frac{2\Lambda}{(n-2)(n-1)},$$

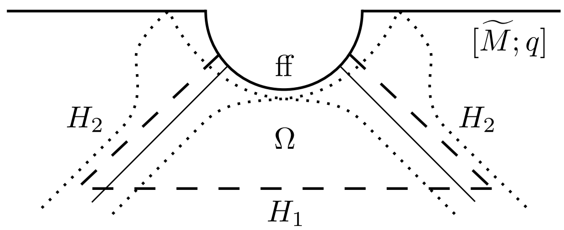

Figure 2.4: Setup of the 'static' asymptotically de Sitter problem. Indicated are the blow-up of $\widetilde{M}$ at q, the front face, the lift of the backward light cone to $[\widetilde{M}; q]$ (solid), and lifts of backward light cones from points nearby p (dotted); moreover, $\Omega \subset M$ (dashed boundary) is a submanifold with corners within M (which is not drawn here). H_1 is a Cauchy hypersurface, and H_2 (which for higher-dimensional generalized static models is connected) is an artificial spacelike hypersurface – once null-geodesics cross H_2 in the outward direction, they can never return to Ω.

with $M_\bullet > 0$ the black hole mass and $\Lambda > 0$ the cosmological constant. The assumption

$$M_\bullet^2 \lambda^{n-3} < \frac{(n-3)^{n-3}}{(n-1)^{n-1}} \tag{2.3.2}$$

guarantees that μ has two unique positive roots $0 < r_- < r_+$. Indeed, let $\widetilde{\mu} = r^{-2}\mu = r^{-2} - 2M_\bullet r^{1-n} - \lambda$. Then $\widetilde{\mu}' = -2r^{-n}(r^{n-3} - (n-1)M_\bullet)$ has a unique positive root

$$r_p = [(n-1)M_\bullet]^{1/(n-3)}, \tag{2.3.3}$$

$\widetilde{\mu}'(r) > 0$ for $r \in (0, r_p)$ and $\widetilde{\mu}'(r) < 0$ for $r > r_p$; moreover, $\widetilde{\mu}(r) < 0$ for $r > 0$ small and $\widetilde{\mu}(r) \to -\lambda < 0$ as $r \to \infty$, thus the existence of the roots $0 < r_- < r_+$ of $\widetilde{\mu}$ is equivalent to the requirement $\widetilde{\mu}(r_p) = \frac{n-3}{n-1} r_p^{-2} - \lambda > 0$, which is equivalent to (2.3.2). In view of the form (2.3.1), we call the coordinates (t, r, ω) *static coordinates.*

Define $\alpha = \mu^{1/2}$, thus $d\alpha = \frac{1}{2}\mu'\alpha^{-1}\, dr$, and let

$$\beta_\pm(r) := \mp \frac{2}{\mu'(r)} \tag{2.3.4}$$

near $r_\pm$, so $\beta_\pm(r_\pm) > 0$ there. Then the metric g can be written as

$$g = \alpha^2\, dt^2 - h, \quad h = \alpha^{-2}\, dr^2 + r^2\, d\omega^2 = \beta_\pm^2\, d\alpha^2 + r^2\, d\omega^2,$$

The singularity of the metric as one approaches $\alpha = 0$ is merely a coordinate singularity.

We thus introduce a new time variable, at first only near $r = r_\pm$, to wit

$$t_* = t - F(\alpha), \quad \partial_\alpha F(\alpha) = -\frac{\beta_\pm}{\alpha} - 2\alpha c(\mu), \tag{2.3.5}$$

with c a smooth function, to be determined momentarily. Then one computes (with $\beta = \beta_\pm$)

$$g = \mu\, dt_*^2 - (\beta + 2\mu c)\, dt_*\, d\mu + (\mu c^2 + \beta c)\, d\mu^2 - r^2\, d\omega^2 \tag{2.3.6}$$

In particular, the determinant of g restricted to the (t_*, μ)-plane equals $-\frac{\beta^2}{4}$, hence g is non-degenerate down to $\mu = 0$. Therefore, g extends as a non-degenerate stationary Lorentzian metric to a neighborhood $M^\circ = \mathbb{R}_{t_*} \times X$ of M_S°, where $X = (r_- - 2\delta, r_+ + 2\delta) \times \mathbb{S}^{n-2}$. We claim that we can choose $c(\mu)$ such that dt_* is timelike on M°: Indeed, with G denoting the dual metric to g, this amounts to requiring

$$G(dt_*, dt_*) = -4\beta^{-2}(\mu c^2 + \beta c) > 0. \tag{2.3.7}$$

This is trivially satisfied if $c = -\beta/2\mu$, which corresponds to undoing the change of coordinates in (2.3.5); however, we want c to be smooth at $\mu = 0$. But for $\mu \geq 0$, (2.3.7) holds provided $-\beta/\mu < c < 0$; hence, we can choose a smooth c verifying (2.3.7) in $\mu \geq 0$ and such that moreover $c = -\beta/2\mu$ in $\mu \geq \mu_1$ for any fixed small $\mu_1 > 0$. Thus, we can choose F as in (2.3.5) with $F = 0$ in $\alpha^2 \geq \mu_1$ (in particular, F is defined globally on X) such that (2.3.7) holds, and

$$t_* = t \text{ in } \mu \geq \mu_1 > 0. \tag{2.3.8}$$

As usual, we compactify M° at future infinity, with $\tau = e^{-t_*}$ as the boundary defining function, to the space $M = [0, \infty)_\tau \times X$. We remark that as in the de Sitter case, there is an equivalent, more geometric way of phrasing the extension of M_S° beyond the horizons to the manifold M thus defined, see [87], which involves compactifying M_S° at future infinity using e^{-t} and at the horizons using α^2 as the boundary defining function, and then performing a (non-homogeneous) blow-up of the corners.

We now again consider a domain

$$\Omega = \{\mathfrak{t}_1 \geq 0, \mathfrak{t}_2 \geq 0\}, \quad \mathfrak{t}_1 = \tau_0 - \tau, \quad \mathfrak{t}_2 = \mu + \delta. \tag{2.3.9}$$

Thus, Ω bounded by the (artificial) Cauchy surface $H_1 = \{\tau = \tau_0\}$, which is spacelike, and by the hypersurface $H_2 = \bigcup_\pm \{r = r_\pm \pm \delta\}$, which has two components, one lying beyond the black hole horizon (r_-) and the other beyond the cosmological horizon (r_+); see Figure 2.5; both components are spacelike in view of

$$G(d\mu, d\mu) = -4\beta^{-2}\mu > 0 \text{ at } \mu = -\delta.$$

Future infinity of Ω is given by $Y := \Omega \cap \partial M = \{\mu \geq -\delta, \tau = 0\}$, and it is the only boundary of Ω from the point of view of b-analysis, since the metric near the H_j is simply a smooth metric up to H_j, rather than a b-metric (cf. the discussion preceding Proposition 2.2.1). The analogue of Proposition 2.2.1 holds in the present context as well, by the same proof.

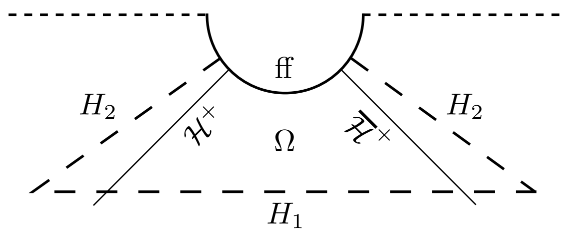

Figure 2.5: Diagrammatic representation of Schwarzschild-de Sitter space. Shown are the black hole horizon $\mathcal{H}^+$ and the cosmological horizon $\overline{\mathcal{H}}^+$, beyond which we put an artificial spacelike hypersurface H_2 with two connected components; the hypersurface H_1 will play the role of a Cauchy hypersurface. The domain Ω is bounded by the hypersurfaces H_1 and H_2. The 'point at future infinity' in the usual Penrose diagrammatic representation is shown blown-up here, since the metric is well-behaved (namely, a Lorentzian b-metric) on the blown-up space.

A crucial new feature of Schwarzschild-de Sitter space as compared to de Sitter space is the presence of *forward/backward trapped rays*, which are null-geodesics that do not escape to either horizon in the forward/backward direction, and trapped rays, which are both forward and backward trapped. In the present spherically symmetric setting, we locate the trapped set by determining when r is constant along the flow. For easier comparison with [43, 114, 124], we consider the flow of the rescaled Hamilton vector field $-r^2 H_G$ on $T_\pm \subset {}^{\mathrm{b}}T^*_Y M$ defined in (2.1.5) (with τ playing the role of x there). Notice that on the characteristic set, where $G = 0$, we have $-r^2 H_G = H_{-r^2 G}$. Introducing coordinates on ${}^{\mathrm{b}}T^*M$ by writing b-covectors as $\sigma \frac{d\tau}{\tau} + \xi\, dr + \eta\, d\omega$ and putting

$$\Delta_r = r^2\mu = r^2(1 - \lambda r^2) - 2M_\bullet r^{5-n},$$

the rescaled dual metric function of g in static coordinates (working away from $r = r_\pm$, see (2.3.8)) is given by

$$p = -r^2 G = \Delta_r \xi^2 - \frac{r^4}{\Delta_r}\sigma^2 + |\eta|^2,$$

and correspondingly the Hamilton vector field is

$$H_p|_{T_\pm} = 2\Delta_r \xi \partial_r - \Big(\partial_r \Delta_r \xi^2 - \partial_r\Big(\frac{r^4}{\Delta_r}\Big)\Big)\partial_\xi + H_{|\eta|^2},$$

since $\sigma = \pm 1$ on $T_\pm$. If $H_p r = 2\Delta_r \xi = 0$, then $\xi = 0$, in which case $H_p^2 r = 2\Delta_r H_p \xi = 2\Delta_r \partial_r (r^4/\Delta_r)$. Recall the definition of the function $\widetilde{\mu} = \mu/r^2 = \Delta_r/r^4$, then we can rewrite this as $H_p^2 r = -2\Delta_r \widetilde{\mu}^{-2}(\partial_r \widetilde{\mu})$. We have already seen that $\partial_r \widetilde{\mu}$ has a single root $r_p \in (r_-, r_+)$, and $(r - r_p)\partial_r \widetilde{\mu} < 0$ for $r \neq r_p$. Therefore, $H_p^2 r = 0$ implies (still assuming $H_p r = 0$) $r = r_p$. We rephrase this to show that the only trapping occurs in the cotangent bundle over $r = r_p$: Let $F(r) = (r - r_p)^2$, then $H_p F = 2(r - r_p)H_p r$ and $H_p^2 F = 2(H_p r)^2 + 2(r - r_p)H_p^2 r$. Thus, if $H_p F = 0$, then either $r = r_p$, in which case $H_p^2 F = 2(H_p r)^2 > 0$ unless $H_p r = 0$, or $H_p r = 0$, in which case $H_p^2 F = 2(r - r_p)H_p^2 r > 0$ unless $r = r_p$. So $H_p F = 0, p = 0$ implies either $H_p^2 F > 0$ or $r = r_p, H_p r = 0$, i.e.

$$(r, \omega; \xi, \eta) \in \Gamma_\hbar := \Big\{(r_p, \omega; 0, \eta) \colon \frac{r^4}{\Delta_r} = |\eta|^2\Big\}, \tag{2.3.10}$$

so $\Gamma_\hbar$ is the only trapping in $T_\pm$, and F is an escape function. (The notation reflects the relation to the semiclassical rescaling of the wave operator associated with the metric g.) The trapped set is spherically symmetric, and its projection to the base $\{r = r_p\}$ is called the *photon sphere*.

We claim that the trapping is hyperbolic in the normal directions to $\Gamma_\hbar$: We compute the linearization of the H_p-flow at $\Gamma_\hbar$ in the normal coordinates $r - r_p$ and ξ to be

$$\begin{aligned} H_p \begin{pmatrix} r - r_p \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 2r_p^4 \widetilde{\mu}|_{r=r_p} \\ 2(n-3)r_p^{-4}(\widetilde{\mu}|_{r=r_p})^{-2} & 0 \end{pmatrix} \begin{pmatrix} r - r_p \\ \xi \end{pmatrix} \\ + \mathcal{O}(|r - r_p|^2 + |\xi|^2), \end{aligned}$$

where we used $\partial_{rr}\widetilde{\mu}|_{r=r_p} = -2(n-3)r_p^{-4}$, which gives $\partial_r \widetilde{\mu} = -2(n-3)r_p^{-4}(r - r_p) + \mathcal{O}(|r -$

$r_p|^2)$. The eigenvalues of the linearization are therefore

$$\pm 2r_p \left(\frac{n-1}{1 - \frac{n-1}{n-3} r_p^2 \lambda} \right)^{1/2},$$

which reduces to the expression given in [114, p. 85] in the case $n = 4$, where $r_p = 3M_\bullet = \frac{3}{2} r_s$ with $r_s = 2M_\bullet$, and $\lambda = \Lambda/3$. In particular, the minimal expansion rate for the flow of H_p at the trapping $\Gamma_\hbar$ in the directions normal to $\Gamma_\hbar$ is

$$\nu_{\min} = 2r_p^{-1} \left(\frac{n-1}{1 - \frac{n-1}{n-3} r_p^2 \lambda} \right)^{1/2} > 0. \tag{2.3.11}$$

(The maximal expansion rate equals $\nu_{\min}$ as well.) The expansion rate of the flow within the trapped set is 0 by spherical symmetry; note that integral curves of H_p on $\Gamma_\hbar$ are simply unit speed geodesics of the round unit sphere $\mathbb{S}^{n-2}$. This shows the *normal hyperbolicity* (in fact, *r-normal hyperbolicity for every r*) of the trapping, which in this setting was first studied in [124]. We refer to [124] and [44, §5] for definitions, and to [43, §2.2] for details on how the spacetime description of trapping and its normally hyperbolic nature relates to the above 'semiclassical' description. The consequences of normally hyperbolic trapping which are relevant in our applications will be explained below in Definition 2.3.1.

By the discussion of §2.1.3, the spacetime trapped set, i.e. the set of points in ${}^{\mathrm{b}}S^*_\Omega M$ that never escape through either horizon along the Hamilton flow, is given by

$$\widetilde{\Gamma} = \{(\tau, r = r_p, \omega; \sigma, \xi = 0, \eta) \colon \sigma^2 = \Psi^2 |\eta|^2\}, \tag{2.3.12}$$

where $\Psi = \alpha r^{-1}$, $\Psi'(r_p) = 0$, in view of (2.3.10). However, every null-geodesic in $\widetilde{\Gamma}$ which is not contained in $\{\tau = 0\}$ escapes to H_1 in either the forward or backward direction, and thus we will only consider $\Gamma := \widetilde{\Gamma} \cap {}^{\mathrm{b}}T^*_Y M$ to be the trapped set, which thus is a subset of the b-cotangent bundle at future infinity. The trapped set Γ is *normally hyperbolically trapped in the b-sense*:

Definition 2.3.1. On a manifold M with boundary Y, equipped with a smooth b-metric g, we say that (M, g) has *normally hyperbolic trapping in the b-sense* at $\Gamma \subset \Sigma \cap {}^{\mathrm{b}}S^*_Y M$, with $\Sigma := G^{-1}(0)$ denoting the characteristic set (G being the dual metric), if the following conditions are satisfied for a fixed choice of sign for the rescaled Hamilton vector field

$V = \pm \mathsf{H}_G$:

(1) $\Gamma \subset \Sigma \cap {}^{\mathrm{b}}S^*_Y M$ is a smooth submanifold disjoint from ${}^{\mathrm{b}}T^*\partial M$,

(2) Γ_+ is a smooth orientable submanifold of $\Sigma \cap {}^{\mathrm{b}}S^*_Y M$ in a neighborhood U_1 of Γ,

(3) Γ_- is a smooth orientable submanifold of Σ transversal to $\Sigma \cap {}^{\mathrm{b}}S^*_Y M$ in U_1,

(4) Γ_+ has codimension 2 in Σ, Γ_- has codimension 1,

(5) Γ_+ and Γ_- intersect transversally in Σ with $\Gamma_+ \cap \Gamma_- = \Gamma$,

(6) the vector field V is tangent to both Γ_+ and Γ_-, and thus to Γ,

(7) Γ_+ is backward trapped for the Hamilton flow (i.e. bicharacteristics in Γ_+ near Γ tend to Γ as the parameter goes to $-\infty$), i.e. is the unstable manifold of Γ, while Γ_- is forward trapped, i.e. is the stable manifold of Γ; in particular, Γ is a trapped set. Quantitatively, let τ denote a fixed boundary defining function, $\phi_+ \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$ a defining function of Γ_+ within ${}^{\mathrm{b}}S^*_Y M$ (thus, Γ_+ is defined within ${}^{\mathrm{b}}S^*M$ by $\tau = 0$, $\phi_+ = 0$), and $\phi_- \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$ a defining function of Γ_- (within ${}^{\mathrm{b}}S^*M$). We then assume that near Γ,
$$V\tau = -c_\partial \tau, \quad c_\partial > 0, \tag{2.3.13}$$
and moreover, with G denoting a homogeneous degree 0 rescaling of G,
$$V\phi_+ = -c_+^2\phi_+ + \mu_+\tau + \nu_+\mathsf{G}, \quad V\phi_- = c_-^2\phi_- + \nu_-\mathsf{G}, \tag{2.3.14}$$
with $c_\pm > 0$ smooth near Γ and $\mu_+, \nu_\pm$ smooth near Γ, and finally
$$\{\phi_+, \phi_-\} = H_{\phi_+}\phi_- > 0 \tag{2.3.15}$$
near Γ.

Note that (2.3.13) is consistent with the stability of Γ_-, and (2.3.14) is consistent with the (in)stability of Γ_- (Γ_+). Furthermore, in condition (7), the tangency of V to $\Gamma_\pm$ implies $V\phi_+ = \alpha_+\phi_+ + \mu_+\tau + \nu_+\mathsf{G}$ and $V\phi_- = \alpha_-\phi_- + \nu_-\mathsf{G}$, thus this condition merely amounts to requiring that α_+ and α_- have a sign.

See Figure 2.6 for an illustration.

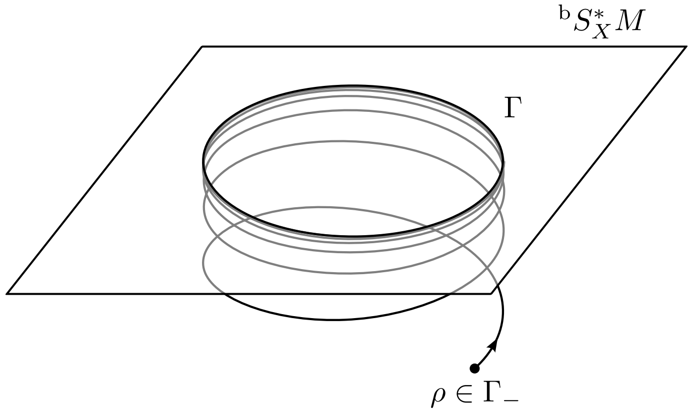

Figure 2.6: An exemplary situation with normally hyperbolic trapping in the b-sense: Shown are the (projection from ${}^{\mathrm{b}}S^*M$ to the base M of the) trapped set Γ, the b-cosphere bundle over X as well as a forward bicharacteristic starting at a point $\rho \in \Gamma_-$.

The fact that the trapped set Γ in the Schwarzschild-de Sitter spacetime satisfies this definition follows from the normally hyperbolic nature of the semiclassical trapped set $\Gamma_\hbar$; the translation between b-trapped sets and semiclassically trapped sets is explained in detail in §3.3.4. We denote the two components of the trapped set Γ by $\Gamma^\pm = \Gamma \cap \Sigma_\pm \subset \Sigma_\pm$, and correspondingly the forward and backward trapped sets have two components $\Gamma^\pm_- \in \Sigma_\pm$ and $\Gamma^\pm_+ \in \Sigma_\pm$. We now have the following analogue of Proposition 2.2.3:

Proposition 2.3.2. *The null-geodesic flow on Ω enjoys the properties (5) and (6) of Proposition 2.2.3, whose notation we continue to use here. The non-trapping statement now is: Ω has normally hyperbolic trapping in the b-sense at $\Gamma \subset {}^{\mathrm{b}}S^*_\Omega M$, and*

> *(7') the metric g is* non-trapping *in the following sense: All bicharacteristics in $\Sigma_\Omega := \Sigma \cap {}^{\mathrm{b}}S^*_\Omega M$ from any point in $\Sigma_\Omega \cap (\Sigma_+ \setminus (L_+ \cup \Gamma^+))$ flow (within Σ_Ω) to ${}^{\mathrm{b}}S^*_{H_1} M \cup L_+ \cup \Gamma^+$ in the forward direction (i.e. either enter ${}^{\mathrm{b}}S^*_{H_1} M$ in finite time or tend to the radial set L_+ or the trapped set Γ^+) and to ${}^{\mathrm{b}}S^*_{H_2} M \cup L_+ \cup \Gamma^+$ in the backward direction, and from any point in $\Sigma_\Omega \cap (\Sigma_- \setminus (L_- \cup \Gamma^-))$ to ${}^{\mathrm{b}}S^*_{H_2} M \cup L_- \cup \Gamma^-$ in the forward direction and to ${}^{\mathrm{b}}S^*_{H_1} M \cup L_- \cup \Gamma^-$ in the backward direction, with tending to $\Gamma^\pm$ allowed in only one of the two directions.*

Again, this follows from the corresponding semiclassical analysis, see [114, §6]. See Figure 2.7.

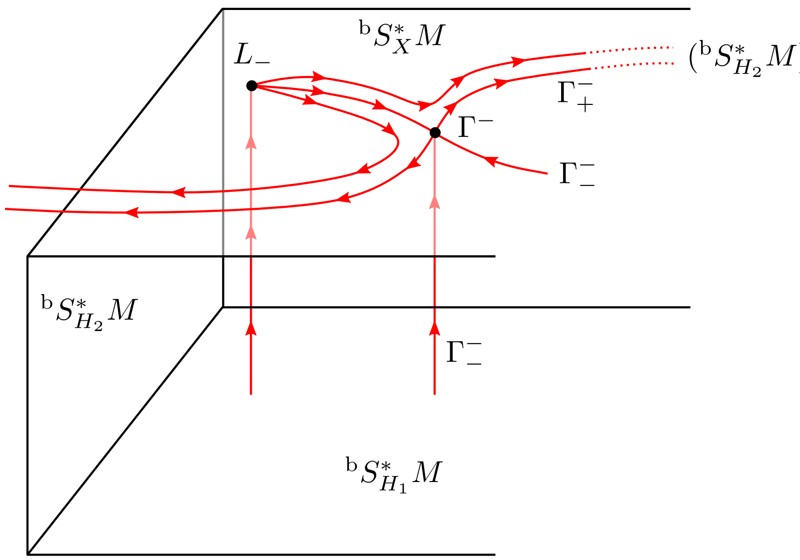

Figure 2.7: Illustration of the null-geodesic flow in the b-cosphere bundle ${}^{\mathrm{b}}S^*M$ of Schwarzschild-de Sitter space. For a description of the flow, see the statement of Proposition 2.3.2. Indicated is the flow in the component Σ_+ of the characteristic set. What is labeled ${}^{\mathrm{b}}S^*_{H_2}M$ is only a part of the cosphere bundle over the event, resp. cosmological horizon, and the component of ${}^{\mathrm{b}}S^*_{H_2}M$ not shown here lies at the cosmological, resp. event horizon. (Thus, the edges and corners in this figure merely mark the ends of the part of ${}^{\mathrm{b}}S^*M$ shown here.) The picture of the flow in Σ_- is analogous, with all arrows reversed and L_-, $\Gamma^-_{(\pm)}$ replaced by L_+, $\Gamma^+_{(\pm)}$; the sign of V in Definition 2.3.1 now is $V = -\mathsf{H}_G$.

2.4 Kerr-de Sitter family

In stationary coordinates, Kerr-de Sitter space with black hole mass $M_\bullet > 0$, cosmological constant $\Lambda > 0$ and angular momentum $a \in \mathbb{R}$ is the 4-dimensional spacetime $M_S^\circ = \mathbb{R}_t \times X_S$, with $X_S = (r_-, r_+)_r \times \mathbb{S}^2$, equipped with the metric

$$g = -\rho^2\Big(\frac{dr^2}{\mu} + \frac{d\theta^2}{\kappa}\Big) - \frac{\kappa\sin^2\theta}{(1+\gamma)^2\rho^2}(a\,dt - (r^2+a^2)\,d\phi)^2 + \frac{\mu}{(1+\gamma)^2\rho^2}(dt - a\sin^2\theta\,d\phi)^2,$$

where we use standard coordinates $\phi \in [0, 2\pi)$, $\theta \in (0,\pi)$ on $\mathbb{S}^2$, and

$$\rho^2 = r^2 + a^2\cos^2\theta,$$

$$\mu = (r^2 + a^2)\Big(1 - \frac{\Lambda r^2}{3}\Big) - 2M_\bullet r,$$
$$\kappa = 1 + \gamma \cos^2\theta,$$
$$\gamma = \frac{\Lambda a^2}{3}.$$

We assume that μ has positive roots $r_- < r_+$ satisfying

$$\mp \mu'(r_\pm) > 0, \tag{2.4.1}$$

which implies that r_- and r_+ are the two largest of the three positive roots of μ (see [114, §6]); we only consider values of the parameters $M_\bullet, \Lambda$ and a for which this non-degeneracy condition holds, see also [114, Equation (6.2)]. If $a = 0$, the Kerr-de Sitter metric reduces to the Schwarzschild-de Sitter metric in $n = 4$ spacetime dimensions. Thus, if $M_\bullet$ and Λ satisfy the non-degeneracy condition (2.3.2) for Schwarzschild-de Sitter spaces with spacetime dimension $n = 4$, the above non-degeneracy holds for Kerr-de Sitter spacetimes with sufficiently small a as well.

Apart from the singularity of the spherical coordinate system at the poles, which can be resolved by working with coordinates which are valid there [114, §6.2],[2] the metric g is again singular at the horizons $r = r_\pm$. We thus introduce new coordinates

$$t_* = t - h(r), \quad \phi_* = \phi - P(r) \tag{2.4.2}$$

near $r = r_\pm$, with

$$h(r) = \pm\frac{1+\gamma}{\mu}(r^2 + a^2) \pm c_\pm(r), \quad P(r) = \pm\frac{1+\gamma}{\mu}a, \tag{2.4.3}$$

where $c_\pm(r)$ is smooth up to $r = r_\pm$, chosen such that dt_* is timelike, see [114, §6.4]; similarly to the Schwarzschild-de Sitter setting, one can choose $c_\pm(r) = -\frac{1+\gamma}{\mu}(r^2 + a^2)$ away from $r = r_\pm$, which undoes the coordinate change, i.e. $t_* = t$ there. Then, we can extend M_S° to a larger spacetime $M^\circ = \mathbb{R}_{t_*} \times X$, $X = (r_- - 2\delta, r_+ + 2\delta) \times \mathbb{S}^2$, extend the metric g as a stationary metric to M°, and compactify M° at future infinity, using $\tau = e^{-t_*}$, to the spacetime $M = [0, \infty)_\tau \times X$, on which g is a Lorentzian b-metric. Kerr-de Sitter space exhibits normally hyperbolic trapping in the b-sense as well, in fact the trapping is still

[2]Our signs of h and P are changed relative to [114, §6].

r-normally hyperbolic for every r, and Proposition 2.3.2 holds; this is the point where the full generality of b-radial sets in Definition 2.2.2 is needed. Furthermore, we can consider domains Ω, extending the static region M_S°, as in the discussion of Schwarzschild-de Sitter space, with an artificial spacelike boundary H_2 placed beyond the event and cosmological horizons, thus H_2 has two connected components, and a Cauchy hypersurface H_1.

Next, we observe that by the nature of the construction of $c_\pm$ in [114, §6.4], one can make $c_\pm$ depend smoothly on the parameters $M_\bullet, \Lambda, a$. Therefore, if we fix $M_\bullet = M_\bullet^0$ and $a = a^0$ and consider the Kerr-de Sitter metric $g_{M_\bullet^0, a^0}$ on M as above, then nearby metrics $g_{M_\bullet, a}$, with $M_\bullet$ and a close to $M_\bullet^0$ and a^0, respectively, are smooth Lorentzian b-metrics on M as well, and the event and cosmological horizons stay within M. Moreover, the (forward/backward) trapped sets, computed in [43, §3.2], depend smoothly on the spacetime parameters as well. Varying the cosmological constant is also be harmless, but may be disregarded as unphysical.

2.5 More general geometries

The geometry of the neighborhoods of the static patch of de Sitter space and of Schwarzschild-de Sitter and Kerr-de Sitter spaces discussed in the preceding sections are the model cases whose natural generalizations we will study beginning in Chapter 5. We thus make the following definition:

Definition 2.5.1. Let M be a manifold with boundary equipped with a Lorentzian b-metric g, and let $\Omega \subset M$ be a domain with corners, bounded by the spacelike hypersurfaces H_1 (considered a Cauchy hypersurface) and H_2 (considered artificial hypersurfaces beyond the horizons). Then:

(1) (Ω, g) is an *exact non-trapping spacetime* if it satisfies conditions (1)-(7) of Propositions 2.2.1 and 2.2.3. If $\widetilde{g}$ is a smooth or conormal perturbation of g (within the class of Lorentzian b-metrics) which is sufficiently small in the sense of Remark 2.2.4, we call $(\Omega, \widetilde{g})$ a *non-trapping spacetime*.

(2) (Ω, g) is an *exact non-trapping spacetime with normally hyperbolic trapping* if it satisfies conditions (1)-(6) of Propositions 2.2.1 and 2.2.3 as well as the non-trapping condition (7') of Proposition 2.3.2. If $\widetilde{g}$ is a smooth or conormal perturbation of g (within the class of Lorentzian b-metrics) which is sufficiently small in the sense of

Remark 2.2.4 and so that $\widetilde{g}$ still only has normally hyperbolic trapping in the b-sense, we call $(\Omega, \widetilde{g})$ a *non-trapping spacetime with normally hyperbolic trapping*.

Thus, the static model of de Sitter space (or rather a neighborhood thereof, see §2.2.1) and its generalization (§2.2.2) are examples of exact non-trapping spacetimes; Schwarzschild-de Sitter and Kerr-de Sitter spaces (§§2.3 and 2.4) are examples of exact non-trapping spacetimes with normally hyperbolic trapping.

As stated, the metrics are assumed to be smooth or conormal, and the forward and backward trapped sets are assumed to be smooth submanifolds. While this is adequate for the study of linear or even semilinear wave-like equations on such spacetimes, as we will see in Chapter 5, the study of quasilinear wave equations in Chapter 9 will require these smoothness assumptions to be relaxed. We remark however that every result we give will only depend on finitely many (b-)derivatives of the metric and the C^k-regularity (for some finite k) of the trapping by 'abstract nonsense,' i.e. simply due to the fact that, for instance, operator bounds on finite regularity spaces only require a finite number of derivatives on the coefficients of the operator. In the specific case of Kerr-de Sitter spacetimes, or more general non-trapping spacetimes with normally hyperbolic trapping whose trapping is r-normally hyperbolic for every r, the regularity of the trapping will be C^r for fixed large r for sufficiently small metric perturbations; see Dyatlov [44] and Hirsch, Shub and Pugh [61]. Therefore, such perturbations satisfy Definition 2.5.1 in this high, but finite regularity sense.

We finish this chapter by stressing a fundamental feature of our analysis that we already indicated in the introduction: Due to the robustness of microlocal techniques, the qualitative properties of (the null-geodesic flow on) spacetimes covered by the above definition are all one needs to assume in order to draw rather strong conclusions about properties of the wave equation (and lower order perturbations thereof), such as asymptotics and decay of solutions.

Chapter 3

Pseudodifferential operators and microlocal analysis

In this chapter, we introduce analytic tools that we will use in the sequel to study waves on geometric classes of spacetimes including those introduced in §2.5: We recall the notions of pseudodifferential operators on Euclidean space in §3.1, on compact manifolds without boundary in §3.2 and on compact manifolds with boundary in §3.3. The study of ps.d.o.s and their mapping properties is intimately tied to the notion of wave front set (§3.1.2), which allows for a very precise understanding of the location, direction and strength of singularities of distributions by analyzing their high frequency behavior in the Fourier domain. We present several standard results relating the singularities of solutions of (pseudo)differential equations $Pu = 0$ to properties of the operator P, such as elliptic regularity (§3.2.1) and real principal type propagation of singularities (§3.2.2). We also briefly discuss complex absorbing potentials (§3.2.3), following [114, §2.5], which is a simple modification of the semiclassical results of Nonnenmacher and Zworski [94] and Datchev and Vasy [32]. We refer to [86, 62, 64, 101, 129, 38] and references therein for detailed accounts of 'classical' microlocal analysis.

We stress that the calculus of ps.d.o.s is very simple by virtue of a principal symbol map, which assigns to an operator P on a manifold X a function on the cotangent bundle of X that captures the behavior of P to leading order; this transforms the qualitative study of many aspects of differential equations into algebraic computations and manipulations. In fact, one can use this calculus in an 'abstract' fashion, since the underlying analytic details of its construction are irrelevant from the point of view of most applications, as we shall see in many instances.

In view of the description of spacetime geometries in Chapter 2, microlocal analysis on manifolds with boundary via so-called b-pseudodifferential operators, introduced by Melrose and Mendoza [88] and discussed in detail by Melrose [82], will play the starring role in our global analysis of wave equations. The regularity analysis of waves near the structures at infinity described in Definitions 2.2.2 and 2.3.1, b-radial sets and normally hyperbolic trapping, requires further work. We will analyze the class of radial sets present in (Kerr-)de Sitter-type spaces in §3.3.1; radial points were first discussed in the context of Euclidean scattering theory in [83], and in the semiclassical setting directly related to the b-setting (see §3.3.4) in [114], as well as on a different class of b-geometries in [8]; we will apply the latter work in §5.5 to the study of nonlinear waves on asymptotically Minkowski spacetimes. Estimates at normally hyperbolic trapping for semiclassical problems were pioneered in [124] and further developed in [44, 42, 94]. This was in turn much preceded by the work of Gérard and Sjöstrand [52] in the analytic category. We give a b-result that suits our purposes in §3.3.2.

We give a number of technical details even in the parts that are well-known, specifically in the development of the calculus on $\mathbb{R}^n$, since, firstly, we shall later need to generalize the ps.d.o.s to operators with non-smooth coefficients, see Chapter 8, and secondly, our treatment of b-operators with conormal (rather than smooth) coefficients in §3.3.5 is based directly on the Euclidean calculus.

To keep the notation simple, we restrict ourselves to operators acting on scalar functions, with the exception of a brief discussion of bundles in §3.2, but all definitions and theorems have analogues for operators mapping between sections of complex vector bundles (which are trivial over $\mathbb{R}^n$), unless stated otherwise. We will explicitly include bundles in the notation when discussing applications in §5.2.2, when studying pseudodifferential operators with rough coefficients in Chapter 8, and in the quasilinear applications in §9.2.

3.1 Calculus on Euclidean space

3.1.1 Symbols, operators and compositions

Consider a differential operator $A = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ on $\mathbb{R}^n$ with smooth coefficients a_α. We can write the action of A on Schwartz functions $u \in \mathscr{S}(\mathbb{R}^n)$ as

$$Au(x) = \sum_{|\alpha|\leq m} a_\alpha(x)\mathcal{F}^{-1}\xi^\alpha \mathcal{F}u = (2\pi)^{-n} \iint e^{i(x-y)\xi} a(x,\xi)u(y)\,dy\,d\xi,$$

where we put $a(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$. The integral on the right hand side makes sense for functions $a(x,\xi)$ which are more general than polynomials in ξ. For instance, the inverse of $P = p(x,D) = I + \Delta$ with $p(x,\xi) = 1 + |\xi|^2$ can be expressed in the above form, with $a(x,\xi) = (1+|\xi|^2)^{-1}$, and we can likewise hope to construct (approximate) inverses of more general, x-dependent elliptic operators (see §3.1.2). To obtain a simple but sufficiently powerful calculus, it is desirable to retain some of the key properties of polynomials $a(x,\xi)$. We only consider the simplest generalization here:

Definition 3.1.1. For $m \in \mathbb{R}$, let $S^m(\mathbb{R}^n_x; \mathbb{R}^n_\xi)$ denote the space of symbols on $\mathbb{R}^n$ of order m, which is the set of all $a \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multiindices α, β, the estimate

$$\sup_x |D_\xi^\alpha D_x^\beta a(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|} \tag{3.1.1}$$

holds with a constant $C_{\alpha\beta} < \infty$. The *left quantization* of a is the operator

$$q_L(a)u \equiv a(x,D)u \equiv \mathrm{Op}(a)u(x) = (2\pi)^{-n} \int e^{i(x-y)\xi} a(x,\xi)u(y)\,dy\,d\xi,$$

defined for $u \in \mathscr{S}(\mathbb{R}^n)$, and we define $\Psi^m(\mathbb{R}^n)$, the space of *pseudodifferential operators on* $\mathbb{R}^n$ *of order* m, to be the space of left quantizations of symbols $a \in S^m(\mathbb{R}^n; \mathbb{R}^n)$.

In particular, polynomials in ξ of order m with uniformly bounded (with all derivatives) coefficients in x are elements of $S^m(\mathbb{R}^n; \mathbb{R}^n)$. We also note the multiplicative property

$$f \in S^m(\mathbb{R}^n; \mathbb{R}^n), g \in S^{m'}(\mathbb{R}^n; \mathbb{R}^n) \Rightarrow fg \in S^{m+m'}(\mathbb{R}^n; \mathbb{R}^n).$$

It is easy to see [86, §2.2] that operators $A = a(x,D)$ in the class $\Psi^m(\mathbb{R}^n)$ define bounded maps $\mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ and thus by duality bounded maps $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$. Their

(Schwartz) kernels

$$K_A(x,y) = (2\pi)^{-n} \int e^{i(x-y)\xi} a(x,\xi)\, d\xi$$

are smooth and rapidly decaying in $|x-y|$ away from the diagonal $x = y$. We think of $S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} S^m(\mathbb{R}^n;\mathbb{R}^n)$ and $\Psi^{-\infty}(\mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} \Psi^m(\mathbb{R}^n)$ as 'trivial' symbols and operators; operators in $\Psi^{-\infty}(\mathbb{R}^n)$ will be shown to be smoothing operators, i.e. they map tempered distributions into smooth functions.

We can also consider differential operators $A = \sum_{|\alpha|\le m} D^\alpha a_\alpha(x)$ written in right reduced form: We then have

$$Au(x) = q_R(a)u \equiv a(D,x)u = (2\pi)^{-n} \int e^{i(x-y)\xi} a(y,\xi) u(y)\, dy\, d\xi$$

with $a(y,\xi) = \sum_{|\alpha|\le m} a_\alpha(y)\xi^\alpha$, and we correspondingly call A the *right quantization* of a; we can again take a to be a symbol in S^m. Intuitively speaking, left quantizations act on functions u by first differentiating and then multiplying by the coefficients of A, while the order is reversed for right quantizations. More generally, we can consider symbols $a \in S^m(\mathbb{R}^n_x, \mathbb{R}^n_y; \mathbb{R}^n_\xi)$, which is to say

$$\sup_{x,y} |D_\xi^\alpha D_x^\beta D_y^\gamma a(x,y,\xi)| \le C_{\alpha\beta\gamma} \langle\xi\rangle^{m-|\alpha|} \tag{3.1.2}$$

for all multiindices α, β, γ, and define their quantization $q(a)$ by

$$q(a)u = (2\pi)^{-n} \int e^{i(x-y)\xi} a(x,y,\xi) u(y)\, dy\, d\xi. \tag{3.1.3}$$

Allowing such symbols increases flexibility, but their quantizations can equivalently expressed as left (or right) quantizations, as we will recall below. Note that if the symbolic order of a is sufficiently negative, $m < -n$, the integral in (3.1.3) converges absolutely; otherwise, it needs to be interpreted as an *oscillatory integral* [62, §1], [86, §2.2].

Proposition 3.1.2. *[86, §2.4]. The range of the quantization map q on $S^m(\mathbb{R}^n_x, \mathbb{R}^n_y; \mathbb{R}^n_\xi)$ is equal to the space $\Psi^m(\mathbb{R}^n)$. That is, for any $a \in S^m(\mathbb{R}^n_x, \mathbb{R}^n_y; \mathbb{R}^n_\xi)$, there exists $a_L \in S^m(\mathbb{R}^n_x; \mathbb{R}^n_\xi)$ such that $q(a) = a_L(x,D)$.*

Proof. The idea of the proof is to expand $a(x,y,\xi)$ in a Taylor series around $y = x$ and use $q((y_j - x_j)a) = q(D_{\xi_j} a)$ for $j = 1,\dots,n$. (For this to make sense, we need to allow polynomial weights in $|x-y|$ for our symbols; see [86, §3].) Thus,

$$a(x,y,\xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_y^\alpha a(x,y,\xi)|_{y=x}(y-x)^\alpha + \sum_{|\alpha|=N} \frac{N}{\alpha!}(y-x)^\alpha \widetilde{r}_N(x,y,\xi)$$

with $\widetilde{r}_N \in S^m$ implies $q(a) = a_{L,N}(x,D) + q(r_N)$ with $r_N \in S^{m-N}$, where

$$a_{L,N}(x,\xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} D_\xi^\alpha \partial_y^\alpha a(x,y,\xi)|_{y=x}.$$

Since r_N can be made to have arbitrarily negative symbol order, the operator $q(r_N)$ will be irrelevant for practical purposes for large N. To finish the proof, we however need to remove r_N completely; we achieve this by asymptotically summing (see below) $\frac{1}{\alpha!}D_\xi^\alpha \partial_y^\alpha a|_{y=x}$, which has symbolic order $m-|\alpha|$, over all multiindices α, which produces a'_L with $q(a)-q_L(a'_L) = q(r_j)$ for all j with $r_j \in S^{m-j}$. Thus, the Schwartz kernel of $R := q(r)$, $r = a - a'_L$, is equal to

$$K_R(x,x+z) = (2\pi)^{-n} \int e^{-iz\zeta} r_j(x,x+z,\zeta)\,d\zeta, \tag{3.1.4}$$

and thus R is the left quantization of b given by

$$b(x,\xi) = \int e^{iz\xi} K_R(x,x+z)\,dz. \tag{3.1.5}$$

One then establishes that $K_R(x,x+z)$ decays superpolynomially in z (uniformly in x) together with all derivatives, and (3.1.5) then gives $b \in S^{-\infty}$. Putting $a_L = a'_L + b$ finishes the proof. □

Thus, in the notation of this proposition, we have $q(a) = a_L(x,D)$, where a_L is an asymptotic sum

$$a_L(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha \partial_y^\alpha a(x,y,\xi)|_{y=x}. \tag{3.1.6}$$

This by definition means that

$$a_L(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} D_\xi^\alpha \partial_y^\alpha a(x,y,\xi)|_{y=x} \in S^{m-N}(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$$

for all N. We can similarly write $q(a) = a_R(D,x)$ as a right reduction of a symbol $a_R \in S^m$

which is an asymptotic sum

$$a_R(x,\xi) \sim \sum_\alpha \frac{(-1)^{|\alpha|}}{\alpha!} D_\xi^\alpha \partial_x^\alpha a(x,y,\xi)|_{y=x}. \tag{3.1.7}$$

We can now prove that $\Psi^*(\mathbb{R}^n) = \bigcup_m \Psi^m(\mathbb{R}^n)$ is a filtered *-algebra:

Proposition 3.1.3. *For $m, m' \in \mathbb{R}$, we have $\Psi^m(\mathbb{R}^n) \circ \Psi^{m'}(\mathbb{R}^n) \subset \Psi^{m+m'}(\mathbb{R}^n)$. In fact, if $a(x,D) \in \Psi^m(\mathbb{R}^n)$ and $b(x,D) \in \Psi^{m'}(\mathbb{R}^n)$, then $a(x,D)b(x,D) = c(x,D)$ with*

$$c(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi). \tag{3.1.8}$$

Moreover, $\Psi^m(\mathbb{R}^n)$ is closed under adjoints, and $a(x,D)^* = c(x,D)$ with

$$c(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha \partial_x^\alpha \bar{a}(x,\xi). \tag{3.1.9}$$

Proof. We can write $b(x,D)$ as a right quantization $b(x,D) = q_R(b_R)$ with symbol $b_R(y,\xi)$ given by (3.1.7), thus $a(x,D)b(x,D) = q(a(x,\xi)b_R(y,\xi))$, which we can write as the left quantization of a symbol c by Proposition 3.1.2. The formula (3.1.8) follows from thus combining (3.1.6) and (3.1.7). For the second part, we have $a(x,D)^* = q_L(a)^* = q_R(\bar{a})$, whose left reduction can be computed using (3.1.6). □

Note in particular that the formula for the leading term (in terms of the symbolic order) is very simple, being simply the product of (the leading terms) of a and b. More precisely:

Definition 3.1.4. For $A = a(x,D) \in \Psi^m(\mathbb{R}^n)$, define its *principal symbol* to be the equivalence class

$$\sigma_m(A) = [a] \in S^m(\mathbb{R}^n; \mathbb{R}^n)/S^{m-1}.$$

Somewhat imprecisely, we will often call any representative of $\sigma_m(A)$ the principal symbol of A.

Thus, (3.1.8) and (3.1.9) show that

$$\sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B), \quad \sigma_m(A^*) = \overline{\sigma_m(A)}.$$

In particular, the noncommutative operation of composing two operators amounts to the multiplication of their principal symbols, which is a commutative operation.

Clearly, one has $\sigma_m(A) = 0$ if and only if $A \in \Psi^{m-1}(\mathbb{R}^n)$. Now, $[A, B] = A \circ B - B \circ A \in \Psi^{m+m'}(\mathbb{R}^n)$ has vanishing principal symbol, so we can ask what is symbol as an operator in $\Psi^{m+m'-1}(\mathbb{R}^n)$ is. Using (3.1.8), but including the next to leading order terms, we calculate:

Proposition 3.1.5. *For $A \in \Psi^m(\mathbb{R}^n)$ and $B \in \Psi^{m'}(\mathbb{R}^n)$ with principal symbols a and b, respectively, we have*

$$\sigma_{m+m'-1}(i[A, B]) = H_a b, \tag{3.1.10}$$

where H_a is the Hamilton vector field of a, defined in (2.1.2).

We briefly discuss the topology of $\Psi^m(\mathbb{R}^n)$: We endow $S^m(\mathbb{R}^n; \mathbb{R}^n)$ with the locally convex topology given by the seminorms $\sup_x \langle\xi\rangle^{-m+|\alpha|} |D_\xi^\alpha D_x^\beta a(x, \xi)|$ (which computes the smallest constant for which (3.1.1) holds) for multiindices α, β. Now, the invertibility of the Fourier transform on the space $\mathscr{S}'(\mathbb{R}^{2n})$ of tempered distributions, which contains $S^m(\mathbb{R}^n; \mathbb{R}^n)$, implies that the map q_L is injective. We can thus endow $\Psi^m(\mathbb{R}^n)$ with the topology induced by q_L, which makes q_L into a topological isomorphism. One can check [86, §2.1] that $S^{-\infty}$ is dense in S^m in the topology of $S^{m'}$ for every $m' > m$, and hence the analogous statement holds for the corresponding spaces of pseudodifferential operators.

Finally, we recall the notion of classical (or one step polyhomogeneous) symbols:

Definition 3.1.6. A symbol $a \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ is called *classical* if it is an asymptotic sum

$$a(x, \xi) \sim \sum_{j \geq 0} a_{m-j}(x, \xi), \tag{3.1.11}$$

where a_{m-j} is positively homogeneous of degree $m-j$, i.e. $a_{m-j}(x, \lambda\xi) = \lambda^{m-j} a_{m-j}(x, \xi)$ for $|\xi| \geq 1$, $\lambda \geq 1$. Left quantizations of classical symbols are called *classical pseudodifferential operators*.

For classical operators $A = a(x, D)$, with a as in (3.1.11), we can identify the principal symbol $\sigma_m(A)$ with the leading order homogeneous part a_m. The proof of Proposition 3.1.2 shows that the class of classical pseudodifferential operators is a filtered *-algebra as well.

3.1.2 Parametrices for elliptic operators; wave front set

As a first application of the calculus, we construct parametrices for elliptic operators.

Definition 3.1.7. Let $m \in \mathbb{R}$. The symbol $a(x,\xi) \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ is *elliptic* at a point $(x_0,\xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus o)$ if there exists constants $c, R > 0$ and a conic (in the ξ variables) neighborhood U of (x_0,ξ_0) in $\mathbb{R}^n \times (\mathbb{R}^n \setminus o)$ such that

$$|a(x,\xi)| \geq c\langle\xi\rangle^m$$

for all $(x,\xi) \in U$, $|\xi| \geq R$. The set of all points (x_0,ξ_0) at which a is elliptic is called the *elliptic set* $\mathrm{Ell}(a)$ of a, and its complement in $\mathbb{R}^n \times (\mathbb{R}^n \setminus o)$ the *characteristic set* $\mathrm{Char}(a)$. We say that a is uniformly elliptic if it is elliptic at every point $(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus o)$, with the constants c and R uniform in x.

Thus, ellipticity at a point (x_0,ξ_0) measures the non-degeneracy of the symbol in a neighborhood of the ray $(x_0, \mathbb{R}_{>0}\xi_0)$. Changing an elliptic symbol of order m by a lower order symbol does not affect ellipticity, hence we say that an operator $A \in \Psi^m(\mathbb{R}^n)$ is (uniformly) elliptic if its principal symbol is; we likewise define the elliptic and characteristic sets of A to be the respective sets for its principal symbol.

Proposition 3.1.8. *If $A \in \Psi^m(\mathbb{R}^n)$ is uniformly elliptic, there exists $Q \in \Psi^{-m}(\mathbb{R}^n)$ such that $PQ - I, QP - I \in \Psi^{-\infty}(\mathbb{R}^n)$.*

Proof. Take $Q \in \Psi^{-m}(\mathbb{R}^n)$ to be a quantization of $\chi(\xi)/\sigma_m(A)$, where χ is a smooth cutoff, equal to 0 in $|\xi| \leq R$ (with R as in definition 3.1.7) and 1 in $|\xi| \geq 2R$. Then $\sigma_0(PQ) = 1$, hence $PQ - I = R \in \Psi^{-1}(\mathbb{R}^n)$. By a simple iterative argument [86, §2.10], one can improve Q so as to remove the error term R up to an error in $\Psi^{-\infty}(\mathbb{R}^n)$. □

This implies that if $u \in \mathscr{S}'(\mathbb{R}^n)$ is a distributional solution to the equation $Au = 0$ and A is uniformly elliptic, then $u = QAu + (I - QP)u = (I - QP)u$ is in fact smooth. This is the statement of *elliptic regularity.*

A statement similar to the above proposition holds for operators which are elliptic only at (thus, near) a point. To make this precise, we need the notion of the wave front set of an operator:

Definition 3.1.9. Let $m \in \mathbb{R}$ and $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$. Then the *essential support* $\operatorname{ess\,supp} a \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus o)$ of a is the complement of the set of all $(x_0,\xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus o)$ for which $a \in S^{-\infty}$ in a conic neighborhood of (x_0,ξ_0); the latter means that for all $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that

$$|a(x,\xi)| \le C_N \langle\xi\rangle^{-N} \tag{3.1.12}$$

for all (x,ξ) such that x is close to x_0 and $\xi/|\xi|$ is close to $\xi_0/|\xi_0|$.

For an operator $A = a(x,D) \in \Psi^m(\mathbb{R}^n)$, we define its *wave front set* as $\mathrm{WF}'(A) := \operatorname{ess\,supp} a$.

By [86, §5.8], the estimate (3.1.12) and the symbolic nature of a imply the same estimate for all derivatives of a.

The operator wave front set $\mathrm{WF}'(A)$ thus measures where the full symbol a of $A = a(x,D)$ is non-trivial. Note that by the asymptotic formulas (3.1.6) and (3.1.7), we could equivalently have defined the wave front set of an operator using its right reduced symbol. By (3.1.8), we have

$$\mathrm{WF}'(A \circ B) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B);$$

moreover, if $\mathrm{WF}'(A) = \emptyset$, then $A \in \Psi^{-\infty}(\mathbb{R}^n)$.

The microlocal version of Proposition 3.1.8 is:

Proposition 3.1.10. *[86, §5.9]. If $A \in \Psi^m(\mathbb{R}^n)$ is elliptic at (x_0,ξ_0), there exists $Q \in \Psi^{-m}(\mathbb{R}^n)$ such that $(x_0,\xi_0) \notin \mathrm{WF}'(PQ - I), \mathrm{WF}'(QP - I)$.*

Guided by the statement of elliptic regularity, we now define:

Definition 3.1.11. Let $u \in \mathscr{D}'(\mathbb{R}^n)$. We define the wave front set $\mathrm{WF}(u)$ as follows: Then $(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus o)$ is *not* contained in $\mathrm{WF}(u)$ if and only if there exists an operator $A \in \Psi^0(\mathbb{R}^n)$ which is elliptic at (x,ξ) and a smooth cutoff $\psi \in \mathcal{C}^\infty_\mathrm{c}(\mathbb{R}^n)$, $\psi(x) \neq 0$, such that $A(\psi u) \in \mathcal{C}^\infty(\mathbb{R}^n)$.

We mention in passing that the wave front set $\mathrm{WF}'(A)$ of the operator A is closely related to the wave front set of the Schwartz kernel of A as a distribution on $\mathbb{R}^{2n}$, see [86, §5.12], [62, §2].

The wave front set has a simple intuitive characterization:

Proposition 3.1.12. *[86, §5.11]. Let $u \in \mathscr{D}'(\mathbb{R}^n)$. Then $(x_0,\xi_0) \notin \mathrm{WF}(u)$ if and only if there exist $\phi \in \mathcal{C}^\infty_\mathrm{c}(\mathbb{R}^n)$, $\phi(x_0) \neq 0$, and $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$ of the form $\chi(\xi) = \widetilde{\chi}(\xi/|\xi|)$ in $|\xi| \ge 1$, where $\widetilde{\chi} \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ with $\widetilde{\chi}(\xi_0/|\xi_0|) = 1$, such that for all N, there exists a constant $C_N > 0$ such that $|\chi(\xi)(\phi u)\hat{\ }(\xi)| \le C_N\langle\xi\rangle^{-N}$ for all $\xi \in \mathbb{R}^n$.*

Proof. Since $u \mapsto \mathcal{F}^{-1}\chi\mathcal{F}\phi u$ is the right quantization of $\chi(\xi)\phi(x) \in S^0(\mathbb{R}^n;\mathbb{R}^n)$, the direction ($\Leftarrow$) is clear. To prove the direction ($\Rightarrow$), one observes that if $A \in \Psi^0$, elliptic at (x_0, ξ_0), and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\psi(x_0) \neq 0$, are such that $A(\psi u) \in \mathcal{C}_c^\infty$, then we also have $B(\psi u) \in \mathcal{C}^\infty$ for all $B \in \Psi^0$ with $\mathrm{WF}'(B)$ contained in the elliptic set of A; this follows from Proposition 3.1.10 and the discussion preceding it. In particular, we can take B to be a right quantization of $\chi\phi$ with χ and ϕ as stated. □

Recall that by definition, the point $x_0 \in \mathbb{R}^n$ is not contained in the *singular support* $\operatorname{sing\,supp} u$ of $u \in \mathscr{D}'(\mathbb{R}^n)$ if $\phi u \in \mathcal{C}^\infty$ for some $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\phi(x_0) \neq 0$. Thus, the wave front set does not only measure the location of singularities, but also their 'co-directions,' i.e. the frequencies which contribute to u near x_0 in a non-trivial manner. By [62, §2], we indeed have $\pi(\mathrm{WF}(u)) = \operatorname{sing\,supp}(u)$, where $\pi\colon \mathbb{R}^n_x \times (\mathbb{R}^n_\xi \setminus o) \to \mathbb{R}^n_x$ is the projection. Therefore, $\mathrm{WF}(u) = \emptyset$ is equivalent to $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Directly from the definitions, one can show [86, §5.10] that for $A \in \Psi^m(\mathbb{R}^n)$, $u \in \mathscr{S}'(\mathbb{R}^n)$, we have

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}(u), \quad \mathrm{WF}(u) \subset \mathrm{WF}(Au) \cup \mathrm{Char}(A).$$

In particular, if A is elliptic at (x, ξ), then $(x, \xi) \in \mathrm{WF}(u)$ if and only if $(x, \xi) \in \mathrm{WF}(Au)$, thus Au is singular (in the sense of wave front sets) if and only if u is. This is the statement of *microlocal elliptic regularity.*

3.1.3 Mapping properties on Sobolev spaces; Sobolev wave front set

We recall the definition of Sobolev spaces: For $s \in \mathbb{N}_0$, we define $H^s(\mathbb{R}^n)$ to consist of all $u \in L^2(\mathbb{R}^n)$ such that $\partial_x^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leq s$, and then for all real s by duality and interpolation. Equivalently,

$$H^s(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) \colon \langle\xi\rangle^s \widehat{u}(\xi) \in L^2(\mathbb{R}^n_\xi)\}, \tag{3.1.13}$$

and the norm on H^s is $\|u\|_{H^s} = \|\langle\xi\rangle^s \widehat{u}\|_{L^2}$. We say that $u \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$ if $\phi u \in H^s(\mathbb{R}^n)$ for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Notice that $\langle D\rangle^m \in \Psi^m(\mathbb{R}^n)$, hence by definition, $\langle D\rangle^m \colon H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ is an isometric isomorphism.

Proposition 3.1.13. *Every $A \in \Psi^m(\mathbb{R}^n)$ defines a bounded map $H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$, $s \in \mathbb{R}$.*

Proof. Replacing A by $\langle D\rangle^{s-m} \circ A \circ \langle D\rangle^{-s}$, it suffices to treat the case $m = s = 0$. If in fact $A \in \Psi^{-\infty}(\mathbb{R}^n)$, the asserted boundedness follows from Schur's lemma and the rapid decay of the Schwartz kernel of A away from the diagonal, see [86, §2.12]. For general $A \in \Psi^0(\mathbb{R}^n)$, one can use Hörmander's square root trick [86, §2.13]. □

As we will see in Chapter 8, one can assume much less regularity than $a \in S^0$ (but some decay in x) to guarantee the boundedness of $a(x, D)$ on L^2: For instance, $\sup_\xi \|a(\cdot, \xi)\|_{H^s} < \infty$ for some $s > n/2$ is sufficient. See also [62, Theorem 18.1.11′] for a related result.

By Proposition 3.1.13, we can equivalently define $H^s(\mathbb{R}^n)$ to consist of all $u \in \mathscr{S}'(\mathbb{R}^n)$ such that $Au \in L^2(\mathbb{R}^n)$ for all $A \in \Psi^s(\mathbb{R}^n)$; more economically, fixing a uniformly elliptic operator $A \in \Psi^s(\mathbb{R}^n)$, we have $u \in H^s(\mathbb{R}^n)$ if and only if $Au \in L^2(\mathbb{R}^n)$. (This is merely a rephrasing of global elliptic regularity.)

Using these mapping properties, we can refine the notion of wave front set given in Definition 3.1.11:

Definition 3.1.14. Fix $s \in \mathbb{R}$, and let $u \in \mathscr{D}'(\mathbb{R}^n)$. Then $(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus o)$ is *not* contained in the H^s-wave front set $\mathrm{WF}^s(u)$ if and only if there exists an operator $A \in \Psi^0(\mathbb{R}^n)$ which is elliptic at (x, ξ) and $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, $\psi(x) \neq 0$, such that $A(\psi u) \in H^s(\mathbb{R}^n)$.

Using elliptic regularity and mapping properties on Sobolev spaces, one can show [86, §5.14] that $\mathrm{WF}^s(u) = \emptyset$ if and only if $u \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$.

We have the following direct analogue of Proposition 3.1.12:

Proposition 3.1.15. *Let $s \in \mathbb{R}$ and $u \in \mathscr{D}'(\mathbb{R}^n)$. Then $(x_0, \xi_0) \notin \mathrm{WF}^s(u)$ if and only if there exist $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, $\phi(x_0) \neq 0$, and $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$ of the form $\chi(\xi) = \widetilde{\chi}(\xi/|\xi|)$ in $|\xi| \geq 1$, where $\widetilde{\chi} \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ with $\widetilde{\chi}(\xi_0/|\xi_0|) = 1$, such that $\mathcal{F}^{-1}\chi\mathcal{F}\phi u \in H^s(\mathbb{R}^n)$, i.e. such that $\int |\chi(\xi)|^2 |(\phi u)\widehat{\ }(\xi)|^2 \langle\xi\rangle^{2s}\, d\xi < \infty$.*

Combining the mapping properties of pseudodifferential operators with the calculus for their wave front sets, one deduces the following microlocal elliptic regularity result:

Proposition 3.1.16. *Let $A \in \Psi^m(\mathbb{R}^n)$ and $u \in \mathscr{D}'(\mathbb{R}^n)$. Suppose A is elliptic at (x, ξ). Then $(x, \xi) \in \mathrm{WF}^s(u)$ if and only if $(x, \xi) \in \mathrm{WF}^{s-m}(Au)$.*

We remark that the proof of this result in fact gives the following quantitative bound: There exist operators $B_1, B_2 \in \Psi^0(\mathbb{R}^n)$, elliptic at (x, ξ), such that for any $N \in \mathbb{R}$ there is a constant $C_N > 0$ such that

$$\|B_1 u\|_{H^{s-m}} \leq C(\|B_2 A u\|_{H^s} + \|u\|_{H^N}) \tag{3.1.14}$$

for all $u \in H^N$, in the strong sense that if the right hand side is finite, then so is the left hand side, and the inequality holds. Here, we think of N as being very negative, so the H^N-norm is very weak. We point out that the qualitative statement of Proposition 3.1.16 is in fact equivalent to the quantitative statement (3.1.14) by the closed graph theorem, see [64, Proof of Theorem 26.1.7] and [115, §4.3], except that we lose control over the constant C. For applications to nonlinear problems however, it is of course crucial to know at least the rough dependence of C on seminorms of A (and on N).

3.1.4 Change of coordinates

Let $\kappa\colon U \to V$ be a diffeomorphism between two open sets $U, V \subset \mathbb{R}^n$. For an operator $A = a(x, D) \in \Psi^m(\mathbb{R}^n)$ whose Schwartz kernel is compactly supported in $U \times U$, we can define the pushforward of A, which is an operator A_κ with Schwartz kernel compactly supported in $V \times V$, by defining $(A_\kappa v) \circ \kappa = A(v \circ \kappa)$, $v \in \mathscr{S}'(\mathbb{R}^n)$. The main result [62, §2.1] is that $A_\kappa \in \Psi^m(\mathbb{R}^n)$, and the full symbol a_κ of A_κ has an asymptotic expansion

$$a_\kappa(\kappa(x), \eta) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha a(x, {}^t\kappa'(x)\eta) D_y^\alpha e^{i\rho_\kappa(y)\eta}|_{y=x}, \tag{3.1.15}$$

where $\rho_\kappa(y) = \kappa(y) - \kappa(x) - \kappa'(x)(y - x)$ vanishes to second order at x. In particular, if we view κ as a change of coordinates on $\mathbb{R}^n$, we see from (3.1.15) that the principal symbol transforms as a function on the cotangent bundle $T^*\mathbb{R}^n$.

Correspondingly, one should really view $\sigma_m(A) \in S^m(T^*\mathbb{R}^n)/S^{m-1}$ (with symbolic behavior in the fiber variables), and moreover the elliptic set of A is an open conic subset of $T^*\mathbb{R}^n \setminus o$, while characteristic sets and wave front sets of operators as well as wave front sets (including H^s-wave front sets) are closed conic subsets of $T^*\mathbb{R}^n \setminus o$. Moreover, the Hamilton vector field $H_{\sigma_m(A)}$ (fixing a representative of the principal symbol, or using the homogeneous representative if A is classical) is invariantly defined as a vector field on $T^*\mathbb{R}^n \setminus o$. This invariant point of view will allow for a very concise description of the standard pseudodifferential calculus on closed manifolds in §3.2.

3.1.5 Radial compactification

In a manner that is entirely analogous to the construction in §2.1.3, we can radially compactify $T^*\mathbb{R}^n$ to $\overline{T}^*\mathbb{R}^n$, and we denote by $S^*\mathbb{R}^n = \partial\overline{T}^*\mathbb{R}^n$ the cosphere bundle, which we thus view as the boundary at fiber infinity of $\overline{T}^*\mathbb{R}^n$. Then, we can view $\mathrm{WF}(u)$ for $u \in \mathscr{D}'(\mathbb{R}^n)$, $\mathrm{WF}'(A)$ for $A \in \Psi^m(\mathbb{R}^n)$, as well as $\mathrm{Ell}(A)$ and $\mathrm{Char}(A)$, which are conic subsets of $T^*\mathbb{R}^n \setminus o$, as subsets of $S^*\mathbb{R}^n$.

We briefly recall the homogeneity discussion in §2.1.3 in the present context: Let us fix $\rho \in \mathcal{C}^\infty(\overline{T}^*\mathbb{R}^n)$ a defining function of $S^*\mathbb{R}^n$, i.e. $\rho > 0$ in $T^*\mathbb{R}^n \subset \overline{T}^*\mathbb{R}^n$, $\rho = 0$ at $S^*\mathbb{R}^n$, and $d\rho|_{S^*\mathbb{R}^n} \neq 0$. We can for instance take $\rho = \langle\xi\rangle^{-1}$. Then, for a classical operator $A \in \Psi^m(\mathbb{R}^n)$ with homogeneous principal symbol $a_m(x,\xi)$, we can write $a_m(x,\xi) = \mathsf{a}(x,\xi)\rho^{-m}$, where $\mathsf{a} \in \mathcal{C}^\infty(S^*\mathbb{R}^n)$. Furthermore, the Hamilton vector field H_{a_m} is homogeneous of degree $(m-1)$; therefore, $\mathsf{H}_{a_m} := \rho^{m-1}H_{a_m} \in \mathcal{V}(S^*\mathbb{R}^n)$ by restriction. However, at points in $S^*\mathbb{R}^n$ where $V = 0$, it is useful to keep information on the behavior of H_{a_m} in the fiber-radial direction, which we do by viewing $\mathsf{H}_{a_m} \in \mathcal{V}_\mathrm{b}(\overline{T}^*\mathbb{R}^n)$, with the relevant information encoded in $V|_{S^*\mathbb{R}^n}$, the restriction to fiber infinity *as a b-vector field.*

3.2 Calculus on compact manifolds without boundary

Using local coordinate charts and partitions of unity, one can construct a calculus for pseudodifferential operators on manifolds starting from the calculus on $\mathbb{R}^n$: Thus, if X is an n-dimensional manifold, we say that an operator $A\colon \mathcal{C}_\mathrm{c}^\infty(X) \to \mathcal{C}^\infty(X)$ is an element of $\Psi^m(X)$ if for every local coordinate chart $\kappa\colon U \subset X \to \mathbb{R}^n$ and $\phi, \psi \in \mathcal{C}_\mathrm{c}^\infty(\kappa(U))$, the operator

$$u \mapsto \psi(\kappa^{-1})^* A \kappa^*(\phi u)$$

is an element of $\Psi^m(\mathbb{R}^n)$. See [64, §18.1] for details. Note that if X is compact, one can freely compose any two such operators; in the non-compact case, one needs to make additional assumptions on the behavior of the Schwartz kernels of elements of $\Psi^m(X)$, for instance proper support in $X \times X$, or appropriate decay conditions away from the diagonal, as for example in the Euclidean case. In this section, we only consider compact manifolds X, and microlocal analysis is cleanest in this setting. The calculus for a class of operators on compact manifolds with boundary discussed in §3.3 is only slightly more delicate and will be very closely related to the non-compact setting on Euclidean space.

Throughout the rest of this section, X denotes a compact n-dimensional manifold without boundary.

The calculus on Euclidean space combined with the discussion of coordinate invariance in §3.1.4 gives the following *pseudodifferential calculus on the closed n-dimensional manifold X*; see also [122, §3.4] for a general overview and [64, §18.1] for details.

(1) *Spaces of operators.* For every $m \in \mathbb{R}$, we have a vector space $\Psi^m(X)$ consisting of bounded operators $\mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$. For $m \in \mathbb{N}_0$, we have $\mathrm{Diff}^m(X) \subset \Psi^m(X)$.

(2) *Algebra property.* The space $\bigcup_{m\in\mathbb{R}} \Psi^m(X)$ is a filtered *-algebra: For $A \in \Psi^m(X)$, $B \in \Psi^{m'}(X)$, we have $A \circ B \in \Psi^{m+m'}(X)$ and $A^* \in \Psi^m(X)$, where we compute the adjoint of A with respect to a fixed volume density on X.

(3) *Principal symbol, ellipticity.* For each $m \in \mathbb{R}$, there is a principal symbol map

$$\sigma_m \colon \Psi^m(X) \to S^m(T^*X)/S^{m-1},$$

with the spaces $S^m(T^*X)$ defined in local coordinates as in Definition 3.1.1. Restricting to classical operators (which in a coordinate chart are classical operators on $\mathbb{R}^n$, see the end of §3.1.1), the symbol map takes values in $S^m_{\mathrm{hom}}(T^*X)$, which is the space of homogeneous functions in the fibers of T^*X. The short sequence

$$0 \to \Psi^{m-1}(X) \to \Psi^m(X) \xrightarrow{\sigma_m} S^m(T^*X)/S^{m-1} \to 0$$

is exact. Thus, σ_m measures if an operator in $\Psi^m(X)$ is in fact of lower order, and moreover every principal symbol a can be quantized, i.e. there is an operator $A \in \Psi^m(X)$ with $\sigma_m(A) = a$. We say that A is *elliptic* at a point $(x, \xi) \in S^*X$ (see §3.1.5) if its principal symbol a is, see Definition 3.1.7; the set of points at which A is elliptic is denoted $\mathrm{Ell}(A)$, and its complement is the (closed) *characteristic set* $\mathrm{Char}(A) \subset S^*X$. If $\mathrm{Ell}(A) = S^*X$, we call A elliptic.

(4) *Properties of the principal symbol map.* For $A \in \Psi^m(X)$, $B \in \Psi^{m'}(X)$, we have

$$\sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B), \quad \sigma_m(A^*) = \overline{\sigma_m(A)}.$$

Furthermore,

$$\sigma_{m+m'-1}(i[A, B]) = H_a b,$$

where a and b are representatives of $\sigma_m(A)$ and $\sigma_{m'}(B)$, respectively.

(5) *Mapping properties.* For $m, s \in \mathbb{R}$, every $A \in \Psi^m(X)$ extends by continuity to a bounded map $A\colon H^s(X) \to H^{s-m}(X)$, where the Sobolev spaces $H^s(X)$ are defined using partitions of unity and the spaces $H^s(\mathbb{R}^n)$. In particular, operators in $\Psi^{-\infty}(X) = \bigcap_{m\in\mathbb{R}} \Psi^m(X)$ are smoothing, i.e. map $\mathcal{C}^{-\infty}(X) \to \mathcal{C}^{\infty}(X)$ continuously, where $\mathcal{C}^{-\infty}(X)$ is the dual space of $\mathcal{C}^{\infty}(X)$ (fixing a volume density on X).

(6) *Operator wave front sets.* Let $A \in \Psi^m(X)$. The set of points (x, ξ) in the cosphere bundle S^*X for which the essential support (Definition 3.1.9) of the full symbol of A in a coordinate chart contains (x, ξ) is well-defined, and is called the wave front set $\mathrm{WF}'(A) \subset S^*X$ of A; it is a closed set. We have $\mathrm{WF}'(A) = \emptyset$ if and only if $A \in \Psi^{-\infty}(X)$. For $A \in \Psi^m(X)$ and $B \in \Psi^{m'}(X)$, we have

$$\mathrm{WF}'(A + B) \subset \mathrm{WF}'(A) \cup \mathrm{WF}'(B), \quad \mathrm{WF}'(A \circ B) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B).$$

(7) *Wave front sets of distributions.* Let $u \in \mathcal{C}^{-\infty}(X)$, $s \in \mathbb{R}$. Then the H^s-wave front set (resp. wave front set) of u, denoted $\mathrm{WF}^s(u) \subset S^*X$ (resp. $\mathrm{WF}(u) \subset S^*X$), is the complement of the set of all $(x, \xi) \in S^*X$ for which there exists an operator $A \in \Psi^0(X)$, elliptic at (x, ξ), such that $Au \in H^s(X)$ (resp. $Au \in \mathcal{C}^{\infty}(X)$). We say that $u \in \mathcal{C}^{-\infty}(X)$ is *in H^s microlocally in a subset $Z \subset S^*X$* if $\mathrm{WF}^s(u) \cap Z = \emptyset$.

Use a more invariant language for defining pseudodifferential operators [62, §2.4], we can define ps.d.o.s to be exactly those operators whose Schwartz kernels are distributions on $X \times X$ conormal to the diagonal $\Delta_X \hookrightarrow X \times X$.

The calculus extends to operators that map sections of a rank $d_{\mathcal{E}}$ vector bundle $\mathcal{E} \to X$ to sections of the rank $d_{\mathcal{F}}$ vector bundle $\mathcal{F} \to X$, and the space of such operators of order m is denoted $\Psi^m(X, \mathcal{E}, \mathcal{F})$, or in the case $\mathcal{F} = \mathcal{E}$ simply $\Psi^m(X, \mathcal{E})$. In local coordinates and local trivializations of the bundles, elements of $\Psi^m(X, \mathcal{E}, \mathcal{F})$ are simply quantizations of symbols of order m which take values in $d_{\mathcal{F}} \times d_{\mathcal{E}}$ matrices, or equivalently, they are $d_{\mathcal{F}} \times d_{\mathcal{E}}$ matrices of scalar symbols. Such operators can be composed in the natural fashion, schematically

$$\Psi^{m'}(X, \mathcal{F}, \mathcal{G}) \circ \Psi^m(X, \mathcal{E}, \mathcal{F}) \subset \Psi^{m+m'}(X, \mathcal{E}, \mathcal{G}),$$

where $\mathcal{G} \to X$ is another vector bundle. If $\mathcal{E}$ and $\mathcal{F}$ are equipped with fiber metrics (not necessarily positive definite) and X comes with a volume density, the adjoint of $A \in$

$\Psi^m(X, \mathcal{E}, \mathcal{F})$ is well-defined, $A^* \in \Psi^m(X, \mathcal{F}, \mathcal{E})$; in general, the adjoint is always well-defined in $\Psi^m(X, \mathcal{F}^* \otimes \Omega^1, \mathcal{E}^* \otimes \Omega^1)$, where $\Omega^1 \to X$ is the rank 1 bundle of (1-)densities on X. In particular, this applies to scalar operators, so the adjoint of $A \in \Psi^m(X)$ naturally is $A^* \in \Psi^m(X, \Omega^1)$, or more symmetrically, the adjoint of $A \in \Psi^m(X, \Omega^{\frac{1}{2}})$ is $A^* \in \Psi^m(X, \Omega^{\frac{1}{2}})$; this symmetry is part of the reason why $\frac{1}{2}$-densities are useful on a technical level. See also the discussion in §6.3. The natural Sobolev spaces of bundle-valued sections $H^s(X, \mathcal{E})$ are defined using partitions of unity on X and local trivializations of $\mathcal{E}$ (using a smooth positive definite inner product on $\mathcal{E}$), and an operator $A \in \Psi^m(X, \mathcal{E}, \mathcal{F})$ defines a continuous map $H^s(X, \mathcal{E}) \to H^{s-m}(X, \mathcal{F})$ for all $s \in \mathbb{R}$.

The principal symbol map now is

$$\sigma_m \colon \Psi^m(X, \mathcal{E}, \mathcal{F}) \to S^m(T^*X, \pi^* \operatorname{Hom}(\mathcal{E}, \mathcal{F}))/S^{m-1},$$

where $\pi \colon T^*X \to X$ denotes the projection, and we have a short exact sequence

$$0 \to \Psi^{m-1}(X, \mathcal{E}, \mathcal{F}) \to \Psi^m(X, \mathcal{E}, \mathcal{F}) \xrightarrow{\sigma_m} S^m(T^*X, \pi^* \operatorname{Hom}(\mathcal{E}, \mathcal{F}))/S^{m-1} \to 0.$$

We have the natural subclass of operators in $\Psi^m(X, \mathcal{E})$ which are *principally scalar*, i.e. whose principal symbol has a scalar (multiple of the identity endomorphism on $\mathcal{E}$) representative, and conversely scalar symbols can be quantized to give principally scalar operators acting on sections of $\mathcal{E}$. The principal symbol of the commutator of two operators $A \in \Psi^m(X, \mathcal{E})$ and $B \in \Psi^{m'}(X, \mathcal{E})$ then equals $\sigma_{m+m'-1}([A, B]) = \frac{1}{i} H_{\sigma_m(A)} \sigma_{m'}(B)$ if A and B are principally scalar, and $\sigma_{m+m'}([A, B]) = [\sigma_m(A), \sigma_m(B)]$ if their principal symbols do not commute (which can only happen if they are principally non-scalar); note the symbolic orders in which we compute the principal symbols here. If the principal symbols do commute, say A is principally scalar but B is not, then the principal symbol $\sigma_{m+m'-1}([A, B])$ involves subprincipal terms of the full symbol of A; see §6.3.3.

The ellipticity of a symbol $a \in S^m(T^*X, \pi^* \operatorname{Hom}(\mathcal{E}, \mathcal{F}))$ at $(x, \xi) \in S^*X$ means that in a conic neighborhood of (x, ξ) and sufficiently far from the zero section of T^*X, the symbol a can be inverted by a symbol $b \in S^{-m}(T^*X, \pi^* \operatorname{Hom}(\mathcal{F}, \mathcal{E}))$. (This is equivalent to the existence of $b \in S^{-m}$ such that $ab - \mathrm{id} \in S^{-1}(T^*X, \pi^* \operatorname{End}(\mathcal{F}))$ and $ba - \mathrm{id} \in S^{-1}(T^*X, \pi^* \operatorname{End}(\mathcal{E}))$.) If the symbol $a \in S^m$ is classical, $a \sim \sum_{j \geq 0} a_{m-j}$, this is simply the requirement that a_m be invertible at (x, ξ).

We can then define the elliptic set, operator wave front set and distributional wave front set as in the scalar setting. We remark that there is a refined notion of wave front set for bundle-valued distributions introduced by Dencker [34], called *polarization set*, which however we will not use here.

3.2.1 Elliptic regularity; Fredholm estimates

We briefly discuss elliptic regularity, mainly to give a simple example of so-called *Fredholm estimates*. There is a direct analogue of microlocal elliptic regularity, Proposition 3.1.16, on manifolds, including the quantitative estimate (3.1.14), so we shall not restate this here.

Thus, consider an (everywhere) elliptic operator $A \in \Psi^m(X)$. The parametrix construction of Proposition 3.1.8 works equally well on a manifold (and only relies on the features of the calculus listed in the previous section); thus, we can find $B \in \Psi^{-m}(X)$ such that $BA - I = R \in \Psi^{-\infty}(X)$. Now, fix $s \in \mathbb{R}$, and let $N \in \mathbb{R}$ be arbitrary, $N < s$. Then, for $u \in H^N(X)$ with $Au \in H^{s-m}(X)$, the mapping properties of B and R on Sobolev spaces imply $u = BAu - Ru \in H^s(X) + \mathcal{C}^\infty(X) = H^s(X)$, and we in fact obtain an estimate

$$\|u\|_{H^s} \leq C(\|Au\|_{H^{s-m}} + \|u\|_{H^N}). \tag{3.2.1}$$

Considering the L^2 adjoint A^* of A, which is elliptic as well, we deduce

$$\|u\|_{H^{s'}} \leq C(\|A^*u\|_{H^{s'-m}} + \|u\|_{H^N}) \tag{3.2.2}$$

for all $s', N \in \mathbb{R}$. Now, if we let $s' = -(s-m)$, thus $s' - m = -s$, the estimates (3.2.1) and (3.2.2) precisely mean by a standard functional analytic argument, see [64, Proof of Theorem 26.1.7], that $A\colon H^s(X) \to H^{s-m}(X)$ is Fredholm, with finite-dimensional nullspace (since the inclusion $H^N(X) \hookrightarrow H^s(X)$ is compact for $N < s$) and closed range, which is equal to the orthogonal complement of the finite-dimensional space $\ker A^* \subset H^{-s+m}(X)$ with respect to the L^2-pairing of $H^{s-m}(X)$ with $H^{-s+m}(X)$. (Of course, by elliptic regularity, $\ker A$ and $\ker A^*$ are contained in $\mathcal{C}^\infty(X)$ and thus independent of s.)

For more complicated operators A which are non-elliptic, one can no longer construct an elliptic parametrix, but estimates of the form (3.2.1) and (3.2.2) may still hold (with changes in the norms on Au and A^*u, and possibly only for a certain range of values of s and s'), and one deduces that A is Fredholm between suitable spaces; see §3.2.3 for an example. Thus, we call the estimates (3.2.1) and (3.2.2) *Fredholm estimates*.

3.2.2 Real principal type propagation of singularities

As the simplest non-elliptic setting, we now consider the case of operators whose principal symbol is real and vanishes non-degenerately at the characteristic set. We will sketch a proof of the celebrated Duistermaat-Hörmander theorem on the propagation of singularities [38, §6], which roughly speaking states that microlocal regularity of solutions u to an equation $Pu = 0$ propagates along null-bicharacteristics of P, i.e. along integral curves of the Hamilton vector field of the principal symbol of P within the characteristic set of P. The main example to keep in mind is the case of wave operators on Lorentzian manifolds, as discussed in Chapter 2: In this case, null-bicharacteristics are null-geodesics, lifted to the cotangent bundle, and the Duistermaat-Hörmander theorem asserts that singularities (in the precise, microlocal sense of wave front sets!) to solutions of the wave equation propagate along light rays.

Thus, let $P \in \Psi^m(X)$ be a classical operator with real homogeneous principal symbol $p_m = \sigma_m(P) \in S^m_{\mathrm{hom}}(T^*X)$. Fix a boundary defining function ρ of fiber infinity in $\overline{T}^*X$. As explained in §3.1.5, we can rescale the Hamilton vector field

$$\mathsf{H}_{p_m} := \rho^{m-1} H_{p_m} \in \mathcal{V}(S^*X),$$

and we also rescale the principal symbol, defining

$$\mathsf{p} = \rho^m p_m \in \mathcal{C}^\infty(S^*X).$$

Recall that within the characteristic set $\mathrm{Char}(P) = \mathsf{p}^{-1}(0) \subset S^*X$, the rescaled vector field H_{p_m} induces a flow, which is merely a rescaling of the Hamilton flow of p_m if we identify the subset $\mathrm{Char}(P) \subset S^*X$ with the corresponding conic subset of $T^*X \setminus o$. We remark that H_{p_m} vanishes (as a vector field) at a point $\zeta \in S^*X$ if and only if H_{p_m} is radial at ζ (i.e. at the ray in $T^*X \setminus o$ associated with ζ), and in this case the integral curve of H_{p_m} through ζ is trivial, i.e. constant.

Now suppose $\zeta_0 \in S^*X$ is such that $\mathsf{H}_{p_m}|_{\zeta_0} \neq 0$, hence the Hamilton flow in $\mathrm{Char}(P)$ starting at ζ_0 is non-trivial; denote by $\gamma\colon [0,T] \to S^*X$ a segment of a null-bicharacteristic, i.e. an integral curve of H_{p_m}, starting at $\gamma(0) = \zeta_0$. We prove the propagation of regularity for u solving $Pu = f$ along forward bicharacteristics, but the analogous statement holds for backward bicharacteristics as well: One can simply replace P by $-P$ and use the forward result, or reverse signs in the proof given below. We point out already here that this symmetry is broken once we allow p_m to have a non-vanishing imaginary part, see §3.2.3.

Theorem 3.2.1. *[38, Theorem 6.1.1′]. Let $s, N \in \mathbb{R}$. Suppose, in the above notation, that $u \in H^N(X)$ is a solution of $Pu = f \in H^N(X)$. If $\zeta_0 \notin \mathrm{WF}^s(u)$ and $\gamma([0,T]) \cap \mathrm{WF}^{s-m+1}(f) = \emptyset$, then $\gamma(T) \notin \mathrm{WF}^s(u)$. Thus, since T was arbitrary, H^s-regularity propagates along null-bicharacteristics. Put differently, $\mathrm{WF}^s(u) \setminus \mathrm{WF}^{s-m+1}(f)$ is the union of maximally extended null-bicharacteristics.*

Quantitatively, suppose $E, B, G \in \Psi^0(X)$ are pseudodifferential operators; assume that E is elliptic at ζ_0, B is elliptic at $\gamma(T)$, and G is elliptic on $\gamma([0,T])$, such that every backward null-bicharacteristic starting at a point in $\mathrm{WF}'(B)$ reaches $\mathrm{Ell}(E)$ in finite time, remaining in $\mathrm{Ell}(G)$. Then

$$\|Bu\|_{H^s} \le C(\|GPu\|_{H^{s-m+1}} + \|Eu\|_{H^s} + \|u\|_{H^N}). \tag{3.2.3}$$

Notice that the propagation estimate (3.2.3) requires control on the H^{s-m+1}-norm of Pu, rather than the H^{s-m}-norm required for elliptic P. We thus say that the propagation estimate loses one derivative (relative to the elliptic setting). See Figure 3.1 for an illustration of the setup for (3.2.3).

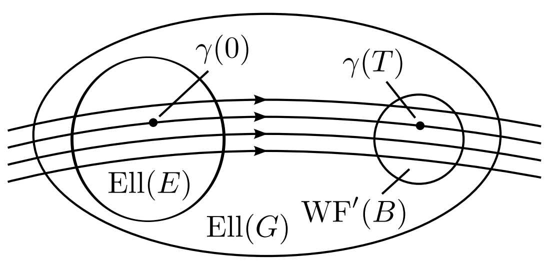

Figure 3.1: Setup for the propagation of singularities and the estimate (3.2.3): We propagate a priori H^s-control of u on the elliptic set of E forward along the null-bicharacteristic flow of P and deduce H^s-regularity of u on the elliptic set of B, assuming H^{s-m+1}-control of Pu on the elliptic set of G.

Proof of Theorem 3.2.1. We only give a brief sketch of the proof along the lines of [122, §4.2]; see [86, §5] (and, using more sophisticated tools, [38, §6]) for details. We give a full proof for rough pseudodifferential operators in §8.5.

We begin by straightening out the flow: Thus, we introduce local coordinates $q = (q_1, q') \in \mathbb{R} \times \mathbb{R}^{2n-2}$ on S^*X near ζ_0 such that $\zeta_0 = (0,0)$, and $\mathsf{H}_{p_m} = \partial_{q_1}$, hence $\gamma(T) =$

$(T, 0)$, and we have microlocal H^s regularity of u in a small neighborhood $U = \{|q_1|, |q'| < 2\delta\}$ of ζ_0. For a *commutant* $C = c(x, D) \in \Psi^{2s-m+1}(X)$, to be chosen later and satisfying $C = C^*$, we compute

$$2 \operatorname{Im}\langle f, Cu\rangle = i(\langle Cu, Pu\rangle - \langle Pu, Cu\rangle) = \operatorname{Re}\big(\langle (i[P, C] + i(P^* - P)C)u, u\rangle\big). \tag{3.2.4}$$

The principal symbol of $i[P, C]$ is given by $H_{p_m} c$. Using the rescaling $\mathsf{c} = \rho^{2s-m+1} c \in \mathcal{C}^\infty(S^*X)$, one can choose c in such a way that

$$\mathsf{H}_{p_m} \mathsf{c} = -\mathsf{b}^2 - M\mathsf{c} + \mathsf{e} \tag{3.2.5}$$

where $\mathsf{b} \in \mathcal{C}^\infty(S^*X)$ is non-negative, and positive in $\{q_1 \in (\delta, 1 + \delta), |q'| < \delta\}$, while $\mathsf{e} \in \mathcal{C}^\infty(S^*X)$ is supported in the neighborhood U where we have a priori control on u; the term involving the fixed but arbitrary parameter $M > 0$ will be used to absorb error terms later on. One can for instance take c to be a product $\mathsf{c}(q_1, q') = \mathsf{c}_1(q_1)\mathsf{c}'(q')$, and then choose c_1 to be exponentially decaying in $q_1 > \delta/2$, while cutting it off near $q_1 = 0$, which produces the error term e. See §8.5 for details in a more general setting. Now

$$H_{p_m} c = \rho^{-m+1} \mathsf{H}_{p_m}(\rho^{-2s+m-1}\mathsf{c}) = \rho^{-2s}(\mathsf{H}_{p_m}\mathsf{c} - (2s - m + 1)\mathsf{c}\rho^{-1}\mathsf{H}_{p_m}\rho) \tag{3.2.6}$$

Let $b = \rho^{-s}\mathsf{b}$, $e = \rho^{-2s}\mathsf{e}$, and let $B \in \Psi^s(X)$ and $E \in \Psi^{2s}(X)$ be quantizations of b and e, respectively. In view of (3.2.5), the commutator calculation (3.2.4) then implies

$$\begin{aligned}\langle B^*Bu, u\rangle + M\langle Cu, u\rangle = \langle Eu, u\rangle + \langle Cu, i(P^* - P)u\rangle + \langle RCu, u\rangle \\ - 2 \operatorname{Im}\langle f, Cu\rangle + \langle R'u, u\rangle,\end{aligned}$$

where $R \in \Psi^0(X)$ comes from the second term on the right hand side of (3.2.6), and $R' \in \Psi^{2s-1}(X)$ is a lower order error term, arising from the fact that (3.2.5) and (3.2.6) are merely equalities of *principal symbols*. The pairing here is the L^2 pairing between Sobolev spaces and their duals. Now, improving the choice of C by arranging $C = D^*D$, we can write the second term on the left as $M\|Du\|^2$, while we can use Cauchy-Schwarz and the Peter-Paul inequality on the terms on the right involving Cu and absorb them into the term $M\|Du\|^2$; notice that we crucially use the fact that P is principally real, thus

$P^* - P \in \Psi^{m-1}(X)$. In effect, we can then drop the terms involving Cu, and obtain

$$\|Bu\|^2 \leq \langle Eu, u\rangle + \|Gf\|^2 + \langle R'u, u\rangle, \tag{3.2.7}$$

with $G \in \Psi^{s-m+1}(X)$ (which we introduce to keep the control required on f microlocal) elliptic on $\mathrm{WF}'(C)$. By construction, the wave front sets of all operators in this expression are localized near $\gamma([0,T])$. Now, if we already have $u \in H^{s-1/2}$ microlocally near $\gamma([0,T])$, the estimate (3.2.7) implies that $Bu \in L^2(X)$, thus $u \in H^s$ microlocally near $\gamma([0,T])$. Thus, starting with the a priori knowledge $u \in H^N(X)$, one can iteratively improve the regularity of u by $1/2$ in each step, until one obtains H^s-regularity as desired.

To make this into a rigorous argument, one needs to justify various integrations by parts; to do this, one regularizes the argument by replacing the commutant C by a family C_ϵ of lower order operators, converging to C strongly as $\epsilon \to 0$, and similarly replacing the other symbols and operators in the proof. This can be conveniently done by fixing a regularization of the identity by operators $J_\epsilon \in \Psi^r(X)$, $r \ll 0$, with principal symbol $(1+\epsilon\rho^{-1})^r$, and putting $C_\epsilon = J_\epsilon C$ etc.; commuting J_ϵ through P generates additional error terms, which are handled as before. □

This proof is the prime example of a *positive commutator argument*: A quantity q which is monotone (possibly modulo error terms) along the Hamilton flow of the operator P gives rise to a microlocal energy estimate like (3.2.3), by commuting a quantization of q through P. We will encounter many more instances of this fundamental principle in the following sections.

We point out that positive commutator arguments for operators acting on sections of a vector bundle $\mathcal{E}$, require the use a Hermitian, i.e. *positive definite*, fiber inner product on $\mathcal{E}$. In applications, the natural fiber inner product is often not positive definite, e.g. the inner product on the form bundle on a manifold equipped with a *Lorentzian* metric. This causes complications in settings where subprincipal terms ($P^* - P$ in the above proof) can only be controlled if they are sufficiently small (thus, the propagation of singularities as above is unaffected by this problem): While this will be largely irrelevant for our applications in the study of radial sets in §3.3.1, it will cause difficulties in the study of normally hyperbolic trapping (see §3.3.2), which we deal with in §§6.3 and 6.4.

3.2.3 Complex absorbing potentials

The propagation of singularities theorem 3.2.1 shows that H^s-regularity propagates once it holds somewhere. Thus, one would ideally like to have a place where one gets H^s-regularity 'for free.' Radial points (with a certain structure) are a natural such device and will be discussed for b-pseudodifferential operators in §3.3.1. A very cheap (even if artificial in most situations) alternative is the use of *complex absorption*: In the notation of the previous section, we consider the operator $P-iQ$, where $Q \in \Psi^m(X)$, the *complex absorbing potential*, is classical and has a real principal symbol q. At places in S^*X where $q \neq 0$, the operator $P-iQ$ is elliptic, and we show that if $q \geq 0$, one can propagate regularity forward along the Hamilton flow of P, while for $q \leq 0$, one can propagate regularity backward along the flow. (Thus, *singularities* propagate backwards where $q \geq 0$, and forward where $q \leq 0$.)

We show this for forward propagation, i.e. with $q \geq 0$: Replace P in the calculation (3.2.4) by $P-iQ$; let us assume $Q = Q_1^* Q_1$ for some $Q_1 \in \Psi^{m/2}(X)$ for simplicity; in general, one would need to use the sharp Gårding inequality [64, §18.1]; see [114, §2.5] for details. The term

$$i((P-iQ)^* - (P-iQ)) = i(P^* - P) - 2Q$$

now has an additional term $2Q \in \Psi^{2m}(X)$, while $i[P-iQ, C] = i[P, C] + [Q, C]$ includes the term $[Q, C]$. Recall that we arranged $i[P, C]$ to be the negative of a square (up to error terms), thus $-2Q = -2Q_1^* Q_1$ has the same sign and can therefore be dropped in the subsequent estimates. As for $[Q, C]$, we need to take care of

$$\mathrm{Re}\langle [Q, C]u, u\rangle = \langle ([Q, C] + [Q, C]^*)u, u\rangle;$$

but $[Q, C]$ has purely imaginary principal symbol, thus $[Q, C] + [Q, C]^* \in \Psi^{2s-1}(X)$ is a lower order operator than the main term $i[P, C]$ and can hence be put into the error term called R' in (3.2.7).

The argument for backward propagation follows from the forward argument by considering the operator $(-P) - i(-Q) = -(P-iQ)$ instead (or one can give an analogous direct proof).

We now have all the necessary ingredients for the simplest *non-elliptic Fredholm problem*: Suppose $P - iQ \in \Psi^m(X)$, with $P = P^*$ and $Q = Q^*$ classical operators with real principal symbols p and q, is *non-trapping* in the following sense: For every point $\zeta \in \mathrm{Char}(P) \setminus \mathrm{Ell}(Q)$, both the forward and the backward bicharacteristic from ζ remain in $\{q \geq 0\}$ until they enter $q > 0$ in finite time. Then, if $u \in H^N(X)$ solves $(P - iQ)u = f \in H^{s-m+1}(X)$, microlocal elliptic regularity implies that u is in H^{s+1} microlocally at the elliptic set of Q; and by the propagation of singularities with complex absorption, we can propagate H^s-regularity of u along forward bicharacteristics of P, starting in $\mathrm{Char}(P) \cap \{q > 0\}$ and propagating forward along the Hamilton flow of p. By the non-trapping assumption, we thus obtain $\mathrm{WF}^s(u) = \emptyset$, i.e. $u \in H^s(X)$. Quantitatively,

$$\|u\|_{H^s} \leq C(\|(P - iQ)u\|_{H^{s-m+1}} + \|u\|_{H^N}).$$

The adjoint $(P - iQ)^* = P + iQ$ satisfies the analogous non-trapping property, but now the sign of Q (and thus q) is switched; thus, for solutions of $(P - iQ)^* u = f$, we propagate microlocal regularity along *backward* bicharacteristics of P. We obtain

$$\|u\|_{H^{s'}} \leq C(\|(P - iQ)^* u\|_{H^{s'-m+1}} + \|u\|_{H^N}).$$

Thus, choosing $s' = -(s-m+1)$, so $s'-m+1 = -s$, we obtain Fredholm estimates analogous to (3.2.1) and (3.2.2). They imply that for any $s \in \mathbb{R}$, the operator $P - iQ \colon \mathcal{X}^s \to \mathcal{Y}^{s-m+1}$ is Fredholm, where

$$\mathcal{X}^s = \{u \in H^s(X) \colon (P - iQ)u \in H^{s-m+1}(X)\}, \quad \mathcal{Y}^{s-m+1} = H^{s-m+1}(X),$$

and the kernels of $P - iQ$ and $(P - iQ)^*$ are both subspaces of $\mathcal{C}^\infty(X)$.

3.3 b-calculus on compact manifolds with boundary

The discussion of the geometry of certain classes of stationary spacetimes in Chapter 2 already demonstrated the usefulness of the language of b-geometry; we now discuss the analytical tools needed to work on such spaces. A full treatment with a slightly different flavor is given in [82], and a nice discussion of the geometric point of view for understanding the kernels of b-operators, compositions etc. is given in [56].

Thus, let M be a smooth compact n-dimensional manifold with boundary $X = \partial M$. In order to motivate the choices of function and operator spaces below, consider first a coordinate patch $(x, y) \in \overline{\mathbb{R}^n_+} = [0, \infty)_x \times \mathbb{R}^{n-1}_y$ near a point in ∂M, with x a local boundary defining function. The perspective on b-analysis that we wish to emphasize here is that it provides tools for uniform analysis on stationary spacetimes; hence, paralleling the discussion in Chapter 2, we introduce $t := -\log x \in (0, \infty)$, with $t \to \infty$ as $x \to 0$. One would for instance like to consider the Laplace-type operator

$$A = D_t^2 + \Delta_y = (xD_x)^2 + \Delta_y \in \mathrm{Diff}_\mathrm{b}^2(\overline{\mathbb{R}^n_+})$$

to be elliptic in the b-sense; notice however that it degenerates as an ordinary differential operator on $\overline{\mathbb{R}^n_+}$ as $x \to 0$. (More generally, the Laplace-Beltrami operator associated with any smooth Riemannian b-metric on X should be elliptic in the b-sense; notice here that since $\mathcal{V}_\mathrm{b}(X)$ is a Lie algebra, a b-metric g induces a covariant derivative of b-vector fields along b-vector fields in view of the Koszul formula, and therefore $\Delta_g \in \mathrm{Diff}_\mathrm{b}^2(X)$ indeed.) Now, $T^*\mathbb{R}^n_{t,y}$ is naturally isomorphic to ${}^\mathrm{b}T^*(\overline{\mathbb{R}^n_+})_{x,y}$: Indeed, with the natural coordinates (t, y, σ, η) on $T^*\mathbb{R}^n$ and (x, y, ξ, η) on ${}^\mathrm{b}T^*\overline{\mathbb{R}^n_+}$, this isomorphism is given by $(t, y, \sigma, \eta) \mapsto (e^{-t}, y, -\sigma, \eta)$. We therefore view the principal symbol $\sigma_{\mathrm{b},2}(A)$ of A as a function on ${}^\mathrm{b}T^*\overline{\mathbb{R}^n_+}$, formally obtained from A by replacing xD_x by ξ and D_y by η. For a general Riemannian b-metric g on M, we thus have $\sigma_{\mathrm{b},2}(\Delta_g) = G \in S^2_\mathrm{hom}({}^\mathrm{b}T^*M)$, where G is the dual metric function; and G is invertible (non-zero) away from the zero section of ${}^\mathrm{b}T^*M$.

Continuing in local coordinates, we note that due to the t-translation invariance of the operator A, it naturally acts on exponentially weighted Sobolev spaces $e^{-rt}H^s(\mathbb{R}^n)$, $r \in \mathbb{R}$. Changing coordinates, we are thus led to define the *weighted b-Sobolev space*

$$H_\mathrm{b}^{s,r}(\overline{\mathbb{R}^n_+}) = \Phi^*(e^{-rt}H^s(\mathbb{R}^n)), \quad \Phi(x, y) = (-\log x, y), \tag{3.3.1}$$

and via partitions of unity, one can define weighted b-Sobolev spaces $H_\mathrm{b}^{s,r}(M) = x^r H_\mathrm{b}^s(M)$ on compact manifolds with boundary; we have

$$H_\mathrm{b}^{s,r}(M) \subset H_\mathrm{b}^{s',r'}(M) \quad \text{if and only if } s' \leq s, r' \leq r.$$

Unweighted spaces are denoted $H_\mathrm{b}^s(M) \equiv H_\mathrm{b}^{s,0}(M)$, and we have the b-$L^2$-space $L^2_\mathrm{b}(M) := H^0_\mathrm{b}(M)$. In local coordinates, one has $u \in L^2_\mathrm{b}(\overline{\mathbb{R}^n_+})$ if and only if $u \in L^2(\overline{\mathbb{R}^n_+}, \frac{dx}{x}\,dy)$; note that $\frac{dx}{x}\,dy$ is a b-density. Then, for $s\in\mathbb{N}_0$, we have $u\in H_{\mathrm{b}}^s(\overline{\mathbb{R}^n_+})$ if and only if

$$V_1\cdots V_\ell u\in L^2_{\mathrm{b}}(\overline{\mathbb{R}^n_+}),\quad V_1,\ldots,V_\ell\in\{x\partial_x,\partial_y\},0\le\ell\le s.$$

Thus, on manifolds, we have $L^2_{\mathrm{b}}(M)=L^2(M,\nu)$ for any fixed non-vanishing b-density $\nu\in\mathcal{C}^\infty(M,{}^{\mathrm{b}}\Omega^1)$, and

$$H_{\mathrm{b}}^{s,r}(M)=\bigl\{u\in x^rL^2_{\mathrm{b}}(M)\colon V_1\cdots V_\ell u\in x^rL^2_{\mathrm{b}}(M),V_1,\ldots,V_\ell\in\mathcal{V}_{\mathrm{b}}(M),0\le\ell\le s\bigr\}$$

for integer s. For non-integer s, one can then equivalently define $H_{\mathrm{b}}^{s,r}(M)$ by duality and interpolation. Picking a different density ν leads to the same space with an equivalent norm.

We have natural space of distributions, $\mathcal{C}^{-\infty}(M):=\dot{\mathcal{C}}^\infty(M)^*$ (fixing a b-1-density for convenience) which contains $H_{\mathrm{b}}^{s,r}(M)$ for all $s,r\in\mathbb{R}$; in fact,

$$\dot{\mathcal{C}}^\infty(M)=\bigcap_{s,r}H_{\mathrm{b}}^{s,r}(M),\quad \mathcal{C}^{-\infty}(M)=\bigcup_{s,r}H_{\mathrm{b}}^{s,r}(M).$$

The space $\mathcal{C}^{-\infty}(M)$ is called the space of *extendible distributions* [64, Appendix B], since the Hahn-Banach theorem shows that it can equivalently be characterized as the space of restrictions of distributions on a closed ambient manifold $\widetilde{M}$, containing M as a submanifold with boundary, to M°. Considering instead $\dot{\mathcal{C}}^{-\infty}(M):=\mathcal{C}^\infty(M)^*$, we obtain the space of *supported distributions*, which can be viewed as the space of distributions on $\widetilde{M}$ that have support in M.

We now present a calculus of b-pseudodifferential operators, a symbolic calculus for quantizations of symbols defined on the b-cotangent bundle of M (and thus for instance allowing for symbolic inversions of elliptic b-differential operators), which is almost entirely analogous to the calculus presented in §3.2.

(1) *Spaces of operators.* For every $m\in\mathbb{R}$, the vector space $\Psi_{\mathrm{b}}^m(M)$ consists of bounded operators $\dot{\mathcal{C}}^\infty(M)\to\dot{\mathcal{C}}^\infty(M)$. For $m\in\mathbb{N}_0$, we have $\mathrm{Diff}_{\mathrm{b}}^m(M)\subset\Psi_{\mathrm{b}}^m(M)$.

(2) *Algebra property.* The space $\bigcup_{m\in\mathbb{R}}\Psi_{\mathrm{b}}^m(M)$ is a filtered *-algebra (fixing a non-vanishing b-density on M to compute adjoints).

(3) *Principal symbol, ellipticity.* For each $m\in\mathbb{R}$, there is a principal symbol map

$$\sigma_{\mathrm{b},m}\colon\Psi_{\mathrm{b}}^m(M)\to S^m({}^{\mathrm{b}}T^*M)/S^{m-1},$$

Restricting to classical operators, the symbol map takes values in $S^m_{\mathrm{hom}}({}^{\mathrm{b}}T^*M)$. The short sequence

$$0 \to \Psi_{\mathrm{b}}^{m-1}(M) \to \Psi_{\mathrm{b}}^m(M) \xrightarrow{\sigma_{\mathrm{b},m}} S^m({}^{\mathrm{b}}T^*M)/S^{m-1} \to 0$$

is exact. We say that A is *elliptic* at a point $(z,\zeta) \in {}^{\mathrm{b}}S^*M$ if its principal symbol a is; the set of points at which A is elliptic is denoted $\mathrm{Ell}(A)$, and its complement is the *characteristic set* $\mathrm{Char}(A) \subset {}^{\mathrm{b}}S^*M$.

(4) *Properties of the principal symbol map.* For $A \in \Psi_{\mathrm{b}}^m(M)$, $B \in \Psi_{\mathrm{b}}^{m'}(M)$, we have

$$\sigma_{\mathrm{b},m+m'}(A \circ B) = \sigma_{\mathrm{b},m}(A)\sigma_{\mathrm{b},m'}(B), \quad \sigma_{\mathrm{b},m}(A^*) = \overline{\sigma_{\mathrm{b},m}(A)}.$$

Furthermore,

$$\sigma_{\mathrm{b},m+m'-1}(i[A,B]) = H_a b,$$

where a and b are representatives of $\sigma_{\mathrm{b},m}(A)$ and $\sigma_{\mathrm{b},m'}(B)$, respectively.

(5) *Mapping properties.* For $m,s,r \in \mathbb{R}$, every $A \in \Psi_{\mathrm{b}}^m(M)$ defines a bounded map $A\colon H^{s,r}(M) \to H^{s-m,r}(M)$. In particular, operators in $\Psi_{\mathrm{b}}^{-\infty}(M) = \bigcap_{m\in\mathbb{R}} \Psi_{\mathrm{b}}^m(M)$ are smoothing *acting between Sobolev spaces with the same weight*, i.e. they map $H^{s,r}(M) \to H^{\infty,r}(M)$ for every $s,r \in \mathbb{R}$.

(6) *Operator wave front sets.* Let $A \in \Psi_{\mathrm{b}}^m(M)$. The set of points (z,ζ) in ${}^{\mathrm{b}}S^*M$ for which the essential support of the full symbol of A in a coordinate chart contains (z,ζ) is well-defined and closed, and is called the wave front set $\mathrm{WF}'_{\mathrm{b}}(A) \subset {}^{\mathrm{b}}S^*M$ of A. We have $\mathrm{WF}'_{\mathrm{b}}(A) = \emptyset$ if and only if $A \in \Psi_{\mathrm{b}}^{-\infty}(M)$. For $A \in \Psi_{\mathrm{b}}^m(M)$ and $B \in \Psi_{\mathrm{b}}^{m'}(M)$, we have

$$\mathrm{WF}'_{\mathrm{b}}(A+B) \subset \mathrm{WF}'_{\mathrm{b}}(A) \cup \mathrm{WF}'_{\mathrm{b}}(B), \quad \mathrm{WF}'_{\mathrm{b}}(A \circ B) \subset \mathrm{WF}'_{\mathrm{b}}(A) \cap \mathrm{WF}'_{\mathrm{b}}(B).$$

(7) *Wave front sets of distributions.* Let $s,r \in \mathbb{R}$, and suppose $u \in H_{\mathrm{b}}^{-\infty,r}(M)$. Then the $H_{\mathrm{b}}^{s,r}$-wave front set of u, denoted $\mathrm{WF}_{\mathrm{b}}^{s,r}(u) \subset {}^{\mathrm{b}}S^*M$, is the complement of the set of all $(z,\zeta) \in {}^{\mathrm{b}}S^*M$ for which there exists an operator $A \in \Psi_{\mathrm{b}}^0(M)$, elliptic at (z,ζ), such that $Au \in H_{\mathrm{b}}^{s,r}(M)$. We say that $u \in H_{\mathrm{b}}^{-\infty,r}(M)$ is *in $H_{\mathrm{b}}^{s,r}$ microlocally in a subset* $Z \subset {}^{\mathrm{b}}S^*M$ if $\mathrm{WF}_{\mathrm{b}}^{s,r}(u) \cap Z = \emptyset$.

Notice that b-pseudodifferential operators only act between b-Sobolev spaces with the same weight; thus one can only define the $H_{\mathrm{b}}^{s,r}$-wave front set for distributions u that are already known to have weight r, which is of course a much more restrictive assumption than merely $u \in \mathcal{C}^{-\infty}(M)$. We moreover point out that the calculus $\Psi_{\mathrm{b}}(M)$ only includes operators which have *smooth* coefficients on M, while operators with conormal coefficients are very natural from the point of view of applications, as discussed in §2.1.2. We indicate how to extend $\Psi_{\mathrm{b}}(M)$ to a calculus $\Psi_{\mathrm{b,bc}}(M)$ allowing for coefficients which are smooth plus conormal in §3.3.5.

The proofs of (microlocal) elliptic regularity, propagation of singularities and complex absorption on closed manifolds given in §§3.2.1, 3.2.2 and 3.2.3 depend purely on the symbol calculus and therefore go through *mutatis mutandis* for b-operators as well; again, in order to prove $H_{\mathrm{b}}^{s,r}$-regularity for a distribution u using these symbolic arguments, one needs to assume a priori that $u \in H_{\mathrm{b}}^{-\infty,r}(M)$: Symbolic arguments cannot lead to improvements in the weight r. Thus, elliptic regularity for $A \in \Psi_{\mathrm{b}}^m(M)$, elliptic at $\alpha \in {}^{\mathrm{b}}S^*M$, states that if $u \in H_{\mathrm{b}}^{-\infty,r}(M)$, then $\alpha \notin \mathrm{WF}_{\mathrm{b}}^{s-m,r}(Au)$ implies $\alpha \notin \mathrm{WF}_{\mathrm{b}}^{s,r}(u)$, while the propagation of singularities states that for $u \in H_{\mathrm{b}}^{-\infty,r}(M)$, the set $\mathrm{WF}_{\mathrm{b}}^{s,r}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{s-m+1,r}(Au)$ is the union of maximally extended null-bicharacteristics of A.

For b-operators, a crucial new feature arises, corresponding to the non-compactness of the translation-invariant picture $\mathbb{R}^n_{t,y}$ introduced at the beginning of this section: Namely, the inclusion $H_{\mathrm{b}}^{N,r}(M) \hookrightarrow H_{\mathrm{b}}^{s,r}(M)$ for $N < s$ is no longer compact, since there is no gain in the weight. This for instance shows that symbolic properties alone are not sufficient to guarantee Fredholm properties of elliptic b-(pseudo)differential operators on M, which in fact do not hold in general; the missing piece is the analysis of a model operator at ∂M, discussed in §3.3.3. (Until §3.3.3, we shall however only study symbolic properties of b-ps.d.o.s.) The map from $\Psi_{\mathrm{b}}(M)$ into the (non-commutative!) algebra of such model operators is a 'non-commutative symbol map'; the non-commutativity of this map is intimately related to the necessity in symbolic calculations to work on spaces with fixed weights.

We now indicate how to construct the above b-calculus by localizing to coordinate charts and using the Euclidean theory to the largest possible extent (after a logarithmic change of coordinates as above); parts of the presentation follow [116].[3] We want to describe a class of ps.d.o.s on $\overline{\mathbb{R}^n_+}$ which has composition and mapping properties analogous to (2)

[3]We again refer to [82] for a more geometric treatment, describing the Schwartz kernels of b-pseudodifferential operators as conormal distributions on the b-double space $[M \times M; \partial M \times \partial M]$ (for ∂M connected) with infinite order of vanishing on the side faces.

and (5) above; working on $\mathbb{R}^n_{t,y}$, $t = -\log x$, we therefore want to define ps.d.o.s A on $\mathbb{R}^n_{t,y}$ which are continuous between exponentially weighted spaces $e^{-rt}H^s(\mathbb{R}^n)$. Equivalently, if $K_A(t, y, t', y')$ denotes the Schwartz kernel of A, we want the operator with Schwartz kernel $e^{-rt}K_A(t, y, t', y')e^{rt'}$ to act between unweighted Sobolev spaces $H^s(\mathbb{R}^n)$. Thus, we are led to require superexponential decay of $K_A(t, y, t', y')$ in $|t - t'|$. Now $\psi(t' - t)K_A \in \Psi^m(\mathbb{R}^n)$, with $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$ identically 1 near 0, satisfies this automatically, and $(1 - \psi(t' - t))K_A \in \Psi^{-\infty}(\mathbb{R}^n)$ has smooth Schwartz kernel on $\mathbb{R}^{2n}$; simplify the presentation by switching freely between (x, y) and (t, y) coordinates, and identifying operators with their Schwartz kernels, we therefore define:

Definition 3.3.1. The *local smooth b-algebra* $\bigcup_m \Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$ consists of operators of the form $A = A' + R \in \Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$; here $K_{A'}(t, y, t', y') = \psi(t' - t)K_B(t, y, t', y')$, with $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$ identically 1 near 0, where K_B is the Schwartz kernel of the left quantization of a symbol $b(t, y; \sigma, \eta) \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ satisfying

$$|(e^t\partial_t)^k \partial_y^\alpha \partial_\zeta^\beta b(t, y; \zeta)| \leq C_{\alpha\beta k\ell}\langle\zeta\rangle^{m-|\beta|} \tag{3.3.2}$$

for all α, β, k, ℓ, while $K_R(t, y, t', y') \in \Psi^{-\infty}(\mathbb{R}^n)$ satisfies

$$|(e^t\partial_t)^k \partial_s^\ell \partial_y^\alpha \partial_{y'}^\beta (K_R(t, y, t + s, y'))| \leq C_{\alpha\beta k\ell M} e^{-M|s|}\langle y - y'\rangle^{-M} \tag{3.3.3}$$

for all $\alpha, \beta, k, \ell, M$.

The weighted t-derivatives simply correspond to the requirement that our operators have smooth coefficients in x; recall $\partial_x = -e^t\partial_t$. A more symmetric definition would use $e^{\langle t\rangle}\partial_t$ rather than $e^t\partial_t$ in order to have an exponential weight as t approaches $\pm\infty$, but since we are only studying a local model for b-operators, thus only work in $\{x < \epsilon\} = \{t > -\log\epsilon\}$, we decree that *all operators and symbols in the local model are compactly supported in x, or equivalently supported in a half-line $t > t_0$*, without making this explicit in the notation. Then, under the identification of (the interior of) $\overline{\mathbb{R}^n_+}$ with $\mathbb{R}^n$ via the logarithmic change of coordinates, we have $\Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}^n_+}) \subset \Psi^m(\mathbb{R}^n)$.

We will prove below that $\bigcup_m \Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$ is indeed an algebra. First, we observe that one can represent elements of $\Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$ in a more convenient way by exploiting the relation of superexponential decay and entire functions via the Fourier transform. Concretely:

Definition 3.3.2. For $m \in \mathbb{R}$, denote by $S^m_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ ('b-symbols') the space of all smooth functions $a(x, y; \xi, \eta)$ which are holomorphic in ξ and satisfy

$$|\partial_x^k \partial_y^\alpha \partial_\xi^\ell \partial_\eta^\beta a(x, y; \xi, \eta)| \leq C_{\alpha\beta k \ell N}(1 + |\operatorname{Re}\xi| + |\eta|)^{m-\ell-|\beta|}, \quad |\operatorname{Im}\xi| \leq N, \tag{3.3.4}$$

for all $\alpha, \beta, k, \ell, N$.

Here and in what follows, we omit the fiber variables in the notation of spaces of symbols; for local considerations, the fiber is always $\mathbb{R}^n$ if n is the dimension of the base.

In (t, y)-coordinates, condition (3.3.4) reads

$$|(e^t\partial_t)^k \partial_y^\alpha \partial_\sigma^\ell \partial_\eta^\beta a(t, y; \sigma, \eta)| \leq C_{\alpha\beta k \ell N}(1 + |\operatorname{Re}\xi| + |\eta|)^{m-\ell-|\beta|}, \quad |\operatorname{Im}\xi| \leq N,$$

which is a stronger requirement than (3.3.2). We show below in Lemma 3.3.6 that any symbol in $S^m(\overline{\mathbb{R}^n_+})$ can be modified by a symbol in $S^{-\infty}(\overline{\mathbb{R}^n_+})$ to yield a symbol in $S^m_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$.

Proposition 3.3.3. *For $m \in \mathbb{R}$, we have $A \in \Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$ if and only if $A = q_L(a)$ for a (uniquely determined) symbol $a \in S^m_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$, i.e. if*

$$K_A(z, z+w) = \int e^{-iw\zeta} a(z, \zeta)\, d\zeta.$$

Proof. We will ignore the tangential variables on $\overline{\mathbb{R}^n_+}$, since they come along for the ride. Thus, we simply assume that we are working on $\overline{\mathbb{R}_+}$, i.e. with $n = 1$. Then, if $A \in \Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}_+})$, thus a forteriori $A \in \Psi^m(\mathbb{R})$ after a change of variables, we obtain the left reduced symbol a from the Schwartz kernel $K_A(t, t')$ using (3.1.5), that is,

$$a(t, \sigma) = \int e^{is\sigma} K_A(t, t+s)\, ds.$$

Writing $A = A' + R$ as in Definition 3.3.1, the estimates (3.3.3) for the Schwartz kernel $K_R(t, t+s)$ of R imply that $\int e^{is\sigma} K_R(t, t+s)\, ds \in S^{-\infty}_{\mathrm{b}}(\overline{\mathbb{R}_+})$ indeed, while for A', we have

$$\int e^{is\sigma} K_{A'}(t, t+s)\, ds = (2\pi)^{-1} \iint e^{is\sigma} e^{-is\lambda} \psi(s) b(t, \lambda)\, d\lambda\, ds = \int \check{\psi}(\sigma - \lambda) b(t, \lambda)\, d\lambda,$$

where $\check{\psi}$ denotes the inverse Fourier transform of ψ. Since ψ is smooth with compact support, we have

$$|\partial_\sigma \check{\psi}(\sigma)| \leq C_N(1 + |\operatorname{Re}\sigma|)^{-N}, \quad |\operatorname{Im}\sigma| \leq N$$

for all N; thus, using the assumption $b \in S^m$, we can estimate for $\sigma \in \mathbb{R}$

$$\left|\int \check\psi(\sigma-\lambda)b(t,\lambda)\,d\lambda\right| \lesssim \langle\sigma\rangle^m \int \frac{\langle\lambda\rangle^m}{\langle\sigma-\lambda\rangle^N\langle\sigma\rangle^m}\,d\lambda \lesssim \langle\sigma\rangle^m,$$

which follows from $\langle\lambda\rangle^m \lesssim \langle\sigma-\lambda\rangle^m + \langle\sigma\rangle^m$ for $m \geq 0$ and $\langle\sigma\rangle^{-m} \lesssim \langle\sigma-\lambda\rangle^{-m} + \langle\lambda\rangle^{-m}$ for $m < 0$, together with the finiteness of $\int_{\mathbb{R}}\langle\lambda\rangle^{-s}\,d\lambda$ for $s > 1$. We similarly obtain an estimate for general $\sigma \in \mathbb{C}$, and for $e^{-t}\partial_t$ and ∂_σ-derivatives. Hence,

$$\int e^{is\sigma}K_{A'}(t,t+s)\,ds \in S^m_{\mathrm{b}}(\overline{\mathbb{R}_+})$$

indeed, proving the direction ($\Rightarrow$). For the converse direction ($\Leftarrow$), given $a \in S^m_{\mathrm{b}}(\overline{\mathbb{R}_+})$, we note that

$$K_A(t,t+s) = (2\pi)^{-1}\int e^{-is\sigma}a(t,\sigma)\,d\sigma$$

is indeed superexponentially decaying in s away from $s=0$ together with all its $e^t\partial_t$ and ∂_s-derivatives, since we can shift the contour of integration from $\mathbb{R}$ to $\operatorname{Im}\sigma = -M\operatorname{sgn}s$, which does not affect the symbol estimates for a by definition of the space $S^m_{\mathrm{b}}(\overline{\mathbb{R}_+})$ but introduces an exponential weight $|e^{-is\sigma}| = e^{-M|s|}$ in the integrand. Thus, the part $(1-\psi(s))K_A(t,t+s)$ satisfies the estimates (3.3.3) of the remainder term, while $\psi(s)K_A(t,t+s)$ is the left quantization of $a'(t,\sigma) := \int \check\psi(\sigma-\lambda)a(t,\lambda)\,d\lambda \in S^m(\overline{\mathbb{R}_+})$. □

Now, we can follow the discussion in §3.1.1, introducing a more general class of 'two-sided' symbols satisfying

$$|\partial_x^k\partial_y^\alpha\partial_{x'}^{k'}\partial_{y'}^{\alpha'}\partial_\xi^\ell\partial_\eta^\beta a(x,y,x',y';\xi,\eta)| \leq C_{\alpha\beta k\ell k'\ell' N}(1+|\operatorname{Re}\xi|+|\eta|)^{m-\ell-|\beta|}, \quad |\operatorname{Im}\xi| \leq N,$$

in analogy to (3.1.2) in the Euclidean setting, and establishing that quantizations of such symbols can uniquely be written as left/right quantizations of symbols in $S^m_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$. Indeed, for the proof of Proposition 3.1.2, we first observe that one can asymptotically sum sequences $a_j(t,y,\sigma,\eta)$ with $a_j \in S^{m-j}_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$, $j = 0,1,\ldots$, by asymptotically summing in the class of standard symbols, obtaining $\widetilde{a}(t,y,\sigma,\eta) \in S^m(\overline{\mathbb{R}^n_+})$ with $\widetilde{a} \sim \sum a_j$; then we define $a \in S^m_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ in terms of its Fourier transform in σ by

$$\int e^{-i\sigma t'}a(t,y,\sigma,\eta)\,d\sigma := \psi(t')\int e^{-i\sigma t'}\widetilde{a}(t,y,\sigma,\eta)\,d\sigma,$$

i.e. cutting off the Fourier transform of $\widetilde{a}$ in σ near $t' = 0$ (which is where the Fourier transform is conormal!), which only changes $\widetilde{a}$ by an element of $S^{-\infty}(\overline{\mathbb{R}^n_+})$; thus $a \sim \sum a_j$. Furthermore, in the last step of the proof of Proposition 3.1.2, in which the arbitrarily good remainder R (in terms of symbolic order) is expressed as the left quantization of a symbol b, we need to show that $b \in S^{-\infty}_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$, rather than merely $b \in S^{-\infty}(\mathbb{R}^n)$; that is, we need to argue that iterated $e^t\partial_t$-derivatives of b enjoy symbolic estimates; this however follows easily from a contour shifting argument applied to (3.1.4).

We can now conclude that the space $\bigcup_m \Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$ of Definition 3.3.1 is indeed an algebra, and its elements act on the exponentially weighted spaces $H^{s,r}_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ defined in (3.3.1). Notice that away from the boundary $\partial\overline{\mathbb{R}^n_+}$, elements of $\Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$ are simply standard pseudodifferential operators on open subsets of $\mathbb{R}^n_+$. Using $\Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$ as a local model for b-ps.d.o.s on M, we then define:

Definition 3.3.4. The b-calculus[4] $\bigcup_m \Psi^m_{\mathrm{b}}(M)$ on the compact manifold M with boundary consists of operators $A \in \Psi^m_{\mathrm{b}}(M)$ characterized as follows:

(1) $A\colon \dot{\mathcal{C}}^\infty(M) \to \dot{\mathcal{C}}^\infty(M)$ continuously,

(2) if U is any coordinate chart with $\Phi\colon U \to \widetilde{U} \subset \mathbb{R}^n$ a diffeomorphism, with $\widetilde{U}$ open in $\mathbb{R}^n$ or in $\overline{\mathbb{R}^n_+}$, then for all $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(U)$, we have $A_\psi := (\Phi^{-1})^*\psi A\psi\Phi^* \in \Psi^m(\mathbb{R}^n)$ or $A_\psi \in \Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$.

(3) If $\psi_U, \psi_V \in \mathcal{C}^\infty(M)$ have disjoint supports in local coordinate charts U, resp. V, then the Schwartz kernel of $\psi_U A\psi_V$ is conormal (i.e. has iterated regularity relative to b-vector fields) on $M \times M$ relative to bounded functions which decay rapidly as x/x' tends to 0 or ∞, i.e. have bounds $C_N(x/x')^N$ in $x/x' < 1$ and $C_N(x'/x)^N$ in $x/x' > 1$, where x, resp. x' denote the pullbacks of the boundary defining function of M from the first, resp. second factor of $M \times M$.

In more detail, (3) means the following:

(3.1) If U and V are disjoint from ∂M, then $\psi_U A\psi_V$ has $\mathcal{C}^\infty(M \times M)$ Schwartz kernel,

(3.2) if U is disjoint from ∂M and V is not (resp. V is disjoint from ∂M and U is not), then $\psi_U A\psi_V$ has a $\mathcal{C}^\infty(M \times M)$ Schwartz kernel vanishing to infinite order at $M \times \partial M$ (resp. $\partial M \times M$),

[4]In the language of [82], this is the 'small' b-calculus, since the Schwartz kernels of its operators vanish to infinite order at the left and right boundary of the b-double space.

(3.3) if U and V both intersect ∂M, with a product decomposition $U \cong [0,\epsilon)_x \times U_0$, $V \cong [0,\epsilon)_x \times V_0$, and with coordinate charts $\Phi_U \colon U \to [0,\epsilon)_x \times \mathbb{R}^{n-1}_y$ and $\Phi_V \colon V \to [0,\epsilon)_{x'} \times \mathbb{R}^{n-1}_{y'}$, then the Schwartz kernel K_{UV} of $(\Phi_U^{-1})^*\psi_U A \psi_V \Phi_V^*$ satisfies

$$|(e^t\partial_t)^k \partial_s^\ell \partial_y^\alpha \partial_{y'}^\beta (K_{UV}(t,y,t+s,y'))| \le C_{\alpha\beta k\ell M} e^{-M|s|}, \tag{3.3.5}$$

where $t = -\log x$ and $t+s = -\log x'$; notice that these are exactly the estimates (3.3.3) apart from the $y-y'$ factor, which is irrelevant here in view of the compact support both in y and y'.

The main task in proving that the space $\bigcup_m \Psi_{\mathrm{b}}^m(M)$ thus defined indeed gives the aforementioned symbolic calculus is checking the composition property; this amounts to showing that compositions of various localized pieces of Schwartz kernels of b-ps.d.o.s behave in the above manner. This is somewhat tedious but straightforward; we refer to [82, §5] for a rather direct treatment and [85] for a geometric proof that generalizes easily to more degenerate calculi.

We end by proving the following simple characterization of $\mathrm{WF}_{\mathrm{b}}^s(u)$, analogous to Proposition 3.1.15 in the Euclidean setting. We work on $\overline{\mathbb{R}^n_+}$, writing points $z \in \overline{\mathbb{R}^n_+}$ as $z = (x,y) \in [0,\infty) \times \mathbb{R}^{n-1}$. For brevity, we write

$$^{\mathrm{b}}D = (xD_x, D_y), \tag{3.3.6}$$

with $D = i^{-1}\partial$ as usual.

Lemma 3.3.5. *Let $u \in H_{\mathrm{b}}^{-\infty}(\overline{\mathbb{R}^n_+})$. Then $\overline{\mathbb{R}^n_+} \times (\mathbb{R}^n \setminus o) \ni (z_0,\zeta_0) \notin \mathrm{WF}_{\mathrm{b}}^s(u)$ if and only if there exists $\phi \in \mathcal{C}_{\mathrm{c}}^\infty(\overline{\mathbb{R}^n_+})$, $\phi(z_0) \neq 0$, and a conic neighborhood K of ζ_0 in $\mathbb{R}^n$ such that*

$$\chi_K(\zeta)\langle\zeta\rangle^s \widehat{\phi u} \in L^2(\mathbb{R}^n), \tag{3.3.7}$$

where χ_K is the characteristic function of K; here, $\widehat{\phi u}$ is the Mellin transform of ϕu in x and the Fourier transform in y.

Proof. It suffices to prove the lemma when χ_K is replaced by $\widetilde{\chi}_K \in \mathcal{C}^\infty(\mathbb{R}^n)$, positively homogeneous away from the origin, where $\widetilde{\chi}_K \equiv 1$ on the ray $\mathbb{R}_{\ge 1}\zeta_0$. Given such a $\widetilde{\chi}_K$ and $\phi \in \mathcal{C}_{\mathrm{c}}^\infty(\overline{\mathbb{R}^n_+})$ so that (3.3.7) holds (with χ_K replaced by $\widetilde{\chi}_K$), the map

$$A \colon v \mapsto (\widetilde{\chi}_K(^{\mathrm{b}}D)\langle ^{\mathrm{b}}D\rangle^s + r(^{\mathrm{b}}D))(\phi v)$$

is an element of $\Psi^s_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$ for an appropriate choice of $r(\zeta) \in S^{-\infty}$ by Lemma 3.3.6 below. Since $r({}^{\mathrm{b}}D)\colon H^{-\infty}_{\mathrm{b}}(\overline{\mathbb{R}^n_+}) \to H^{\infty}_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$, we conclude that $\widehat{Au} \in L^2(\mathbb{R}^n)$, which by Plancherel's theorem gives $(z_0,\zeta_0) \notin \mathrm{WF}^s_{\mathrm{b}}(u)$, as desired.

For the converse direction, given $A \in \Psi^s_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$, $\sigma_{\mathrm{b},s}(A)(z_0,\zeta_0) \neq 0$, with $Au \in L^2_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$, take $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\overline{\mathbb{R}^n_+})$ and $\widetilde{\chi}_K \in \mathcal{C}^\infty(\mathbb{R}^n)$ with $\phi(z_0) \neq 0, \widetilde{\chi}_K(\zeta_0) \neq 0$ such that A is elliptic on $\mathrm{WF}'_{\mathrm{b}}(B)$, where $B = (\widetilde{\chi}_K({}^{\mathrm{b}}D)\langle{}^{\mathrm{b}}D\rangle^s + r({}^{\mathrm{b}}D))\phi \in \Psi^s_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$, again with an appropriately chosen $r \in S^{-\infty}$. A straightforward application of the symbol calculus gives the existence of $C \in \Psi^0_{\mathrm{lb}}(\overline{\mathbb{R}^n_+}), R' \in \Psi^{-\infty}_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$ such that $B = CA - R'$; thus $Bu = C(Au) - R'u \in L^2_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$. Since $r({}^{\mathrm{b}}D)\colon H^{-\infty}_{\mathrm{b}} \to H^{\infty}_{\mathrm{b}}$, we conclude that $\chi_K(\zeta)\langle\zeta\rangle^s\widehat{\phi u} \in L^2(\mathbb{R}^n)$. □

To complete the proof, we show that $S^m + S^{-\infty} = S^m_{\mathrm{b}}$ on $\overline{\mathbb{R}^n_+}$:

Lemma 3.3.6. *For any symbol $a \in S^m((\overline{\mathbb{R}_+})_x \times \mathbb{R}^{n-1}_y; \mathbb{R}_\xi \times \mathbb{R}_\eta)$, there is a symbol $\widetilde{a} \in S^m_{\mathrm{b}}$ with $a - \widetilde{a} \in S^{-\infty}$.*

Proof. Fix $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ identically 1 near 0 and put

$$\widetilde{a}(x,y;\xi,\eta) = \mathcal{F}_{\xi\to t}\big(\phi(t)(\mathcal{F}^{-1}_{\xi\to t}a)(x,y;t,\eta)\big).$$

Then $\widetilde{a} \in S^m_{\mathrm{b}}$ by the proof of Proposition 3.3.3. Moreover, $\mathcal{F}^{-1}_{\xi\to t}(a - \widetilde{a})$ is smooth and rapidly decaying, thus the lemma follows. □

Lastly, we note that the operator with full symbol $\langle\zeta\rangle^s$ is not a b-ps.d.o. unless $s \in 2\mathbb{N}$. By the preceding Lemma, this can be fixed by changing $\langle\zeta\rangle^s$ by a symbol of order $-\infty$; more precisely:

Corollary 3.3.7. *For each $s \in \mathbb{R}$, there is $\Lambda_s \in \Psi^s_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$ with full symbol $\lambda_s(\zeta) \in S^s_{\mathrm{b}}(\overline{\mathbb{R}^n_+} \times \mathbb{R}^n)$, $\lambda_s(\zeta) \neq 0$ for all $\zeta \in \mathbb{R}^n$, such that $\lambda_s - \langle\zeta\rangle^s \in S^{-\infty}(\overline{\mathbb{R}^n_+} \times \mathbb{R}^n)$.*

Proof. The only statement left to be proved is that λ_s can be arranged to be non-vanishing. Let $\widetilde{\lambda}_s \in S^s_{\mathrm{b}}$ be the symbol constructed in Lemma 3.3.6. Since $\widetilde{\lambda}_s$ differs from the positive function $\langle\zeta\rangle^s \in S^s \setminus S^{s-1}$ by a symbol of order $S^{-\infty}$, it is automatically positive for large $|\zeta|$; thus we can choose $C = C(s)$ large such that $\lambda_s(\zeta) = \widetilde{\lambda}_s(\zeta) + C(s)e^{-\zeta^2}$ is positive for all $\zeta \in \mathbb{R}^n$. Since $e^{-\zeta^2} \in S^{-\infty}_{\mathrm{b}}$, the proof is complete. □

3.3.1 Radial points

The theorem on the propagation of singularities for an equation $\mathcal{P}u = f$, $\mathcal{P} \in \Psi_{\mathrm{b}}^m(M)$, only gives information about the wave front set of u over the interior, and, separately from this, over the boundary, since bicharacteristics of $\mathcal{P}$ either lie completely within the boundary or do not intersect it at all. Hence, for the global analysis of non-elliptic b-operators, one needs additional structure in order to connect these two pieces. Generalized b-radial sets, see Definition 2.2.2, are such a structure, and we now discuss the propagation of singularities near them.

We recall the setup, compressing both choices of signs (corresponding to source/sink behavior within ${}^{\mathrm{b}}\overline{T}_X^*M$): Let $\mathcal{P} \in \Psi_{\mathrm{b}}^m(M)$ be an operator with real principal symbol p, and assume that dp does not vanish where p does, i.e. at $\Sigma = p^{-1}(0)$, and is linearly independent of $d\tau$, τ a boundary defining function of M, at $\{\tau = 0, p = 0\} = \Sigma \cap {}^{\mathrm{b}}S_X^*M$. Thus, Σ is a smooth submanifold of ${}^{\mathrm{b}}S^*M$ transversal to ${}^{\mathrm{b}}S_X^*M$. For the generalized radial set L, assume that $L = L_+ \cup L_-$ with $L_\pm$ smooth disjoint submanifolds of ${}^{\mathrm{b}}S_X^*M$, given by $\mathcal{L}_\pm \cap {}^{\mathrm{b}}S_X^*M$ where $\mathcal{L}_\pm$ are smooth disjoint submanifolds of Σ transversal to ${}^{\mathrm{b}}S_X^*M$, defined locally near ${}^{\mathrm{b}}S_X^*M$. Fix a defining function $\widehat{\rho}$ of fiber infinity ${}^{\mathrm{b}}S^*M \subset {}^{\mathrm{b}}\overline{T}^*M$, then we assume that

$$\mathsf{H}_p = \widehat{\rho}^{m-1} H_p$$

is tangent to $\mathcal{L}_\pm$; we require

$$\begin{gathered} \widehat{\rho}^{-1}\mathsf{H}_p\widehat{\rho}|_{L\pm} = \mp\beta_0, \qquad -\tau^{-1}\mathsf{H}_p\tau|_{L_\pm} = \mp\widetilde{\beta}\beta_0, \\ \beta_0, \widetilde{\beta} \in \mathcal{C}^\infty(L_\pm),\ \beta_0, \widetilde{\beta} > 0, \end{gathered} \tag{3.3.8}$$

and, for a homogeneous degree zero quadratic defining function ρ_0 of $\mathcal{L}$ within Σ, that

$$\mp \mathsf{H}_p\rho_0 - \beta_1\rho_0 \geq 0 \tag{3.3.9}$$

within ${}^{\mathrm{b}}S_X^*M$, modulo cubic vanishing terms at $L_\pm$, with $\beta_1 > 0$. Then L_- is a source and L_+ is a sink within ${}^{\mathrm{b}}S_X^*M$, but at L_- there is also a stable, and at L_+ an unstable, manifold, namely $\mathcal{L}_-$, resp. $\mathcal{L}_+$. In order to simplify the statements, we assume that

$$\widetilde{\beta} \text{ is constant on } L_\pm;\ \widetilde{\beta} = \beta > 0; \tag{3.3.10}$$

we refer to [114, Equation (2.5)-(2.6)], and the discussion throughout that paper, as well as the numerology in §8.5.4, where a general $\widetilde{\beta}$ is allowed, at the cost of either $\sup\widetilde{\beta}$ or $\inf\widetilde{\beta}$ playing a role in various statements depending on signs. Finally, we assume that $\mathcal{P}-\mathcal{P}^*\in\Psi_{\mathrm{b}}^{m-2}(M)$ for convenience; see Remark 3.3.10 for the general case.

Proposition 3.3.8. *Suppose $\mathcal{P}$ is as above.*

If $s\geq s'$, $s'-(m-1)/2>\beta r$, and if $u\in H_{\mathrm{b}}^{-\infty,r}(M)$ then $L_\pm$ (and thus a neighborhood of $L_\pm$) is disjoint from $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$ provided $L_\pm\cap\mathrm{WF}_{\mathrm{b}}^{s-m+1,r}(\mathcal{P}u)=\emptyset$, $L_\pm\cap\mathrm{WF}_{\mathrm{b}}^{s',r}(u)=\emptyset$, and in a neighborhood of $L_\pm$, $\mathcal{L}_\pm\cap\{\tau>0\}$ are disjoint from $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$.

On the other hand, if $s-(m-1)/2<\beta r$, and if $u\in H_{\mathrm{b}}^{-\infty,r}(M)$ then $L_\pm$ (and thus a neighborhood of $L_\pm$) is disjoint from $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$ provided $L_\pm\cap\mathrm{WF}_{\mathrm{b}}^{s-m+1,r}(\mathcal{P}u)=\emptyset$ and a punctured neighborhood of $L_\pm$, with $L_\pm$ removed, in $\Sigma\cap{}^{\mathrm{b}}S_X^*M$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$.

Thus, if the a priori regularity s' of u at $L_\pm$ exceeds a certain threshold value, we can propagate H^s-regularity from the interior into the boundary. In the low regularity regime, the threshold value gives an upper bound for the amount of regularity u can have. Roughly speaking then, the threshold regularity is precisely the regularity of certain conormal solutions of $\mathcal{P}u\in H_{\mathrm{b}}^{\infty,r}$, and having higher a priori regularity excludes these, while they are generally present below the threshold regularity.

Remark 3.3.9. The decay order r plays the role of $-\operatorname{Im}\sigma$ in [114] in view of the Mellin transform in the dilation invariant setting identifying weighted b-Sobolev spaces with weight r with semiclassical Sobolev spaces on the boundary on the line $\operatorname{Im}\sigma=-r$, see [114, Equation (3.8)-(3.9)] and §3.3.4. Thus, the numerology in this proposition is a direct translation of that in [114, Propositions 2.3-2.4]. See [114, Remark 4.5] for further information on the conceptual reason behind the threshold numerology in the semiclassical setting.

Remark 3.3.10. The natural assumption is that the principal symbol of $\frac{1}{2i}(\mathcal{P}-\mathcal{P}^*)\in\Psi_{\mathrm{b}}^{m-1}(M)$ at $L_\pm$ is

$$\pm\widehat{\beta}\beta_0\widehat{\rho}^{\,-m+1},\quad \widehat{\beta}\in\mathcal{C}^\infty(L_\pm). \tag{3.3.11}$$

If $\widehat{\beta}$ vanishes, Proposition 3.3.8 is valid without a change; otherwise it shifts the threshold quantity $s-(m-1)/2-\beta r$ in Proposition 3.3.8 to $s-(m-1)/2-\beta r+\widehat{\beta}$ if $\widehat{\beta}$ is constant, with modifications as in [114, Proof of Propositions 2.3-2.4] otherwise.

Remark 3.3.11. While the assumptions listed above for Proposition 3.3.8 are not stable under perturbations of the operator $\mathcal{P}\in\Psi_{\mathrm{b}}^m(M)$, the estimates derived from it are, as the positive commutator proof below relies on the positivity of certain Hamilton derivatives, and positivity is an open condition.

Proof of Proposition 3.3.8. We remark first that $\mathsf{H}_p\rho_0$ vanishes quadratically on $\mathcal{L}_\pm$ since H_p is tangent to $\mathcal{L}_\pm$ and ρ_0 itself vanishes there quadratically. Further, this quadratic expression is positive definite near $\tau = 0$ because it is such at $\tau = 0$. Correspondingly, we can strengthen (3.3.9) to

$$\mp \mathsf{H}_p\rho_0 - \frac{\beta_1}{2}\rho_0 \tag{3.3.12}$$

being non-negative modulo cubic terms vanishing at $\mathcal{L}_\pm$ in a neighborhood of $\tau = 0$.

Notice next that, using (3.3.12) in the first case and (3.3.8) in the second, and that $L_\pm$ is defined in Σ by $\tau = 0$, $\rho_0 = 0$, there exist $\delta_0 > 0$ and $\delta_1 > 0$ such that

$$\alpha \in \Sigma,\ \rho_0(\alpha) < \delta_0,\ \tau(\alpha) < \delta_1,\ \rho_0(\alpha) \neq 0 \Rightarrow (\mp\mathsf{H}_p\rho_0)(\alpha) > 0$$

and

$$\alpha \in \Sigma,\ \rho_0(\alpha) < \delta_0,\ \tau(\alpha) < \delta_1 \Rightarrow (\pm\tau^{-1}\mathsf{H}_p\tau)(\alpha) > 0.$$

Similarly to [114, Proof of Propositions 2.3-2.4], which is not in the b-setting, and [8, Proof of Proposition 4.4], which is but concerns only sources/sinks (corresponding to Minkowski type spaces), we consider commutants

$$C \in \tau^{-r}\Psi_{\mathrm{b}}^{s-(m-1)/2}(M) = \Psi_{\mathrm{b}}^{s-(m-1)/2,-r}(M)$$

with principal symbol

$$c = \phi(\rho_0)\phi_0(p_0)\phi_1(\tau)\widehat{\rho}^{-s+(m-1)/2}\tau^{-r},\ p_0 = \widehat{\rho}^m p,$$

where $\phi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ is identically 1 near 0, $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ is identically 1 near 0 with $\phi' \leq 0$ in $[0,\infty)$ and ϕ supported in $(-\delta_0, \delta_0)$, while $\phi_1 \in \mathcal{C}_c^\infty(\mathbb{R})$ is identically 1 near 0 with $\phi_1' \leq 0$ in $[0,\infty)$ and ϕ_1 supported in $(-\delta_1, \delta_1)$, so that

$$\alpha \in \operatorname{supp} d(\phi \circ \rho_0) \cap \operatorname{supp}(\phi_1 \circ \tau) \cap \Sigma \Rightarrow \mp(\mathsf{H}_p\rho_0)(\alpha) > 0,$$

and $\pm\tau^{-1}\mathsf{H}_p\tau$ remains positive on $\operatorname{supp}(\phi_1 \circ \tau) \cap \operatorname{supp}(\phi \circ \rho_0)$.

The main contribution then comes from the weights, which give

$$\mathsf{H}_p(\widehat{\rho}^{-s+(m-1)/2}\tau^{-r}) = \mp(-s+(m-1)/2+\beta r)\beta_0\widehat{\rho}^{-s+(m-1)/2}\tau^{-r},$$

where the sign of the factor in parentheses on the right hand side being negative, resp. positive, gives the first, resp. the second, case of the statement of the proposition. Further, the sign of the term in which $\phi_1(\tau)$, resp. $\phi(\rho_0)$, gets differentiated, yielding $\pm\tau\widetilde{\beta}\beta_0\phi_1'(\tau)$, resp. $\phi'(\rho_0)\mathsf{H}_p\rho_0$, is, when $s-(m-1)/2-\beta r>0$, the opposite, resp. the same, of these terms, while when $s-(m-1)/2-\beta r<0$, it is the same, resp. the opposite, of these terms. Correspondingly,

$$\begin{aligned}\sigma_{2s}(i[\mathcal{P},C^*C]) = \mp 2\Big(-\beta_0\Big(s-\frac{m-1}{2}-\beta r\Big)\phi\phi_0\phi_1 - \beta_0\widetilde{\beta}\tau\phi\phi_0\phi_1' \\ \mp(\mathsf{H}_p\rho_0)\phi'\phi_0\phi_1 + m\beta_0 p_0\phi\phi_0'\phi_1\Big)\phi\phi_0\phi_1\widehat{\rho}^{-2s}\tau^{-2r}.\end{aligned}$$

We can regularize using using $S_\epsilon\in\Psi_{\mathrm{b}}^{-\delta}(M)$ for $\epsilon>0$, uniformly bounded in $\Psi_{\mathrm{b}}^0(M)$, converging to Id in $\Psi_{\mathrm{b}}^{\delta'}(M)$ for $\delta'>0$, with principal symbol $(1+\epsilon\widehat{\rho}^{-1})^{-\delta}$, as in [114, Proof of Propositions 2.3-2.4], where the only difference was that the calculation was on $X=\partial M$, and thus the pseudodifferential operators were standard ones, rather than b-pseudodifferential operators. The a priori regularity assumption on $\mathrm{WF}_{\mathrm{b}}^{s',r}(u)$ arises as the regularizer has the opposite sign as compared to the contribution of the weights, thus the amount of regularization one can do is limited. The positive commutator argument then proceeds completely analogously to [114, Proof of Propositions 2.3-2.4], except that, as in [114], one has to assume a priori bounds on the term with the sign opposite to that of $s-(m-1)/2-\beta r$, of which there is exactly one for either sign (unlike in [114], in which only $s-(m-1)/2+\beta\,\mathrm{Im}\,\sigma<0$ has such a term), thus on $\Sigma\cap\mathrm{supp}(\phi_1'\circ\tau)\cap\mathrm{supp}(\phi\circ\rho_0)$ when $s-(m-1)/2-\beta r>0$ and on $\Sigma\cap\mathrm{supp}(\phi_1\circ\tau)\cap\mathrm{supp}(\phi'\circ\rho_0)$ when $s-(m-1)/2-\beta r<0$.

Using the openness of the complement of the wave front set we can finally choose ϕ and ϕ_1 (satisfying the support conditions, among others) so that the a priori assumptions are satisfied, choosing ϕ_1 first and then shrinking the support of ϕ in the first case, with the choice being made in the opposite order in the second case. This completes the proof of the proposition. □

We will give full details for the proof in the case that the operator $\mathcal{P}$ has non-smooth coefficients, see Theorem 8.5.10.

Another type of generalized b-radial set shows up on (asymptotically) Minkowski spaces, the difference being that there is no relative sign difference between $\widehat{\rho}^{-1}\mathsf{H}_p\widehat{\rho}$ and $\tau^{-1}\mathsf{H}_p\tau$ in (3.3.8): Thus, the (generalized) radial set in this case is a source/sink not only within ${}^{\mathrm{b}}\overline{T}^*_X M$, but also in the direction transverse to the boundary. In this case, one can in particular obtain microlocal regularity for free at the radial set provided the a priori regularity is sufficiently high, since all terms in the positive commutator computation are positive, so no regularity needs to be required elsewhere. (For saddle points as in Proposition 3.3.8, the intuitive statement is: If no singularities flow into the saddle, none flow out of it. For the propagation out of sources/sinks on the other hand, there is no place from which singularities could propagate.) See [8, §4] and Proposition 5.5.3 for details.

We end this section by presenting a very simple toy example in which a threshold behavior as in Proposition 3.3.8 can be observed; our applications of course provide much better examples, but they will be less explicit. To wit, on $\mathbb{R}$, consider the operator $\mathcal{P}$ defined by $\mathcal{P}u(x) = xu(x)$; within the characteristic set $T^*\mathbb{R} \setminus o$, $\mathcal{P}$ has radial points at the two boundary points of $\overline{T}^*_0\mathbb{R}$, and the value of the threshold regularity is $(0-1)/2 = -1/2$ in this case, 0 being the order of $\mathcal{P}$. Concretely, suppose $u \in H^{s'}_{\mathrm{loc}}(\mathbb{R})$ is such that $\mathcal{P}u \in \mathcal{C}^\infty(\mathbb{R})$. Then one can write

$$u = u_-(x - i0)^{-1} + u_+(x + i0)^{-1} + u_s$$

with $u_\pm \in \mathbb{C}$ and $u_s \in \mathcal{C}^\infty(\mathbb{R})$. Now $(x \pm i0)^{-1} \in H^{-1/2-0}_{\mathrm{loc}}(\mathbb{R})$; therefore, if we assume *a priori* that the regularity of u is $s' > -1/2$, then we can conclude that $u \in \mathcal{C}^\infty(\mathbb{R})$; on the other hand, if we only have $s' < -1/2$ a priori, then we can only obtain $u \in H^{-1/2-0}_{\mathrm{loc}}(\mathbb{R})$, but u can have non-trivial conormal behavior at $x = 0$.

3.3.2 Normally hyperbolic trapping

One needs separate microlocal regularity results at (normally hyperbolically) trapped sets, see Definition 2.3.1. Generalizing the geometric setting considered there slightly, suppose $\mathcal{P} \in \Psi^m_{\mathrm{b}}(M)$, $\mathcal{P} - \mathcal{P}^* \in \Psi^{m-2}_{\mathrm{b}}(M)$. Let p be the principal symbol of $\mathcal{P}$, which is thus a homogeneous degree m function on ${}^{\mathrm{b}}T^*M \setminus o$, which we assume to be real-valued. Let $\widehat{\rho}$ denote a defining function of ${}^{\mathrm{b}}S^*M$, and let

$$\widehat{p}_0 = \widehat{\rho}^m p,$$

so $\Sigma = \widehat{p_0}^{-1}(0) \subset {}^{\mathrm{b}}S^*M$ is the characteristic set. We define the rescaled Hamilton vector field as $\mathsf{H}_p = \widehat{\rho}^{m-1}H_p \in \mathcal{V}_{\mathrm{b}}({}^{\mathrm{b}}\overline{T}^*M)$. We assume that $\mathcal{P}$ has normally hyperbolic trapping in the b-sense according to Definition 2.3.1, replacing G by p, at the set $\Gamma \subset \Sigma$, with forward (resp. backward) trapped set Γ_- (resp. Γ_+), and we adopt the notation used there: In particular, we have the defining function $\phi_- \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$ of Γ_- within ${}^{\mathrm{b}}S^*M$ and the defining function $\phi_+ \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$ of Γ_+ within ${}^{\mathrm{b}}S^*_XM$, in a neighborhood U_1 of Γ, which satisfy

$$\mathsf{H}_p\phi_+ = -c_+^2\phi_+ + \mu_+\tau + \nu_+\widehat{p_0}, \quad \mathsf{H}_p\phi_- = c_-^2\phi_- + \nu_-\widehat{p_0}, \tag{3.3.13}$$

with $c_\pm > 0$ smooth near Γ and $\mu_+, \nu_\pm$ smooth near Γ, and

$$\{\phi_+, \phi_-\} = H_{\phi_+}\phi_- > 0 \tag{3.3.14}$$

near Γ, while the boundary defining function τ of M satisfies

$$\mathsf{H}_p\tau = -c_\partial\tau, \quad c_\partial > 0. \tag{3.3.15}$$

Here we recall from [44, Lemma 5.1], see also [42, Lemma 2.4], that in the closely related semiclassical setting, one can arrange for any (small) $\epsilon > 0$ that

$$0 < \nu_{\min} - \epsilon < c_\pm^2 < \nu_{\max} + \epsilon, \tag{3.3.16}$$

where $\nu_{\min}$ and $\nu_{\max}$ are the minimal and maximal normal expansion rates; see [44, Equations (5.1) and (5.2)] for the definition of the latter, with $\nu_{\min}$ also given in (9.2.4) in Chapter 9, and see also the discussion prior to Theorem 9.2.9. Note that in these works of Dyatlov our $c_\pm^2$ is denoted by $c_\pm$. In particular, if M is replaced by $[0,\infty) \times X$, and if $\mathcal{P}$ is dilation invariant, then the semiclassical and the b-settings are equivalent; since in our general case $c_\pm|_{{}^{\mathrm{b}}S^*_XM}$ is what matters, we can replace $\mathcal{P}$ by $N(\mathcal{P})$, and in particular (3.3.16) applies, with the expansion rate calculated using $p|_{{}^{\mathrm{b}}T^*_XM}$.

Let $U_0 \subset \overline{U_0} \subset U_1$ be a neighborhood of Γ such that the Poisson bracket in (3.3.14) as well as $c_\pm$ have positive lower bounds. There is an asymmetry between the roles of $\phi_\pm$ and τ, and thus we consider the parabolic defining function of Γ_+

$$\rho_+ = \phi_+^2 + \mathsf{M}\tau \tag{3.3.17}$$

with $\mathsf{M} > 0$ to be chosen. Then near Γ,

$$\begin{aligned}\widehat{\rho}_+ = \mathsf{H}_p\rho_+ &= -2c_+^2\phi_+^2 + 2\mu_+\phi_+\tau + 2\nu_+\phi_+\widehat{p}_0 - \mathsf{M}c_\partial\tau \\ &= -2c_+^2\phi_+^2 - (\mathsf{M}c_\partial - 2\mu_+\phi_+)\tau + 2\nu_+\phi_+\widehat{p}_0 \\ &\leq -\widetilde{c}_+^2\rho_+ + 2\nu_+\phi_+\widehat{p}_0, \qquad \widetilde{c}_+ > 0,\end{aligned} \tag{3.3.18}$$

if $\mathsf{M} > 0$ is chosen sufficiently large, consistently with the forward trapped nature of Γ_-. (Here the term with $\widehat{p}_0$ is considered harmless as one essentially restricts to the characteristic set, $\widehat{p}_0 = 0$.) Also, note that one can use the reciprocal $\widehat{\rho} = |\sigma|^{-1}$ of the principal symbol σ of τD_τ as the local defining function of ${}^{\mathrm{b}}S^*M$ as fiber-infinity in ${}^{\mathrm{b}}T^*M$ near Γ. (Indeed, in the semiclassical setting, see §3.3.4, after Mellin transforming this problem, $|\sigma|^{-1}$ plays the role of the semiclassical parameter h, which in that case *commutes* with the operator.) Then

$$\mathsf{H}_p\widehat{\rho} = -c_f\widehat{\rho}\tau \tag{3.3.19}$$

with c_f smooth.

We briefly pause to address the differences and similarities between generalized b-radial sets and normally hyperbolic trapping in the b-sense: Recall from the previous section that the b-radial set, locally defined within Σ by $\tau = \rho = 0$, $\rho_0 = 0$, has a stable ($\tau = 0$) and unstable ($\rho = 0, \rho_0 = 0$) manifold, with $\rho = \rho_0 = 0$ defining the b-radial set within the stable manifold, and ρ_0 is its defining function within $\tau = 0$ which gives an advantageous sign when differentiated along H_p, see (3.3.9). Now, the functions ρ and τ correspond to the two fundamental properties of b-spaces, namely regularity and decay, and this is a fundamental reason why one can prove microlocal regularity estimates near radial sets in ordinary *weighted b-Sobolev spaces*: These spaces consist precisely of those functions which land in L^2_{b} when one applies appropriate quantizations of ρ and τ to them. The analogue for trapped sets, which have a stable ($\phi_- = 0$) and unstable ($\tau = \phi_+ = 0$) manifold within Σ, is to engineer spaces whose elements are mapped into L^2_{b} when one applies quantizations of ϕ_+, ϕ_- and τ (or rather $\tau^{1/2}$, in accordance with (3.3.17)) to them; thus, elements of such spaces 'degenerate' in a controlled manner at $\phi_+ = \phi_- = 0$ and $\tau = 0$.

We therefore introduce spaces which we call *normally isotropic at* Γ.[5] Concretely, let

[5]Note that ${}^{\mathrm{b}}T^*M$ is *not* a symplectic manifold (in a natural way) since the symplectic form on ${}^{\mathrm{b}}T^*_{M^\circ}M$ does not extend smoothly to ${}^{\mathrm{b}}T^*M$. Thus, the word 'normally isotropic' is not completely justified; we use it since it reflects that in the analogous semiclassical setting, see [124], the set Γ is symplectic, and the origin in the symplectic orthocomplement $(T_\alpha\Gamma)^\perp$ of $T_\alpha\Gamma$, which is also symplectic, is isotropic within $(T_\alpha\Gamma)^\perp$.

$Q_\pm \in \Psi_{\mathrm{b}}^0(M)$ have principal symbol $\phi_\pm$, $\widehat{P}_0 \in \Psi_{\mathrm{b}}^0(M)$ have principal symbol $\widehat{p}_0$, and let $Q_0 \in \Psi_{\mathrm{b}}^0(M)$ be elliptic, with real principal symbol for convenience, on U_0^c (and thus nearby).

Definition 3.3.12. The *(global) b-normally isotropic space at* Γ *of order* s, $\mathcal{H}_{\mathrm{b},\Gamma}^s$, is defined by the norm

$$\|u\|_{\mathcal{H}_{\mathrm{b},\Gamma}^s}^2 = \|Q_0 u\|_{H_{\mathrm{b}}^s}^2 + \|Q_+ u\|_{H_{\mathrm{b}}^s}^2 + \|Q_- u\|_{H_{\mathrm{b}}^s}^2 + \|\tau^{1/2} u\|_{H_{\mathrm{b}}^s}^2 + \|\widehat{P}_0 u\|_{H_{\mathrm{b}}^s}^2 + \|u\|_{H_{\mathrm{b}}^{s-1/2}}^2, \quad (3.3.20)$$

Its dual space relative to L_{b}^2 is denoted by $\mathcal{H}_{\mathrm{b},\Gamma}^{*,-s}$, is[6]

$$\mathcal{H}_{\mathrm{b},\Gamma}^{*,-s} = Q_0 H_{\mathrm{b}}^{-s} + Q_+ H_{\mathrm{b}}^{-s} + Q_- H_{\mathrm{b}}^{-s} + \tau^{1/2} H_{\mathrm{b}}^{-s} + \widehat{P}_0 H_{\mathrm{b}}^{-s} + H_{\mathrm{b}}^{-s+1/2}.$$

Note that microlocally away from Γ, $\mathcal{H}_{\mathrm{b},\Gamma}^s$ is just the standard H_{b}^s space, while $\mathcal{H}_{\mathrm{b},\Gamma}^{*,-s}$ is H_{b}^{-s}, since at least one of Q_0, $Q_\pm$ and τ is elliptic. Moreover, $\Psi_{\mathrm{b}}^k(M) \ni A\colon \mathcal{H}_{\mathrm{b},\Gamma}^s \to \mathcal{H}_{\mathrm{b},\Gamma}^{s-k}$ is continuous since $[Q_+, A] \in \Psi_{\mathrm{b}}^{k-1}(M)$ etc.; the analogous statement also holds for the dual spaces. We also note the inclusions

$$\begin{gathered} H_{\mathrm{b}}^s(M) \subset \mathcal{H}_{\mathrm{b},\Gamma}^s(M) \subset H_{\mathrm{b}}^{s-1/2}(M) \cap H_{\mathrm{b}}^{s,-1/2}(M), \\ H_{\mathrm{b}}^{s+1/2}(M) + H_{\mathrm{b}}^{s,1/2}(M) \subset \mathcal{H}_{\mathrm{b},\Gamma}^{*,s}(M) \subset H_{\mathrm{b}}^s(M). \end{gathered} \quad (3.3.21)$$

Further, the last term in (3.3.20) can be replaced by $\|u\|_{H_{\mathrm{b}}^{s-1}}^2$ as $i[Q_+, Q_-] = B^*B + R$, $B \in \Psi_{\mathrm{b}}^{-1/2}(M)$ elliptic at Γ, $R \in \Psi_{\mathrm{b}}^{-2}(M)$: Indeed, this gives

$$\|Bu\|_{H_{\mathrm{b}}^s}^2 \lesssim \|Q_- u\|_{H_{\mathrm{b}}^s}^2 + \|Q_+ u\|_{H_{\mathrm{b}}^s}^2 + \|u\|_{H_{\mathrm{b}}^{s-1}}^2$$

after integration by parts, which thus controls the $H_{\mathrm{b}}^{s-1/2}$-norm of u microlocally near Γ.

Remark 3.3.13. The notation $\mathcal{H}_{\mathrm{b},\Gamma}^s(M)$ is justified for the space is independent of the particular defining functions $\phi_\pm$ chosen; near Γ any other choice would replace $\phi_\pm$ by smooth non-degenerate linear combinations plus a multiple of τ and of $\widehat{p}_0$, denote these by $\widetilde{\phi}_\pm$, and

[6] We refer to [91, Appendix A] for a general discussion of the underlying functional analysis. In particular, Lemma A.3 there essentially gives the density of $\dot{\mathcal{C}}^\infty(M)$ in $\mathcal{H}_{\mathrm{b},\Gamma}^s(M)$: One can simply drop the subscript 'e' in the statement of that lemma to conclude that $H_{\mathrm{b}}^\infty(M)$ (so in particular $H_{\mathrm{b}}^s(M)$) is dense in $\mathcal{H}_{\mathrm{b},\Gamma}^s(M)$, and then the density of $\dot{\mathcal{C}}^\infty(M)$ in $H_{\mathrm{b}}^{s'}(M)$ for any s' completes the argument. The completeness of $\mathcal{H}_{\mathrm{b},\Gamma}^s(M)$ follows from the continuity of $\Psi_{\mathrm{b}}^0(M)$ on $H_{\mathrm{b}}^{s-1/2}(M)$.

thus the corresponding $\widetilde{Q}_\pm$ can be expressed as

$$B_+Q_+ + B_-Q_- + B_\partial\tau + \widehat{B}\widehat{P}_0 + B_0Q_0 + R, \quad B_\pm, B_0, B_\partial, \widehat{B} \in \Psi_b^0(M),\ R \in \Psi_b^{-1}(M),$$

so the new norm can be controlled by the old norm and vice versa in view of the non-degeneracy.

The propagation of singularities result at Γ then is:

Theorem 3.3.14. *With $\mathcal{P}, \mathcal{H}^s_{b,\Gamma}, \mathcal{H}^{*,s}_{b,\Gamma}$ as above, for any neighborhood U of Γ and for any N there exist $B_0 \in \Psi_b^0(M)$ elliptic at Γ and $B_1, B_2 \in \Psi_b^0(M)$ with $\mathrm{WF}_b'(B_j) \subset U$, $j = 0, 1, 2$, $\mathrm{WF}_b'(B_2) \cap \Gamma_+ = \emptyset$ and $C > 0$ such that*

$$\|B_0u\|_{\mathcal{H}^s_{b,\Gamma}} \leq \|B_1\mathcal{P}u\|_{\mathcal{H}^{*,s-m+1}_{b,\Gamma}} + \|B_2u\|_{H_b^s} + C\|u\|_{H_b^{-N}}, \tag{3.3.22}$$

i.e. if all the functions on the right hand side are in the indicated spaces, then $B_0u \in \mathcal{H}^s_{b,\Gamma}$, and the inequality holds. The same conclusion also holds if we require $\mathrm{WF}_b'(B_2) \cap \Gamma_- = \emptyset$ instead of $\mathrm{WF}_b'(B_2) \cap \Gamma_+ = \emptyset$.

Finally, if $r < 0$, then, with $\mathrm{WF}_b'(B_2) \cap \Gamma_+ = \emptyset$, the analogue of estimate (3.3.22) on weighted b-Sobolev spaces is

$$\|B_0u\|_{H_b^{s,r}} \leq \|B_1\mathcal{P}u\|_{H_b^{s-m+1,r}} + \|B_2u\|_{H_b^{s,r}} + C\|u\|_{H_b^{-N,r}}, \tag{3.3.23}$$

while if $r > 0$, then, with $\mathrm{WF}_b'(B_2) \cap \Gamma_- = \emptyset$,

$$\|B_0u\|_{H_b^{s,r}} \leq \|B_1\mathcal{P}u\|_{H_b^{s-m+1,r}} + \|B_2u\|_{H_b^{s,r}} + C\|u\|_{H_b^{-N,r}}. \tag{3.3.24}$$

Remark 3.3.15. Note that the weighted versions (3.3.23)-(3.3.24) use *standard* weighted b-Sobolev spaces. This corresponds to non-trapping semiclassical estimates if the subprincipal symbol has the correct, definite, sign at Γ.

Proof. We may assume that $U \subset U_0$ is disjoint from a neighborhood of $\mathrm{WF}_b'(Q_0)$, and thus ignore the Q_0 term in the definition of $\mathcal{H}^s_{b,\Gamma}$. We first prove that there exist B_0, B_1, B_2 as above and $B_3 \in \Psi_b^0(M)$ with $\mathrm{WF}_b'(B_3) \subset U$ such that

$$\|B_0u\|_{\mathcal{H}^s_{b,\Gamma}} \leq \|B_1\mathcal{P}u\|_{\mathcal{H}^{*,s-m+1}_{b,\Gamma}} + \|B_2u\|_{H_b^s} + \|B_3u\|_{H_b^{s-1}} + C\|u\|_{H_b^{-N}}. \tag{3.3.25}$$

An iterative argument will then prove the theorem.

We start by pointing out that for any $\widetilde{B}_0 \in \Psi_{\mathrm{b}}^0(M)$ and any $\widetilde{B}_3 \in \Psi_{\mathrm{b}}^0(M)$ elliptic on $\mathrm{WF}_{\mathrm{b}}'(\widetilde{B}_0)$, we have

$$\|\widehat{P}_0\widetilde{B}_0 u\|_{H_{\mathrm{b}}^s} \leq C\|\widetilde{B}_0\mathcal{P}u\|_{H_{\mathrm{b}}^{s-m}} + C'\|\widetilde{B}_3 u\|_{H_{\mathrm{b}}^{s-1}}, \tag{3.3.26}$$

by simply using that $\widehat{P}_0$ is an elliptic multiple of $\mathcal{P}$ modulo $\Psi_{\mathrm{b}}^{-1}(M)$. Since $\|\widetilde{B}_0\mathcal{P}u\|_{H_{\mathrm{b}}^{s-m}} \leq C\|\widetilde{B}_0\mathcal{P}u\|_{\mathcal{H}_{\mathrm{b},\Gamma}^{*,s-m}}$, the $\widehat{P}_0$ contribution to $\|\widetilde{B}_0 u\|_{\mathcal{H}_{\mathrm{b},\Gamma}^s}$ in (3.3.25) is thus automatically controlled.

So let $\chi_0(t) = e^{-\digamma/t}$ for $t > 0$, $\chi_0(t) = 0$ for $t \leq 0$, with $\digamma > 0$ (large) to be specified, $\chi \in \mathcal{C}_{\mathrm{c}}^\infty([0,\infty))$ be identically 1 near 0 with $\chi' \leq 0$, and indeed with $\chi'\chi = -\chi_1^2$, $\chi_1 \geq 0$, $\chi_1 \in \mathcal{C}_{\mathrm{c}}^\infty([0,\infty))$, and let $\psi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R})$ be identically 1 near 0. As we use the Weyl quantization,[7] we write $\mathcal{P}$ as the Weyl quantization of $p = p_0 + \widehat{\rho}p_1$, with $\widehat{\rho}p_1$ of order $m-1$. Let

$$a = \widehat{\rho}^{-s+(m-1)/2}\chi_0(\rho_+ - \phi_-^2 + \kappa)\chi(\rho_+)\psi(\widehat{p}_0), \tag{3.3.27}$$

$\kappa > 0$ small. Notice that if χ is supported in $[0, R]$, then on $\operatorname{supp} a$, we have

$$\rho_+ \leq R,\ \phi_-^2 \leq \rho_+ + \kappa = R + \kappa,$$

so a is localized near Γ if R and κ are taken sufficiently small. In particular, the argument of χ_0 is bounded above by $R + \kappa$, so given any $\mathsf{M}_0 > 0$ one can take $\digamma > 0$ large so that

$$\chi_0'\chi_0 - \mathsf{M}_0\chi_0^2 = b^2\chi_0'\chi_0,$$

with b smooth and $b \geq 1/2$ on the range of the argument of χ_0.

In fact, we also need to regularize, namely introduce

$$a_\epsilon = (1 + \epsilon\widehat{\rho}^{-1})^{-2}a,\ \epsilon \in [0,1], \tag{3.3.28}$$

which is a symbol of order $s-(m-1)/2-2$ for $\epsilon > 0$, and is uniformly bounded in symbols of order $s-(m-1)/2$ as ϵ varies in $[0,1]$. In order to avoid more cumbersome notation below, we ignore the regularizer and work directly with a; since the regularizer gives the same kind

[7] The Weyl quantization is in fact irrelevant: If $A \in \Psi_{\mathrm{b}}^m(X)$ and the principal symbol of A is real, then the real part of the subprincipal symbol is defined independently of choices, which suffices below.

of contributions to the commutator as the weight $\widehat{\rho}^{-s+(m-1)/2}$, these contributions can be dominated in exactly the same way.

Put $W = \widehat{\rho}^{m-2} H_{\widehat{\rho} p_1}$, which is a smooth vector field near ${}^{\mathrm{b}}S^*M$ as $\widehat{\rho} p_1$ is order $m-1$, then $W\widehat{\rho} = -c_{f,1}\tau\widehat{\rho}$ similarly to (3.3.19), and $W\tau = c_{\partial,1}\tau$ by the tangency of W to $\tau = 0$; so with $p = p_0 + \widehat{\rho} p_1$ as above, we have

$$\begin{aligned}\frac{1}{4}H_p(a^2) = &- (-\widehat{\rho}_+/2 + c_-^2\phi_-^2 + \nu_-\phi_-\widehat{p}_0 - \widehat{\rho}\phi_+(W\phi_+) - \widehat{\rho}\mathsf{M}c_{\partial,1}\tau + \widehat{\rho}\phi_-(W\phi_-)) \\ &\qquad\times \widehat{\rho}^{-2s}(\chi_0\chi_0')(\rho_+ - \phi_-^2 + \kappa)\chi(\rho_+)^2\psi(\widehat{p}_0)^2 \\ &+ \frac{1}{4}(-2s+m-1)\widehat{\rho}^{-2s}(-c_f - c_{f,1})\tau\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\chi(\rho_+)^2\psi(\widehat{p}_0)^2 \\ &+ \frac{1}{2}\widehat{\rho}^{-2s}(\widehat{\rho}_+ + \widehat{\rho} W\rho_+)(\chi'\chi)(\rho_+)\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\psi(\widehat{p}_0)^2 \\ &+ \frac{m}{2}(-c_f - c_{f,1})\widehat{\rho}^{-2s}(\widehat{p}_0)\tau\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\chi(\rho_+)^2(\psi\psi')(\widehat{p}_0).\end{aligned} \tag{3.3.29}$$

A key point is that the second term on the right hand side, given by the weight $\widehat{\rho}^{-2s+m-1}$ being differentiated, can be absorbed into the first by making $F > 0$ large so that $\widehat{\rho}_+\chi_0'(\rho_+ - \phi_-^2 + \kappa)$ dominates

$$|-2s+m-1||c_f|\tau\chi_0(\rho_+ - \phi_-^2 + \kappa)$$

on $\operatorname{supp} a$, which can be arranged as $|-2s+m-1||c_f|\tau$ is bounded by a sufficiently large multiple of $\widehat{\rho}_+$ there. Thus,

$$\frac{1}{4}H_p(a^2) = -c_+^2 a_+^2 - c_-^2 a_-^2 - a_\partial^2 + 2g_+a_+ + 2g_-a_- + e + \widetilde{e} + 2a_+j_+p + 2a_-j_-p \tag{3.3.30}$$

with

$$\begin{aligned} a_\pm &= \widehat{\rho}^{-s}\phi_\pm\sqrt{(\chi_0\chi_0')(\rho_+ - \phi_-^2 + \kappa)}\chi(\rho_+)\psi(\widehat{p}_0), \\ a_\partial &= \widehat{\rho}^{-s}\tau^{1/2}\Big((\mathsf{M}(c_\partial/2) - \mu_+\phi_+ - \widehat{\rho}\mathsf{M}c_{\partial,1})(\chi_0\chi_0')(\rho_+ - \phi_-^2 + \kappa) \\ &\qquad - \frac{1}{4}(-2s+m-1)(-c_f - c_{f,1})\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\Big)^{1/2}\chi(\rho_+)\psi(\widehat{p}_0), \\ g_\pm &= \pm\frac{1}{2}\widehat{\rho}^{-s+1}((W\phi_\pm) - \nu_\pm\widehat{\rho}^{m-1}p_1)\sqrt{(\chi_0\chi_0')(\rho_+ - \phi_-^2 + \kappa)}\chi(\rho_+)\psi(\widehat{p}_0), \\ e &= -\frac{1}{2}\widehat{\rho}^{-2s}(\widehat{\rho}_+ + \widehat{\rho} W\rho_+)\chi_1(\rho_+)^2\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\psi(\widehat{p}_0)^2, \\ \widetilde{e} &= \frac{m}{2}\widehat{\rho}^{-2s}(\widehat{p}_0)(-c_f - c_{f,1})\tau\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\chi(\rho_+)^2(\psi\psi')(\widehat{p}_0), \end{aligned}$$

$$j_\pm = \pm\frac{1}{2}\nu_\pm\widehat{\rho}^{-s+m}\sqrt{(\chi_0\chi_0')(\rho_+ - \phi_-^2 + \kappa)}\chi(\rho_+)\psi(\widehat{p}_0);$$

the square root in a_∂ is that of a non-negative quantity and is $\mathcal{C}^\infty$ for M large (so that $\mu_+\phi_+$ can be absorbed into $\mathsf{M}(c_\partial/2)$) and $\digamma$ large (so that a small multiple of χ_0' can be used to dominate χ_0), as discussed earlier. Moreover,

$$\operatorname{supp} e \subset \operatorname{supp} a,\ \operatorname{supp} e \cap \Gamma_+ = \emptyset,$$
$$\operatorname{supp} \widetilde{e} \subset \operatorname{supp} a,\ \operatorname{supp} \widetilde{e} \cap \Sigma = \emptyset.$$

This gives, with the various operators being Weyl quantizations of the corresponding lower case symbols,

$$\begin{aligned}\frac{i}{4}[\mathcal{P}, A^*A] = &-(C_+A_+)^*(C_+A_+) - (C_-A_-)^*(C_-A_-) - A_\partial^*A_\partial \\ &+ G_+^*A_+ + A_+^*G_+ + G_-^*A_- + A_-^*G_- \\ &+ E + \widetilde{E} + A_+^*J_+\mathcal{P} + \mathcal{P}^*J_+^*A_+ + A_-^*J_-\mathcal{P} + \mathcal{P}^*J_-^*A_- + F\end{aligned} \tag{3.3.31}$$

where $A \in \Psi_{\mathrm{b}}^{s-(m-1)/2}(M)$, $A_\pm, A_\partial \in \Psi_{\mathrm{b}}^s(M)$, $G_\pm \in \Psi_{\mathrm{b}}^{s-1}(M)$, $E \in \Psi_{\mathrm{b}}^{2s}(M)$, $\widetilde{E} \in \Psi_{\mathrm{b}}^{2s}(M)$, $J_\pm \in \Psi_{\mathrm{b}}^{s-m}(M)$, $F \in \Psi_{\mathrm{b}}^{2s-2}(M)$ with $\mathrm{WF}_{\mathrm{b}}'(F) \subset \operatorname{supp} a$.

Thus, first using $\mathcal{P} - \mathcal{P}^* \in \Psi_{\mathrm{b}}^{m-2}(M)$,

$$\begin{aligned}\|C_+A_+u\|^2 &+ \|C_-A_-u\|^2 + \|A_\partial u\|^2 \\ &\le |\langle Eu, u\rangle| + |\langle \widetilde{E}u, u\rangle| + |\langle A\mathcal{P}u, Au\rangle| + 2\|A_+u\|\|G_+u\| + 2\|A_-u\|\|G_-u\| \\ &\quad + 2|\langle J_+\mathcal{P}u, A_+u\rangle| + 2|\langle J_-\mathcal{P}u, A_-u\rangle| + C_1\|\widetilde{F}_1u\|^2_{H_{\mathrm{b}}^{s-1}} + C_1\|u\|^2_{H_{\mathrm{b}}^{-N}},\end{aligned} \tag{3.3.32}$$

where we took $\widetilde{F}_1 \in \Psi_{\mathrm{b}}^0(M)$ elliptic on $\mathrm{WF}_{\mathrm{b}}'(F)$ and with $\mathrm{WF}_{\mathrm{b}}'(\widetilde{F}_1)$ near Γ. Noting that $\mathrm{WF}_{\mathrm{b}}'(\widetilde{E}) \cap \Sigma = \emptyset$, elliptic regularity yields

$$|\langle \widetilde{E}u, u\rangle| \le C\|B_1\mathcal{P}u\|^2_{H_{\mathrm{b}}^{s-m}} + C\|u\|^2_{H_{\mathrm{b}}^{-N}}$$

if $B_1 \in \Psi_{\mathrm{b}}^0(M)$ is elliptic on $\operatorname{supp}\widetilde{e}$. Let $\Lambda \in \Psi_{\mathrm{b}}^{(m-1)/2}(M)$ be elliptic with real principal symbol λ, and let $\Lambda^- \in \Psi_{\mathrm{b}}^{-(m-1)/2}(M)$ be a parametrix for it so that $\Lambda\Lambda^- - \mathrm{Id} = R_0 \in$

$\Psi_{\mathrm{b}}^{-\infty}(M)$. Then

$$\begin{aligned}|\langle A\mathcal{P}u, Au\rangle| &\leq |\langle \Lambda^- A\mathcal{P}u, \Lambda^* Au\rangle\| + |\langle R_0 A\mathcal{P}u, Au\rangle| \\ &\leq \frac{1}{2\epsilon}\|\Lambda^- A\mathcal{P}u\|^2_{\mathcal{H}^{*,0}_{\mathrm{b},\Gamma}} + \frac{\epsilon}{2}\|\Lambda^* Au\|^2_{\mathcal{H}^0_{\mathrm{b},\Gamma}} + C'\|u\|^2_{H_{\mathrm{b}}^{-N}}\end{aligned}$$

As $\Lambda^* A \in \Psi_{\mathrm{b}}^s(M)$, for sufficiently small $\epsilon > 0$, the term $\frac{\epsilon}{2}\|\Lambda^* Au\|^2_{\mathcal{H}^0_{\mathrm{b},\Gamma}}$ can be absorbed into $\|C_+A_+u\|^2 + \|C_-A_-u\|^2 + \|A_\partial u\|^2$ plus $\|\widetilde{B}_0\widehat{P}_0 u\|^2_{H_{\mathrm{b}}^s}$, and as discussed above, the latter already has the control required for (3.3.25). (The point here is that $A_+^*C_+^*C_+A_+ - \epsilon A^*\Lambda Q_+^* Q_+\Lambda^* A$ has principal symbol $c_+^2 a_+^2 - \epsilon a^2\phi_+^2\lambda^2$ which can be written as the square of a real symbol for $\epsilon > 0$ small in view of the main difference in vanishing factors in the two terms being that χ_0' in a_+^2 is replaced by χ_0 in a, and thus the corresponding operator can be expressed as $\widetilde{C}^*\widetilde{C}$ for suitable $\widetilde{C}$, modulo an element of $\Psi_{\mathrm{b}}^{2s-2}(M)$, with the latter contributing to the H_{b}^{s-1} error term on the right hand side of (3.3.25).) On the other hand, taking $B_1 \in \Psi_{\mathrm{b}}^0(M)$ elliptic on $\mathrm{WF}_{\mathrm{b}}'(A)$, as $\Lambda^- A \in \Psi_{\mathrm{b}}^{s-m+1}(M)$,

$$\|\Lambda^- A\mathcal{P}u\|^2_{\mathcal{H}^{*,0}_{\mathrm{b},\Gamma}} \leq C''\|B_1\mathcal{P}u\|^2_{\mathcal{H}^{*,s-m+1}_{\mathrm{b},\Gamma}} + C''\|u\|^2_{H_{\mathrm{b}}^{-N}}.$$

Similarly, to deal with the $J_\pm$ terms on the right hand side of (3.3.32), one writes

$$\begin{aligned}|\langle J_\pm\mathcal{P}u, A_\pm u\rangle| &\leq \frac{1}{2\epsilon}\Big(\|B_1\mathcal{P}u\|^2_{H_{\mathrm{b}}^{s-m}} + C''\|u\|^2_{H_{\mathrm{b}}^{-N}}\Big) + \frac{\epsilon}{2}\|A_\pm u\|^2_{L^2} \\ &\leq \frac{1}{2\epsilon}\Big(\|B_1\mathcal{P}u\|^2_{\mathcal{H}^{*,s-m}_{\mathrm{b},\Gamma}} + C''\|u\|^2_{H_{\mathrm{b}}^{-N}}\Big) + \frac{\epsilon}{2}\|A_\pm u\|^2_{L^2},\end{aligned}$$

while the $G_\pm$ terms can be estimated by

$$\epsilon\|A_+u\|^2 + \epsilon^{-1}\|G_+u\|^2 + \epsilon\|A_-u\|^2 + \epsilon^{-1}\|G_-u\|^2,$$

and for $\epsilon > 0$ sufficiently small, the $\|A_\pm u\|^2$ terms in both cases can be absorbed into the left hand side of (3.3.32) while the $G_\pm$ into the error term. This gives, with $\widetilde{F}_2$ having properties as $\widetilde{F}_1$,

$$\begin{aligned}&\|C_+A_+u\|^2 + \|C_-A_-u\|^2 + \|A_\partial u\|^2 \\ &\quad\leq |\langle Eu, u\rangle| + C\|B_1\mathcal{P}u\|^2_{\mathcal{H}^{*,s-m+1}_{\mathrm{b},\Gamma}} + C_2\|\widetilde{F}_2 u\|^2_{H_{\mathrm{b}}^{s-1}} + C_2\|u\|^2_{H_{\mathrm{b}}^{-N}}.\end{aligned}$$

By the remark before the statement of the theorem, if $B_0 \in \Psi_{\mathrm{b}}^0(M)$ is such that $\chi_0(\rho_+ -$

$\phi_-^2 + \kappa)\chi(\rho_+)\psi(p) > 0$ on $\mathrm{WF}'_{\mathrm{b}}(B_0)$, $\|B_0 u\|^2_{H_{\mathrm{b}}^{s-1/2}}$ can be added to the left hand side at the cost of changing the constant in front of $\|\widetilde{F}_2 u\|^2_{H_{\mathrm{b}}^{s-1}} + \|u\|^2_{H_{\mathrm{b}}^{-N}}$ on the right hand side. Taking such $B_0 \in \Psi_{\mathrm{b}}^0(M)$, and B_1 elliptic on $\mathrm{WF}'_{\mathrm{b}}(A)$ as before, $B_2 \in \Psi_{\mathrm{b}}^0(M)$ elliptic on $\mathrm{WF}'_{\mathrm{b}}(E)$ but with $\mathrm{WF}'_{\mathrm{b}}(B_2)$ disjoint from Γ_+, we conclude that

$$\|B_0 u\|^2_{\mathcal{H}^s_{\mathrm{b},\Gamma}} \leq C\|B_1 \mathcal{P} u\|^2_{\mathcal{H}^{*,s-m+1}_{\mathrm{b},\Gamma}} + C\|B_2 u\|^2_{H^s_{\mathrm{b}}} + C\|\widetilde{F}_2 u\|^2_{H^{s-1}_{\mathrm{b}}} + C\|u\|^2_{H^{-N}_{\mathrm{b}}},$$

proving (3.3.25), up to redefining B_j by multiplication by a positive constant. Recall that unless one makes sufficient a priori assumptions on the regularity of u, one actually needs to regularize, but as mentioned after (3.3.28), the regularizer is handled in exactly the same manner as the weight.

Now in general, with χ as before, but supported in $[0, 1]$ instead of $[0, R]$, let $\chi_R = \chi(\cdot/R)$ and write $a = a_{R,\kappa}$ to emphasize its dependence on these quantities. When R and κ are decreased, $\operatorname{supp} a_{R,\kappa}$ also decreases in Σ in the strong sense that $0 < R < R'$ and $0 < \kappa < \kappa'$ imply that $a_{R',\kappa'}$ is elliptic on $\operatorname{supp} a_{R,\kappa}$ within Σ, and indeed globally if the cutoff ψ is suitably adjusted as well. Thus, if $u \in H_{\mathrm{b}}^{-N}$, say, one uses first (3.3.25) with $s = -N + 1$, and with B_j given by the proof above, so the $B_3 u$ term is a priori bounded, to conclude that $B_0 u \in \mathcal{H}^s_{\mathrm{b},\Gamma}$ and the estimate holds, so in particular, u is in $H_{\mathrm{b}}^{-N+1/2}$ microlocally near Γ (concretely, on the elliptic set of B_0). Now one decreases κ and R by an arbitrarily small amount and applies (3.3.25) with $s = -N + 3/2$; the $B_3 u$ term is now a priori bounded by the microlocal membership of u in $H_{\mathrm{b}}^{-N+1/2}$, and one concludes that $B_0 u \in \mathcal{H}^{-N+3/2}_{\mathrm{b},\Gamma}$, so in particular u is microlocally in H_{b}^{-N+1}. Proceeding inductively, one deduces the first statement of the theorem, (3.3.22).

If one reverses the role of Γ_+ and Γ_- in the statement of the theorem, one simply reverses the roles of $\rho_+ = \phi_+^2 + \mathsf{M}\tau$ and ϕ_-^2 in the definition of a in (3.3.27). This reverses the signs of all terms on the right hand side of (3.3.29) whose sign mattered below, and thus the signs of the first three terms on the right hand side of (3.3.31), which then does not affect the rest of the argument.

In order to prove (3.3.23), one simply adds a factor τ^{-2r} to the definition of a in (3.3.27). This adds a factor τ^{-2r} to every term on the right hand side of (3.3.31), as well as an additional term

$$\frac{r}{2}\tau^{-2r}\widehat{\rho}^{-2s} c_\partial \chi_0(\rho_+ - \phi_-^2 + \kappa)^2 \chi(\rho_+)^2 \psi(p)^2,$$

which for $r < 0$ has the same sign as the terms whose sign was used above, and indeed can be written as the negative of a square. Thus (3.3.30) becomes

$$\begin{aligned}\frac{1}{4}H_p(a^2) = -c_+^2 a_+^2 - c_-^2 a_-^2 - a_\partial^2 - a_r^2 \\ + 2g_+a_+ + 2g_-a_- + e + \widetilde{e} + 2j_+a_+p + 2j_-a_-p\end{aligned} \tag{3.3.33}$$

with

$$a_r = \sqrt{\frac{-r}{2}}\tau^{-r}\widehat{\rho}^{-s}c_\partial^{1/2}\chi_0(\rho_+ - \phi_-^2 + \kappa)\chi(\rho_+)\psi(p),$$

and all other terms as above apart from the additional factor of τ^{-r} in the definition of $a_\pm$, etc. Since a_r is actually elliptic at Γ when $r \neq 0$, this proves the desired estimate (and one does not need to use the improved properties given by the Weyl calculus!). When the role of Γ_+ and Γ_- is reversed, there is an overall sign change, and thus $r > 0$ gives the advantageous sign; the rest of the argument is unchanged. □

Remark 3.3.16. The estimate (3.3.22) can be strengthened by adding the term $\|B_0\widehat{P}_0 u\|_{H_{\mathrm{b}}^{s+1}}$ to the left hand side, which is controlled by elliptic regularity, likewise for (3.3.23)–(3.3.24). A more natural way of phrasing such an improvement is to use 'coisotropic, normally isotropic' spaces $\widetilde{\mathcal{H}}_{\mathrm{b},\Gamma}^s$ and $\widetilde{\mathcal{H}}_{\mathrm{b},\Gamma}^{*,s}$ in the estimate (3.3.22), where the squared norm on $\widetilde{\mathcal{H}}_{\mathrm{b},\Gamma}^s$ is defined by

$$\|u\|_{\widetilde{\mathcal{H}}_{\mathrm{b},\Gamma}^s}^2 = \|Q_0 u\|_{H_{\mathrm{b}}^s}^2 + \|Q_+ u\|_{H_{\mathrm{b}}^s}^2 + \|Q_- u\|_{H_{\mathrm{b}}^s}^2 + \|\tau^{1/2} u\|_{H_{\mathrm{b}}^s}^2 + \|\widehat{P}_0 u\|_{H_{\mathrm{b}}^{s+1/2}}^2 + \|u\|_{H_{\mathrm{b}}^{s-1/2}}^2,$$

i.e. strengthening the norm of $\widehat{P}_0 u$ by a half, which strengthens the space and weakens its dual. To obtain the necessary elliptic estimate (3.3.26) with the strengthened norms on the terms involving $\widetilde{B}_0$, but keeping the norm on $\widetilde{B}_3 u$ (which is required for the iterative argument at the end of the proof), one can choose $\widetilde{B}_0$ with $\mathrm{WF}_{\mathrm{b}}'(I - \widetilde{B}_0) \cap \Gamma = \emptyset$ so that $\widetilde{B}_3$ can be chosen to be microsupported away from Γ. Then $\|\widetilde{B}_3 u\|_{H_{\mathrm{b}}^{s-1/2}} \leq C\|\widetilde{B}_3 u\|_{\mathcal{H}_{\mathrm{b},\Gamma}^{s-1/2}}$ is controlled using the estimate (3.3.22) (with $s - 1/2$ in place of s), noting that the norm on $B_1\mathcal{P}u$ in this case is $\|B_1\mathcal{P}u\|_{\mathcal{H}_{\mathrm{b},\Gamma}^{*,s-m+1/2}} \leq C\|B_1\mathcal{P}u\|_{\widetilde{\mathcal{H}}_{\mathrm{b},\Gamma}^{*,s-m+1}}$, and the error term being measured in $H_{\mathrm{b}}^{s-3/2} \supset H_{\mathrm{b}}^{s-1}$, as required.

3.3.3 Normal operator family; Fredholm analysis for b-operators

We only discuss normal operators of b-differential operators here; for the case of general b-ps.d.o.s, see [82, §4].

The normal operator of a b-differential operator A is a model operator at the boundary $X = \partial M$ of M, obtained by freezing coefficients of A at X. In order to do this in a natural way globally on M, we follow [114, §3.1] and trivialize the inward pointing normal bundle ${}_+NX \subset T_X M$ by a choice of an inward pointing vector field V, which fixes the differential of a global boundary defining function x. This gives an identification ${}_+NX \cong [0,\infty)_x \times X =: M_I$. Thus, b-differential operators on M_I which are invariant under dilations in x (equivalently: translations in $t = -\log x$) have the form

$$\sum_{j+|\alpha|\leq m} b_\alpha(y)(xD_x)^j D_y^\alpha,$$

where y are local coordinates in X, and the space of such operators is denoted $\mathrm{Diff}_{\mathrm{b,I}}^m(M_I)$. Now, writing $A \in \mathrm{Diff}_{\mathrm{b}}^m(M)$ as

$$A = \sum_{j+|\alpha|\leq m} a_{j\alpha}(x,y)(xD_x)^j D_y^\alpha,$$

we define the *normal operator* of A, denoted $N(A)$, as the operator

$$N(A) := \sum_{j+|\alpha|\leq m} a_{j\alpha}(0,y)(xD_x)^j D_y^\alpha \in \mathrm{Diff}_{\mathrm{b,I}}^m(M_I). \tag{3.3.34}$$

We can (non-canonically) identify a collar neighborhood of X in M with a neighborhood of $\{0\}_x \times X$ in M_I; transferring $A \in \mathrm{Diff}_{\mathrm{b}}^m(M)$ to this neighborhood and extending it arbitrarily to an operator on M_I, we then have $A - N(A) \in x\mathrm{Diff}_{\mathrm{b}}^m(M_I)$.

Since $N(A)$ is dilation-invariant in x, i.e. translation-invariant in $t = -\log x$, it is naturally studied by conjugating it by the Mellin transform in x and considering the *normal operator family*

$$\widehat{N}(A)(\sigma) \equiv \widehat{A}(\sigma) := \sum_{j+|\alpha|\leq m} a_{j\alpha}(0,y)\sigma^j D_y^\alpha, \tag{3.3.35}$$

which is a family of operators in $\mathrm{Diff}^m(X)$ depending holomorphically on σ. Thus, $\widehat{A}(\sigma)$ is the operator on X acting on $u \in \mathcal{C}^\infty(X)$ by choosing any extension $f \in \mathcal{C}^\infty(M)$ of u and defining $\widehat{A}(\sigma)u = (x^{-i\sigma}Ax^{i\sigma}f)|_{\partial M}$; see also [110, §2]. We remark that the normal operator (family) enjoys many naturality properties: For instance, if $A, B \in \mathrm{Diff}_{\mathrm{b}}(M)$, then $N(A\circ B) = N(A)\circ N(B)$, similarly for the normal operator families; moreover, if $\nu = \left|\frac{dx}{x}\right|\nu'$ is the product decomposition of a smooth non-vanishing b-1-density on M near X, with ν'

a 1-density on X (depending on x), then

$$N(A^*)(\sigma) = N(A)^*(\overline{\sigma}), \tag{3.3.36}$$

where the adjoint of A^* is computed relative to ν, and the adjoint of $N(A)$ relative to $\nu'|_X$.

One can study $\widehat{A}(\sigma)$ as a large parameter family of differential operators, or perform a semiclassical rescaling and thus recast the normal operator family as a semiclassical operator; we give details and examples in §3.3.4. For now, we content ourselves with showing how invertibility properties of $N(A)$ on weighted b-Sobolev spaces imply Fredholm properties of A for elliptic operators $A \in \mathrm{Diff}_{\mathrm{b}}^m(M)$; completely analogous arguments will apply in the non-elliptic settings discussed in Chapter 5. Thus, let us assume that $s' < s \in \mathbb{R}$ and the weight $r \in \mathbb{R}$ are such that $N(A)\colon H_{\mathrm{b}}^{s',r}(M_I) \to H_{\mathrm{b}}^{s'-m,r}(M_I)$ is invertible. (Since we consider only elliptic A here, this is only a restriction on the weight; the regularity orders s and $s' < s$ can be chosen arbitrarily. Moreover, as we shall see in §3.3.4, the mapping properties of $N(A)$ on such weighted spaces are determined by $\widehat{N}(A)(\sigma)$ for $\operatorname{Im}\sigma = -r$; combined with (3.3.36), this will imply that $N(A^*)$ is invertible as a map $H_{\mathrm{b}}^{s'',-r}(M_I) \to H_{\mathrm{b}}^{s''-m,-r}(M_I)$ for $s'' \in \mathbb{R}$.) We can then combine the elliptic regularity estimate

$$\|u\|_{H_{\mathrm{b}}^{s,r}(M)} \lesssim \|Au\|_{H_{\mathrm{b}}^{s-m,r}(M)} + \|u\|_{H_{\mathrm{b}}^{s',r}(M)} \tag{3.3.37}$$

with the invertibility of the normal operator, to wit

$$\|v\|_{H_{\mathrm{b}}^{s',r}(M_I)} \lesssim \|N(A)v\|_{H_{\mathrm{b}}^{s'-m,r}(M_I)}, \tag{3.3.38}$$

in the following way: Choose a cutoff $\chi \in \mathcal{C}^\infty(M)$, identically 1 near ∂M and vanishing outside a collar neighborhood of ∂M, then

$$\|u\|_{H_{\mathrm{b}}^{s',r}(M)} \lesssim \|\chi u\|_{H_{\mathrm{b}}^{s',r}(M_I)} + \|(1-\chi)u\|_{H_{\mathrm{b}}^{s',r}(M)}$$

under an identification of the collar neighborhood of ∂M with a neighborhood of $\{0\} \times X$ in M_I as above. Since $(1-\chi)u$ is supported away from the boundary, we have $\|(1-\chi)u\|_{H_{\mathrm{b}}^{s',r}(M)} \lesssim \|(1-\chi)u\|_{H_{\mathrm{b}}^{s',r'}(M)}$ for any $r' \in \mathbb{R}$. To deal with χu, we use (3.3.38) and obtain

$$\|\chi u\|_{H_{\mathrm{b}}^{s',r}(M_I)} \lesssim \|N(A)\chi u\|_{H_{\mathrm{b}}^{s'-m,r}(M_I)}$$

$$
\begin{aligned}
&\leq \|\chi A u\|_{H_{\mathrm{b}}^{s'-m,r}(M_I)} + \|\chi(N(A)-A)u\|_{H_{\mathrm{b}}^{s'-m,r}(M_I)} + \|[N(A),\chi]u\|_{H_{\mathrm{b}}^{s'-m,r}(M_I)} \\
&\lesssim \|Au\|_{H_{\mathrm{b}}^{s'-m,r}(M)} + \|\chi u\|_{H_{\mathrm{b}}^{s',r-1}(M)} + \|u\|_{H_{\mathrm{b}}^{s'-1,r'}(M)},
\end{aligned} \tag{3.3.39}
$$

where we used in the last step that $N(A) - A \in x\mathrm{Diff}_{\mathrm{b}}^m(M_I)$, and that $[N(A), \chi] \in \mathrm{Diff}_{\mathrm{b}}^{m-1}(M_I)$ is supported away from ∂M_I. Plugging this into (3.3.37) gives

$$
\|u\|_{H_{\mathrm{b}}^{s,r}(M)} \lesssim \|Au\|_{H_{\mathrm{b}}^{s-m,r}(M)} + \|u\|_{H_{\mathrm{b}}^{s',r-1}(M)}, \tag{3.3.40}
$$

and now the inclusion $H_{\mathrm{b}}^{s',r-1}(M) \hookrightarrow H_{\mathrm{b}}^{s,r}(M)$ is compact. Together with an analogous estimate for the adjoint of A, we hence obtain *Fredholm estimates* for A analogous to those in the boundaryless setting in §3.2.1, implying that $\ker A \subset H_{\mathrm{b}}^{s,r}(M)$ is finite-dimensional, and $\operatorname{ran} A \subset H_{\mathrm{b}}^{s-m,r}(M)$ has finite codimension and is equal to the annihilator of $\ker A^* \subset H_{\mathrm{b}}^{-s+m,-r}(M)$.

3.3.4 Mellin transform and semiclassical analysis

We proceed to describe mapping properties of normal operators, i.e. general dilation-invariant operators on the model space $M_I = [0,\infty)_x \times X$. We first study function spaces: Recall from [114, §3] that the Mellin transform

$$
(\mathcal{M}u)(\sigma, y) = \int_0^\infty x^{-i\sigma} u(x,y)\,\frac{dx}{x} \tag{3.3.41}
$$

gives an isometric isomorphism of $L^2(M_I, \frac{dx}{x}\,d\mu)$ with $L^2(\mathbb{R}_\sigma, L^2(X, d\mu))$ by Plancherel's theorem, where $d\mu$ is a 1-density on X, and its inverse is

$$
(\mathcal{M}^{-1}v)(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} x^{i\sigma} v(\sigma,y)\,d\sigma;
$$

more generally, for $u \in x^r L_{\mathrm{b}}^2(M_I)$, $\mathcal{M}u(\cdot - ir)$ is well-defined as an element of $L^2(\mathbb{R}_\sigma, L^2(X))$, and the inverse Mellin transform becomes

$$
(\mathcal{M}_r^{-1}v)(x,y) = \frac{1}{2\pi}\int_{\operatorname{Im}\sigma=-r} x^{i\sigma} v(\sigma,y)\,d\sigma. \tag{3.3.42}
$$

If $u \in x^r L_{\mathrm{b}}^2(M_I)$ has compact support in x, the Mellin transform $\mathcal{M}u$ extends to a holomorphic function in $\operatorname{Im}\sigma > -r$ with values in $L^2(X)$, satisfying $\sup_{-r<\alpha<C} \|\mathcal{M}u(\cdot +$

$i\alpha)\|_{L^2(\mathbb{R}_\sigma,L^2(X))} < \infty$ for all $C < \infty$, and $\mathcal{M}u(\cdot + i\alpha)$ extends continuously to $\alpha = -r$ in the topology of $L^2(\mathbb{R}_\sigma, L^2(X))$. Since $\mathcal{M}$ intertwines differentiation xD_x and multiplication by σ, we obtain similar statements for weighted b-Sobolev spaces, namely

$$H_{\mathrm{b}}^{k,r}(M_I) \ni u \mapsto \mathcal{M}u(\cdot - ir) \in \bigcap_{j=0}^{k} \langle\sigma\rangle^{-j} L^2(\mathbb{R}_\sigma, H^{k-j}(X)) = \langle\sigma\rangle^{-k} L^2(\mathbb{R}_\sigma, H^k_{\langle\sigma\rangle^{-1}}(X)) \tag{3.3.43}$$

is an isometric isomorphism, where $H_h^k(X)$ is the semiclassical Sobolev space on X, i.e. $u \in H_h^k(X)$ if up to k *semiclassical derivatives* of u are in $L^2(X)$, where semiclassical derivatives are ordinary derivatives weighted by h, i.e. $h\partial_y$ with y coordinates on X. Thus $H_h^k(X) = H^k(X)$ as a space, and the norms are equivalent for h bounded away from 0, but not as $h \to 0$. By interpolation and duality, the isomorphism (3.3.43) extends to all $k \in \mathbb{R}$. For $u \in H_{\mathrm{b}}^{s,r}(M_I)$ compactly supported in x, the Mellin transform $\mathcal{M}u$ is holomorphic in $\operatorname{Im}\sigma > -r$ as before, with spaces changed according to (3.3.43).

Now, a dilation-invariant operator $A \in \mathrm{Diff}_{\mathrm{b,I}}^m(M_I)$ acts on $u \in H_{\mathrm{b}}^{s,r}(M_I)$ by

$$\mathcal{M}(Au)(\sigma) = \widehat{A}(\sigma)(\mathcal{M}u)(\sigma), \quad \operatorname{Im}\sigma = -r.$$

Hence, in view of (3.3.43), mapping properties of $\widehat{A}(\sigma)$ on semiclassical Sobolev spaces imply mapping properties of A on weighted b-Sobolev spaces. It is therefore convenient to rescale $\widehat{A}(\sigma)$ (which is a differential operator on X of order m large parameter σ) to a semiclassical operator: We introduce $h = |\sigma|^{-1}$ and $z = h\sigma$, so $z \in \mathbb{C}$ has $|z| = 1$, and consider

$$A_{h,z} := h^m \widehat{A}(h^{-1}z) \in \mathrm{Diff}_\hbar^m(X), \tag{3.3.44}$$

where $\mathrm{Diff}_\hbar^m(X)$ is the algebra of semiclassical differential operators, generated by semiclassical vector fields hV, $V \in \mathcal{V}(X)$. Concretely then, suppose $A_{h,z}$ satisfies the estimate

$$\|v\|_{H_h^s(X)} \lesssim h^{-\ell} \|A_{h,z}v\|_{H_h^{s-(m-\ell)}}, \quad \operatorname{Im} z = -hr, \tag{3.3.45}$$

where ℓ is the 'loss of derivatives' of $A_{h,z}$ relative to elliptic operators, for which $\ell = 0$,[8]

[8]Indeed, if A is elliptic as a b-operator, then $A_{h,z}$ is elliptic as a semiclassical operator for $\operatorname{Im} z = \mathcal{O}(h)$, and elliptic regularity gives $\|v\|_{H_h^s(X)} \lesssim \|A_{h,z}v\|_{H_h^{s-m}(X)} + h^N \|v\|_{H_h^{-N}(X)}$ for any N. The error term can be absorbed in the left hand side for $h > 0$ sufficiently small, giving (3.3.45) for small h, corresponding to the invertibility of $\widehat{A}(\sigma)$ for $|\operatorname{Im}\sigma| < C$ (for arbitrary, fixed $C > 0$) and $|\operatorname{Re}\sigma| \gg 1$. The full statement (3.3.45) then requires the invertibility of $\widehat{A}(\sigma)$ for σ in the remaining, compact part of the line $\operatorname{Im}\sigma = -r$.

then $\|u\|_{H_{\mathrm{b}}^{s,r}(M_I)} \lesssim \|Au\|_{H_{\mathrm{b}}^{s-(m-\ell)}(M_I)}$.

One can consider semiclassical pseudodifferential operators on closed manifolds, analogously to ordinary ps.d.o.s on closed manifolds, except that due to the additional semiclassical parameter h, symbolic expansions include additional powers of h; see [129] for details. In particular, the principal symbol of a semiclassical operator in $\Psi_\hbar^m(X)$ is a well-defined function in $S^m(T^*X)$, with no homogeneity properties in compact subsets of T^*X; notice that we are not taking a quotient here. Then, ellipticity, wave front sets etc. can be defined for semiclassical operators and distributions, see [115, §4.4].

The central feature of the relation between dilation-invariant b-operators $A \in \Psi_{\mathrm{b}}^m(M_I)$ on $M_I = [0,\infty)_x \times X$ and their semiclassical rescalings $A_{h,z}$ then is the following, recalling that $T^*X \subset {}^{\mathrm{b}}T_X^*M_I$ in a natural fashion here, since we are given a boundary defining function x, see §2.1.3: The set ${}^{\mathrm{b}}S_X^*M_I \setminus {}^{\mathrm{b}}S^*X$ can be identified with $T_+ \cup T_- \equiv (\frac{dx}{x} + T^*X) \cup (-\frac{dx}{x} + T^*X)$ (which in turn can be identified with two copies of T^*X) since ${}^{\mathrm{b}}T_X^*M_I = \operatorname{span}\{\frac{dx}{x}\} \oplus T^*X$, and each $\mathbb{R}_+$-orbit outside ${}^{\mathrm{b}}T^*X$ intersects $T_+ \cup T_-$ in a unique point; see also (2.1.5). This provides the connection between the b- and the semiclassical perspectives, i.e. between b-analysis on ${}^{\mathrm{b}}T_X^*M_I$ and semiclassical analysis on T^*X: In fact, if $a = \sigma_{\mathrm{b},m}(A)$, which is a homogeneous degree m function, then $\widehat{\rho}^m a$, where $\widehat{\rho}$ can be taken as the reciprocal of the absolute value of the symbol of xD_x in this region (which is well-defined, independent of choices), i.e. $\widehat{\rho} = |\sigma|^{-1} = h$ in the above notation, gives a function on $\{\pm 1\} \times T^*X$. We thus see that $\widehat{\rho}^m a$ (which can be identified with $a|_{T_\pm}$, thus with a function on T^*X) is the semiclassical principal symbol $a_{h,z}$ (depending in addition on the parameter $z = h\sigma$) of the rescaled operator family $A_{h,z}$; that is, $h^m a(h^{-1}z, \eta) = a_{h,z}(\eta)$.

In particular, if a conic set is disjoint from T^*X in ${}^{\mathrm{b}}T^*M_I$, then its image under the semiclassical identification lies in a compact subset of T^*X. Thus, for $A \in \Psi_{\mathrm{b}}^m(M_I)$ dilation invariant, the large parameter principal symbol and wave front set of the Mellin conjugate $\widehat{A}(\sigma)$ of A are exactly those of A under the above identification of $\sigma\frac{dx}{x} + \varpi \in {}^{\mathrm{b}}T^*M_I$, $\varpi \in {}^{\mathrm{b}}T^*X$, with $(\sigma, \varpi) \in \mathbb{R} \times T^*X$, and then the analogous statement also holds for $\widehat{A}$ considered as an element of $\Psi_\hbar(X)$ under the semiclassical identification.

Radially compactifying $T_\pm \cong T^*X$ as in §2.1.3, we moreover see that the rescaled Hamilton vector field of a, restricted to $\overline{T}_\pm$ as a b-vector field on $\overline{T}_\pm$, is equal to the Hamilton vector field of $\widehat{\rho}^m a$ on $\overline{T}^*X$. Notice however that this loses information about H_a as a b-vector field in the direction transverse to ${}^{\mathrm{b}}\overline{T}_X^*M_I$. This is easily recovered: Formula

(2.1.3) yields (dropping the dependence on variables in T^*X)

$$x^{-1}H_a x = \partial_\sigma a(\sigma) = \partial_\sigma\big(|\sigma|^m a(|\sigma|^{-1}\sigma)\big) = mza_{h,z} + \partial_z a_{h,z}$$

at $\sigma = \pm 1$, so $x^{-1}H_a x = \partial_z a_{h,z}$ at the semiclassical characteristic set $a_{h,z}^{-1}(0)$ (which for $z = \pm 1$ equals the intersection of the b-characteristic set $a^{-1}(0)$ with $T_\pm$). The knowledge of $x^{-1}H_a x$ is therefore equivalent to that of $\partial_z a_{h,z}$. Let us rephrase this from a different perspective for $A \in \Psi_{\mathrm{b}}^m(M_I)$ dilation-invariant with real principal symbol a, and let us in fact assume $A - A^* \in \Psi_{\mathrm{b}}^{m-2}(M_I)$: Introducing $A_r := x^{-r}Ax^r$, $r \in \mathbb{R}$, we have $\widehat{A_r}(\sigma) = \widehat{A}(\sigma - ir)$. In view of

$$A_r - A_r^* = A - A^* + x^{-r}[A, x^r] + [A^*, x^r]x^{-r},$$

we have

$$\sigma_{\mathrm{b},m-1}\Big(\frac{1}{2i}(A_r - A_r^*)\Big) = -rx^{-1}H_a x.$$

Hence, for $\sigma = \sigma_0 - ir \in \mathbb{C}$, $\sigma_0, r \in \mathbb{R}$, we compute the large parameter principal symbol (with r fixed and σ_0 the large parameter)

$$\sigma_{m-1}\Big(\frac{1}{2i}(\widehat{A}(\sigma) - \widehat{A}(\sigma)^*)\Big) = \sigma_{m-1}\Big(\frac{1}{2i}(\widehat{A_r}(\sigma_0) - \widehat{A_r}(\sigma_0)^*)\Big) = (\operatorname{Im}\sigma)x^{-1}H_a x, \tag{3.3.46}$$

or in the semiclassical rescaling, allowing $z = h\sigma$ (with $h = |\sigma|^{-1}$) to be complex,

$$\sigma_{\hbar,m-1}\Big(\frac{1}{2ih}(A_{h,z} - A_{h,z}^*)\Big) = (\operatorname{Im} z)x^{-1}H_a x.$$

In particular, at generalized b-radial sets as discussed in §§2.2.1 and 3.3.1, the numerology (3.3.8) in the normal-to-boundary direction translates directly to the numerology in the semiclassical setting, see [114, Propositions 2.3 and 2.4] for the general statement and [114, §4.4], specifically the displayed equation after [114, Equation (4.12)], for the concrete computation on static de Sitter space.[9]

At normally hyperbolic trapping in the b-sense, the sign of the Hamilton derivative of x (analogous to (3.3.15)), i.e. the normal-to-boundary dynamics at the trapped set, corresponds to a sign condition on the derivative of the semiclassical principal symbol in

[9]Note that computing the semiclassical principal symbol *at fiber infinity*, which is the location of the radial set, is equivalent to computing the principal symbol in (3.3.46) in the standard sense, i.e. without large parameter.

the parameter z, see [124, Theorem 1] and [114, §6.4]. In the notation of §3.3.2, notice that if we merely assume the normal hyperbolicity within ${}^{\mathrm{b}}S^*_X M$, in the sense of the above identification with semiclassical analysis on T^*X, as in [124, §1.2], then [124, Lemma 4.1], as corrected in [123], gives defining functions $\phi^0_\pm$ of $\Gamma_\pm$ *within* ${}^{\mathrm{b}}S^*_X M$; taking an arbitrary extension in case of ϕ_+, and an extension which is a defining function of Γ_- in case of ϕ_-, we thus infer that the b-setting considered in §3.3.2 is indeed in one-to-one correspondence with the semiclassical setting of [44, 124], including the precise numerology.

Anticipating Chapter 4, where we prove global energy estimates for wave equations on b-spacetimes, we mention a further conceptual analogy between b- and semiclassical analysis: These estimates rely on the timelike nature of the boundary defining function, in which case they imply the global forward solvability of linear wave equations in $H^{0,r}_{\mathrm{b}}$ for sufficiently negative r, i.e. in growing spaces. In the semiclassical setting, having a timelike boundary defining function implies the absence of resonances (poles of the inverse normal operator family acting on suitable function spaces) in $\operatorname{Im}\sigma > -r$ for sufficiently negative r, which in the dilation-invariant setting also guarantees forward solvability in $H^{0,r}_{\mathrm{b}}$ for $r \ll 0$ by the Paley-Wiener theorem, see [114, Lemma 3.1].

Lastly, we point out the role of *high energy estimates* for the normal operator family of an operator $A \in \mathrm{Diff}^m_{\mathrm{b}}(M)$: By this, we mean estimates of the form (3.3.45) which are however only valid for $h < h_0$ sufficiently small; thus, undoing the semiclassical rescaling, these are operator norm estimates for the inverse normal operator family $\widehat{A}(\sigma)^{-1}$ which are polynomial in $|\sigma|$ as $|\operatorname{Re}\sigma| \to \infty$, with $\operatorname{Im}\sigma$ bounded. The exponent of the bound (as well as the precise function spaces on which one inverts $\widehat{A}(\sigma)$) determines how many derivatives one loses (relative to the order of the operator) when inverting $N(A)$ (on matching function spaces). The polynomial nature of the bound allows to deduce asymptotic expansions to solutions of $Au = f$ of the form (1.0.3) via a contour shifting argument, see [114, Lemma 3.1], which we will use frequently in this thesis, see for example Theorem 5.2.3.

Let us summarize our general discussion of b-analysis: The analysis of a b-(pseudo)differential operator P has two ingredients, corresponding to the two orders, smoothness and decay, of the Sobolev spaces:

(1) *b-regularity analysis.* This provides the framework for understanding PDEs at high b-frequencies, which in non-degenerate situations involves the b-principal symbol and perhaps a subprincipal term (as in elliptic regularity and the propagation of singularities in various context, see §§3.3.1 and 3.3.2). This is sufficient in order to control

solutions u in $H_{\mathrm{b}}^{s,r}$ modulo $H_{\mathrm{b}}^{s',r}$, $s' < s$, i.e. modulo a space with higher regularity, but no additional decay. Since for the inclusion $H_{\mathrm{b}}^{s,r} \to H_{\mathrm{b}}^{s',r'}$ to be compact one needs both $s > s'$ and $r > r'$, this does not control the problem modulo relatively compact errors.

(2) *Normal operator analysis.* This provides a framework for understanding the decay properties of solutions of the PDE. The b-regularity analysis, in non-degenerate situations, gives control of this family $\widehat{P}(\sigma)$ in a Fredholm sense, uniformly as $|\sigma| \to \infty$ with $\operatorname{Im}\sigma$ bounded [114]. However, in any such strip, $\widehat{P}(\sigma)^{-1}$ will still typically have finitely many poles σ_j; these poles, called *resonances*, dictate the asymptotic behavior of solutions of the PDE.

In order for P to be a Fredholm operator, one needs to work in spaces such as $H_{\mathrm{b}}^{s,r}$, where r is such that there are no resonances σ_j with $\operatorname{Im}\sigma_j = -r$, see [82, §6]. We will see this general perspective in action at many places in the sequel, in particular in Chapters 5 and 9.

3.3.5 Conormal coefficients

Motivated by the discussion of smoothness and conormality in §2.1.2, we show how to extend the b-pseudodifferential calculus with smooth coefficients to allow for weighted conormal coefficients. In the local model $\overline{\mathbb{R}^n_+}$ with coordinates x, y as usual, we thus want to allow coefficients $a(x,y) \in \mathcal{C}^\infty(\mathbb{R}^n_+)$ satisfying

$$|(x\partial_x)^j \partial_y^\alpha a(x,y)| \leq C_{j\alpha} x^\gamma$$

for all α, j, where $\gamma \in \mathbb{R}$ is a fixed weight; such functions a are called *conormal of order* γ. If $\gamma > 0$, then a extends by continuity to $x = 0$ and equals 0 there; in fact, a is in the Hölder class $C^{k,\alpha}(\overline{\mathbb{R}^n_+})$ for $k + \alpha < \gamma$. Of course, away from $x = 0$, conormal functions are smooth. In terms of $t = -\log x$, the conormality condition becomes

$$|\partial_t^j \partial_y^\alpha a(t,y)| \leq C_{j\alpha} e^{-\gamma t}$$

for all α, j, which again shows the direct connection of conormal coefficient operators on $\overline{\mathbb{R}^n_+}$ with the uniform calculus on Euclidean space.

Definition 3.3.17. For $m, \gamma \in \mathbb{R}$, denote by $S^{m,\gamma}_{\mathrm{bc}}(\overline{\mathbb{R}^n_+})$ ('b-conormal symbols') the space of all functions $a(x,y;\xi,\eta) \in \mathbb{R}^n_+ \times \mathbb{R}^n$ which are holomorphic in ξ and satisfy

$$|(x\partial_x)^k \partial_y^\alpha \partial_\xi^\ell \partial_\eta^\beta a(x,y;\xi,\eta)| \leq C_{\alpha\beta k\ell N} x^\gamma (1 + |\operatorname{Re}\xi| + |\eta|)^{m-\ell-|\beta|}, \quad |\operatorname{Im}\xi| \leq N, \tag{3.3.47}$$

for all $\alpha, \beta, k, \ell, N$. Define $\Psi^{m,\gamma}_{\mathrm{lbc}}(\overline{\mathbb{R}^n_+})$, the space of *local conormal b-operators*, as the space of (left) quantizations of such symbols.

As in the considerations following Definition 3.3.1, we *only consider symbols and operators with compact support in* x, and we shall restrict attention to weights $\gamma \geq 0$ for convenience. Then $\Psi^{m,\gamma}_{\mathrm{lbc}}(\overline{\mathbb{R}^n_+})$ can be viewed as a subspace of $\Psi^m(\mathbb{R}^n)$ after the change of coordinates from (x,y) to (t,y). In order to understand compositions of operators in $\Psi^{m,\gamma}_{\mathrm{lbc}}(\overline{\mathbb{R}^n_+})$, one also needs to consider two-sided symbols a subject to the condition

$$\begin{aligned} &|(x\partial_x)^k \partial_y^\alpha (x'\partial_{x'})^{k'} \partial_{y'}^{\alpha'} \partial_\xi^\ell \partial_\eta^\beta a(x,y,x',y';\xi,\eta)| \\ &\qquad \leq C_{\alpha\beta k\ell k'\ell' N} x^{\gamma_1} (x')^{\gamma_2} (1 + |\operatorname{Re}\xi| + |\eta|)^{m-\ell-|\beta|}, \quad |\operatorname{Im}\xi| \leq N, \end{aligned}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$ are weights. As before, one can extend the proof of Proposition 3.1.2 to show that a quantization of such a two-sided symbols can be written in a unique way as the left quantization of a b-conormal symbol of order m with weight $\gamma = \gamma_1 + \gamma_2$; this then shows that for $A \in \Psi^{m,\gamma}_{\mathrm{lbc}}(\overline{\mathbb{R}^n_+})$ and $B \in \Psi^{m',\gamma'}_{\mathrm{lbc}}(\overline{\mathbb{R}^n_+})$, one has $A \circ B \in \Psi^{m+m',\gamma+\gamma'}_{\mathrm{lbc}}(\overline{\mathbb{R}^n_+})$. Since $\Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}^n_+}) \subset \Psi^{m,0}_{\mathrm{lbc}}(\overline{\mathbb{R}^n_+})$, we automatically understand compositions of conormal b-operators with standard b-operators as well: Namely, such compositions simply produce local conormal b-operators.

The analytic continuation of b-conormal symbols in ξ ensures, as before, that elements of $\Psi^{m,\gamma}_{\mathrm{lbc}}(\overline{\mathbb{R}^n_+})$ act on weighted b-Sobolev spaces; concretely, since elements of $\Psi^m(\mathbb{R}^n)$ act on standard Sobolev spaces on $\mathbb{R}^n$, which are the push-forwards of unweighted b-Sobolev spaces on $\overline{\mathbb{R}^n_+}$, see (3.3.1), we have

$$\Psi^{m,\gamma}_{\mathrm{lbc}}(\overline{\mathbb{R}^n_+}) \ni A \colon H_{\mathrm{b}}^{s,r}(\overline{\mathbb{R}^n_+}) \to H_{\mathrm{b}}^{s-m,r+\gamma}(\overline{\mathbb{R}^n_+}).$$

We can transport the local conormal b-algebra $\bigcup_{m\in\mathbb{R},\gamma\geq 0} \Psi^{m,\gamma}_{\mathrm{lbc}}(\overline{\mathbb{R}^n_+})$ to manifolds in a way that is analogous to Definition 3.3.4; we thus obtain spaces $\Psi^{m,\gamma}_{\mathrm{bc}}(M)$ of b-ps.d.o.s on M of order m with weight γ. (Since we have ensured that operators in the space $\Psi^{m,\gamma}_{\mathrm{bc}}(M)$ act on weighted b-Sobolev spaces, we can in fact allow γ to be arbitrary, but the case of $\gamma \leq 0$

is of no interest for us here.) The only non-trivial adjustment concerns point (3.3) there: Namely, condition (3.3.5) for the Schwartz kernel localized off-diagonally near $\partial M\times\partial M$ now becomes

$$|\partial_t^k\partial_{t'}^\ell\partial_y^\alpha\partial_{y'}^\beta K_{UV}(t,y,t',y')|\leq C_{\alpha\beta k\ell M}e^{-M|s|},$$

i.e. the exponential weight in the t-derivative is removed. We can now define:

Definition 3.3.18. We define the algebra $\Psi_{\mathrm{b,bc}}(M)=\bigcup_{m\in\mathbb{R},\gamma>0}\Psi_{\mathrm{b,bc}}^{m,\gamma}(M)$ of *b-conormal operators* by requiring that elements $A\in\Psi_{\mathrm{b,bc}}^{m,\gamma}(M)$ have the form $A=B+C$ with $B\in\Psi_{\mathrm{b}}^m(M)$ and $C\in\Psi_{\mathrm{bc}}^{m,\gamma}(M)$.

Since we are assuming that the weight γ is positive, the principal symbol of $A\in\Psi_{\mathrm{b,bc}}^{m,\gamma}(M)$ is well-defined, and is a sum of a smooth and a conormal (of order γ) symbol, in particular it is smooth in ${}^{\mathrm{b}}T^*_{M^\circ}M$, continuous up to ${}^{\mathrm{b}}T^*_{\partial M}M$, and its restriction to ${}^{\mathrm{b}}T^*_{\partial M}M$ is smooth. Hence, the notions of operator wave front set, ellipticity etc. are defined for b-conormal operators as well, and one has a symbolic calculus, elliptic regularity, propagation of singularities etc.

Furthermore, if we write $A=B+C$ as in the above definition, we can define the normal operator of A to be the normal operator of the smooth part, so $N(A):=N(B)$, and we then have $A-N(A)\in\Psi_{\mathrm{b,bc}}^{m,\min(1,\gamma)}(M)$ near ∂M: If $\gamma<1$, the normal operator $N(A)$ only equals A up to an error of size x^γ. Therefore, for an elliptic b-conormal operator A whose normal operator is invertible on some weighted Sobolev space, the argument presented in §3.3.3 establishing the Fredholm property of A goes through once we replace the weight $r-1$ by $r-\gamma$ in (3.3.39) and (3.3.40).

While many of our later results about wave equations on spacetimes equipped with smooth Lorentzian b-metrics will also apply to b-metrics which are merely smooth plus conormal (of positive order), we will usually not make this explicit, except in §5.2.2. Note that we are not hiding any serious technicalities here, since it will be apparent from the arguments involving $\Psi_{\mathrm{b}}(M)$ that they go through with operators in $\Psi_{\mathrm{b,bc}}(M)$ as well after almost entirely notational modifications.

Chapter 4

Energy estimates for b-operators

On manifolds M with boundary, we prove local and global energy estimates for b-operators which equal the wave operator to leading order. Thus, we consider $\mathcal{P} \in \mathrm{Diff}_{\mathrm{b}}^2(M)$ and assume that there is a Lorentzian b-metric g on M such that

$$\mathcal{P} - \Box_g \in \mathrm{Diff}_{\mathrm{b}}^1(M).$$

The reason for the interest in energy estimates is the fact that initial value problems for wave equations do not mesh well with microlocal analysis on the spacetime level, i.e. working directly on M rather than foliating M by spacelike hypersurfaces; for instance, Cauchy hypersurfaces force a lower bound on the level of Sobolev regularity one can work with. More to the point, the failure of microlocal analysis to reproduce physical space energy estimates (that is on M rather than on ${}^{\mathrm{b}}S^*M$), which are closely related to the finite speed of propagation for the wave equation, is due to the fact that the wave operator is indistinguishable microlocally, i.e. at high frequencies, from any other real principal type (possibly pseudodifferential) operator, whereas finite speed of propagation holds only for hyperbolic differential operators. (Thus, this 'failure' really is a *feature*, as it allows for a very general, unified treatment of many central aspects of large classes of operators, e.g. even in the context of wave equations transcending physical space methods when studying Feynman propagators [51].) We remark that complex absorption may be used to obtain 'forward' solutions in the sense of their singularity structure, but in general, this does not produce forward solutions in the sense of supports; see §5.2.1 for more on this.

Thus, we use energy estimates near spacelike hypersurfaces in order to obtain regularity for solutions of the Cauchy problem for $\mathcal{P}$ nearby, from which point on microlocal analysis, propagation of singularities etc. is directly applicable. Furthermore, very crude global energy estimates imply the existence of solutions in low regularity, exponentially growing spaces, which are then improved in regularity using microlocal high frequency analysis and in decay using normal operator analysis, as outlined at the end of §3.3.3.

Concretely, we work on domains with corners $\Omega \subset M$ bounded by a artificial spacelike surfaces as well as by 'future infinity' ∂M, motivated by (2.2.5) and Proposition 2.2.1; we recall the setup below. In §4.1, we prove energy estimates near the artificial surfaces, uniform up to ∂M, in fact providing a rather general setting for b-energy estimates on domains with corners, while §4.2 provides global estimates on Ω on sufficiently weighted spaces.

4.1 Local energy estimates

Assume that $U \subset M$ is open, and we have two functions $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{C}^\infty(M)$, both of which, restricted to U, are timelike (in particular have non-zero differential) near their respective 0-level sets H_1 and H_2, and

$$\Omega = \mathfrak{t}_1^{-1}([0,\infty)) \cap \mathfrak{t}_2^{-1}([0,\infty)) \subset U.$$

Notice that the timelike assumption forces $d\mathfrak{t}_j$ to not lie in $N^*X = N^*\partial M$ (for its image in the b-cosphere bundle would be zero), and thus if the H_j intersect X, they do so transversally. We assume that the H_j intersect only away from X, and that they do so transversally, i.e. the differentials of $\mathfrak{t}_j$ are independent at the intersection. Then Ω is a manifold with corners with boundary hypersurfaces H_1, H_2 and X (all intersected with Ω). We however keep thinking of Ω as a domain in M.

On a manifold with corners, such as Ω, one can consider supported and extendible distributions; see [64, Appendix B.2] for the smooth boundary setting, with only simple changes needed for the corners setting, which is discussed e.g. in [110, §3] and indicated in §3.3. Here we consider Ω as a domain in M, and thus its boundary face $X \cap \Omega$ is regarded as having a different character from the $H_j \cap \Omega$, i.e. the support/extendibility considerations do not arise at X – all distributions are regarded as acting on a subspace of $\mathcal{C}^\infty$ functions on Ω vanishing at X to infinite order, i.e. they are automatically extendible distributions at X. On the other hand, at H_j we consider both extendible distributions, acting on $\mathcal{C}^\infty$ functions vanishing to infinite order at H_j, and supported distributions, which act on all $\mathcal{C}^\infty$ functions (as far as conditions at H_j are concerned). For example, the space of supported distributions at H_1 extendible at H_2 (and at X, as we always tacitly assume) is the dual space of the subspace of $\mathcal{C}^\infty(\Omega)$ consisting of functions vanishing to infinite order at H_2 and X (but not necessarily at H_1). An equivalent way of characterizing this space of distributions is that they are restrictions of elements of the dual of $\dot{\mathcal{C}}^\infty(M)$ (consisting of $\mathcal{C}^\infty$ functions on M vanishing to infinite order at X) with support in $\mathfrak{t}_1 \geq 0$ to $\mathcal{C}^\infty$ functions on Ω which vanish to infinite order at X and H_2, i.e. in the terminology of [64], restriction to $\Omega \setminus (H_2 \cup X)$. The main interest is in spaces induced by the Sobolev spaces $H_{\mathrm{b}}^{s,r}(M)$. For instance,

$$H_{\mathrm{b}}^{s,r}(M)^{\bullet,-},$$

with the first superscript on the right denoting whether supported ($\bullet$) or extendible ($-$) distributions are discussed at H_1, and the second the analogous property at H_2, consists of restrictions of elements of $H_{\mathrm{b}}^{s,r}(M)$ with support in $\mathfrak{t}_1 \geq 0$ to $\Omega \setminus (H_2 \cup X)$. (Notice that the Sobolev norm is of completely different nature at X than at the H_j, namely the derivatives are based on complete, rather than incomplete, vector fields: $\mathcal{V}_{\mathrm{b}}(M)$ is being restricted to Ω, so one obtains vector fields tangent to X but not to the H_j.) Then elements of $\mathcal{C}^\infty(\Omega)$ with the analogous vanishing conditions, so in the example vanishing to infinite order at H_1 and X, are dense in $H_{\mathrm{b}}^{s,r}(M)^{\bullet,-}$; further the dual of $H_{\mathrm{b}}^{s,r}(M)^{\bullet,-}$ is $H_{\mathrm{b}}^{-s,-r}(M)^{-,\bullet}$ with respect to the L^2 (sesquilinear) pairing.

First we work locally. For this purpose it is convenient to introduce another function $\widetilde{\mathfrak{t}}_j$, not necessarily timelike, and consider

$$\Omega_{[t_0,t_1]} = \mathfrak{t}_j^{-1}([t_0,\infty)) \cap \widetilde{\mathfrak{t}}_j^{-1}((-\infty,t_1]), \quad \Omega_{(t_0,t_1)} = \mathfrak{t}_j^{-1}((t_0,\infty)) \cap \widetilde{\mathfrak{t}}_j^{-1}((-\infty,t_1)),$$

and similarly on half-open, half-closed intervals. Thus, $\Omega_{[t_0,t_1]}$ becomes smaller as t_0 becomes larger or t_1 becomes smaller.

We then consider energy estimates on $\Omega_{[T_0,T_1]}$. In order to set up the following arguments, choose

$$T_- < T_-' < T_0, \quad T_1 < T_+' < T_+,$$

and assume that $\Omega_{[T_-,T_+]}$ is compact, $\Omega_{[T_0,T_1]}$ is non-empty, and $\mathfrak{t}_j$ is timelike on $\Omega_{[T_-,T_+]}$. The energy estimates propagate estimates in the direction of either increasing or decreasing

$\mathfrak{t}_j$. With the extendible/supported character of distributions at $\widetilde{\mathfrak{t}}_j = T_+$ being irrelevant for this matter in the case being considered and thus dropped from the notation, so $(-)$ refers to extendibility at $\mathfrak{t}_j = T_0$, consider

$$\mathcal{P} \colon H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_+]})^- \to H_{\mathrm{b}}^{s-2,r}(\Omega_{[T_0,T_+]})^-, \quad s, r \in \mathbb{R}.$$

The energy estimate, with backward propagation in $\mathfrak{t}_j$, from $\widetilde{\mathfrak{t}}_j^{-1}([T'_+, T_+])$, in this setting takes the form:

Lemma 4.1.1. *Let $r \in \mathbb{R}$. There is $C > 0$ such that for $u \in H_{\mathrm{b}}^{2,r}(\Omega_{[T_0,T_+]})^-$,*

$$\|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_1]})^-} \leq C\Big(\|\mathcal{P}u\|_{H_{\mathrm{b}}^{0,r}(\Omega_{[T_0,T_+]})^-} + \|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_+]} \cap \widetilde{\mathfrak{t}}_j^{-1}([T'_+,T_+]))^-}\Big). \tag{4.1.1}$$

This also holds with $\mathcal{P}$ replaced by $\mathcal{P}^*$, acting on the same spaces.

Remark 4.1.2. The lemma is also valid if one has several boundary hypersurfaces, i.e. if one replaces $\mathfrak{t}_j^{-1}([t_0, \infty))$ by $\mathfrak{t}_j^{-1}([t_{j,0}, \infty)) \cap \mathfrak{t}_k^{-1}([t_{k,0}, \infty))$ in the definition of $\Omega_{[t_0,t_1]}$, and/or $\widetilde{\mathfrak{t}}_j^{-1}((-\infty, t_1])$ by $\widetilde{\mathfrak{t}}_j^{-1}((-\infty, t_{j,1}]) \cap \widetilde{\mathfrak{t}}_k^{-1}((-\infty, t_{k,1}])$, i.e. regarding $\mathfrak{t}_j$ and/or $\widetilde{\mathfrak{t}}_j$ as vector valued, and propagating backwards in $\mathfrak{t}_{j_0}$ for some fixed j_0, under the additional hypothesis that $\mathfrak{t}_{j_0}$ is timelike in $\Omega_{[t_0,t_1]}$, and all $\mathfrak{t}_j$, $j \neq j_0$, are timelike near their respective zero sets, with the same timelike character at $\mathfrak{t}_{j_0}$. (One can also have more than two such functions.) To see this, replace $\chi(\mathfrak{t}_j)$ by $\chi_{j_0}(\mathfrak{t}_{j_0})\chi_k(\mathfrak{t}_k)$, and analogously with $\widetilde{\chi}$ in the definition of V in (4.1.2), where χ_k is the characteristic function of $[t_{k,0}, \infty)$, while letting $W = G({}^{\mathrm{b}}d\mathfrak{t}_{j_0}, \cdot)$. Then $\chi'\widetilde{\chi}\tau^\alpha A^\sharp$ is replaced by $\chi'_j\chi_k\widetilde{\chi}_j\widetilde{\chi}_k\tau^\alpha A^\sharp + \chi_j\chi'_k\widetilde{\chi}_j\widetilde{\chi}_k\tau^\alpha A^\sharp$, etc., and our additional hypothesis guarantees that the matrix $A^\sharp$ is indeed positive definite: The contribution from differentiating χ_{j_0} is positive definite by the timelike nature of $d\mathfrak{t}_{j_0}$, while the contribution from differentiating χ_j, $j \neq j_0$, giving δ-distributions at the hypersurfaces $\mathfrak{t}_j^{-1}(t_{j,0})$, is positive definite by the second part of the above additional hypothesis and can therefore be dropped as in the proof of Lemma 4.1.1 below. Thus χ'_{j_0} can still be used to dominate χ_{j_0}; and the terms in which $\widetilde{\chi}_j$ is differentiated have support where $\widetilde{\mathfrak{t}}_j$ is in $(T'_{+,j}, T_{+,j})$, so the control region on the right hand side of (4.1.1) is the union of these sets.

In our application this situation arises as we need domains of the form Ω_j, $j = 1, 2, 3$, described in Figure 4.1.

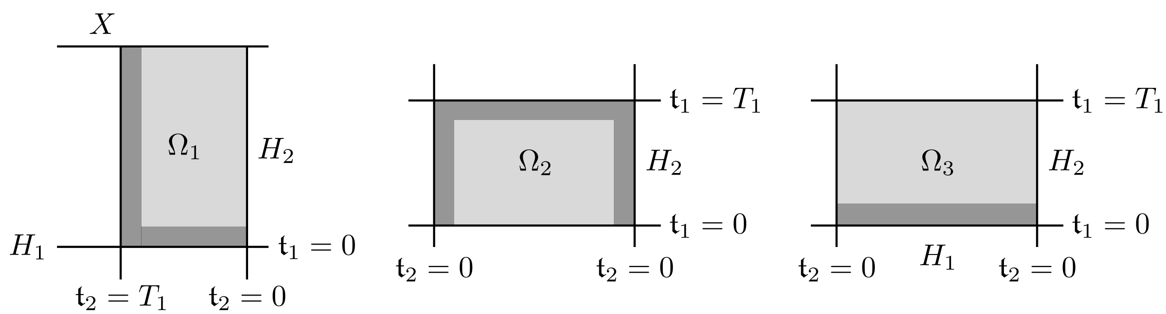

Figure 4.1: Domains on which we will use the energy estimate (4.1.1), with a priori control on the dark shaded regions. *Left:* $\Omega_1 = \mathfrak{t}_1^{-1}([0,\infty))\cap\mathfrak{t}_2^{-1}([T_0,T_1])$, written as $\Omega_1 = \mathfrak{t}_2^{-1}([0,\infty))\cap\big((-\mathfrak{t}_1)^{-1}((-\infty,0])\cap\mathfrak{t}_2^{-1}((-\infty,T_1])\big)$, propagating estimates backwards in $\mathfrak{t}_2$. *Middle:* $\Omega_2 = \mathfrak{t}_1^{-1}([0,T_1])\cap\mathfrak{t}_2^{-1}([0,\infty))$, written as $\Omega_2 = \mathfrak{t}_1^{-1}([0,\infty))\cap\big(\mathfrak{t}_1^{-1}([-\infty,T_1))\cap(-\mathfrak{t}_2)^{-1}((-\infty,0])\big)$, propagating estimates backwards in $\mathfrak{t}_2$. *Right:* The same domain, now written as $\Omega_3 = \big((-\mathfrak{t}_1)^{-1}([-T_1,\infty))\cap\mathfrak{t}_2([0,\infty))\big)\cap(-\mathfrak{t}_1)^{-1}((-\infty,0])$, propagating estimates backwards in $-\mathfrak{t}_1$. Equivalently, $\Omega_3 = \mathfrak{t}_1^{-1}([0,\infty))\cap\big(\mathfrak{t}_1^{-1}((-\infty,T_1])\cap(-\mathfrak{t}_2)^{-1}((-\infty,0])\big)$, and we propagate in the forward direction in $\mathfrak{t}_1$ using the estimate (4.1.4).

Proof of Lemma 4.1.1. To see (4.1.1), one proceeds as in [114, §3.3] and considers

$$V = -i\chi(\mathfrak{t}_j)\widetilde{\chi}(\widetilde{\mathfrak{t}}_j)\tau^\alpha W \tag{4.1.2}$$

with $W = G(d\mathfrak{t}_j,\cdot)$ a timelike vector field and with $\chi,\widetilde{\chi}\in\mathcal{C}^\infty(\mathbb{R})$, both non-negative, to be specified. Then choosing a *Riemannian* b-metric $\widetilde{g}$ with respect to which we compute adjoints,

$$-i(V^*\Box_g - \Box_g^* V) = {}^{\mathrm{b}}d_{\widetilde{g}}^* C^\flat\, {}^{\mathrm{b}}d,$$

with the subscript on the right making the dependence of the adjoint on the metric $\widetilde{g}$ explicit, and with

$$C^\flat = \chi'\widetilde{\chi}\tau^\alpha A^\sharp + \chi\widetilde{\chi}'\tau^\alpha\widetilde{A}^\sharp + \chi\widetilde{\chi}\tau^\alpha R^\flat$$

where $A^\sharp, \widetilde{A}^\sharp$ and $R^\flat$ are bundle endomorphisms of ${}^{\mathbb{C}\mathrm{b}}T^*M$, and $A^\sharp$ is positive definite because W and (the b-vector field dual to) $d\mathfrak{t}_j$ have the same timelike orientation. Proceeding further, replacing $\Box_g$ by $\mathcal{P}$, one has

$$\begin{aligned} -i(V^*\mathcal{P} - \mathcal{P}^*V) &= {}^{\mathrm{b}}d_{\widetilde{g}}^* C^\sharp\, {}^{\mathrm{b}}d + (\widetilde{E}_1)_{\widetilde{g}}^*\tau^\alpha\chi\widetilde{\chi}\,{}^{\mathrm{b}}d + {}^{\mathrm{b}}d_{\widetilde{g}}^*\tau^\alpha\chi\widetilde{\chi}\widetilde{E}_2, \\ C^\sharp &= \chi'\widetilde{\chi}\tau^\alpha A^\sharp + \chi\widetilde{\chi}'\tau^\alpha\widetilde{A}^\sharp + \chi\widetilde{\chi}\tau^\alpha\widetilde{R}^\sharp, \end{aligned} \tag{4.1.3}$$

with $\widetilde{E}_j$ bundle maps from the trivial bundle over M to ${}^{\mathbb{C}\mathrm{b}}T^*M$, $A^\sharp$, $\widetilde{A}^\sharp$ as before, and $\widetilde{R}^\sharp$ a bundle endomorphism of ${}^{\mathbb{C}\mathrm{b}}T^*M$, as follows by expanding

$$-i(V^*(\mathcal{P}-\Box_g)-(\mathcal{P}-\Box_g)^*V)$$

using that $\mathcal{P}-\Box_g\in\mathrm{Diff}^1_{\mathrm{b}}(M)$. We regard the second term on the right hand side of the definition of $C^\sharp$ in (4.1.3) as the one requiring a priori control by $\|u\|_{H^{1,r}_{\mathrm{b}}(\Omega_{[T_0,T_+]}\cap\widetilde{\mathfrak{t}}_j^{-1}([T'_+,T_+]))^-}$; we achieve this by making $\widetilde{\chi}$ supported in $(-\infty,T_+)$, identically 1 near $(-\infty,T'_+]$, so $d\widetilde{\chi}$ is supported in (T'_+,T_+). Now making $\chi'\geq 0$ large relative to χ on $\mathrm{supp}(\chi\widetilde{\chi})$, as in[10] [114, Equation (3.27)], allows one to dominate all terms without derivatives of χ. In order to obtain a non-degenerate estimate up to $\mathfrak{t}_j=T_0$, one cuts off χ at $\mathfrak{t}_j=T_0$ using the Heaviside function, so χ' gives a (positive!) δ-distribution there. Applying (4.1.3) to v, pairing with v and integrating by parts, the δ-distributions have the same sign as $\chi' A^\sharp$ and can thus be dropped. Put differently, without the sharp cutoff, one again computes the same pairing, but this time on the domain $\Omega_{[T_0,T_+]}$, thus picking up boundary terms with the correct sign in the integration by parts, so these terms can be dropped. This proves the energy estimate (4.1.1) when one takes $\alpha=-2r$. □

We refer to the proof of the analogous Proposition 8.6.1, in which we discuss the case of non-smooth metrics, for further details.

Propagating in the forward direction in $\mathfrak{t}_j$, with a priori bounds in $\mathfrak{t}_j^{-1}([T_-,T'_-])$, where now $(-)$ denotes the character of the space at T_1, so $(-)$ refers to extendibility at $\mathfrak{t}_j=T_1$, we have

$$\|u\|_{H^{1,r}_{\mathrm{b}}(\Omega_{[T_0,T_1]})^-}\leq C\Big(\|\mathcal{P}u\|_{H^{0,r}_{\mathrm{b}}(\Omega_{[T_-,T_1]})^-}+\|u\|_{H^{1,r}_{\mathrm{b}}(\Omega_{[T_-,T_1]}\cap\mathfrak{t}_j^{-1}([T_-,T'_-]))^-}\Big) \tag{4.1.4}$$

In particular, for u supported in $\mathfrak{t}_j\geq T_0$, the last estimate becomes, with the first superscript on the right denoting whether supported $(\bullet)$ or extendible $(-)$ distributions are discussed at $\mathfrak{t}=T_0$, the second superscript the same at $\mathfrak{t}=T_1$,

$$\|u\|_{H^{1,r}_{\mathrm{b}}(\Omega_{[T_0,T_1]})^{\bullet,-}}\leq C\|\mathcal{P}u\|_{H^{0,r}_{\mathrm{b}}(\Omega_{[T_0,T_1]})^{\bullet,-}},$$

[10] In [114, Equation (3.27)] the sign of χ' is opposite, as the estimate is propagated in the opposite direction.

when

$$\mathcal{P}\colon H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-} \to H_{\mathrm{b}}^{s-2,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$$

and $u \in H_{\mathrm{b}}^{2,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$. To summarize, we state both this and (4.1.1) in terms of these supported spaces:

Corollary 4.1.3. *Let $r, \widetilde{r} \in \mathbb{R}$. For $u \in H_{\mathrm{b}}^{2,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, one has*

$$\|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \leq C\|\mathcal{P}u\|_{H_{\mathrm{b}}^{0,r}(\Omega_{[T_0,T_1]})^{\bullet,-}}, \tag{4.1.5}$$

while for $v \in H_{\mathrm{b}}^{2,\widetilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}$, the estimate

$$\|v\|_{H_{\mathrm{b}}^{1,\widetilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}} \leq C\|\mathcal{P}^*v\|_{H_{\mathrm{b}}^{0,\widetilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}} \tag{4.1.6}$$

holds.

A duality argument, combined with propagation of singularities, thus gives:

Lemma 4.1.4. *Let $s \geq 0$, $r \in \mathbb{R}$. Then there is $C > 0$ with the following property: If $f \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, then there exists $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ such that $\mathcal{P}u = f$ and*

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \leq C\|f\|_{H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}}.$$

Remark 4.1.5. As in Remark 4.1.2, the lemma remains valid in more generality, provided that the $\mathfrak{t}_j$ have linearly independent differentials on their joint zero set, and similarly for the $\widetilde{\mathfrak{t}}_j$. The place where this linear independence is used (the energy estimate above does not need this) is for the continuous Sobolev extension map, valid on manifolds with corners, see [110, §3].

Proof. We work on the slightly bigger region $\Omega_{[T'_-,T'_+]}$, applying the energy estimates with T_0 replaced by T'_-, T_1 replaced by T'_+. First, by the supported property at $\mathfrak{t}_j = T_0$, one can regard f as an element of $H_{\mathrm{b}}^{s-1,r}(\Omega_{[T'_-,T_1]})^{\bullet,-}$ with support in $\Omega_{[T_0,T_1]}$. Let

$$\widetilde{f} \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-} \subset H_{\mathrm{b}}^{-1,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}$$

be an extension of f, so $\widetilde{f}$ is supported in $\Omega_{[T_0,T'_+]}$, and restricts to f; by the definition of spaces of extendible distributions as quotients of spaces of distributions on a larger space, see [64, Appendix B.2], we can assume

$$\|\widetilde{f}\|_{H_{\mathrm{b}}^{s-1,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}} \leq 2\|f\|_{H_{\mathrm{b}}^{s-1,r}(\Omega_{[T'_-,T_1]})^{\bullet,-}}. \tag{4.1.7}$$

By (4.1.1) applied with $\mathcal{P}$ replaced by $\mathcal{P}^*$, $\widetilde{r}=-r$,

$$\|\phi\|_{H_{\mathrm{b}}^{1,\widetilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}} \leq C\|\mathcal{P}^*\phi\|_{H_{\mathrm{b}}^{0,\widetilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}},$$

for $\phi \in H_{\mathrm{b}}^{2,\widetilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}$. Correspondingly, by the Hahn-Banach theorem, there exists

$$\widetilde{u} \in (H_{\mathrm{b}}^{0,\widetilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet})^* = H_{\mathrm{b}}^{0,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}$$

such that

$$\langle \mathcal{P}\widetilde{u},\phi\rangle = \langle \widetilde{u},\mathcal{P}^*\phi\rangle = \langle \widetilde{f},\phi\rangle, \qquad \phi \in H_{\mathrm{b}}^{2,\widetilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet},$$

and

$$\|\widetilde{u}\|_{H_{\mathrm{b}}^{0,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}} \leq C\|\widetilde{f}\|_{H_{\mathrm{b}}^{-1,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}}. \tag{4.1.8}$$

One can regard $\widetilde{u}$ as an element of $H_{\mathrm{b}}^{0,r}(\Omega_{[T_-,T'_+]})^{\bullet,-}$ with support in $\Omega_{[T'_-,T'_+]}$, with $\widetilde{f}$ similarly extended; then $\langle \mathcal{P}\widetilde{u},\phi\rangle = \langle \widetilde{f},\phi\rangle$ for $\phi \in \dot{\mathcal{C}}_{\mathrm{c}}^\infty(\Omega_{(T_-,T'_+)})$ (here the dot over $\mathcal{C}^\infty$ refers to infinite order vanishing at $X=\partial M$!), so $\mathcal{P}\widetilde{u}=\widetilde{f}$ in distributions. Since $\widetilde{u}$ vanishes on $\Omega_{(T_-,T'_-)}$, and

$$\widetilde{f} \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_-,T'_+]})^{\bullet,-},$$

propagation of singularities applied on $\Omega_{(T_-,T'_+)}$ (which has only the boundary $\partial M = X$) gives that $\widetilde{u} \in H_{\mathrm{b,loc}}^{s,r}(\Omega_{(T_-,T'_+)})$ (i.e. here we are ignoring the two boundaries, $\mathfrak{t}_j = T_-, T'_+$, not making a uniform statement there, but we are not ignoring $\partial M = X$). In addition, for $\chi,\widetilde{\chi} \in \mathcal{C}_{\mathrm{c}}^\infty(\Omega_{(T_-,T'_+)})$, $\widetilde{\chi}\equiv 1$ on $\operatorname{supp}\chi$, we have the estimate

$$\|\chi\widetilde{u}\|_{H_{\mathrm{b}}^{s,r}(\Omega_{[T_-,T'_+]})} \leq C\Big(\|\widetilde{\chi}\mathcal{P}\widetilde{u}\|_{H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_-,T'_+]})} + \|\widetilde{\chi}\widetilde{u}\|_{H_{\mathrm{b}}^{0,r}(\Omega_{[T_-,T'_+]})}\Big). \tag{4.1.9}$$

In view of the support property of $\widetilde{u}$, this gives that restricting to $\Omega_{(T_-,T_1]}$, we obtain an element of $H_{\mathrm{b}}^{s,r}(\Omega_{(T_-,T_1]})^-$, with support in $\Omega_{[T_0,T_1]}$, i.e. an element of $H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$. The desired estimate follows from (4.1.8), controlling the second term of the right hand side of (4.1.9), and (4.1.7) as well as using $\mathcal{P}\widetilde{u}=\widetilde{f}$. □

At this point, u given by Lemma 4.1.4 is not necessarily unique. However:

Lemma 4.1.6. *Let $s, r \in \mathbb{R}$. If $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ is such that $\mathcal{P}u = 0$, then $u = 0$.*

Proof. Propagation of singularities, as in the proof of Lemma 4.1.4, regarding u as a distribution on (T_-, T_1) with support in $[T_0, T_1)$ gives that $u \in H_{\mathrm{b,loc}}^{\infty,r}(\Omega_{(T_-,T_1)})$. Taking $T_0 < T_1' < T_1$, letting $u' = u|_{[T_0,T_1']}$, (4.1.5) shows that $u' = 0$. Since T_1' is arbitrary, this shows $u = 0$. □

Corollary 4.1.7. *Let $s \geq 0$, $r \in \mathbb{R}$. Then there is $C > 0$ with the following property: If $f \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, then there exists a unique $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ such that $\mathcal{P}u = f$. Further, this unique u satisfies*

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \leq C\|f\|_{H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}}.$$

Proof. Existence is Lemma 4.1.4, uniqueness is linearity plus Lemma 4.1.6, which together with the estimate in Lemma 4.1.4 prove the corollary. □

Corollary 4.1.8. *Let $s \geq 0$, $r, \widetilde{r} \in \mathbb{R}$.*

For $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ with $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$,

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \leq C\|\mathcal{P}u\|_{H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}}, \tag{4.1.10}$$

while for $v \in H_{\mathrm{b}}^{s,\widetilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}$ with $\mathcal{P}^*v \in H_{\mathrm{b}}^{s-1,\widetilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}$,

$$\|v\|_{H_{\mathrm{b}}^{s,\widetilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}} \leq C\|\mathcal{P}^*v\|_{H_{\mathrm{b}}^{s-1,\widetilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}}. \tag{4.1.11}$$

Remark 4.1.9. Again, this estimate remains valid for vector valued $\mathfrak{t}_j$ and $\widetilde{\mathfrak{t}}_j$, see Remarks 4.1.2 and 4.1.5, under the linear independence condition of the latter.

Proof of Corollary 4.1.8. It suffices to consider (4.1.10). Let $f = \mathcal{P}u \in H_{\mathrm{b}}^{-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, and let $u' \in H_{\mathrm{b}}^{0,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ be given by Corollary 4.1.7. In view of the uniqueness statement of Corollary 4.1.7, $u = u'$. Then the estimate of Corollary 4.1.7 proves the claim. □

This yields the following propagation of singularities type result:

Proposition 4.1.10. *Let $s \geq 0$, $r \in \mathbb{R}$.*

If $u \in H_{\mathrm{b}}^{-\infty,-\infty}(\Omega_{[T_0,T_1]})^{\bullet,-}$ with $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, then $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$.

If instead $u \in H_{\mathrm{b}}^{-\infty,-\infty}(\Omega_{[T_0,T_1]})^{-,-}$ *with* $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{-,-}$ *and for some* $\widetilde{T}_0 > T_0$, $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]} \setminus \Omega_{(\widetilde{T}_0,T_1]})^{-,-}$, *then* $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{-,-}$.

Remark 4.1.11. One can 'mix and match' the two parts of the proposition in the setting of Remark 4.1.2, with say a supportedness condition at $\widetilde{\mathfrak{t}}_j$, and only an extendibility assumption at $\widetilde{\mathfrak{t}}_k$, but with $H_{\mathrm{b}}^{s,r}$ membership assumption on u in $\Omega_{[T_0,T_1]} \setminus \widetilde{\mathfrak{t}}_k^{-1}((-\infty,\widetilde{T}_1))$, $\widetilde{T}_1 < T_1$, with a completely analogous argument. For instance, in the setting of Figure 4.1, one gets the regularity under supportedness assumptions at H_1, just extendibility at $\mathfrak{t}_2 = T_1$, but a priori regularity for $\mathfrak{t}_2 \in (\widetilde{T}_1, T_1)$.

Proof of Proposition 4.1.10. Applying the existence part of Corollary 4.1.7, we let $u' \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ be the unique solution in $H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ of $\mathcal{P}u' = f$ where $f = \mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$. Then $u, u' \in H_{\mathrm{b}}^{-\infty,-\infty}(\Omega_{[T_0,T_1]})^{\bullet,-}$ and $\mathcal{P}(u-u') = 0$. Applying Lemma 4.1.6, we conclude that $u = u'$, which completes the proof of the first part.

For the second part, let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ be supported in (T_0,∞), identically 1 near $[\widetilde{T}_0,\infty)$, and consider $u' = (\chi \circ \mathfrak{t}_j)u \in H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, with the support property arising from the vanishing of χ near T_0. Then $\mathcal{P}u' = [\mathcal{P}, (\chi \circ \mathfrak{t}_j)]u + (\chi \circ \mathfrak{t}_j)\mathcal{P}u$. Now the first term on the right hand side is in $H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ as on the support of $d\chi$, which is in $\Omega_{[T_0,T_1]} \setminus \Omega_{(\widetilde{T}_0,T_1]}$, u is in $H_{\mathrm{b}}^{s,r}$, and the commutator is first order, while the second term is in the desired space since $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{-,-}$, and as for u itself, the cutoff improves the support property. Thus, the first part of the lemma is applicable, giving that $\chi u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$. Since $(1-\chi)u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{-,-}$ by the a priori assumption, the conclusion follows. □

4.2 Global energy estimates

We keep the notation from the previous section. We now consider, for $s \geq 0$,

$$\mathcal{P} \colon H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-2,r}(\Omega)^{\bullet,-}$$

and

$$\mathcal{P}^* \colon H_{\mathrm{b}}^{s,r}(\Omega)^{-,\bullet} \to H_{\mathrm{b}}^{s-2,r}(\Omega)^{-,\bullet}.$$

We now prove global energy estimates; *we assume that* Ω *is such that there is a boundary defining function* τ *of* M *with* $\frac{d\tau}{\tau}$ *timelike on* Ω, of the same timelike character as $\mathfrak{t}_2$, opposite to $\mathfrak{t}_1$. (As explained in [114, §7], in this case there is $C > 0$ such that for $\operatorname{Im}\sigma > C$,

$\widehat{P}(\sigma)$ is necessarily invertible.) This in particular holds for the spacetimes described in Definition 2.5.1.

The energy estimate is:

Lemma 4.2.1. *There exists $r_0 < 0$ such that for $r \leq r_0$, $-\widetilde{r} \leq r_0$, there is $C > 0$ such that for $u \in H_{\mathrm{b}}^{2,r}(\Omega)^{\bullet,-}$, $v \in H_{\mathrm{b}}^{2,\widetilde{r}}(\Omega)^{-,\bullet}$, one has*

$$\begin{aligned} \|u\|_{H_{\mathrm{b}}^{1,r}(\Omega)^{\bullet,-}} &\leq C\|\mathcal{P}u\|_{H_{\mathrm{b}}^{0,r}(\Omega)^{\bullet,-}}, \\ \|v\|_{H_{\mathrm{b}}^{1,\widetilde{r}}(\Omega)^{-,\bullet}} &\leq C\|\mathcal{P}^*v\|_{H_{\mathrm{b}}^{0,\widetilde{r}}(\Omega)^{-,\bullet}}. \end{aligned} \tag{4.2.1}$$

Proof. We run the argument of Lemma 4.1.1 globally on Ω using a timelike vector field (e.g. starting with $W = G(\frac{d\tau}{\tau}, .))$ that has, as a multiplier, a sufficiently large positive power $\alpha = -2r$ of τ, i.e. replacing (4.1.2) by

$$V = -i\tau^\alpha W.$$

Then the term with τ^α differentiated (which in (4.1.3) is included in the $\widetilde{R}^\sharp$ term), and thus possessing a factor of α, is used to dominate the other, 'error,' terms in (4.1.3), completing the proof of the lemma as in Lemma 4.1.1. □

This can be used as in Lemma 4.1.4 to show the solvability of $\mathcal{P}u = f \in H_{\mathrm{b}}^{-1,r}(\Omega)^{\bullet,-}$ by $u \in H_{\mathrm{b}}^{0,r}(\Omega)^{\bullet,-}$.

In order to improve regularity, one needs further assumptions on the null-bicharacteristic flow in Ω. We thus assume from now on that Ω *is a non-trapping spacetime, possibly with normally hyperbolic trapping*, according to Definition 2.5.1: One then uses the propagation of singularities, which includes the use of the radial point estimate in Proposition 3.3.8, noting that we are automatically above the weight-regularity-threshold for large negative weights, and in the trapping case in addition Theorem 3.3.14, specifically the estimate (3.3.23). We then obtain the following analogues of Corollaries 4.1.7 and 4.1.8.

Corollary 4.2.2. *There is $r_0 < 0$ such that for $r \leq r_0$ and for $s \geq 0$ there is $C > 0$ with the following property: If $f \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}$, then there exists a unique $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$ such that $\mathcal{P}u = f$. Further, this unique u satisfies*

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}} \leq C\|f\|_{H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}}.$$

Corollary 4.2.3. *There is* $r_0 < 0$ *such that if* $r < r_0$, $-\tilde{r} < r_0$ *and* $s \geq 0$ *then there is* $C > 0$ *such that the following holds: For* $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$ *with* $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}$, *one has*

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}} \leq C\|\mathcal{P}u\|_{H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}}, \tag{4.2.2}$$

while for $v \in H_{\mathrm{b}}^{s,\tilde{r}}(\Omega)^{-,\bullet}$ *with* $\mathcal{P}^*v \in H_{\mathrm{b}}^{s-1,\tilde{r}}(\Omega)^{-,\bullet}$, *one has*

$$\|v\|_{H_{\mathrm{b}}^{s,\tilde{r}}(\Omega)^{-,\bullet}} \leq C\|\mathcal{P}^*v\|_{H_{\mathrm{b}}^{s-1,\tilde{r}}(\Omega)^{-,\bullet}}. \tag{4.2.3}$$

We restate Corollary 4.2.2 as an invertibility statement.

Theorem 4.2.4. *There is* $r_0 < 0$ *with the following property: Suppose* $s \geq 0$, $r \leq r_0$, *and let*

$$\mathcal{X}^{s,r} = \{u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-} \colon \mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}\}, \quad \mathcal{Y}^{s,r} = H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-},$$

where $\mathcal{P}$ *is a priori a map* $\mathcal{P}\colon H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-2,r}(\Omega)^{\bullet,-}$. *Then*

$$\mathcal{P}\colon \mathcal{X}^{s,r} \to \mathcal{Y}^{s-1,r}$$

is a continuous, invertible map, with continuous inverse.

Remark 4.2.5. Note that $\mathcal{Y}^{s,r}$, $\mathcal{X}^{s,r}$ are complete, in the case of $\mathcal{X}^{s,r}$ with the natural norm being $\|u\|^2_{\mathcal{X}^{s,r}} = \|u\|^2_{H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}} + \|\mathcal{P}u\|^2_{H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}}$, as follows by the continuity of $\mathcal{P}$ as a map $H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-2,r}(\Omega)^{\bullet,-}$ and the completeness of the b-Sobolev spaces $H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$.

This will be the starting point for the global analysis of linear and nonlinear waves, starting with the discussion of generalized static models in §5.2.1.

Chapter 5

Semilinear wave equations

5.1 Introduction

The purpose of this chapter is to show how the microlocal analysis of Chapter 3 and the energy estimates of Chapter 4 can be combined to give the global solvability of linear and semilinear wave equations on many classes of spacetimes, in particular those covered by Definition 2.5.1. The study of much more general quasilinear equations requires technically more sophisticated tools and is deferred to Chapters 8 and 9.

Concretely, we consider semilinear wave equations in contexts such as asymptotically de Sitter and Kerr-de Sitter spaces, as well as asymptotically Minkowski spaces. The word 'asymptotically' here does *not* mean that the asymptotic behavior has to be that of exact de Sitter, etc., spaces, or even a perturbation of these at infinity; much more general infinities, that nonetheless possess a similar structure as far as the underlying analysis is concerned, are allowed, such as spacetimes covered by Definition 2.5.1. Recent progress [114] and [8] allows one to set up the analysis of the associated linear problem *globally* as a Fredholm problem, concretely using the framework of Melrose's b-pseudodifferential operators, discussed in §3.3, on appropriate compactifications M of these spaces. This allows one to use the contraction mapping theorem to solve semilinear equations with small data in many cases since typically the semilinear terms can be considered perturbations of the linear problem. That is, as opposed to solving an evolution equation on time intervals of some length, possibly controlling this length in some manner, and iterating the solution using (almost) conservation laws, we solve the equation globally in one step.

As Fredholm analysis means that one has to control the linear operator L modulo compact errors, which in these settings means modulo terms which are *both* smoother and more decaying, see §3.3.3, the underlying linear analysis involves both arguments based on the principal symbol of the wave operator and on its (b-)normal operator family $\widehat{N}(L)(\sigma)$ on ∂M. At the principal symbol level one encounters real principal type phenomena as well as radial points of the Hamilton flow at the boundary of the compactified underlying space M; these allow for the usual (for wave equations) loss of one (b-)derivative relative to elliptic problems. Physically, in the de Sitter and Kerr-de Sitter type settings, radial points correspond to a red shift effect. (In Kerr-de Sitter spaces there is an additional loss of derivatives dues to trapping.) On the other hand, the b-normal operator family enters via the poles σ_j, also called *resonances*, of the meromorphic inverse $\widehat{N}(L)(\sigma)^{-1}$; these poles determine the decay/growth rates of solutions of the linear problem at ∂M, namely $\operatorname{Im}\sigma_j > 0$ means growing while $\operatorname{Im}\sigma_j < 0$ means decaying solutions. Translated into the nonlinear setting, taking powers of solutions of the linear equation means that growing linear solutions become even more growing, thus the nonlinear problem is uncontrollable, while decaying linear solutions become even more decaying, thus the nonlinear effects become negligible at infinity. Correspondingly, the location of these resonances becomes crucial for nonlinear problems. We note that in addition to providing solvability of semilinear problems, our results can also be used to obtain the *asymptotic expansion* of the solution.

In short, we present a *systematic approach* to the analysis of semilinear wave and Klein-Gordon equations: Given an appropriate structure of the space at infinity and given that the location of the resonances fits well with the nonlinear terms, see the discussion below, one can solve (suitable) semilinear equations. Thus, the main purpose of this chapter is to present the first step towards a general theory for the global study of linear and nonlinear wave-type equations; the semilinear applications we give are meant to show how far we can get in the nonlinear regime using relatively simple means, and lend themselves to meaningful comparisons with existing literature, see the discussion below. The approach readily generalizes to the analysis of *quasilinear* equations, provided one understands the necessary (b-)analysis in the setting of non-smooth metrics; see Chapters 8 and 9 for such a generalization in both the de-Sitter and Kerr-de Sitter type settings.

We now describe our setting in more detail. We consider semilinear wave equations of the form

$$(\Box_g - \lambda)u = f + q(u, du)$$

on a manifold M where q is (typically; more general functions are also considered) a polynomial vanishing at least quadratically at $(0,0)$, so contains no constant or linear terms, which should be included either in f or in the operator on the left hand side. The derivative du is measured relative to the metric structure (e.g. when constructing polynomials in it). Here g and λ fit in one of the following scenarios, which we state slightly informally, with references to the precise theorems.

(1) A neighborhood of the backward light cone from future infinity in an asymptotically de Sitter space, i.e. of a 'static' asymptotically de Sitter space, or more general non-trapping spacetimes in the sense of Definition 2.5.1. In order to solve the semilinear equation, if $\lambda > 0$, one can allow q to be an arbitrary polynomial with quadratic vanishing at the origin, or indeed a more general function. If $\lambda = 0$ and q depends on du only, the same conclusion holds. Further, in either case, one obtains an expansion of the solution at infinity. See Theorems 5.2.6 and 5.2.17, and Corollary 5.2.9.

(2) Kerr-de Sitter space, including a neighborhood of the event horizon, or more general non-trapping spacetimes with normally hyperbolic trapping in the sense of Definition 2.5.1. In the main part of the section we assume $\lambda > 0$, and allow $q = q(u)$ with quadratic vanishing at the origin. We also obtain an expansion at infinity. See Theorems 5.3.6 and 5.3.10, and Corollary 5.3.9. However, in §5.3.3 we briefly discuss non-linearities involving derivatives which are appropriately behaved at the trapped set.

(3) Global *even* asymptotically de Sitter spaces. These are in some sense the easiest examples as they correspond, via extension across the conformal boundary, to working on a manifold without boundary. Here $\lambda = (n-1)^2/4 + \sigma^2$. If $\operatorname{Im}\sigma < 0$ is sufficiently small and the dimension satisfies $n \geq 6$, quadratic vanishing of q suffices; if $n \geq 4$ then cubic vanishing is sufficient. If q does not involve derivatives, $\operatorname{Im}\sigma \geq 0$ small also works, and if $\operatorname{Im}\sigma > 0, n \geq 5$, or $\operatorname{Im}\sigma = 0$, $n \geq 6$, then quadratic vanishing of q is sufficient. (The equation is unchanged if one replaces σ by $-\sigma$. The process of extending across the boundary, however, breaks this symmetry, and in §5.4 we mostly consider $\operatorname{Im}\sigma \leq 0$.) See Theorems 5.4.10, 5.4.12 and 5.4.15. Using the results from 'static' asymptotically de Sitter spaces, quadratic vanishing of q in fact suffices for all $\lambda > 0$, and indeed $\lambda \geq 0$ if $q = q(du)$, but the decay estimates for solutions are lossy relative to the *decay* of the forcing. See Theorem 5.4.17.

(4) Non-trapping Lorentzian scattering (generalized asymptotically Minkowski) spaces, $\lambda = 0$. If $q = q(du)$, we allow q with quadratic vanishing at 0 if $n \geq 5$; cubic if $n \geq 4$. If $q = q(u)$, we allow q with quadratic vanishing if $n \geq 6$; cubic if $n \geq 4$. Further, for $q = q(du)$ quadratic satisfying a null condition, $n = 4$ also works. See Theorems 5.5.13, 5.5.15 and 5.5.21.

See [111, 114, 117] for relating analysis on 'global' and 'static' problems, and see Chapter 2 for a discussion of the settings (1)–(3). We refer to [8, §3] and to §5.5.1 here for a definition of asymptotically Minkowski spaces, but roughly they are manifolds with boundary M with Lorentzian metrics g on the interior M° conformal to a b-metric $\widehat{g}$ as $g = \tau^{-2}\widehat{g}$, with τ a boundary defining function[11] (so these are Lorentzian *scattering* metrics in the sense of Melrose [83], i.e. symmetric cotensors in the second power of the scattering cotangent bundle, and of signature $(1, n-1)$), with a real $\mathcal{C}^\infty$ function v defined on M with dv, $d\tau$ linearly independent at $S = \{v = 0,\ \tau = 0\}$, and with a specific behavior of the metric at S which reflects that of Minkowski space on its radial compactification near the boundary of the light cone at infinity so that S is the light cone at infinity in this greater generality. Concretely, the specific form is

$$\tau^2 g = \widehat{g} = v\frac{d\tau^2}{\tau^2} - \Big(\frac{d\tau}{\tau} \otimes \alpha + \alpha \otimes \frac{d\tau}{\tau}\Big) - \widetilde{h},$$

where α is a smooth one form on M, equal to $\frac{1}{2}\,dv$ at S, $\widetilde{h}$ is a smooth 2-cotensor on M, which is positive definite on the annihilator of $d\tau$ and dv (which is a codimension 2 space). The difference between the de Sitter-type and Minkowski settings is in part this conformal factor τ^{-2}, which however does not affect the behavior of the null-bicharacteristics so one can consider those of $\widehat{g}$ on ${}^{\mathrm{b}}S^*M$ instead; more importantly, at the spherical conormal bundle ${}^{\mathrm{b}}SN^*S$ of S, the nature of the radial points is source/sink rather than a saddle point (as in the static de Sitter context) of the flow. One also makes a non-trapping assumption in the asymptotically Minkowski setting.

We now indicate the specific ways in which these settings fit into the b-framework, and how the various restrictions described above arise:

(1) Asymptotically 'static' de Sitter. Due to a zero resonance for the linear problem when $\lambda = 0$, which moves to the lower half plane for $\lambda > 0$, in this setting $\lambda > 0$ works in

[11] In §5.5 we switch to ρ as the boundary defining function for consistency with [8].

general; $\lambda = 0$ works if q depends on du but not on u. The relevant function spaces are L^2-based b-Sobolev spaces on the bordification (partial compactification) of the space, or analogous spaces plus a finite expansion. Further, the semilinear terms involving du have coefficients corresponding to the b-structure, i.e. b-objects are used to create functions from the differential forms, or equivalently b-derivatives of u are used.

(2) Kerr-de Sitter space. This is an extension of (1), i.e. the framework is essentially the same, with the difference being that there is now trapping corresponding to the photon sphere. This makes first order terms in the non-linearity non-perturbative, unless they are well-adapted to the trapping. Thus, we assume $\lambda > 0$. The relevant function spaces are as in the asymptotically de Sitter setting.

(3) Global *even* asymptotically de Sitter spaces. In order to get reasonable results, one needs to measure regularity relatively finely, using the module of vector fields tangent to what used to be the conformal boundary in the extension. The relevant function spaces are thus Sobolev spaces with additional (finite) conormal regularity. Further, du has coefficients corresponding to the 0-structure of Mazzeo and Melrose, in the same sense the b-structure was used in (1). The range of λ here is limited by the process of extension across the boundary; for non-linearities involving u only, the restriction amounts to (at least very slowly) decaying solutions for the linear problem (without extension across the conformal boundary).

    Another possibility is to view global de Sitter space as a union of static patches. Here, the b-Sobolev spaces on the static parts translate into 0-Sobolev spaces on the global space, which have weights that are shifted by a dimension-dependent amount relative to the weights of the b-spaces. This approach allows for most of the non-linearities that we can deal with on static parts; however, the resulting decay estimates on u are quite lossy relative to the decay of the forcing term f.

(4) Non-trapping Lorentzian scattering (generalized asymptotically Minkowski) spaces, $\lambda = 0$. Note that if $\lambda > 0$, the type of the equation changes drastically; it naturally fits into Melrose's scattering algebra rather than the b-algebra which can be used for $\lambda = 0$. While the results here are quite robust and there are no issues with trapping, they are more involved as one needs to keep track of regularity relative to the module of vector fields tangent to the light cone at infinity. The relevant function spaces are b-Sobolev spaces with additional b-conormal regularity corresponding to

the aforementioned module. Further, du has coefficients corresponding to Melrose's scattering structure. These spaces, in the special case of Minkowski space, are related to the spaces used in [69], using the infinitesimal generators of the Lorentz group, but while the analysis in [69] takes place in an $L^\infty L^2$ setting, we remain purely in a (weighted) L^2 based setting, as the latter is more amenable to the tools of microlocal analysis.

We reiterate that while the way de Sitter, Minkowski, etc., type spaces fit into it differs somewhat, the underlying linear framework is that of L^2-based b-analysis, on manifolds with boundary, except that in the global view of asymptotically de Sitter spaces one can eliminate the boundary altogether.

In order to underline the generality of the method, we emphasize that, corresponding to cases (1) and (2), in b-settings in which one can work on standard b-Sobolev spaces, the restrictions on the solvability of the semilinear equations are simply given firstly by the presence of resonances for the Mellin transformed normal operator family in $\operatorname{Im}\sigma \geq 0$, which would allow growing solutions to the equation, making the non-linearity non-perturbative, with an exception if $\operatorname{Im}\sigma = 0$, in which case the nonlinear iterative arguments produce growth unless the non-linearity has a special structure; and secondly by the losses at high energy estimates for this Mellin transformed operator and the closely related b-principal symbol estimates when one has trapping: These losses cause the difference in the trapping setting for non-linearities with or without derivatives. In particular, the results are necessarily optimal in the non-trapping setting of (1), as shown even by an ODE, see Remark 5.2.11. In the trapping setting, the treatment of non-linearities with derivatives requires a more powerful approach, see Chapter 9, though when there are no derivatives in the non-linearity, we already have no restrictions on the non-linearity, and to this extent our result is optimal.

On Lorentzian scattering spaces more general function spaces are used, and it is not in principle clear whether the results are optimal, but at least comparison with the work of Klainerman and Christodoulou for perturbations of Minkowski space [19, 69, 70] gives consistent results; see the comments below. On global asymptotically de Sitter spaces, the framework of [114] and [115] is very convenient for the linear analysis, but it is not clear to what extent it gives optimal results in the nonlinear setting. The reason why more precise function spaces become necessary is the following: There are two basic properties of spaces of functions on manifolds with boundaries, namely differentiability and decay. Whether one can have both at the same time for the linear analysis depends on the (Hamiltonian) dynamical nature of radial points: when defining functions of the corresponding boundaries of the compactified cotangent bundle have opposite character (stable vs. unstable), which in particular means that the radial point is a saddle, one can have both at the same time, otherwise not; see Propositions 3.3.8 and 5.5.3 for details. For nonlinear purposes, the most convenient setting, in which we are in (1), is if one can work with spaces of arbitrarily high regularity and fast decay, and corresponds to saddle points of the flow in the above sense. In (4) however, working in higher regularity spaces, which is necessary in order to be able to make sense of the non-linearity, requires using faster growing (or at least less decaying) weights, which is problematic when dealing with non-linearities (e.g. polynomials) since multiplication gives even worse growth properties then. Thus, to make the nonlinear analysis work, the function spaces we use need to have more structure; it is a module regularity that is used to capture some weaker regularity in order to enable work in spaces with acceptable weights.

While all results are stated for the scalar equation, analogous results hold in many cases for operators on vector bundles, such as the d'Alembertian (or Klein-Gordon operator) on differential forms, since the linear arguments work in general for operators with scalar principal symbol whose subprincipal symbol satisfies appropriate estimates at radial sets (which are automatic, for sufficiently high regularity, on de Sitter and Kerr-de Sitter spacetimes), though of course for semilinear applications the presence of resonances in the closed upper half plane has to be checked, see §5.2.2 and Remark 5.3.5. This already suffices to obtain the well-posedness of the semilinear equations on asymptotically de Sitter that we consider in this chapter; for semilinear equations on asymptotically Kerr-de Sitter spaces, one moreover needs suitable high energy estimates in the presence of trapping for operators acting on vector bundles, and while these are not automatic, we prove them for natural vector bundles on Kerr-de Sitter space in Chapter 6. On asymptotically Minkowski spaces, the absence of poles of an asymptotically hyperbolic resolvent has to be checked in addition, see Theorem 5.5.4, and the numerology depends crucially on the delicate balance of weights and regularity, as alluded to above. (On *perturbations* of Minkowski space, this follows from the appropriate behavior of poles of the resolvent of the Laplacian on forms on *exact* hyperbolic space.) We will study resonances for waves on bundles in Chapters 6 and 7, and will point out the ramifications of the results proved there for applications to nonlinear equations, see in particular Remark 7.5.3.

While the basic ingredients of the necessary linear b-analysis were analyzed in [114], the solvability framework was only discussed in the dilation invariant setting, and in general the asymptotic expansion results were slightly lossy in terms of derivatives in the non-dilation-invariant case. We remedy these issues here, providing a full Fredholm framework. The key technical tools are the propagation of b-singularities at b-radial points which are saddle points of the flow in ${}^{\mathrm{b}}S^*M$, see Proposition 3.3.8, as well as the b-normally hyperbolic versions, proved in §3.3.2, of the semiclassical normally hyperbolic trapping estimates [42, 44, 94, 124]; the rest of the Fredholm setup is discussed in §5.2.1 in the non-trapping and §5.3.1 in the normally hyperbolic trapping setting. The analogue of Proposition 3.3.8 for sources/sinks was already proved in [8, §4]; our Lorentzian scattering metric Fredholm discussion, which relies on this, is in §5.5.1.

We emphasize that our analysis would be significantly less cumbersome in terms of technicalities if we were not including Cauchy hypersurfaces and solved a globally well-behaved problem by imposing sufficiently rapid decay at past infinity instead (it is standard to convert a Cauchy problem into a forward solution problem). Cauchy hypersurfaces are only necessary for us if we deal with a problem ill-behaved in the past because complex absorption does not force appropriate forward supports even though it does so at the level of singularities; otherwise we can work with appropriate (weighted) Sobolev spaces. The latter is the case with Lorentzian scattering spaces, which thus provide an ideal example for our setting. It can also be done in the global setting of asymptotically de Sitter spaces, as in setting (3) above, essentially by realizing these as the boundary of the appropriate compactification of a Lorentzian scattering space, see [117]. In the case of Kerr-de Sitter black holes, in the presence of dilation invariance, one has access to a similar luxury: Complex absorption does the job as in [114]; the key aspect is that it needs to be imposed *outside* the static region we consider. For a general Lorentzian b-metric with a normally hyperbolic trapped set, this may not be easy to arrange, and we do work by adding Cauchy hypersurfaces, even at the cost of the resulting (rather artificial in terms of PDE theory) technical complications. We remark that Cauchy hypersurfaces are somewhat ill-behaved for L^2 based estimates, which we use, but match $L^\infty L^2$ estimates quite well, which explains the large role they play in existing hyperbolic theory, such as [69] or [64, Chapter 23.2].

We also explain the role that the energy estimates (as opposed to microlocal energy estimates) play: These mostly enter to deal with the artificially introduced boundaries; if other methods (like complex absorption) were used to truncate the flow, their role reduces to checking that in certain cases, when the microlocal machinery only guarantees Fredholm properties of the underlying linear operators, the potential finite dimensional kernel and cokernel are indeed trivial. Asymptotically Minkowski spaces illustrate this best, as the Hamilton flow is globally well-behaved there; see §5.5.1.

The other key technical tool is the algebra property of b-Sobolev spaces and other spaces with additional conormal regularity. These are stated in the respective sections; the case of the standard b-Sobolev spaces reduces to the algebra property of the standard Sobolev spaces on $\mathbb{R}^n$. Given the algebra properties, the results are proved by applying the contraction mapping theorem to the linear operator.

In summary, the plan of this chapter is the following: In each of the sections below we consider one of these settings, and first describe the Sobolev spaces on which one has invertibility for the linear problems of interest, then analyze the algebra properties of these Sobolev spaces and finally prove the solvability of the semilinear equations by checking that the hypotheses of the contraction mapping theorem are satisfied.

5.1.1 Previous and related work

The degree to which these nonlinear problems have been studied differ, with the Minkowski problem (on perturbations of Minkowski space, as opposed to our more general setting) being the most studied. There semilinear and indeed even quasilinear equations are well understood due to the work of Christodoulou [19] and Klainerman [69, 70], with their book on the global stability of Einstein's equation [20] being one of the main achievements. (We also refer to the work of Lindblad and Rodnianski [74, 75] simplifying some of the arguments, of Bieri and Zipser [10] relaxing some of the decay conditions, of Wang [119] obtaining asymptotic expansions, and of Lindblad [73] for results on a class of quasilinear equations. Hörmander's book [63] provides further references in the general area. There are numerous works on the *linear* problem, and estimates this yields for the nonlinear problems, such as Strichartz estimates; here we refer to the recent work of Metcalfe and Tataru [92] for a parametrix construction in low regularity, and references therein.) Here we obtain results comparable to these (when restricted to the semilinear setting), on a larger class of manifolds, see Remark 5.5.18. For non-linearities which do not involve derivatives, slightly stronger results have been obtained, in a slightly different setting, in [21]; see Remark 5.5.19. On the other hand, there is little work on the asymptotically de Sitter and Kerr-de Sitter settings. The paper by Baskin [7] has roughly comparable generality in terms of the setting, though on *exact* de Sitter space Yagdjian [126, 125] has studied a large class of semilinear equations with no derivatives. Baskin's result is for a semilinear equation with no derivatives and a single exponent, using his parametrix construction [6], namely u^p with[12] $p = 1 + \frac{4}{n-2}$, and for $\lambda > (n-1)^2/4$. In the same setting, $p > 1 + \frac{4}{n-1}$ works for us, and thus Baskin's setting is in particular included. Yagdjian works with the explicit solution operator (derived using special functions) in exact de Sitter space, again with no derivatives in the non-linearity. While there are some exponents that his results cover (for $\lambda > (n-1)^2/4$, all $p > 1$ work for him) that ours do not directly (but indirectly, via the static model, we in fact obtain such results), the range $(\frac{(n-1)^2}{4} - \frac{1}{4}, \frac{(n-1)^2}{4})$ is excluded by him while covered by our work for sufficiently large p. However, we point out that the microlocal, high regularity approach that we take in this chapter (as well as in Chapters 8 and 9) does not apply to low regularity non-linearities covered by the results of Baskin and Yagdjian. In the (asymptotically) Kerr-de Sitter setting, to our knowledge, there has been no similar semilinear work. Fully general stability results for Einstein's equations on de Sitter space specifically are available by the works of Friedrich [50, 49, 48], Anderson [2], Rodnianski and Speck [98], Ringström [97] and Speck [102].

There is more work on the linear problem in de Sitter, de Sitter-Schwarzschild and Kerr-de Sitter spaces: We refer to [114] for more details; some references are Polarski [96], Yagdjian and Galstian [127], Sá Barreto and Zworski [5], Bony and Häfner [13], Vasy [111], Baskin [6], Dafermos and Rodnianski [26], Melrose, Sá Barreto and Vasy [87] and Dyatlov [40, 39, 41]. Also, while it received more attention, the linear problem on Kerr space does not fit directly into our setting; see the introduction of [114] for an explanation and for further references, [27] for more background and additional references, and the recent work of Dafermos, Rodnianski and Shlapentokh-Rothman [31] on scalar wave decay on all subextremal Kerr spacetimes, building on their earlier works [30, 29, 100] and following pioneering work by Kay and Wald [67, 118] in the Schwarzschild setting. Tataru and Tohaneanu [105, 106] proved decay and Price's law for slowly rotating Kerr using local energy decay estimates, and Strichartz estimates were proved by Marzuola, Metcalfe, Tataru and Tohaneanu [78]. There is further work by Donninger, Schlag and Soffer [36] on L^∞ estimates on Schwarzschild black holes, following L^∞ estimates of Dafermos and Rodnianski [28, 25] and of Blue and Soffer [12] on non-rotating charged black holes giving L^6 estimates. There are also nonlinear results on Kerr spacetimes: Tohaneanu [109] and Luk [76] studied

[12] The dimension of the spacetime in Baskin's paper is $n + 1$; we continue using our notation above.

semilinear forward problems on Kerr, and Dafermos, Holzegel and Rodnianski [24] gave a scattering construction of dynamical black holes.

There is also physics literature on the subject, starting with Carter's discovery of Kerr-de Sitter spacetime [17, 16], either using explicit solutions in special cases, or numerical calculations, see in particular [128], and references therein. We also refer to the paper of Dyatlov and Zworski [46] connecting recent mathematical advances with the physics literature.

5.2 Generalized static models

In this section we discuss solving semilinear wave equations on asymptotically de Sitter spaces from the 'static perspective,' i.e. in neighborhoods (in a blown-up space) of the backward light cone from a fixed point at future conformal infinity; see Figure 2.4. The first ingredient is extending the linear theory from that of [114] in various ways, which is the subject of §5.2.1. Following this, we use this extension to solve semilinear equations and to obtain their asymptotic behavior.

5.2.1 The linear Fredholm framework

The goal of this section is to fully extend the results of [114] on linear estimates for wave equations for b-metrics to non-dilation-invariant settings. Namely, while the results of [114] on linear estimates for wave equations for b-metrics are optimally stated when the metrics and thus the corresponding operators are dilation-invariant, i.e. when near $\tau = 0$ the normal operator can be identified with the operator itself, see [114, Lemma 3.1], the estimates for Sobolev derivatives are lossy for general b-metrics in [114, Proposition 3.5], essentially because one should not treat the difference between the normal operator and the actual operator purely as a perturbation. We first strengthen the linear results in [114] in the non-dilation-invariant setting using the analysis of b-radial points which are saddle points of the Hamilton flow, see §3.3.1. This is then used to set up a Fredholm framework for the linear problem. If one is mainly interested in the dilation invariant case, one can use [114, Lemma 3.1] in place of Theorem 5.2.3 below, either adding the boundary corresponding to H_2 below, or still using complex absorption as was done in [114].

Complex absorption

In order to have good Fredholm properties we either need a complete Hamilton flow, or need to 'stop it' in a manner that gives suitable estimates; one may want to do the latter to avoid global assumptions on the flow on the ambient space. The microlocally best behaved version is given by complex absorption, discussed in §3.2.3; it is microlocal, works easily with Sobolev spaces of arbitrary order, and makes the operator elliptic in the absorbing region, giving rise to very convenient analysis. The main downside of complex absorption is that it does not automatically give forward mapping properties for the support of solutions in wave equation-like settings, even though at the level of singularities, it does have the desired forward property. It was used extensively in [114] – in the dilation invariant setting, the bicharacteristics on $M_I = X \times [0,\infty)_\tau$ are controlled (by the invariance) as $\tau \to \infty$ as well as when $\tau \to 0$, and thus one need not use complex absorption there, instead decay as $\tau \to \infty$ (corresponding to growth as $\tau \to 0$ on these dilation invariant spaces) gives the desired forward property; complex absorption was only used to cut off the flow *within* the boundary X. Here we want to localize in τ as well, and while complex absorption can achieve this, it loses the forward *support* character of the problem. However, as it is conceptually much cleaner, we discuss Fredholm properties using it first before turning to adding artificial (spacelike) boundary hypersurfaces instead.

So suppose $\mathcal{P} \in \Psi_{\mathrm{b}}^m(M)$, M a manifold with boundary $X = \partial M$, and let p be the principal symbol of $\mathcal{P}$. Assume that the characteristic set Σ of $\mathcal{P}$ has the form

$$\Sigma = \Sigma_+ \cup \Sigma_-,$$

with each of $\Sigma_\pm$ being a union of connected components, and that $\mathcal{P}$ has a (generalized) radial set $L = L_+ \cup L_-$ with $L_\pm \subset \Sigma_\pm$; we adopt the notation used there, see in particular (3.3.8), (3.3.9) and (3.3.10). Adding complex absorption, we now consider $\mathcal{P} - i\mathcal{Q} \in \Psi_{\mathrm{b}}^m(M)$, $\mathcal{Q} \in \Psi_{\mathrm{b}}^m(M)$, with real principal symbol q, being the complex absorption similarly to [114, §§2.2 and 2.8]; we assume that $\mathrm{WF}_{\mathrm{b}}'(\mathcal{Q}) \cap L = \emptyset$. Here the semiclassical version, discussed in [114] with further references there, is a close parallel to our b-setting; it is equivalent to the b-setting in the special case that $\mathcal{P}$, $\mathcal{Q}$ are dilation-invariant, for then the Mellin transform gives rise exactly to the semiclassical problem, see §3.3.4.

$$\mp q \geq 0 \text{ near } \Sigma_\pm.$$

Under these sign conditions on q, we showed in §3.2.3 (which translates directly to the b-setting) that estimates can be propagated in the backward direction along the Hamilton flow on Σ_+ and in the forward direction for Σ_-, or, phrased as a wave front set statement (the property of being singular propagates in the opposite direction as the property of being regular!), $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$ is invariant in $\Sigma_+ \setminus \mathrm{WF}^{s-m+1,r}(\mathcal{P} - i\mathcal{Q})u$ under the forward Hamilton flow, and in $\Sigma_- \setminus \mathrm{WF}^{s-m+1,r}((\mathcal{P} - i\mathcal{Q})u)$ under the backward flow.

In analogy with Definition 2.5.1, we say that $\mathcal{P}-i\mathcal{Q}$ is *non-trapping* if all bicharacteristics in Σ from any point in $\Sigma \setminus (L_+ \cup L_-)$ flow to $\mathrm{Ell}(q) \cup L_+ \cup L_-$ in both the forward and backward directions (i.e. either enter $\mathrm{Ell}(q)$ in finite time or tend to $L_+ \cup L_-$). Notice that as $\Sigma_\pm$ are closed under the Hamilton flow, bicharacteristics in $\mathcal{L}_\pm \setminus (L_+ \cup L_-)$ necessarily enter the elliptic set of $\mathcal{Q}$ in the forward (in Σ_+), resp. backward (in Σ_-) direction. Indeed, by the non-trapping hypothesis, these bicharacteristics have to reach the elliptic set of $\mathcal{Q}$ as they cannot tend to L_+, resp. L_-: for $\mathcal{L}_+$ and $\mathcal{L}_-$ are unstable, resp. stable manifolds, and these bicharacteristics cannot enter the boundary (which is preserved by the flow), so cannot lie in the stable, resp. unstable, manifolds of $L_+ \cup L_-$, which are within ${}^{\mathrm{b}}S^*_X M$. Similarly, bicharacteristics in $(\Sigma \cap {}^{\mathrm{b}}S^*_X M) \setminus (L_+ \cup L_-)$ necessarily reach the elliptic set of $\mathcal{Q}$ in the backward (in Σ_+), resp. forward (in Σ_-) direction. Then for s, r satisfying

$$s - (m-1)/2 > \beta r$$

one has an estimate

$$\|u\|_{H_{\mathrm{b}}^{s,r}} \leq C\|(\mathcal{P} - i\mathcal{Q})u\|_{H_{\mathrm{b}}^{s-m+1,r}} + C\|u\|_{H_{\mathrm{b}}^{s',r}}, \tag{5.2.1}$$

provided one assumes $s' < s$,

$$s' - (m-1)/2 > \beta r, \quad u \in H_{\mathrm{b}}^{s',r}.$$

Indeed, this is a simple consequence of the fact that $u \in H_{\mathrm{b}}^{s',r}$ and $(\mathcal{P} - i\mathcal{Q})u \in H_{\mathrm{b}}^{s-m+1,r}$ imply $u \in H_{\mathrm{b}}^{s,r}$. This implication in turn holds as on the elliptic set of $\mathcal{Q}$ one has the stronger statement $u \in H_{\mathrm{b}}^{s+1,r}$ under these conditions, and then using real-principal type propagation of regularity in the *backward* direction on Σ_+ and the *forward* direction on Σ_-, one can propagate the microlocal membership of $H_{\mathrm{b}}^{s,r}$ (i.e. the absence of the corresponding wave front set) in the backward, resp. forward, direction on Σ_+, resp. Σ_-. Since bicharacteristics in $\mathcal{L}_\pm \setminus (L_+ \cup L_-)$ necessarily enter the elliptic set of $\mathcal{Q}$ in the forward, resp. backward direction, and thus one has $H_{\mathrm{b}}^{s,r}$ membership along them by what we have shown, Proposition 3.3.8 extends this membership to $L_\pm$, and hence to a neighborhood of these, and by our non-trapping assumption every bicharacteristic enters either this neighborhood of $L_\pm$ or the elliptic set of $\mathcal{Q}$ in finite time in the backward, resp. forward, direction, so by the real principal type propagation of singularities we have the claimed microlocal membership everywhere. This implies (5.2.1) either via the closed graph theorem, or directly if one uses the quantitative versions of elliptic regularity, propagation of singularities etc., see also the discussion at the end of §3.1.3.

Reversing the direction in which one propagates estimates, one also has a similar estimate for the adjoint $\mathcal{P}^* + i\mathcal{Q}^*$, except now one needs to have

$$s - (m-1)/2 < \beta r$$

in order to propagate through the saddle points in the opposite direction, i.e. from within ${}^{\mathrm{b}}S^*_X M$ to $\mathcal{L}_\pm$. Then for $s' < s$,

$$\|u\|_{H_{\mathrm{b}}^{s,r}} \leq C\|(\mathcal{P}^* + i\mathcal{Q}^*)u\|_{H_{\mathrm{b}}^{s-m+1,r}} + C\|u\|_{H_{\mathrm{b}}^{s',r}}. \tag{5.2.2}$$

As already pointed out in §3.3, the issue with these estimates is that $H_{\mathrm{b}}^{s,r}$ does not include compactly into the error term $H_{\mathrm{b}}^{s',r}$ on the right hand side due to the lack of additional decay. We thus further assume that there are no poles of the inverse of the Mellin transformed normal operator family $(\mathcal{P} - i\mathcal{Q})\widehat{\ }(\sigma)$ (see §3.3.3) on the line $\operatorname{Im}\sigma = -r$. Then using the Mellin transform, which is an isomorphism between weighted b-Sobolev spaces and semiclassical Sobolev spaces (see §3.3.4), and the estimates for $(\mathcal{P} - i\mathcal{Q})\widehat{\ }(\sigma)$ (including the high energy, i.e. semiclassical, estimates, all of which is discussed in detail in [114, §2] — the high energy assumptions of [114, §2] hold by our assumptions on the b-flow at ${}^{\mathrm{b}}S^*_X M$, and which imply that for all but a discrete set of r the aforementioned lines do not contain such poles), we obtain that on $\mathbb{R}_+ \times \partial M$

$$\|v\|_{H_{\mathrm{b}}^{s,r}} \leq C\|N(\mathcal{P} - i\mathcal{Q})v\|_{H_{\mathrm{b}}^{s-m+1,r}}$$

when

$$s - (m-1)/2 > \beta r.$$

Again, we have an analogous estimate for $N(\mathcal{P}^* + i\mathcal{Q}^*)$:

$$\|v\|_{H_{\mathrm{b}}^{s,r}} \leq C\|N(\mathcal{P}^* + i\mathcal{Q}^*)v\|_{H_{\mathrm{b}}^{s-m+1,r}},$$

provided $-r$ is not the imaginary part of a pole of the inverse of $(\mathcal{P}^* + i\mathcal{Q}^*)\widehat{\ }$, and provided

$$s - (m-1)/2 < \beta r.$$

As $(\mathcal{P}^* + i\mathcal{Q}^*)\widehat{\ }(\sigma) = ((\mathcal{P} - i\mathcal{Q})\widehat{\ })^*(\overline{\sigma})$, see (3.3.36), the requirement on $-r$ is the same as r not being the imaginary part of a pole of the inverse of $(\mathcal{P} - i\mathcal{Q})\widehat{\ }$.

We apply these results using the same argument that led up to (3.3.40); in the present context, the estimate (5.2.1) requires control of $(\mathcal{P} - i\mathcal{Q})u$ in a b-Sobolev space whose regularity is 1 stronger than what would be needed for elliptic operators, and correspondingly the norm of the second term in (3.3.39) needs to be increased by 1. Thus,

$$\|u\|_{H_{\mathrm{b}}^{s,r}} \leq C\|(\mathcal{P} - i\mathcal{Q})u\|_{H_{\mathrm{b}}^{s-m+1,r}} + C\|u\|_{H_{\mathrm{b}}^{s'+1,r-1}}, \tag{5.2.3}$$

where now the inclusion $H_{\mathrm{b}}^{s,r} \to H_{\mathrm{b}}^{s'+1,r-1}$ is compact when we choose, as we may, $s' < s-1$, requiring, however, $s' - (m-1)/2 > \beta r$, so that the radial point estimate can be applied to $N(\mathcal{P} - i\mathcal{Q})$. Recall that this argument required that s, r, s' satisfied the requirements preceding (5.2.1), and that $-r$ is not the imaginary part of any pole of $(\mathcal{P} - i\mathcal{Q})\widehat{\ }$.

Analogous estimates hold for $(\mathcal{P} - i\mathcal{Q})^*$ where now we write $\widetilde{s}$, $\widetilde{r}$ and $\widetilde{s}'$ for the Sobolev orders for the eventual application:

$$\|u\|_{H_{\mathrm{b}}^{\widetilde{s},\widetilde{r}}} \leq C\|(\mathcal{P} - i\mathcal{Q})^*u\|_{H_{\mathrm{b}}^{\widetilde{s}-m+1,\widetilde{r}}} + C\|u\|_{H_{\mathrm{b}}^{\widetilde{s}'+1,\widetilde{r}-1}}, \tag{5.2.4}$$

provided $\widetilde{s}$, $\widetilde{r}$ in place of s and r satisfy the requirements stated before (5.2.2), and provided $-\widetilde{r}$ is not the imaginary part of a pole of $(\mathcal{P}^* + i\mathcal{Q}^*)\widehat{\ }$ (i.e. $\widetilde{r}$ of $(\mathcal{P} - i\mathcal{Q})\widehat{\ }$). Note that we *do not* have a stronger requirement for $\widetilde{s}'$, unlike for s' above, since upper bounds for s imply those for $s' \leq s$.

The estimates (5.2.3) and (5.2.4) are 'Fredholm estimates' as in §3.2.1; we thus obtain Fredholm properties of $\mathcal{P} - i\mathcal{Q}$ (see also [114, §2.6] for the functional analytic argument in the present context), in particular solvability, modulo a (possible) finite dimensional obstruction, in $H_{\mathrm{b}}^{s,r}$ if

$$s-(m-1)/2-1>\beta r. \tag{5.2.5}$$

Concretely, we take $\widetilde{s}=m-1-s$, $\widetilde{r}=-r$, $s'<s-1$ sufficiently close to $s-1$ so that $s'-(m-1)/2>\beta r$ (which is possible by (5.2.5)). Thus, $s-(m-1)/2>\beta r$ means $\widetilde{s}-(m-1)/2=(m-1)/2-s<-\beta r=\beta\widetilde{r}$, so the space on the left hand side of (5.2.3) is dual to that in the first term on the right hand side of (5.2.4), and the same for the equations interchanged, and notice that the condition on the poles of the inverse of the Mellin transformed normal operators is the same for both $\mathcal{P}-i\mathcal{Q}$ and $\mathcal{P}^*+i\mathcal{Q}^*$: $-r$ is not the imaginary part of a pole of $(\mathcal{P}-i\mathcal{Q})\widehat{\ }$. This yields:

Proposition 5.2.1. *Suppose that $\mathcal{P}$ is non-trapping. Suppose $s,r\in\mathbb{R}$, $s-(m-1)/2-1>\beta r$, $-r$ is not the imaginary part of a pole of $(\mathcal{P}-i\mathcal{Q})\widehat{\ }$ and let*

$$\mathcal{X}^{s,r}=\{u\in H_{\mathrm{b}}^{s,r}(M)\colon(\mathcal{P}-i\mathcal{Q})u\in H_{\mathrm{b}}^{s-1,r}(M)\},\quad \mathcal{Y}^{s,r}=H_{\mathrm{b}}^{s,r}(M),$$

where $\mathcal{P}-i\mathcal{Q}$ is a priori a map

$$\mathcal{P}-i\mathcal{Q}\colon H_{\mathrm{b}}^{s,r}(M)\to H_{\mathrm{b}}^{s-2,r}(M).$$

Then

$$\mathcal{P}-i\mathcal{Q}\colon\mathcal{X}^{s,r}\to\mathcal{Y}^{s-1,r}$$

is Fredholm.

We remark that $\mathcal{Y}^{s,r}$, $\mathcal{X}^{s,r}$ are complete, in the case of $\mathcal{X}^{s,r}$ with the natural norm being $\|u\|^2_{\mathcal{X}^{s,r}}=\|u\|^2_{H_{\mathrm{b}}^{s,r}(M)}+\|(\mathcal{P}-i\mathcal{Q})u\|^2_{H_{\mathrm{b}}^{s-1,r}(M)}$. See Remark 4.2.5.

Initial value problems

As already mentioned, the main issue with this argument using complex absorption that it does not guarantee the forward nature (in terms of supports) of the solution for a wave-like equation, although it does guarantee the correct microlocal structure. So now we assume that $\mathcal{P}\in\mathrm{Diff}_{\mathrm{b}}^2(M)$ with

$$\mathcal{P}-\Box_g\in\mathrm{Diff}_{\mathrm{b}}^1(M) \tag{5.2.6}$$

for a Lorentzian b-metric g, as in Chapter 4. Then one can run an argument completely analogous to the above, obtaining Fredholm properties of $\mathcal{P}$ using energy estimates by restricting the domain we consider to be a manifold Ω with corners, where the new boundary hypersurfaces are spacelike with respect to g, i.e. given by level sets of timelike functions. Such a possibility was mentioned in [114, Remark 2.6], though it was not described in detail as it was not needed there, essentially because the existence/uniqueness argument for forward solutions was given only for dilation invariant operators. The main difference between using complex absorption and adding boundary hypersurfaces is that the latter limit the Sobolev regularity one can use, with the most natural choice coming from energy estimates. However, a posteriori one can improve the result to better Sobolev spaces using propagation of singularities type results.

We assume that $\Omega \subset M$, equipped with the b-metric g, is a non-trapping spacetime in the sense of Definition 2.5.1, and that $\mathcal{P} \in \mathrm{Diff}_{\mathrm{b}}^2(M)$ satisfies $\mathcal{P} - \Box_g \in \mathrm{Diff}_{\mathrm{b}}^1(M)$. We proved global energy estimates and b-regularity on weighted spaces for $\mathcal{P}$ in §4.2, see in particular Theorem 4.2.4, giving the invertibility of $\mathcal{P}\colon \mathcal{X}^{s,r} \to \mathcal{Y}^{s-1,r}$ for $s \geq 0$, $r \ll 0$, with

$$\mathcal{X}^{s,r} = \{u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-} \colon \mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}\}, \quad \mathcal{Y}^{s,r} = H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}.$$

Correspondingly, the normal operator family $\widehat{\mathcal{P}}(\sigma)$ is a family of operators on

$$Y := \Omega \cap X, \quad X = \partial M,$$

and the semiclassical analysis of $\widehat{\mathcal{P}}(\sigma)$ therefore takes place on supported/extendible Sobolev spaces. Concretely, $\widehat{\mathcal{P}}(\sigma)\colon \mathcal{X}_\partial^s \to \mathcal{Y}_\partial^{s-1}$ is Fredholm, with

$$\mathcal{X}_\partial^s = \{u \in H^s(Y)^- \colon \widehat{\mathcal{P}}(\sigma)u \in H^{s-1}(Y)^-\}, \quad \mathcal{Y}_\partial^s = H^s(Y)^-,$$

for $s > 1/2 - \beta \operatorname{Im} \sigma$, $s \geq 0$, the latter requirement coming from the use of energy estimates near the Cauchy hypersurface ∂Y, and one has non-trapping high energy estimates on semiclassical Sobolev spaces. (Note here that the space $\mathcal{X}_\partial^s$ only depends on the principal symbol of $\widehat{\mathcal{P}}(\sigma)$, which is independent of σ, cf. the discussion around [114, Equation (2.22)] and in §A.2.)

Remark 5.2.2. Using normal operators as in the discussion leading to Proposition 5.2.1, one would get the following statement: Suppose $s > 1$, $s - 3/2 > \beta r$. Then with $\mathcal{X}^{s,r}$,

$\mathcal{Y}^{s,r}$ as above, $\mathcal{P}\colon \mathcal{X}^{s,r} \to \mathcal{Y}^{s,r}$ is Fredholm. Here the main loss, which is an artifact of combining local energy estimates with the b-theory, is that one needs to assume $s > 1$; this is done since in the argument one needs to take s' with $s'+1 < s$ in order to transition the normal operator estimates from $N(\mathcal{P})u$ to $\mathcal{P}u$ and still have a compact inclusion, but the normal operator estimates need $s' \geq 0$ as they are again based on energy estimates due to the boundary H_2; in the semiclassical setting of the normal operator analysis, the latter are proved in [114, Proposition 3.8] when combined with semiclassical propagation of singularities, see [115, §4.4]. Using the direct global energy estimate eliminates this loss. In particular, in the complex absorption setting, this problem does not arise, but on the other hand, one need not have the forward support property of the solution.

The methods of [114] are immediately applicable to obtain an expansion of the solutions; the main point of the following theorem is the elimination of the losses in differentiability in [114, Proposition 3.5] due to Proposition 3.3.8.

Theorem 5.2.3. *(Strengthened version of [114, Proposition 3.5].) Let $\Omega \subset M$, equipped with the b-metric g, be a non-trapping spacetime as above, with τ a boundary defining function with $d\tau/\tau$ timelike, $\mathfrak{t}_1 = \tau_0 - \tau$ as in (2.2.5), and $\mathcal{P}$ as in (5.2.6).*

Let σ_j be the poles of $\widehat{\mathcal{P}}^{-1}$, and let ℓ be such that $\operatorname{Im}\sigma_j + \ell \notin \mathbb{N}$ for all j. Let $\phi \in \mathcal{C}^\infty(\mathbb{R})$ be such that $\operatorname{supp}\phi \subset (0,\infty)$, and $\phi \circ \mathfrak{t}_1 \equiv 1$ near $Y = X \cap \Omega$. Then for $s > 3/2 + \beta\ell$, there are $m_{jl} \in \mathbb{N}$ such that solutions of $\mathcal{P}u = f$ with $f \in H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$, and with $u \in H_{\mathrm{b}}^{s_0,r_0}(\Omega)^{\bullet,-}$, $s \geq s_0 \geq 1$, $s_0 - 1/2 > \beta r_0$ satisfy that for some $a_{jl\kappa} \in \mathcal{C}^\infty(Y)$,

$$u' = u - \sum_j \sum_{l\in\mathbb{N}} \sum_{\kappa \leq m_{jl}} \tau^{i\sigma_j + l}(\log\tau)^\kappa (\phi\circ\mathfrak{t}_1) a_{jl\kappa} \in H_{\mathrm{b}}^{s,\ell}(\Omega)^{\bullet,-}, \tag{5.2.7}$$

where the sum is understood to be over a finite set with $-\operatorname{Im}\sigma_j + l < \ell$. Here the (semi)norms of both $a_{jl\kappa}$ in $\mathcal{C}^\infty(Y)$ and u' in $H_{\mathrm{b}}^{s,\ell}(\Omega)^{\bullet,-}$ are bounded by a constant times that of f in $H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$.

The analogous result also holds if f possesses an expansion modulo $H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$ of the form

$$f = f' + \sum_j \sum_{\kappa \leq m_j'} \tau^{\alpha_j}(\log\tau)^\kappa (\phi\circ\mathfrak{t}_1) a_{j\kappa},$$

with $f' \in H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$ and $a_{j\kappa} \in \mathcal{C}^\infty(Y)$, where terms corresponding to the expansion of the f are added to (5.2.7) in the sense of the extended union of index sets [82, §5.18], recalled *below in Definition 5.2.12.*

Thus, on static de Sitter space, in terms of the time coordinate $t_* = -\log \tau$ as in §2.2.1 (which extends across the cosmological horizon), the expansion (5.2.7) yields (in the case $\ell < 0$) exponential decay in t_* up to a finite-dimensional space of resonances. See also Theorem 6.1.1 for a formulation of the above theorem (albeit in the Kerr-de Sitter setting) in these terms.

Remark 5.2.4. Here the factor $\phi \circ \mathfrak{t}_1$ is added to cut off the expansion away from H_1, thus assuring that u' is in the indicated space (a supported distribution).

Also, the sum over l is generated by the lack of dilation invariance of $\mathcal{P}$. If we take ℓ such that $-\operatorname{Im}\sigma_j > \ell - 1$ for all j, then all the terms in the expansion arise directly from the resonances, thus $l = 0$ and $m_{j0} + 1$ is the order of the pole of $\widehat{\mathcal{P}}^{-1}$ at σ_j, with the $a_{j0\kappa}$ being resonant states.

Proof of Theorem 5.2.3. First assume that $-\operatorname{Im}\sigma_j > \ell$ for every j; thus there are no terms subtracted from u in (5.2.7). We proceed as in [114, Proposition 3.5], but use the propagation of singularities, in particular Propositions 3.3.8 and 4.1.10, to eliminate the losses. See Figure 5.1.

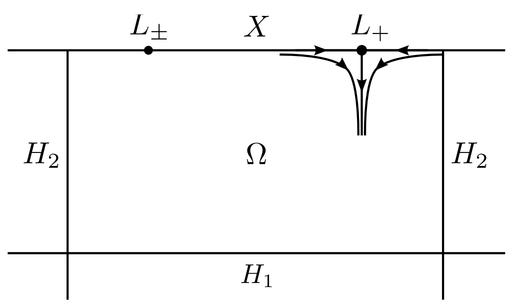

Figure 5.1: Setup for the discussion of the forward problem on non-trapping spacetimes. Near the spacelike hypersurfaces H_1 and H_2, which are the replacement for the complex absorbing operator $\mathcal{Q}$, we use standard (non-microlocal) energy estimates, and away from them, we use b-microlocal propagation results, including at the radial sets $L_\pm$. The bicharacteristic flow, in fact its projection to the base, is only indicated near L_+; near L_-, the directions of the flowlines are reversed.

First, by the propagation of singularities, using $s_0 - 1/2 > \beta r_0$ and $s \geq s_0$, $s \geq 0$,

$$u \in H_{\mathrm{b}}^{s,r_0}(\Omega)^{\bullet,-}.$$

Thus, as $\mathcal{P} - N(\mathcal{P}) \in \tau\mathrm{Diff}_{\mathrm{b}}^2(M)$,

$$N(\mathcal{P})u = f - \widetilde{f}, \quad \widetilde{f} = (\mathcal{P} - N(\mathcal{P}))u \in H_{\mathrm{b}}^{s-2,r_0+1}(\Omega)^{\bullet,-} \tag{5.2.8}$$

Applying[13] [114, Lemma 3.1] (using $s \geq s_0 \geq 1$), which is the lossless version of [114, Proposition 3.5] in the dilation invariant case, one obtains (5.2.7) with ℓ replaced by $\ell' = \min(\ell, r_0 + 1)$ except that $u = u' \in H_{\mathrm{b}}^{s-1,\ell'}(\Omega)^{\bullet,-}$ corresponding to the $\widetilde{f}$ term in $N(\mathcal{P})u$ rather than $u = u' \in H_{\mathrm{b}}^{s,\ell'}(\Omega)^{\bullet,-}$ as desired. However, using $\mathcal{P}u = f \in H_{\mathrm{b}}^{s-1,\ell'}(\Omega)^{\bullet,-}$, we deduce by the propagation of singularities, using $s - 1 > \beta\ell' + 1/2$, $s \geq 0$, that $u = u' \in H_{\mathrm{b}}^{s,\ell'}(\Omega)^{\bullet,-}$. If $\ell \leq r_0 + 1$, we have proved (5.2.7). Otherwise we iterate, replacing r_0 by $r_0 + 1$. We thus reach the conclusion, (5.2.7), in finitely many steps.

If there are j such that $-\operatorname{Im}\sigma_j < \ell$, then in the first step, when using [114, Lemma 3.1], we obtain the partial expansion u_1 corresponding to $\ell' = \min(\ell, r_0 + 1)$ in place of ℓ; here we may need to decrease ℓ' by an arbitrarily small amount to make sure that ℓ' is not $-\operatorname{Im}\sigma_j$ for any j. Further, the terms of the partial expansion are annihilated by $N(\mathcal{P})$, so u' satisfies

$$\mathcal{P}u' = \mathcal{P}u - N(\mathcal{P})u_1 - (\mathcal{P} - N(\mathcal{P}))u_1 \in H_{\mathrm{b}}^{s-1,\ell'}(\Omega)^{\bullet,-}$$

as $(\mathcal{P} - N(\mathcal{P}))u_1 \in H_{\mathrm{b}}^{\infty,r_0+1}(\Omega)^{\bullet,-}$ in fact due to the conormality of u_1 and $\mathcal{P} - N(\mathcal{P}) \in \tau\mathrm{Diff}_{\mathrm{b}}^2(M)$. Correspondingly, the propagation of singularities result is applicable as above to conclude that $u' \in H_{\mathrm{b}}^{s,\ell'}(\Omega)^{\bullet,-}$. If $\ell \leq r_0 + 1$ we are done. Otherwise we have better information on $\widetilde{f}$ in the next step, namely

$$\widetilde{f} = (\mathcal{P} - N(\mathcal{P}))u = (\mathcal{P} - N(\mathcal{P}))u' + (\mathcal{P} - N(\mathcal{P}))u_1$$

with the first term in $H_{\mathrm{b}}^{s-2,r_0+1}(\Omega)^{\bullet,-}$ (same as in the case first considered above, without relevant resonances), while the expansion of u_1 shows that $(\mathcal{P} - N(\mathcal{P}))u_1$ has a similar expansion, but with an extra power of τ (i.e. $\tau^{i\sigma_j}$ is shifted to $\tau^{i\sigma_j+1}$). We can now apply [114, Lemma 3.1] again; in the case of the terms arising from the partial expansion, u_1, there are now new terms corresponding to shifting the powers $\tau^{i\sigma_j}$ to $\tau^{i\sigma_j+1}$, as stated in the referred Lemma, and possibly causing logarithmic terms if $\sigma_j - i$ is also a pole of

[13] In [114], Lemma 3.1 is stated on the normal operator space M_I, which does not have a boundary face corresponding to H_2, i.e. $S_2 \times [0, \infty)$, with complex absorption instead. However, given the analysis on Y discussed above, all the arguments go through essentially unchanged: This is a Mellin transform and contour deformation argument.

$\widehat{\mathcal{P}}^{-1}$. Iterating in the same manner proves the theorem when $f \in H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$. When f has an expansion modulo $H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$, the same argument works; [114, Lemma 3.1] gives the terms with the extended union, which then further generate additional terms due to $\mathcal{P} - N(\mathcal{P})$, just as the resonance terms did. □

There is one problem with this theorem for the purposes of semilinear equations: The resonant terms with $\operatorname{Im}\sigma_j \geq 0$ which give rise to unbounded, or at most just bounded, terms in the expansion which become larger when one takes powers of these, or when one iteratively applies $\mathcal{P}^{-1}$ (with the latter being the only issue if $\operatorname{Im}\sigma_j = 0$ and the pole is simple). See Remark 5.2.11.

Concretely, we now consider an asymptotically de Sitter-like space $(\widetilde{M}, \widetilde{g})$ and blow up a point q at the future boundary $\widetilde{X}_+$, as discussed in §2.2.2, to obtain the analogue $M = [\widetilde{M}; q]$ of the static model of de Sitter space with the pullback-metric g, which is a b-metric near the front face; let $\mathcal{P} = \Box_g - \lambda$. The metric of the asymptotically de Sitter space, frozen at q, induces a de Sitter metric, g_0, which is well defined at the front face of the blow up M (but away from its side faces) as a b-metric. In particular, the resonances in the 'static region' of any asymptotically de Sitter space are the same as in the static model of actual de Sitter space.

On actual de Sitter space, the poles of $\widehat{\mathcal{P}}^{-1}$ are those on the hyperbolic space in the interior of the light cone equipped by a potential, as described in [111, Lemma 7.11], or indeed in [114, Proposition 4.2] where essentially the present notation is used.[14] As shown in Corollary 7.18 of [111], converted to our notation, the only possible poles are at

$$i\widehat{s}_\pm(\lambda) - i\mathbb{N}, \qquad \widehat{s}_\pm(\lambda) = -\frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda}, \tag{5.2.9}$$

and for $\lambda = 0$, the highest resonance $s_+(0) = 0$ is simple. (We will give a direct, robust proof of the latter fact in Chapter 7, see Theorem 7.5.1, which does not recover the entire set of possible poles though; however, for nonlinear applications, control of the highest resonance in this way is already sufficient to obtain existence and exponential decay to a zero resonant state.) In particular, when $\lambda = m^2$, $m > 0$, we conclude:

Lemma 5.2.5. *For $m > 0$, $\mathcal{P} = \Box_g - m^2$, g induced by an asymptotically de Sitter metric as above, all poles of $\widehat{\mathcal{P}}^{-1}$ have strictly negative imaginary part.*

[14] In [111, Lemma 7.11], $-\sigma^2$ plays the same role as σ^2 here or in [114, Proposition 4.2].

In other words, for small mass $m > 0$, there are no resonances σ of the Klein-Gordon operator with $\operatorname{Im}\sigma \geq -\epsilon_0$ for some $\epsilon_0 > 0$. Therefore, the expansion of u as in (5.2.7) no longer has a constant term. Let us fix such $m > 0$ and $\epsilon_0 > 0$, which ensures that for $0 < \epsilon < \epsilon_0$, the only term in the asymptotic expansion (5.2.7), when $s > 1/2 + \epsilon$ and $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$, is the 'remainder' term $u' \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$. Here we use that $\beta = 1$ in de Sitter space, hence on an asymptotically de Sitter space, see (2.2.12), and in the semiclassical setting [114, §4.4], in particular the second displayed equation after Equation (4.16) there which computes β in accordance with Remark 3.3.9.

Being interested in finding *forward solutions* to (nonlinear) wave equations on *generalized static de Sitter spaces*, we now define the forward solution operator

$$S_{\mathrm{KG}} \colon H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \tag{5.2.10}$$

using Theorems 4.2.4 and 5.2.3.

5.2.2 Operators on bundles; conormal metrics

As already alluded to in §5.1, we point out that the above arguments, leading up to Theorem 5.2.3, go through without changes on general non-trapping spacetimes (M, g) for second order b-differential operators $\mathcal{P}$ acting on sections of a finite rank complex vector bundle $\mathcal{E}$ over M if $\sigma_{\mathrm{b},2}(\mathcal{P}) = G \cdot \mathrm{id}$ (in particular, $\mathcal{P}$ is principally scalar), which generalizes (5.2.6), yielding resonance expansions for forward solutions of $\mathcal{P}u = f$ as in Theorem 5.2.3; note here that the energy estimates developed in Chapter 4 work with bundles as well by the same proofs, and the microlocal energy estimates, both in the b- and in the semiclassical (normal operator) setting, are symbolic arguments that only rely on the principal symbol, except at radial points, where the subprincipal symbol enters through the threshold regularity; see also [114, Remark 2.1].

More precisely, in order to make sense of adjoints and integration by parts in positive commutator estimates, which we use both for standard and for microlocal energy estimates, equip $\mathcal{E}$ with an arbitrary Hermitian inner product and any smooth b-connection, which gives a notion of differentiating sections of $\mathcal{E}$ along b-vector fields; over Ω (which is compact), all choices of inner products are equivalent. We can then define the b-Sobolev space $H_{\mathrm{b}}^s(\Omega, \mathcal{E})$ for $s \in \mathbb{N}_0$ to consist of all sections of $\mathcal{E}$ over Ω which are square integrable (with respect to the volume density $|dg|$ induced by the metric g) together with all of its b-derivatives up to order s, using the b-connection on $\mathcal{E}$ to define the latter, and extend this to all $s \in \mathbb{R}$ by duality and interpolation, or via the use of b-pseudodifferential operators. Weighted b-Sobolev spaces $H_{\mathrm{b}}^{s,r}(\Omega, \mathcal{E})^{\bullet,-}$ of extendible/supported distributions are defined as in the scalar setting. Likewise, we can define Sobolev spaces (including semiclassical versions of these) of sections of $\mathcal{E}$ over $\Omega \cap \partial M$ with extendible/supported character at the boundary.

We can also generalize the class of metrics we work in, namely we can use asymptotically stationary metrics as discussed in §§2.1.2 and 3.3.5: Namely, we can allow g to be a Lorentzian b-metric such that for some smooth Lorentzian b-metric g', we have

$$g - g' \in H_{\mathrm{b}}^{\infty,r}(\Omega, S^{2\mathrm{b}}T^*M) \text{ for some } r > 0. \tag{5.2.11}$$

We can of course similarly relax the requirements on the lower order terms of $\mathcal{P}$; thus, we require

$$\mathcal{P} \in \mathrm{Diff}_{\mathrm{b}}^2(M, \mathcal{E}) + H_{\mathrm{b}}^{\infty,r}(\Omega)\mathrm{Diff}_{\mathrm{b}}^2(M, \mathcal{E}), \quad \sigma_{\mathrm{b},2}(\mathcal{P}) = G.$$

We again stress that this is an invariant condition, since different choices of the boundary defining function merely rescale the weight r. Now, as long as g satisfies the geometric and dynamical requirements of a non-trapping spacetime in Definition 2.5.1, our proofs again go through: The microlocal arguments now require the use of the b-conormal calculus developed in §3.3.5. We point out the only serious change in the proof of Theorem 5.2.3: In the contour shifting argument, we can only shift the line over which we integrate in order to compute the inverse Mellin transform by at most the amount $\min(r, 1)$, rather than 1, the reason being that $\mathcal{P} - N(\mathcal{P}) \in \tau^{\min(r,1)}\mathrm{Diff}_{\mathrm{b}}^2(M, \mathcal{E})$ now.

In the nonlinear theorems developed below, we can likewise allow the coefficients of nonlinearities to be smooth plus conormal (in the sense of H_{b}^{∞}, no decay relative to smooth coefficients is needed for the conormal coefficients) rather than merely smooth, and the proofs go through unchanged; see Theorems 9.1.15 and 9.2.2 for details in the quasilinear setting.

As already mentioned above, one needs to control the resonances in the closed upper half plane in order to obtain global nonlinear well-posedness results: If there are no resonances in $\operatorname{Im}\sigma \geq 0$, any (polynomial) non-linearity works, furthermore a simple resonance at $\sigma = 0$ is allowed as well, provided the non-linearity annihilates the corresponding resonant states.

5.2.3 A class of semilinear equations

Continuing to work on generalized static de Sitter models, let us fix $m > 0$ and $\epsilon_0 > 0$ as above for statements about semilinear equations involving the Klein-Gordon operator; for equations involving the wave operator only, let $-\epsilon_0$ be equal to the largest imaginary part of all *non-zero* resonances of $\Box_g$.

Theorem 5.2.6. *Let $0 \leq \epsilon < \epsilon_0$ and $s > 3/2 + \epsilon$. Moreover, let $q\colon H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \times H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T_\Omega^* M)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ be a continuous function with $q(0,0) = 0$ such that there exists a continuous non-decreasing function $L\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying*

$$\|q(u, {}^{\mathrm{b}}du) - q(v, {}^{\mathrm{b}}dv)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R,$$

where we use the norms corresponding to the map q. Then there is a constant $C_L > 0$ so that the following holds: If $L(0) < C_L$, then for small $R > 0$, there exists $C > 0$ such that for all $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with norm $\leq C$, the equation

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du) \tag{5.2.12}$$

has a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, with norm $\leq R$, that depends continuously on f.

More generally, suppose

$$q\colon H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \times H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T_\Omega^* M)^{\bullet,-} \times H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

satisfies $q(0,0,0) = 0$ and

$$\|q(u, {}^{\mathrm{b}}du, w) - q(u', {}^{\mathrm{b}}du', w')\| \leq L(R)(\|u - u'\| + \|w - w'\|)$$

provided $\|u\| + \|w\|, \|u'\| + \|w'\| \leq R$, where we use the norms corresponding to the map q, for a continuous non-decreasing function $L\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$. Then there is a constant $C_L > 0$ so that the following holds: If $L(0) < C_L$, then for small $R > 0$, there exists $C > 0$ such that for all $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with norm $\leq C$, the equation

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du, \Box_g u) \tag{5.2.13}$$

has a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, *with* $\|u\|_{H_{\mathrm{b}}^{s,\epsilon}} + \|\Box_g u\|_{H_{\mathrm{b}}^{s-1,\epsilon}} \leq R$, *that depends continuously on* f.

Further, if $\epsilon > 0$ *and the non-linearity is of the form* $q({}^{\mathrm{b}}du)$, *with*

$$q\colon H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T^*_\Omega M)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

having a small Lipschitz constant near 0, *then for small* $R > 0$, *there exists* $C > 0$ *such that for all* $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ *with* $\|f\| \leq C$, *the equation*

$$\Box_g u = f + q({}^{\mathrm{b}}du)$$

has a unique solution u *with* $u - (\phi \circ \mathfrak{t}_1)c = u' \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, *where* $c \in \mathbb{C}$, *that depends continuously on* f, *in the sense that* $c \in \mathbb{C}$ *and* $u' \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ *depend continuously on* f. *Here,* $\phi \in \mathcal{C}^\infty(\mathbb{R})$ *with support in* $(0, \infty)$ *and* $\mathfrak{t}_1$ *are as in Theorem 5.2.3. In fact, the statement even holds for non-linearities* $q(u, {}^{\mathrm{b}}du)$ *provided*

$$q\colon (\mathbb{C}(\phi \circ \mathfrak{t}_1) \oplus H_{\mathrm{b}}^{s,\epsilon}(\Omega)) \times H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T^*_\Omega M)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

has a small Lipschitz constant near 0.

Note that when one writes e.g. $q(u, {}^{\mathrm{b}}du)$, one could instead, at least locally, write

$$a(u, x\partial_x u, \partial_{y_1} u, \ldots, \partial_{y_{n-1}} u),$$

where x is a local boundary defining function and the y_j are local coordinates on the boundary; however, the ${}^{\mathrm{b}}du$ notation is more concise and invariant.

Proof of Theorem 5.2.6. To prove the first part, let S_{KG} be the forward solution operator for $\Box_g - m^2$ as in (5.2.10). We want to apply the Banach fixed point theorem to the operator $T_{\mathrm{KG}}\colon H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, $T_{\mathrm{KG}}u = S_{\mathrm{KG}}(f + q(u, {}^{\mathrm{b}}du))$.

Let $C_L = \|S_{\mathrm{KG}}\|^{-1}$, then we have the estimate

$$\|T_{\mathrm{KG}}u - T_{\mathrm{KG}}v\| \leq \|S_{\mathrm{KG}}\| L(R') \|u - v\| \leq C_0 \|u - v\| \tag{5.2.14}$$

for $\|u\|, \|v\| \leq R$ and a constant $C_0 < 1$, granted that $L(R) \leq C_0 \|S_{\mathrm{KG}}\|^{-1}$, which holds for small $R > 0$ by assumption on L. Then, T_{KG} maps the R-ball in $H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ into itself if

$\|S_{\mathrm{KG}}\|(\|f\| + L(R)R) \leq R$, i.e. if $\|f\| \leq R(\|S_{\mathrm{KG}}\|^{-1} - L(R))$. Put

$$C = R(\|S_{\mathrm{KG}}\|^{-1} - L(R)).$$

Then the existence of a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ with norm $\leq R$ to the PDE (5.2.12) with $\|f\|_{H_{\mathrm{b}}^{s-1,\epsilon}} \leq C$ follows from the Banach fixed theorem.

To prove the continuous dependence of u on f, suppose we are given $u_j \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, $j = 1, 2$, with norms $\leq R$, $f_j \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with norms $\leq C$, such that

$$(\Box_g - m^2)u_j = f_j + q(u_j, {}^{\mathrm{b}}du_j), \quad j = 1, 2.$$

Then

$$(\Box_g - m^2)(u_1 - u_2) = f_1 - f_2 + q(u_1, {}^{\mathrm{b}}du_1) - q(u_2, {}^{\mathrm{b}}du_2),$$

hence

$$\|u_1 - u_2\| \leq \|S_{\mathrm{KG}}\|(\|f_1 - f_2\| + L(R)\|u_1 - u_2\|),$$

which in turn gives

$$\|u_1 - u_2\| \leq \frac{\|f_1 - f_2\|}{1 - C_0}.$$

This completes the proof of the first part.

For the more general statement, we use that one can think of $\Box_g$ in the non-linearity as a first order operator. Concretely, we work on the coisotropic space

$$\mathcal{X} = \{u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} : \Box_g u \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}\}$$

with norm

$$\|u\|_{\mathcal{X}} = \|u\|_{H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}} + \|\Box_g u\|_{H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}}.$$

This is a Banach space: Indeed, if (u_k) is a Cauchy sequence in $\mathcal{X}$, then $u_k \to u$ in $H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, and $\Box_g u_k \to v$ in $H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$; in particular, $\Box_g u_k \to \Box_g u$ and $\Box_g u_k \to v$ in $\tau^\epsilon H_{\mathrm{b}}^{s-2}(\Omega)^{\bullet,-}$, thus $\Box_g u = v \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$, which was to be shown. We then define $T_{\mathrm{KG}} : \mathcal{X} \to \mathcal{X}$ by $T_{\mathrm{KG}}u = S_{\mathrm{KG}}(f + q(u, {}^{\mathrm{b}}du, \Box_g u))$ and obtain the estimate

$$\begin{aligned}\|T_{\mathrm{KG}}u - T_{\mathrm{KG}}v\|_{\mathcal{X}} &= \|T_{\mathrm{KG}}u - T_{\mathrm{KG}}v\|_{H_{\mathrm{b}}^{s,\epsilon}} + \|q(u, {}^{\mathrm{b}}du, \Box_g u) - q(v, {}^{\mathrm{b}}dv, \Box_g v)\|_{H_{\mathrm{b}}^{s-1,\epsilon}} \\ &\leq (\|S_{\mathrm{KG}}\| + 1)L(R)(\|u - v\|_{H_{\mathrm{b}}^{s,\epsilon}} + \|\Box_g u - \Box_g v\|_{H_{\mathrm{b}}^{s-1,\epsilon}})\end{aligned}$$

$$= (\|S_{\mathrm{KG}}\| + 1)L(R)\|u - v\|_{\mathcal{X}} \leq C_0\|u - v\|_{\mathcal{X}}$$

for $u, v \in \mathcal{X}$ with norms $\leq R$, with $C_0 < 1$ if $R > 0$ is small enough, provided we require $L(0) < C_L := (\|S_{\mathrm{KG}}\| + 1)^{-1}$. Then, for $u \in \mathcal{X}$ with norm $\leq R$,

$$\|T_{\mathrm{KG}}u\|_{\mathcal{X}} \leq (\|S_{\mathrm{KG}}\| + 1)(\|f\|_{H_{\mathrm{b}}^{s-1,\epsilon}} + L(R)R) \leq R$$

if $\|f\| \leq C$, $C > 0$ small. Thus, T_{KG} is a contraction on $\mathcal{X}$, and we obtain the solvability of equation (5.2.13). The continuous dependence of the solution on the forcing term f is proved as above.

For the third part, we use the forward solution operator $S\colon H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-} \to \mathcal{Y} := \mathbb{C} \oplus H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ for $\Box_g$. Clearly, $\mathcal{Y}$ is a Banach space with norm $\|(c, u')\|_{\mathcal{Y}} = |c| + \|u'\|_{H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}}$; see §5.2.4 for related and more general statements. We will apply the Banach fixed point theorem to the operator $T\colon \mathcal{Y} \to \mathcal{Y}$, $Tu = S(f + q(u, {}^{\mathrm{b}}du))$: We again have an estimate like (5.2.14), since ${}^{\mathrm{b}}du \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T^*_{\Omega}M)^{\bullet,-}$ for $u \in \mathcal{Y}$, and for small $R > 0$, T maps the R-ball around 0 in $\mathcal{Y}$ into itself if the norm of f in $H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ is small, as above. The continuous dependence of the solution on the forcing term is proved as above. $\square$

The following basic statement ensures that there are interesting non-linearities q that satisfy the requirements of the theorem; see also §5.2.4.

Lemma 5.2.7. *Let $s > n/2$, then $H_{\mathrm{b}}^s(\mathbb{R}^n_+)$ is an algebra. In particular, $H_{\mathrm{b}}^s(N)$ is an algebra on any compact n-dimensional manifold N with boundary which is equipped with a b-metric.*

Proof. The first statement is the special case $k = 0$ of Lemma 5.4.4 after a logarithmic change of coordinates, which gives an isomorphism $H_{\mathrm{b}}^s(\mathbb{R}^n_+) \cong H^s(\mathbb{R}^n)$; the lemma is well-known in this case, see e.g. [108, Chapter 13.3]. The second statement follows by localization and from the coordinate invariance of H_{b}^s. $\square$

More and related statements will be given in §5.4.2.

Remark 5.2.8. The algebra property of $H_{\mathrm{b}}^s(N)$ for $s > \dim(N)/2$ is a special case of the fact that for any $F \in \mathcal{C}^\infty(\mathbb{R})$, for real valued u, or $F \in \mathcal{C}^\infty(\mathbb{C})$, for complex valued u, with $F(0) = 0$, the composition map $H_{\mathrm{b}}^s(N) \ni u \mapsto F \circ u \in H_{\mathrm{b}}^s(N)$ is well-defined and continuous, see for example [108, Chapter 13.10]. In the real valued u case, if $F(0) \neq 0$, then writing $F(t) = F(0) + tF_1(t)$ shows that $F \circ u \in \mathbb{C} + H_{\mathrm{b}}^s(N)$. If $r > 0$, then $H_{\mathrm{b}}^{s,r}(N) \subset H_{\mathrm{b}}^s(N)$ shows that $F_1(u) \in H_{\mathrm{b}}^s(N)$, thus $F \circ u = F(0) + uF_1(u) \in \mathbb{C} + H_{\mathrm{b}}^{s,r}(N)$; and if F vanishes to order k at 0, then $F(t) = t^k F_k(t)$, so $F \circ u = u^k(F_k \circ u)$, and the multiplicative properties of $H_{\mathrm{b}}^{s,r}(N)$ show that $F \circ u \in H_{\mathrm{b}}^{s,kr}(N)$. The argument is analogous for complex valued u, indeed for $\mathbb{R}^L$-valued u, using Taylor's theorem on F at the origin.

As a corollary of Theorem 5.2.6, we have:

Corollary 5.2.9. *If $s > n/2$, the hypotheses of Theorem 5.2.6 hold for non-linearities $q(u) = cu^p$, $p \geq 2$ integer, $c \in \mathbb{C}$, as well as $q(u) = q_0 u^p$, $q_0 \in H_{\mathrm{b}}^s(M)$.*

If $s - 1 > n/2$, the hypotheses of Theorem 5.2.6 hold for non-linearities q

$$q(u, {}^{\mathrm{b}}du) = \sum_{2 \leq j+|\alpha| \leq d} q_{j\alpha} u^j \prod_{k \leq |\alpha|} X_{\alpha,k} u, \tag{5.2.15}$$

where$q_{j,\alpha} \in \mathbb{C} + H_{\mathrm{b}}^s(M)$, $X_{\alpha,k} \in \mathcal{V}_{\mathrm{b}}(M)$.

Thus, in either case, for $m > 0$, $0 \leq \epsilon < \epsilon_0$, $s > 3/2 + \epsilon$, and for small $R > 0$, there exists $C > 0$ such that for all $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with norm $\leq C$, the equation

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du) \tag{5.2.16}$$

has a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, with norm $\leq R$, that depends continuously on f.

The analogous conclusion also holds for $\Box_g u = f + q(u, {}^{\mathrm{b}}du)$ provided $\epsilon > 0$ and

$$q(u, {}^{\mathrm{b}}du) = \sum_{2 \leq j+|\alpha| \leq d, |\alpha| \geq 1} q_{j\alpha} u^j \prod_{k \leq |\alpha|} X_{\alpha,k} u, \tag{5.2.17}$$

with the solution being in $\mathbb{C}(\phi \circ \mathfrak{t}_1) \oplus H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, $\phi \circ \mathfrak{t}_1$ identically 1 near $X \cap \Omega$, vanishing near H_1.

For such polynomial non-linearities, the Lipschitz constant $L(R)$ in the statement of Theorem 5.2.6 actually satisfies $L(0) = 0$.

Remark 5.2.10. In this chapter, we do not yet prove that one obtains smooth (i.e. conormal) solutions if the forcing term is smooth (conormal); see Theorem 9.1.15 for such a result in the more general quasilinear setting on generalized static models, and, more robustly, Theorem 9.2.2, using Nash-Moser iteration.

Since in Theorem 5.2.6, we allow q to depend on $\Box_g u$, we can also solve certain quasilinear equations (rather unnatural ones though) if $s > \max(1/2 + \epsilon, n/2 + 1)$: Suppose for example that $q' \colon H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1}(\Omega)^{\bullet,-}$ is continuous with $\|q'(u) - q'(v)\| \leq L'(R)\|u - v\|$

for $u, v \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ with norms $\leq R$, where $L' \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is locally bounded, then we can solve the equation

$$(1+q'(u))(\Box_g - m^2)u = f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

provided the norm of f is small. Indeed, put $q(u,w) = -q'(u)(w - m^2 u)$, then $q(u, \Box_g u) = -q'(u)(\Box_g - m^2)u$, and the PDE becomes

$$(\Box_g - m^2)u = f + q(u, \Box_g u),$$

which is solvable by Theorem 5.2.6, since, with $\|\cdot\| = \|\cdot\|_{H_{\mathrm{b}}^{s-1,\epsilon}}$, for $u, u' \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, $w, w' \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|u\| + \|w\|, \|u'\| + \|w'\| \leq R$, we have

$$\begin{aligned}
&\|q(u,w) - q(u',w')\| \\
&\quad \leq \|q'(u) - q'(u')\| \|w - m^2 u\| + \|q'(u')\| \|w - w' - m^2(u - u')\| \\
&\quad \leq L'(R)((1+m^2)R + m^2 R)\|u - u'\| + L'(R)R\|w - w'\| \\
&\quad \leq L(R)(\|u - u'\| + \|w - w'\|)
\end{aligned}$$

with $L(R) \to 0$ as $R \to 0$. By a similar argument, one can also allow q' to depend on ${}^{\mathrm{b}}du$ and $\Box_g u$.

Remark 5.2.11. Recalling the discussion following Theorem 5.2.3, let us emphasize the importance of $\widehat{P}(\sigma)^{-1}$ having no poles in the closed upper half plane by looking at the explicit example of the operator $\mathcal{P} = \partial_x$ in 1 dimension. In terms of $\tau = e^{-x}$, we have $\mathcal{P} = -\tau\partial_\tau$, thus $\widehat{P}(\sigma) = -i\sigma$, considered as an operator on the boundary (which is a single point) at $+\infty$ of the radial compactification of $\mathbb{R}$; hence $\widehat{P}(\sigma)^{-1}$ has a simple pole at $\sigma = 0$, corresponding to constants being annihilated by $\mathcal{P}$. Now suppose we want to find a forward solution of $u' = u^2 + f$, where $f \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R})$. In the first step of the iterative procedure described above, we will obtain a constant term; the next step gives a term that is linear in x (x being the antiderivative of 1), i.e. in $\log\tau$, then we get quadratic terms and so on, therefore the iteration does not converge (for general f), which is of course to be expected, since solutions to $u' = u^2 + f$ in general blow up in finite time. On the other hand, if $\mathcal{P} = \partial_x + 1$, then $\widehat{P}(\sigma)^{-1} = (1 - i\sigma)^{-1}$, which has a simple pole at $\sigma = -i$, which means that forward solutions u of $u' + u = u^2 + f$ with f as above can be constructed iteratively, and the first term of the expansion of u at $+\infty$ is $c\tau^{i(-i)} = ce^{-x}$, $c \in \mathbb{C}$.

5.2.4 Semilinear equations with polynomial non-linearities

We stay in the setting of generalized static models. With polynomial non-linearities as in (5.2.15), we can use the second part of Theorem 5.2.3 to obtain an asymptotic expansion for the solution; see Remark 5.2.18 and, in a slightly different setting, §5.3.2 for details on this. Here, we instead define a space that encodes asymptotic expansions directly in such a way that we can run a fixed point argument directly. To describe the exponents appearing in the expansion, we use index sets as introduced by Melrose, see [82].

Definition 5.2.12. An *index set* is a discrete subset $\mathscr{E}$ of $\mathbb{C} \times \mathbb{N}_0$ satisfying the conditions

(1) $(z, k) \in \mathscr{E} \Rightarrow (z, j) \in \mathscr{E}$ for $0 \leq j \leq k$,

(2) If $(z_j, k_j) \in \mathscr{E}$, $|z_j| + k_j \to \infty \Rightarrow \operatorname{Re} z_j \to \infty$.

For any index set $\mathscr{E}$, define

$$w_{\mathscr{E}}(z) = \begin{cases} \max\{k \in \mathbb{N}_0 \colon (z, k) \in \mathscr{E}\}, & (z, 0) \in \mathscr{E} \\ -\infty & \text{otherwise.} \end{cases}$$

For two index sets $\mathscr{E}, \mathscr{E}'$, define their extended union by

$$\mathscr{E} \overline{\cup} \mathscr{E}' = \mathscr{E} \cup \mathscr{E}' \cup \{(z, l + l' + 1) \colon (z, l) \in \mathscr{E}, (z, l') \in \mathscr{E}'\}$$

and their product by $\mathscr{E}\mathscr{E}' = \{(z + z', l + l') \colon (z, l) \in \mathscr{E}, (z', l') \in \mathscr{E}'\}$. We shall write $\mathscr{E}^k$ for the k-fold product of $\mathscr{E}$ with itself. Lastly, a *positive index set* is an index set $\mathscr{E}$ with the property that $\operatorname{Re} z > 0$ for all $z \in \mathbb{C}$ with $(z, 0) \in \mathscr{E}$.

Remark 5.2.13. To ensure that the class of polyhomogeneous conormal distributions with a given index set $\mathscr{E}$ is invariantly defined, Melrose [82] in addition requires that $(z, k) \in \mathscr{E}$ implies $(z + j, k) \in \mathscr{E}$ for all $j \in \mathbb{N}_0$. In particular, this is a natural condition in non-dilation-invariant settings as in Theorem 5.2.3. A convenient way to enforce this condition in all relevant situations is to enlarge the index set corresponding to the poles of the inverse of the normal operator accordingly; see the statement of Theorem 5.2.17. Observe though that this condition is not needed in the dilation-invariant cases of the solvability statements below.

Since we want to capture the asymptotic behavior of solutions near $X \cap \Omega$, we fix a cutoff $\phi \in \mathcal{C}^\infty(\mathbb{R})$ with support in $(0,\infty)$ such that $\phi \circ \mathfrak{t}_1 \equiv 1$ near $X \cap \Omega$ (we already used such a cutoff in Theorem 5.2.3), and make the following definition.

Definition 5.2.14. Let $\mathscr{E}$ be an index set, and let $s, r \in \mathbb{R}$. For $\epsilon > 0$ with the property that there is no $(z,0) \in \mathscr{E}$ with $\operatorname{Re} z = \epsilon$, define the space $\mathcal{X}^{s,r,\epsilon}_{\mathscr{E}}$ to consist of all tempered distributions v on M with support in $\bar\Omega$ such that

$$v' = v - \sum_{\substack{(z,k)\in\mathscr{E} \\ \operatorname{Re} z<\epsilon}} \tau^z (\log\tau)^k (\phi\circ\mathfrak{t}_1) v_{z,k} \in H^{s,\epsilon}_{\mathrm{b}}(\Omega)^{\bullet,-} \tag{5.2.18}$$

with $v_{z,k} \in H^r(X \cap \Omega)$.

Observe that the terms $v_{z,k}$ in the expansion (5.2.18) are uniquely determined by v, since $\epsilon > \operatorname{Re} z$ for all $z \in \mathbb{C}$ for which $(z,0)$ appears in the sum (5.2.18); then also v' are uniquely determined by v. Therefore, we can use the isomorphism

$$\mathcal{X}^{s,r,\epsilon}_{\mathscr{E}} \cong \Big(\bigoplus_{\substack{(z,k)\in\mathscr{E} \\ \operatorname{Re} z<\epsilon}} H^r(X\cap\Omega) \Big) \oplus H^{s,\epsilon}_{\mathrm{b}}(\Omega)^{\bullet,-}$$

to give $\mathcal{X}^{s,r,\epsilon}_{\mathscr{E}}$ the structure of a Banach space.

Lemma 5.2.15. *Let $\mathscr{P}, \mathscr{F}$ be positive index sets, and let $\epsilon > 0$. Define $\mathscr{E}'_0 = \mathscr{P}\overline{\cup}\mathscr{F}$ and recursively $\mathscr{E}'_{N+1} = \mathscr{P}\overline{\cup}\big(\mathscr{F} \cup \bigcup_{k\geq 2} (\mathscr{E}'_N)^k\big)$; put $\mathscr{E}_N = \{(z,k) \in \mathscr{E}'_N \colon 0 < \operatorname{Re} z \leq \epsilon\}$. Then there exists $N_0 \in \mathbb{N}$ such that $\mathscr{E}_N = \mathscr{E}_{N_0}$ for all $N \geq N_0$; moreover, the limiting index set $\mathscr{E}_\infty(\mathscr{P},\mathscr{F},\epsilon) := \mathscr{E}_{N_0}$ is finite.*

Proof. Writing $\pi_1 \colon \mathbb{C} \times \mathbb{N}_0 \to \mathbb{C}$ for the projection, one has

$$\pi_1\mathscr{E}_1 = \Big\{ z \colon 0 < \operatorname{Re} z \leq \epsilon, z = \sum_{j=1}^k z_j \colon k \geq 1, z_j \in \pi_1\mathscr{E}_0 \Big\},$$

and it is then clear that $\pi_1\mathscr{E}_N = \pi_1\mathscr{E}_1$ for all $N \geq 1$. Since $\mathscr{E}_0$ is a positive index set, there exists $\delta > 0$ such that $\operatorname{Re} z \geq \delta$ for all $z \in \mathscr{E}_0$; hence $\pi_1\mathscr{E}_\infty = \pi_1\mathscr{E}_1$ is finite.

To finish the proof, we need to show that for all $z \in \mathbb{C}$, the number $w_{\mathscr{E}_N}(z)$ stabilizes.

Defining $p(z) = w_{\mathscr{P}}(z)+1$ for $z \in \pi_1\mathscr{P}$ and $p(z) = 0$ otherwise, we have a recursion relation

$$w_{\mathscr{E}_N}(z) = p(z) + \max\left\{w_{\mathscr{F}}(z), \max_{\substack{z=z_1+\cdots+z_k\\ k\geq 2, z_j\in\pi_1\mathscr{E}_\infty}} \left\{\sum_{j=1}^{k} w_{\mathscr{E}_{N-1}}(z_j)\right\}\right\}, \quad N \geq 1. \tag{5.2.19}$$

For each z_j appearing in the sum, we have $\operatorname{Im} z_j \leq \operatorname{Im} z - \delta$. Thus, we can use (5.2.19) with z replaced by such z_j and N replaced by $N-1$ to express $w_{\mathscr{E}_N}(z)$ in terms of a finite number of $p(z_\alpha)$ and $w_{\mathscr{F}}(z_\alpha)$, $\operatorname{Im} z_\alpha \leq \operatorname{Im} z$, and a finite number of $w_{\mathscr{E}_{N-2}}(z_\beta)$, $z_\beta \leq \operatorname{Im} z - 2\delta$. Continuing in this way, after $N_0 = \lfloor(\operatorname{Im} z)/\delta\rfloor + 1$ steps we have expressed $w_{\mathscr{E}_{N(z)}}$ in terms of a finite number of $p(z_\gamma)$ and $w_{\mathscr{F}}(z_\gamma)$, $\operatorname{Im} z_\gamma \leq \operatorname{Im} z$, only, and this expression is independent of N as long as $N \geq N_0$. □

Definition 5.2.16. Let $\mathscr{P}, \mathscr{F}$ be positive index sets, and let $\epsilon > 0$ be such that there is no $(z, 0) \in \mathscr{E}_\infty(\mathscr{P}, \mathscr{F}, \epsilon)$ with $\operatorname{Re} z = \epsilon$, with $\mathscr{E}_\infty(\mathscr{P}, \mathscr{F}, \epsilon)$ as defined in the statement of Lemma 5.2.15. Then for $s, r \in \mathbb{R}$, define the Banach spaces

$$\mathcal{X}^{s,r,\epsilon}_{\mathscr{P},\mathscr{F}} := \mathcal{X}^{s,r,\epsilon}_{\mathscr{E}_\infty(\mathscr{P},\mathscr{F},\epsilon)},$$
$$^{0}\mathcal{X}^{s,r,\epsilon}_{\mathscr{P},\mathscr{F}} := \mathcal{X}^{s,r,\epsilon}_{\mathscr{E}_\infty(\mathscr{P},\mathscr{F},\epsilon)\cup\{(0,0)\}}.$$

Note that the spaces ${}^{(0)}\mathcal{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$ are Banach algebras for $s > n/2$, up to rescaling their norms, or equivalently in the sense that there is a constant $C > 0$ such that $\|uv\| \leq C\|u\|\|v\|$ for all $u, v \in {}^{(0)}\mathcal{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$. Moreover, $\mathcal{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$ interacts well with the forward solution operator S_{KG} of $\Box_g - m^2$ in the sense that $u \in \mathcal{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$, $k \geq 2$, with $\mathscr{P}$ being related to the poles of $\widehat{\mathcal{P}}(\sigma)^{-1}$, where $\mathcal{P} = \Box_g - m^2$, as will be made precise in the statement of Theorem 5.2.17 below, implies $S_{\mathrm{KG}}(u^k) \in \mathcal{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$.

We can now state the result giving an asymptotic expansion of the solution of $(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du)$ for polynomial non-linearities q.

Theorem 5.2.17. *Let $\epsilon > 0, s > \max(3/2 + \epsilon, n/2 + 1)$, and q as in (5.2.15). Moreover, if $\sigma_j \in \mathbb{C}$ are the poles of the inverse family $\widehat{\mathcal{P}}(\sigma)^{-1}$, where $\mathcal{P} = \Box_g - m^2$, and $m_j + 1$ is the order of the pole of $\widehat{\mathcal{P}}(\sigma)^{-1}$ at σ_j, let $\mathscr{P} = \{(i\sigma_j + k, \ell) \colon 0 \leq \ell \leq m_j, k \in \mathbb{N}_0\}$. Assume that $\epsilon \neq \operatorname{Re}(i\sigma_j)$ for all j, and that moreover $m > 0$, which implies that $\mathscr{P}$ is a positive index set by Lemma 5.2.5. Finally, let $\mathscr{F}$ be a positive index set.*

Then for small enough $R > 0$, there exists $C > 0$ such that for all $f \in \mathcal{X}^{s-1,s-1,\epsilon}_{\mathscr{F}}$ with *norm* $\leq C$*, the equation*

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du)$$

has a unique solution $u \in \mathcal{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$*, with norm* $\leq R$*, that depends continuously on* f*; in particular,* u *has an asymptotic expansion with remainder term in* $H^{s,\epsilon}_{\mathrm{b}}(\Omega)^{\bullet,-}$*.*

Further, if the polynomial non-linearity is of the form $q({}^{\mathrm{b}}du)$*, then for small* $R > 0$*, there exists* $C > 0$ *such that for all* $f \in \mathcal{X}^{s-1,s-1,\epsilon}_{\mathscr{F}}$ *with norm* $\leq C$*, the equation*

$$\Box_g u = f + q({}^{\mathrm{b}}du)$$

has a unique solution $u \in {}^0\mathcal{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$*, with norm* $\leq R$*, that depends continuously on* f*.*

Proof. By Theorem 5.2.3 and the definition of the space $\mathcal{X} = \mathcal{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$, we have a forward solution operator $S_{\mathrm{KG}} \colon \mathcal{X} \to \mathcal{X}$ of $\Box_g - m^2$. Thus, we can apply the Banach fixed point theorem to the operator $T \colon \mathcal{X} \to \mathcal{X}$, $Tu = S_{\mathrm{KG}}(f + q(u, {}^{\mathrm{b}}du))$, where we note that $q \colon \mathcal{X} \to \mathcal{X}$, which follows from the definition of $\mathcal{X}$ and the fact that q is a polynomial only involving terms of the form $u^j \prod_{k \leq |\alpha|} X_{\alpha,k} u$ for $j + |\alpha| \geq 2$. This condition on q also ensures that T is a contraction on a sufficiently small ball in $\mathcal{X}_+$.

For the second part, writing ${}^0\mathcal{X} = {}^0\mathcal{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$, we have a forward solution operator $S \colon \mathcal{X} \to {}^0\mathcal{X}$. But $q({}^{\mathrm{b}}du) \colon {}^0\mathcal{X} \to \mathcal{X}$, since ${}^{\mathrm{b}}d$ annihilates constants, and we can thus finish the proof as above.

The continuous dependence of the solution on the right hand side is proved as in the proof of Theorem 5.2.6. $\Box$

Note that $\epsilon > 0$ is (almost) unrestricted here, and thus we can get arbitrarily many terms in the asymptotic expansion if we work with arbitrarily high Sobolev spaces.

The condition that the polynomial $q(u, {}^{\mathrm{b}}du)$ does not involve a linear term is very important as it prevents logarithmic terms from stacking up in the iterative process used to solve the equation. Also, adding a term νu to $q(u, {}^{\mathrm{b}}du)$ effectively changes the Klein-Gordon parameter from $-m^2$ to $\nu - m^2$, which will change the location of the poles of $\widehat{P}(\sigma)^{-1}$; in the worst case, if $\nu > m^2$, this would even cause a pole to move to $\operatorname{Im}\sigma > 0$, corresponding to a resonant state that blows up exponentially in time. Lastly, let us remark that the form (5.2.17) of the non-linearity is not sufficient to obtain an expansion beyond leading order, since in the iterative procedure, logarithmic terms would stack up in the next-to-leading order term of the expansion.

Remark 5.2.18. Instead of working with the spaces ${}^{(0)}\mathcal{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$, which have the expansion built in, one could alternatively first prove the existence of a solution u in a (slightly) decaying b-Sobolev space, which then allows one to regard the polynomial non-linearity as a perturbation of the linear operator $\Box_g - m^2$; then an iterative application of the dilation-invariant result [114, Lemma 3.1] gives an expansion of the solution to the nonlinear equation. We will follow this idea in the discussion of polynomial non-linearities on asymptotically Kerr-de Sitter spaces in the next section.

5.3 Kerr-de Sitter space

In this section we analyze semilinear waves on Kerr-de Sitter space, and more generally on non-trapping spacetimes with normally hyperbolic trapping, see Definition 2.5.1. The effect of the trapping is a loss of derivatives for the linear estimates in general, but we show that at least derivatives with principal symbol vanishing on the trapped set are well-behaved. We then use these results to solve semilinear equations in the rest of the section.

For concreteness, we focus on Kerr-de Sitter spaces, see however Remark 5.3.5.

5.3.1 Linear Fredholm theory

The linear theorem in the case of normally hyperbolic trapping for $\mathcal{P} = \Box_g - \lambda$ is the following:

Theorem 5.3.1. *(Strengthened version of [114, Theorem 1.4].) Let M be a manifold with a dilation-invariant b-metric g as above, with boundary X, and let τ be the boundary defining function, $\mathcal{P}$ as in (5.2.6); suppose that $\Omega \subset M$ is a domain as above (see also (2.2.5)), and Ω is a non-trapping spacetime with normally hyperbolic trapping. Let $\phi \in \mathcal{C}^\infty(\mathbb{R})$ be as in Theorem 5.2.3. Then there exist $C' > 0$, $\varkappa > 0$, $\beta \in \mathbb{R}$ such that for $0 \leq \ell < C'$ and $s > 1/2 + \beta\ell$, $s \geq 0$, solutions $u \in H_{\mathrm{b}}^{-\infty,-\infty}(\Omega)^{\bullet,-}$ of $(\Box_g - \lambda)u = f$ with $f \in H_{\mathrm{b}}^{s-1+\varkappa,\ell}(\Omega)^{\bullet,-}$ satisfy that for some $a_{j\kappa} \in \mathcal{C}^\infty(\Omega \cap X)$ (which are the resonant states) and $\sigma_j \in \mathbb{C}$ (which are the resonances),*

$$u' = u - \sum_j \sum_{\kappa \leq m_j} \tau^{i\sigma_j} (\log \tau)^\kappa (\phi \circ \mathfrak{t}_1) a_{j\kappa} \in H_{\mathrm{b}}^{s,\ell}(\Omega)^{\bullet,-}. \tag{5.3.1}$$

Here the (semi)norms of both $a_{j\kappa}$ in $\mathcal{C}^\infty(\Omega \cap X)$ and u' in $H_{\mathrm{b}}^{s,\ell}(\Omega)^{\bullet,-}$ are bounded by a *constant times that of f in $H_{\mathrm{b}}^{s-1+\varkappa,\ell}(\Omega)^{\bullet,-}$.*

The same conclusion holds for sufficiently small (not necessary dilation-invariant!) perturbations of the metric as a symmetric bilinear form on ${}^{\mathrm{b}}TM$ provided the trapping is normally hyperbolic.

In the non-dilation-invariant setting, one could similarly proceed precisely as the proof of Theorem 5.2.3; now, in order to regain the regularity lost by treating $\mathcal{P} - N(\mathcal{P})$ as a perturbation as in (5.2.8) in the course of the contour shifting argument, while we are still in $\operatorname{Im}\sigma > 0$ (i.e. working on growing spaces), we need to appeal to Theorem 3.3.14, more specifically (3.3.23), in addition to the radial point and propagation estimates. However, an inspection of the argument, see in particular (5.2.8), reveals that a forcing term $f \in H_{\mathrm{b}}^{s-1+\varkappa,\ell}$ only yields a solution with remainder term $u' \in H_{\mathrm{b}}^{s-1,\ell}$, which is insufficient for our simple contraction mapping arguments for nonlinear equations.

Again, as mentioned after the statement of Theorem 5.2.3, the above theorem states exponential decay in $t_* := -\log\tau$, see (2.3.5) and (2.4.2), up to a finite-dimensional space of resonances.

In order to state the analogue of Theorem 5.2.3 when one has normally hyperbolic trapping in the b-sense at $\Gamma \subset {}^{\mathrm{b}}S^*_X M$, see Definition 2.3.1 and Proposition 2.3.2, we will employ the non-trapping estimates on normally isotropic function spaces $\mathcal{H}^s_{\mathrm{b},\Gamma}(M)$ and $\mathcal{H}^{*,s}_{\mathrm{b},\Gamma}(M)$, see Definition 3.3.12, established in Theorem 3.3.14. In particular, we now do not require g to be dilation-invariant. Now, if $\Omega \subset M$, as in §5.2, is such that ${}^{\mathrm{b}}S^*_{H_j}\Omega \cap \Gamma = \emptyset$, $j = 1, 2$, then spaces such as

$$\mathcal{H}^{*,s}_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-}$$

are not only well-defined, but are standard H^s_{b}-spaces near the H_j. The relations between normally isotropic and b-Sobolev spaces analogous to (3.3.21) also hold for the corresponding spaces over Ω.

Notice that elements of $\Psi^p_{\mathrm{b}}(M)$ only map $\mathcal{H}^s_{\mathrm{b},\Gamma}(M)$ to $\mathcal{H}^{*,s-p-1}_{\mathrm{b},\Gamma}(M)$, with the issues being at Γ corresponding to (3.3.21) (thus there is no distinction between the behavior on the Ω vs. the M-based spaces). However, if $A \in \Psi^p_{\mathrm{b}}(M)$ has principal symbol vanishing on Γ then

$$A\colon \mathcal{H}^s_{\mathrm{b},\Gamma}(M) \to H^{s-p}_{\mathrm{b}}(M), \quad A\colon H^s_{\mathrm{b}}(M) \to \mathcal{H}^{*,s-p}_{\mathrm{b},\Gamma}(M), \tag{5.3.2}$$

as A can be expressed as $A_+Q_+ + A_-Q_- + A_\partial\tau + \widehat{A}\widehat{P} + A_0Q_0 + R$, $A_\pm, A_0, A_\partial, \widehat{A} \in \Psi^0_{\mathrm{b}}(M)$, $R \in \Psi^{-1}_{\mathrm{b}}(M)$, with the second mapping property following by duality as $\Psi^p_{\mathrm{b}}(M)$ is closed under adjoints, and the principal symbol of the adjoint vanishes wherever that of the original operator does. Correspondingly, if $A_j \in \Psi_{\mathrm{b}}^{m_j}(M)$, $j = 1, 2$, have principal symbol vanishing at Γ then $A_1 A_2 u : \mathcal{H}^s_{\mathrm{b},\Gamma}(M) \to \mathcal{H}^{*,s-m_1-m_2}_{\mathrm{b},\Gamma}(M)$.

We consider $\mathcal{P}$ as a map

$$\mathcal{P}\colon \mathcal{H}^s_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-} \to \mathcal{H}^{s-2}_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-},$$

and let

$$\mathcal{X}^s_\Gamma = \{u \in \mathcal{H}^s_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-} :\ \mathcal{P}u \in \mathcal{Y}^{s-1}_\Gamma\}, \quad \mathcal{Y}^s_\Gamma = \mathcal{H}^{*,s}_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-}.$$

While $\mathcal{X}^s_\Gamma$ is complete,[15] it is a slightly exotic space, unlike $\mathcal{X}^s$ in Theorem 4.2.4 which is a coisotropic space depending on Σ (and thus the principal symbol of $\mathcal{P}$) only, since elements of $\Psi^p_{\mathrm{b}}(M)$ only map $\mathcal{H}^s_{\mathrm{b},\Gamma}(M)$ to $\mathcal{H}^{*,s-p-1}_{\mathrm{b},\Gamma}(M)$ as remarked earlier. Correspondingly, $\mathcal{X}^s_\Gamma$ actually depends on $\mathcal{P}$ modulo $\Psi^0_{\mathrm{b}}(M)$ plus first order pseudodifferential operators of the form $A_1 A_2$, $A_1 \in \Psi^0_{\mathrm{b}}(M)$, $A_2 \in \Psi^1_{\mathrm{b}}(M)$, both with principal symbol vanishing at Γ – here the operators should have Schwartz kernels supported away from the H_j; near H_j (but away from Γ), one should say $\mathcal{P}$ matters modulo $\mathrm{Diff}^1_{\mathrm{b}}(M)$, i.e. only the principal symbol of $\mathcal{P}$ matters.

We then have:

Theorem 5.3.2. *Suppose $s \geq 3/2$, and that the inverse of the Mellin transformed normal operator $\widehat{\mathcal{P}}(\sigma)^{-1}$ has no poles with $\operatorname{Im}\sigma \geq 0$. Then*

$$\mathcal{P}\colon \mathcal{X}^s_\Gamma \to \mathcal{Y}^{s-1}_\Gamma$$

is invertible, giving the forward solution operator.

Proof. First, with $r < -1/2$, thus with dual spaces having weight $\tilde{r} > 1/2$, Theorem 4.2.4 holds without changes as Theorem 3.3.14 gives non-trapping estimates in this case on the standard b-Sobolev spaces. In particular, if $r \ll 0$, $\operatorname{Ker}\mathcal{P}$ is trivial even on $H_{\mathrm{b}}^{s-1/2,r}(\Omega)^{\bullet,-}$, hence certainly on its subspace $\mathcal{H}^s_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-}$. Similarly, $\operatorname{Ker}\mathcal{P}^*$ is trivial on $H_{\mathrm{b}}^{s,\tilde{r}}(\Omega)^{-,\bullet}$,

[15] Also, elements of $\mathcal{C}^\infty(\Omega)$ vanishing to infinite order at H_1 and $X \cap \Omega$ are dense in $\mathcal{X}^s_\Gamma$. Indeed, in view of [91, Lemma A.3] the only possible issue is at Γ, thus the distinction between Ω and M may be dropped. To complete the argument, one proceeds as in the quoted lemma, using the ellipticity of σ at Γ, letting $\Lambda_n \in \Psi_{\mathrm{b}}^{-\infty}(M)$, $n \in \mathbb{N}$, be a quantization of $\phi(\sigma/n)a$, $a \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$ supported in a neighborhood of Γ, identically 1 near Γ, $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$, noting that $[\Lambda_n, \mathcal{P}] \in \Psi_{\mathrm{b}}^{-\infty}(M)$ is uniformly bounded in $\Psi^0_{\mathrm{b}}(M) + \tau\Psi^1_{\mathrm{b}}(M)$ in view of (2.1.3), and thus for $u \in \mathcal{X}^s_\Gamma$, $\mathcal{P}\Lambda_n u = \Lambda_n \mathcal{P}u + [\mathcal{P}, \Lambda_n]u \to \mathcal{P}u$ in $\mathcal{H}^{*,s-1}_{\mathrm{b},\Gamma}$ since $[\mathcal{P}, \Lambda_n]$ is uniformly bounded $H_{\mathrm{b}}^{s-1/2} \cap H_{\mathrm{b}}^{s,-1/2} \to H_{\mathrm{b}}^{s-1/2} \cap H_{\mathrm{b}}^{s-1,1/2}$, and thus $\mathcal{H}^s_{\mathrm{b},\Gamma} \to \mathcal{H}^{*,s-1}_{\mathrm{b},\Gamma}$ by (3.3.21).

$\widetilde{r} \gg 0$. Therefore, if $r < -1/2$ and $f \in H_{\mathrm{b}}^{-1,r}(\Omega)^{\bullet,-}$, there exists $u \in H_{\mathrm{b}}^{0,r}(\Omega)^{\bullet,-}$ with $\mathcal{P}u = f$. Further, making use of the non-trapping estimates in Theorem 3.3.14, if $r < 0$ and $f \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}$, then the argument of Theorem 5.2.3 improves this statement to $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$.

In particular, if $f \in \mathcal{H}_{\mathrm{b},\Gamma}^{*,s-1}(\Omega)^{\bullet,-} \subset H_{\mathrm{b}}^{s-1,0}(\Omega)^{\bullet,-}$, then $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$ for $r < 0$. This can be improved using the argument of Theorem 5.2.3. Indeed, with $-1 \leq r < 0$ arbitrary, $\mathcal{P} - N(\mathcal{P}) \in \tau\mathrm{Diff}_{\mathrm{b}}^2(M)$ implies as in (5.2.8) that

$$N(\mathcal{P})u = f - \widetilde{f}, \quad \widetilde{f} = (\mathcal{P} - N(\mathcal{P}))u \in H_{\mathrm{b}}^{s-2,r+1}(\Omega)^{\bullet,-}.$$

But $f \in \mathcal{H}_{\mathrm{b},\Gamma}^{*,s-1}(\Omega)^{\bullet,-} \subset H_{\mathrm{b}}^{s-1,0}(\Omega)^{\bullet,-}$, hence the right hand side is in $H_{\mathrm{b}}^{s-2,0}(\Omega)^{\bullet,-}$; thus the dilation-invariant result, [114, Lemma 3.1], gives $u \in H_{\mathrm{b}}^{s-1,0}(\Omega)^{\bullet,-}$. This can then be improved further since in view of $\mathcal{P}u = f \in \mathcal{H}_{\mathrm{b},\Gamma}^{*,s-1}(\Omega)^{\bullet,-}$, propagation of singularities, most crucially Theorem 3.3.14, yields $u \in \mathcal{H}_{\mathrm{b},\Gamma}^{s}(\Omega)^{\bullet,-}$. This completes the proof of the theorem. $\square$

This result shows the importance of controlling the resonances in $\operatorname{Im}\sigma \geq 0$. For the wave operator on exact 4-dimensional Kerr-de Sitter space, Dyatlov's analysis [39, 40] shows that the zero resonance of $\Box_g$ is the only one in $\operatorname{Im}\sigma \geq 0$, the residue at 0 having constant functions as its range; in §7.5, we will prove a generalization of this result to perturbations of higher-dimensional Schwarzschild-de Sitter spacetimes that also covers the case of differential form-valued waves. (The very precise analysis of [41], relying on the exact form of the Kerr-de Sitter metric, could presumably be used in nonlinear applications as well, giving a much more precise resonance expansion of solutions, but we will not consider this here.) For the Klein-Gordon operator $\Box_g - m^2$, the statement is even better from our perspective as there are no resonances in $\operatorname{Im}\sigma \geq 0$ for $m > 0$ small. This is pointed out in [40]; we give a direct proof based on perturbation theory.

Lemma 5.3.3. *Let $\mathcal{P} = \Box_g$ on exact Kerr-de Sitter space. Then for small $m > 0$, all poles of $(\widehat{\mathcal{P}}(\sigma) - m^2)^{-1}$ have strictly negative imaginary part.*

Proof. By the perturbation theory results in Appendix A, the inverse family of $\widehat{\mathcal{P}}(\sigma) - \lambda$ has a simple pole at $\sigma(\lambda)$ coming with a single resonant state $\phi(\lambda)$ and a dual state $\psi(\lambda)$, with analytic dependence on λ, where $\sigma(0) = 0, \phi(0) \equiv 1, \psi(0) = 1_{\{\mu > 0\}}$ (see also Theorem 7.5.1), where we use the notation of §2.4. Differentiating $\widehat{\mathcal{P}}(\sigma(\lambda))\phi(\lambda) = \lambda\phi(\lambda)$ with respect to λ

and evaluating at $\lambda = 0$ gives

$$\sigma'(0)\partial_\sigma\widehat{\mathcal{P}}(0)\phi(0) + \widehat{\mathcal{P}}(0)\phi'(0) = \phi(0).$$

Pairing this with $\psi(0)$, which is orthogonal to $\operatorname{Ran}\widehat{\mathcal{P}}(0)$, yields

$$\sigma'(0) = \frac{\langle\psi(0), \phi(0)\rangle}{\langle\psi(0), \partial_\sigma\widehat{\mathcal{P}}(0)\phi(0)\rangle},$$

Since $\phi(0) = 1$ and $\psi(0) = 1_{\{\mu>0\}}$, this implies

$$\operatorname{sgn}\operatorname{Im}\sigma'(0) = -\operatorname{sgn}\operatorname{Im}\langle\psi(0), \partial_\sigma\widehat{\mathcal{P}}(0)\phi(0)\rangle. \tag{5.3.3}$$

To find the latter quantity, we note that the only terms in the expression of the d'Alembertian that could possibly yield a non-zero contribution here are terms involving τD_τ and either D_r, D_ϕ or D_θ. Concretely, using the explicit form of the dual metric, see [114, Equation (6.1)],[16] G in the new coordinates $t_* = t - h(r), \phi_* = \phi - P(r), \tau = e^{-t_*}$, with $h(r), P(r)$ as in (2.4.3),

$$\begin{aligned} G = -\rho^{-2}\Big(&\mu(\partial_r + h'(r)\tau\partial_\tau - P'(r)\partial_{\phi_*})^2 + \frac{(1+\gamma)^2}{\kappa\sin^2\theta}(-a\sin^2\theta\tau\partial_\tau + \partial_{\phi_*})^2 + \kappa\partial_\theta^2 \\ &- \frac{(1+\gamma)^2}{\mu}(-(r^2+a^2)\tau\partial_\tau + a\partial_{\phi_*})^2\Big), \end{aligned}$$

and its determinant $|\det G|^{1/2} = (1+\gamma)^2\rho^{-2}(\sin\theta)^{-1}$, we see that the only non-zero contribution to the right hand side of (5.3.3) comes from the term

$$\begin{aligned} -(1+\gamma)^2\rho^{-2}(\sin\theta)^{-1}&D_r\big((1+\gamma)^{-2}\rho^2\sin\theta\rho^{-2}\mu h'(r)\big)\tau D_\tau \\ &= i\rho^{-2}\partial_r(\mu h'(r))\tau D_\tau \end{aligned}$$

of the d'Alembertian. Mellin transforming this amounts to replacing τD_τ by σ; then differentiating the result with respect to σ gives

$$\langle\psi(0), \partial_\sigma\widehat{\mathcal{P}}(0)\phi_*(0)\rangle = i\int_{\mu>0}\rho^{-2}\partial_r(\mu h'(r))\,d\mathrm{vol}$$

[16]What we call $t, t_*, \phi, \phi_*, \mu, h(r), P(r)$ here is denoted $\widetilde{t}, t, \widetilde{\phi}, \phi, \widetilde{\mu}, -h(r), -P(r)$ in [114].

$$= i \int_0^\pi \int_0^{2\pi} \int_{r_-}^{r_+} (1+\gamma)^{-2} \sin\theta \, \partial_r(\mu h'(r)) \, dr \, d\phi_* \, d\theta$$
$$= \frac{4\pi i}{(1+\gamma)^2} [(\mu h'(r))|_{r_+} - (\mu h'(r))|_{r_-}]. \tag{5.3.4}$$

Since the singular part of $h'(r)$ at $r_\pm$ (which are the roots of μ) is $h'(r) = \pm\frac{1+\gamma}{\mu}(r^2+a^2)$, the right hand side of (5.3.4) is positive up to a factor of i; thus $\operatorname{Im}\sigma'(0) < 0$ as claimed. □

In other words, for small mass $m > 0$, there are no resonances σ of the Klein-Gordon operator with $\operatorname{Im}\sigma \geq -\epsilon_0$ for some $\epsilon_0 > 0$. Therefore, the expansion of u as in (5.3.1) no longer has a constant term. Correspondingly, for $\epsilon \in \mathbb{R}$, $\epsilon \leq \epsilon_0$, Theorem 5.3.1 gives the forward solution operator

$$S_{\mathrm{KG,I}} \colon H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \tag{5.3.5}$$

in the dilation-invariant setting. Further, Theorem 5.3.2 is applicable and gives the forward solution operator

$$S_{\mathrm{KG}} \colon \mathcal{H}_{\mathrm{b},\Gamma}^{*,s-1}(\Omega)^{\bullet,-} \to \mathcal{H}_{\mathrm{b},\Gamma}^{s}(\Omega)^{\bullet,-} \tag{5.3.6}$$

on normally isotropic spaces, without the assumption of dilation-invariance.

For semilinear applications, for non-linearities without derivatives, it is important that the loss of derivatives $\varkappa$ in the space $H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}$ is ≤ 1. This is not explicitly specified in the paper of Wunsch and Zworski [124], though their proof directly gives that, for small $\epsilon > 0$, $\varkappa$ can be taken proportional to ϵ, and there is $\epsilon_0' > 0$ such that $\varkappa \in (0,1]$ for $\epsilon < \epsilon_0'$; see especially the part before [124, §4.4]. We reduce $\epsilon_0 > 0$ above if needed so that $\epsilon_0 \leq \epsilon_0'$; then (5.3.5) holds with $\varkappa = c\epsilon \in (0,1]$ if $\epsilon < \epsilon_0$, where $c > 0$. In fact however, one does not need to go through the proof of [124], for the Phragmén-Lindelöf theorem allows one to obtain the same conclusion from their final result:

Lemma 5.3.4. *Suppose $h \colon U \to E$ is a holomorphic function on the half strip $U = \{z \in \mathbb{C} \colon 0 \leq \operatorname{Im} z \leq c, \operatorname{Re} z \geq 1\}$ which is continuous on $\overline{U}$, with values in a Banach space E, and suppose moreover that there are constants $A, C > 0$ such that*

$$\|h(z)\| \leq C|z|^{k_1}, \quad \operatorname{Im} z = 0,$$
$$\|h(z)\| \leq C|z|^{k_2}, \quad \operatorname{Im} z = c,$$
$$\|h(z)\| \leq C\exp(A|z|), \quad z \in \overline{U}.$$

Then there is a constant $C' > 0$ *such that*

$$\|h(z)\| \leq C'|z|^{k_1\left(1-\frac{\operatorname{Im} z}{c}\right)+k_2\frac{\operatorname{Im} z}{c}}$$

for all $z \in \overline{U}$.

Proof. Consider the function $f(z) = z^{k_1 - i\frac{k_2-k_1}{c}z}$, which is holomorphic on a neighborhood of $\overline{U}$. Writing $z \in \overline{U}$ as $z = x + iy$ with $x, y \in \mathbb{R}$, one has

$$\begin{aligned} |f(z)| &= |z|^{k_1} \exp\left(\operatorname{Im}\left(\frac{k_2-k_1}{c} z \log z\right)\right) \\ &= |z|^{k_1}|z|^{\frac{k_2-k_1}{c}\operatorname{Im} z} \exp\left(\frac{k_2-k_1}{c} x \arctan(y/x)\right). \end{aligned}$$

Noting that $|x\arctan(y/x)| = y|(x/y)\arctan(y/x)|$ is bounded by c for all $x + iy \in \overline{U}$, we conclude that

$$e^{-|k_2-k_1|}|z|^{k_1\left(1-\frac{\operatorname{Im} z}{c}\right)+k_2\frac{\operatorname{Im} z}{c}} \leq |f(z)| \leq e^{|k_2-k_1|}|z|^{k_1\left(1-\frac{\operatorname{Im} z}{c}\right)+k_2\frac{\operatorname{Im} z}{c}}.$$

Therefore, $f(z)^{-1}h(z)$ is bounded by a constant C' on $\partial\overline{U}$, and satisfies an exponential bound for $z \in U$. By the Phragmén-Lindelöf theorem, $\|f(z)^{-1}h(z)\|_E \leq C'$, and the claim follows. □

Since for any $\delta > 0$, we can bound $|\log z| \leq C_\delta |z|^\delta$ for $|\operatorname{Re} z| \geq 1$, we obtain that the inverse family $R(\sigma) = \widehat{\mathcal{P}}(\sigma)^{-1}$ of the normal operator of $\Box_g$ on (asymptotically) Kerr-de Sitter spaces as in [114], here in the setting of artificial boundaries as opposed to complex absorption, satisfies a bound

$$\|R(\sigma)\|_{|\sigma|^{-(s-1)}H^{s-1}_{|\sigma|^{-1}}(X\cap\Omega)\to|\sigma|^{-s}H^s_{|\sigma|^{-1}}(X\cap\Omega)} \leq C_\delta|\sigma|^{-1+\varkappa'+\delta}$$

for any $\delta > 0$, $\operatorname{Im}\sigma \geq -c\varkappa'$ and $|\operatorname{Re}\sigma|$ large. Therefore, as mentioned above, by the proof of Theorem 5.3.1, i.e. [114, Theorem 1.4], in particular using [114, Lemma 3.1], we can assume $\varkappa \in (0, 1]$ in the dilation-invariant result, Theorem 5.3.1, if we take $C' > 0$ small enough, i.e. if we do not go too far into the lower half plane $\operatorname{Im}\sigma < 0$, which amounts to only taking terms in the expansion (5.3.1) which decay to at most some fixed order, which we may assume to be less than $-\operatorname{Im}\sigma_j$ for all resonances σ_j.

Remark 5.3.5. As in §5.2.2, we can again consider general non-trapping spacetimes with normally hyperbolic trapping, equipped with metrics which have a conormal part as well, and moreover treat operators acting on vector bundles. The b-regularity analysis again only relies on principal symbol considerations (plus contributions from the subprincipal symbol at radial points, which only shifts the regularity requirement for forward solutions); but in order to obtain the resonance expansion with exponentially decaying remainder as in Theorem 5.3.1, one needs in addition high energy estimates for the normal operator family in a strip below the real axis. In the scalar setting, these are well understood [44, 42, 94, 124], but for operators on bundles, they require additional work; see Chapter 6 for their proof for wave operators on subbundles of the tensor bundle on Kerr-de Sitter spaces. Furthermore, one needs to know the location of resonances: If there are none in $\operatorname{Im}\sigma \geq 0$, our methods for semilinear equations in this section go through; if there is a simple resonance at $\sigma = 0$, as is the case for the scalar wave equation on Kerr-de Sitter space, we cannot prove any semilinear results with the methods employed in the present chapter. However, with more machinery, we can even handle very general quasilinear wave equations in this case, see Chapters 8 and 9.

5.3.2 A class of semilinear equations; polynomial non-linearities

In the following semilinear applications, let us fix $\varkappa \in (0,1]$ and ϵ_0 as explained before Lemma 5.3.4, so that we have the forward solution operator $S_{\mathrm{KG,I}}$ as in (5.3.5). We then have statements paralleling Theorems 5.2.6, 5.2.17 and Corollary 5.2.9, see Theorems 5.3.6, 5.3.10 and Corollary 5.3.9, respectively.

Theorem 5.3.6. *Suppose (M,g) is dilation-invariant. Let $-\infty < \epsilon < \epsilon_0, s > 1/2 + \beta\epsilon$, $s \geq 1$, and let $q\colon H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}$ be a continuous function with $q(0) = 0$ such that there exists a continuous non-decreasing function $L\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying*

$$\|q(u) - q(v)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R.$$

Then there is a constant $C_L > 0$ so that the following holds: If $L(0) < C_L$, then for small $R > 0$, there exists $C > 0$ such that for all $f \in H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}$ with norm $\leq C$, the equation

$$(\Box_g - m^2)u = f + q(u)$$

has a unique solution $u \in H_b^{s,\epsilon}(\Omega)^{\bullet,-}$*, with norm* $\leq R$*, that depends continuously on* f*.*

More generally, suppose

$$q\colon H_b^{s,\epsilon}(\Omega)^{\bullet,-} \times H_b^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-} \to H_b^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}$$

satisfies $q(0,0) = 0$ *and*

$$\|q(u,w) - q(u',w')\| \leq L(R)(\|u-u'\| + \|w-w'\|)$$

provided $\|u\| + \|w\|, \|u'\| + \|w'\| \leq R$*, where we use the norms corresponding to the map* q*, for a continuous non-decreasing function* $L\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$*. Then there is a constant* $C_L > 0$ *so that the following holds: If* $L(0) < C_L$*, then for small* $R > 0$*, there exists* $C > 0$ *such that for all* $f \in H_b^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}$ *with norm* $\leq C$*, the equation*

$$(\Box_g - m^2)u = f + q(u, \Box_g u)$$

has a unique solution $u \in H_b^{s,\epsilon}(\Omega)^{\bullet,-}$*, with* $\|u\|_{H_b^{s,\epsilon}} + \|\Box_g u\|_{H_b^{s-1+\varkappa,\epsilon}} \leq R$*, that depends continuously on* f*.*

Proof. We use the proof of the first part of Theorem 5.2.6, where in the current setting the solution operator $S_{\mathrm{KG,I}}$ maps $H_b^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-} \to H_b^{s,\epsilon}(\Omega)^{\bullet,-}$, and the contraction map is $T\colon H_b^{s,\epsilon}(\Omega)^{\bullet,-} \to H_b^{s,\epsilon}(\Omega)^{\bullet,-}$, $Tu = S_{\mathrm{KG,I}}(f + q(u))$.

For the general statement, we follow the proof of the second part of Theorem 5.2.6, where we now instead use the space

$$\mathcal{X} = \{u \in H_b^{s,\epsilon}(\Omega)^{\bullet,-} : \Box_g u \in H_b^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}\}$$

with norm

$$\|u\|_{\mathcal{X}} = \|u\|_{H_b^{s,\epsilon}} + \|\Box_g u\|_{\tau^\epsilon H_b^{s-1+\varkappa}}.$$

which is a Banach space by the same argument as in the proof of Theorem 5.2.6. ☐

We have a weaker statement in the general, non-dilation-invariant case, where we work in unweighted spaces.

Theorem 5.3.7. *Let* $s \geq 1$*, and suppose* $q\colon H_b^s(\Omega)^{\bullet,-} \to H_b^s(\Omega)^{\bullet,-}$ *is a continuous function with* $q(0) = 0$ *such that there exists a continuous non-decreasing function* $L\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$

satisfying

$$\|q(u) - q(v)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R.$$

Then there is a constant $C_L > 0$ *so that the following holds: If* $L(0) < C_L$*, then for small* $R > 0$*, there exists* $C > 0$ *such that for all* $f \in H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$ *with norm* $\leq C$*, the equation*

$$(\Box_g - m^2)u = f + q(u)$$

has a unique solution $u \in H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$*, with norm* $\leq R$*, that depends continuously on* f*.*

An analogous statement holds for non-linearities $q = q(u, \Box_g u)$ *which are continuous maps* $q\colon H_{\mathrm{b}}^s(\Omega)^{\bullet,-} \times H_{\mathrm{b}}^s(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$*, vanish at* $(0,0)$ *and have a small Lipschitz constant near* 0*.*

Proof. Since

$$S_{\mathrm{KG}}\colon H_{\mathrm{b}}^s(\Omega)^{\bullet,-} \subset \mathcal{H}_{\mathrm{b},\Gamma}^{*,s-1/2}(\Omega)^{\bullet,-} \to \mathcal{H}_{\mathrm{b},\Gamma}^{s+1/2}(\Omega)^{\bullet,-} \subset H_{\mathrm{b}}^s(\Omega)^{\bullet,-},$$

by (3.3.21) and (5.3.6), this follows again from the Banach fixed point theorem. □

Remark 5.3.8. The proof of Theorem 5.3.2 shows that equations on function spaces with negative weights (i.e. growing near infinity) behave as nicely as equations on the static part of asymptotically de Sitter spaces, discussed in §5.2. However, naturally occurring non-linearities (e.g., polynomials) will not be continuous nonlinear operators on such growing spaces.

Corollary 5.3.9. *If* $s > n/2$*, the hypotheses of Theorem 5.3.7 hold for non-linearities* $q(u) = cu^p$*,* $p \geq 2$ *integer,* $c \in \mathbb{C}$*, as well as* $q(u) = q_0 u^p$*,* $q_0 \in H_{\mathrm{b}}^s(M)$*.*

Thus for small $m > 0$ *and* $R > 0$*, there exists* $C > 0$ *such that for all* $f \in H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$ *with norm* $\leq C$*, the equation*

$$(\Box_g - m^2)u = f + q(u)$$

has a unique solution $u \in H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$*, with norm* $\leq R$*, that depends continuously on* f*.*

If f satisfies stronger decay assumptions, then u does as well. More precisely, writing

$$\mathcal{P} = \Box_g - m^2,$$

the inverse normal operator family $\widehat{\mathcal{P}}(\sigma)^{-1}$ has poles only in $\operatorname{Im}\sigma < 0$ for small $m > 0$ (cf.

Lemma 5.3.3 and [40, 114]). Then, defining the spaces $\mathcal{X}^{s,r,\epsilon}_{\mathscr{F}}$ and $\mathcal{X}^{s,r,\epsilon}_{\mathscr{P},\mathscr{F}}$ analogously to the corresponding spaces in §5.2.4, we have the following result:

Theorem 5.3.10. *Fix $0 < \epsilon < \min\{C', 1/2\}$ and let $s \gg s' \geq \max(1/2 + \beta\epsilon, n/2, 1 + \varkappa)$. (A concrete bound for s will be given in the course of the proof, see equation 5.3.8.) Let*

$$q(u) = \sum_{p=2}^{d} q_p u^p, \quad q_p \in H^s_{\mathrm{b}}(M).$$

Moreover, if $\sigma_j \in \mathbb{C}$ are the poles of $\mathcal{P}(\sigma)^{-1}$, and $m_j + 1$ is the order of the pole of $\mathcal{P}(\sigma)^{-1}$ at σ_j, let $\mathscr{P} = \{(i\sigma_j + k, \ell) : 0 \leq \ell \leq m_j, k \in \mathbb{N}_0\}$. Assume that $\epsilon \neq \mathrm{Re}(i\sigma_j)$ for all j, and that $m > 0$ is so small that $\mathscr{P}$ is a positive index set. Finally, let $\mathscr{F}$ be a positive index set.

Then for small enough $R > 0$, there exists $C > 0$ such that for all $f \in \mathcal{X}^{s,s,\epsilon}_{\mathscr{F}}$ with norm $\leq C$, the equation

$$(\Box_g - m^2)u = f + q(u) \tag{5.3.7}$$

has a unique solution $u \in \mathcal{X}^{s',s',\epsilon}_{\mathscr{P},\mathscr{F}}$, with norm $\leq R$, that depends continuously on f; in particular, u has an asymptotic expansion with remainder in $H^{s',\epsilon}_{\mathrm{b}}(\Omega)^{\bullet,-}$.

Proof. Let us write $\mathcal{P} = \Box_g - m^2$. Let $\delta < 1/2$ be such that $0 < 2\delta < \mathrm{Re}\, z$ for all $(z, 0) \in \mathscr{F}$, then $f \in H^{s,2\delta}_{\mathrm{b}}(\Omega)^{\bullet,-}$. Now, for $u \in H^{s,\delta}_{\mathrm{b}}(\Omega)^{\bullet,-}$, consider $Tu := S_{\mathrm{KG}}(f + q(u))$. First of all, $f + q(u) \in H^{s,2\delta}_{\mathrm{b}}(\Omega)^{\bullet,-} \subset H^s_{\mathrm{b}}(\Omega)^{\bullet,-}$, thus the proof of Theorem 5.3.2 shows that we have $Tu \in H^{s+1,r}_{\mathrm{b}}(\Omega)^{\bullet,-}$, $r < 0$ arbitrary. Therefore,

$$N(\mathcal{P})u = f + q(u) + (N(\mathcal{P}) - \mathcal{P})u \in H^{s,2\delta}_{\mathrm{b}}(\Omega)^{\bullet,-} + H^{s-1,r+1}_{\mathrm{b}}(\Omega)^{\bullet,-} \subset H^{s-1,2\delta}_{\mathrm{b}}(\Omega)^{\bullet,-},$$

and thus if $\delta > 0$ is sufficiently small, namely, $\delta < \inf\{-\mathrm{Im}\,\sigma_j\}/2$, Theorem 5.3.1 implies $u \in H^{s-\varkappa,2\delta}_{\mathrm{b}}(\Omega)^{\bullet,-}$. Since we can choose $\varkappa = c\delta$ for some constant $c > 0$, we obtain

$$Tu \in \bigcap_{r>0} H^{s+1,r}_{\mathrm{b}}(\Omega)^{\bullet,-} \cap H^{s-c\delta,2\delta}_{\mathrm{b}}(\Omega)^{\bullet,-} \subset \bigcap_{r'>0} H^{s,2\delta-2c\delta^2/(1+c\delta)-r'}_{\mathrm{b}}(\Omega)^{\bullet,-}$$

by interpolation. In particular, choosing $\delta > 0$ even smaller if necessary, we obtain $Tu \in H^{s,\delta}_{\mathrm{b}}(\Omega)^{\bullet,-}$. Applying the Banach fixed point theorem to the map T thus gives a solution $u \in H^{s,\delta}_{\mathrm{b}}(\Omega)^{\bullet,-}$ to the equation (5.3.7).

For this solution u, we obtain

$$N(\mathcal{P})u = \mathcal{P}u + (N(\mathcal{P}) - \mathcal{P})u \in H_{\mathrm{b}}^{s,2\delta} + H_{\mathrm{b}}^{s-2,\delta+1} \subset H_{\mathrm{b}}^{s-2,2\delta}$$

since q only has quadratic and higher terms. Hence Theorem 5.3.1 implies that $u = u_1 + u'$, where u_1 is an expansion with terms coming from poles of $\widehat{\mathcal{P}}^{-1}$ whose decay order lies between δ and 2δ, and $u' \in H_{\mathrm{b}}^{s-1-\varkappa,2\delta}(\Omega)^{\bullet,-}$. This in turn implies that $f + q(u)$ has an expansion with remainder term in $H_{\mathrm{b}}^{s-1-\varkappa,\min\{4\delta,\epsilon\}}(\Omega)^{\bullet,-}$, thus

$$N(\mathcal{P})u \in H_{\mathrm{b}}^{s-3-\varkappa,\min\{4\delta,\epsilon\}}(\Omega)^{\bullet,-} \text{ plus an expansion,}$$

and we proceed iteratively, until, after k more steps, we have $4 \cdot 2^k\delta \geq \epsilon$, and then u has an expansion with remainder term $H_{\mathrm{b}}^{s-3-2k-\varkappa,\epsilon}(\Omega)^{\bullet,-}$ provided we can apply Theorem 5.3.1 in the iterative procedure, i.e. provided $s - 3 - 2k - \varkappa =: s' > \max(1/2 + \beta\epsilon, n/2, 1 + \varkappa)$. This is satisfied if

$$s > \max(1/2 + \beta\epsilon, n/2, 1 + \varkappa) + 2\lceil \log_2(\epsilon/\delta) \rceil + \varkappa - 1. \tag{5.3.8}$$

□

5.3.3 Semilinear equations with derivatives in the non-linearities

Theorem 5.3.2 allows one to solve even semilinear equations with derivatives in some cases. For instance, in the case of 4-dimensional Schwarzschild-de Sitter space, within $\Sigma \cap {}^{\mathrm{b}}S_X^*M$, Γ is given by $r = r_p$, $\sigma_1(D_r) = 0$, where $r_p = \frac{3}{2}r_s = 2M_\bullet$ is the radius of the photon sphere, see e.g. [114, §6.4], and similarly in higher dimensions, see §2.3 and equation (2.3.3) for the radius r_p of the photon sphere in general. Thus, nonlinear terms such as $(r - r_p)(\partial_r u)^2$ are allowed for $s > \frac{n}{2} + 1$ since $\partial_r : \mathcal{H}_{\mathrm{b},\Gamma}^s(M) \to H_{\mathrm{b}}^{s-1}(M)$, with the latter space being an algebra, while multiplication by $r - r_p$ maps this space to $\mathcal{H}_{\mathrm{b},\Gamma}^{*,s-1}$ by (5.3.2). Thus, a straightforward modification of Theorem 5.3.7, applying the fixed point theorem on the normally isotropic spaces directly, gives well-posedness.

5.4 Asymptotically de Sitter spaces: global approach

We can approach the problem of solving nonlinear wave equations on global asymptotically de Sitter spaces in two ways: Either, we proceed as in the previous two sections, first showing invertibility of the linear operator on suitable spaces and then applying the contraction mapping principle to solve the nonlinear problem; or we use the solvability results from §5.2 for backward light cones from points at future conformal infinity and glue the solutions on all these 'static' parts together to obtain a global solution. The first approach, which we will follow in §§5.4.1-5.4.4, has the disadvantage that the conditions on the non-linearity that guarantee the existence of solutions are quite restrictive, however in case the conditions are met, one has good decay estimates for solutions. The second approach on the other hand, detailed in §5.4.5, allows many of the non-linearities, suitably reinterpreted, that work on 'static' asymptotically de Sitter spaces (i.e. backward light cones), but the decay estimates for solutions are quite weak relative to the decay of the forcing term because of the gluing process.

5.4.1 The linear framework

Let g be the metric on an n-dimensional asymptotically de Sitter space X, see §2.2.2,[17] and let μ denote a defining function of the boundaries at future and past infinity. Then, following [114, §4], the operator[18]

$$P_\sigma = \mu^{-1/2}\mu^{i\sigma/2-(n+1)/4}\left(\Box_g - \left(\frac{n-1}{2}\right)^2 - \sigma^2\right)\mu^{-i\sigma/2+(n+1)/4}\mu^{-1/2} \qquad (5.4.1)$$

extends non-degenerately to an operator on a closed manifold $\widetilde{X}$ which contains the compactification $\overline{X}$ of the asymptotically de Sitter space as a submanifold with boundary Y, where $Y = Y_- \cup Y_+$ has two connected components, which we call the boundary of X at past, resp. future, infinity; non-degenerately here means that near $Y_\pm$, P_σ fits into the framework of [114]. Here, $\mu = 0$ is the defining function of Y, and $\mu > 0$ is the interior of the asymptotically de Sitter space. Moreover, null-bicharacteristics of P_σ tend to $Y_\pm$ as $t \to \pm\infty$.

Following [117], let us in fact assume that $\widetilde{X} = \overline{C_-} \cup \overline{X} \cup \overline{C_+}$ is the union of the compactifications of asymptotically de Sitter space X and two asymptotically hyperbolic caps $C_\pm$; one might need to take two copies of X to construct $\widetilde{X}$, see [117]. See Figure 5.2.

[17] We use a slightly different notation here to make the notation less cumbersome: The focus here is on the global space rather than on its static patches.

[18] P_σ in our notation corresponds to P_σ^* in [114], the latter operator being the one for which one solves the forward problem.

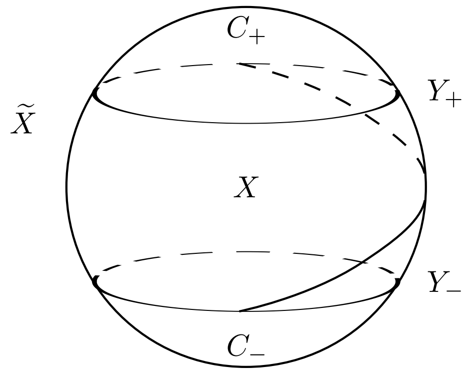

Figure 5.2: Embedding of the asymptotically de Sitter space X as an open subset of the closed manifold $\widetilde{X}$, here drawn as a sphere. The boundary at future (past) infinity of the compactification $\overline{X}$ is Y_+ (Y_-). One obtains $\widetilde{X}$ from $\overline{X}$ by adding asymptotically hyperbolic caps $C_\pm$. Also shown is a null-geodesic on X.

Then P_σ is the restriction to X of an operator $\widetilde{P}_\sigma \in \mathrm{Diff}^2(\widetilde{X})$, which is Fredholm as a map

$$\widetilde{P}_\sigma\colon \widetilde{\mathcal{X}}^s \to \widetilde{\mathcal{Y}}^{s-1}, \quad \widetilde{\mathcal{X}}^s = \{u \in H^s \colon \widetilde{P}_\sigma u \in H^{s-1}\}, \quad \widetilde{\mathcal{Y}}^{s-1} = H^{s-1},$$

where $s \in \mathcal{C}^\infty(S^*\widetilde{X})$, monotone along the bicharacteristic flow, is such that $s|_{N^*Y_-} > 1/2 - \operatorname{Im}\sigma$, $s|_{N^*Y_+} < 1/2 - \operatorname{Im}\sigma$, and s is constant near $S^*Y_\pm$.[19] The spaces H^s are variable order Sobolev spaces as in [8, §1 and Appendix].

Restricting our attention to X, we define the space $H^s(X)^{\bullet,-}$ to be the completion in $H^s(X)$ of the space of $\mathcal{C}^\infty$ functions that vanish to infinite order at Y_-; thus the superscripts indicate that distributions in $H^s(X)^{\bullet,-}$ are supported distributions near Y_- and extendible distributions near Y_+. Then, define the spaces

$$\mathcal{X}^s = \{u \in H^s(X)^{\bullet,-} \colon P_\sigma u \in H^{s-1}(X)^{\bullet,-}\}, \quad \mathcal{Y}^{s-1} = H^{s-1}(X)^{\bullet,-}.$$

Denote by t a global time function on X, e.g. $t = \pm\mu^{-1}$ near $Y_\pm$, so that $t \to \pm\infty$ along bicharacteristics tending to $Y_\pm$.

Theorem 5.4.1. *Fix $\sigma \in \mathbb{C}$ and $s \in \mathcal{C}^\infty(S^*\overline{X})$ as above. Then $P_\sigma\colon \mathcal{X}^s \to \mathcal{Y}^{s-1}$ is invertible, and $P_\sigma^{-1}\colon H^{s-1}(X)^{\bullet,-} \to H^s(X)^{\bullet,-}$ is the forward solution operator of P_σ.*

[19] The choice of signs here is opposite to the one in [117], since here we are going to construct the forward solution operator on X.

Proof. First, let us assume $\operatorname{Re}\sigma \gg 0$ so semiclassical/large parameter estimates are applicable to $\widetilde{P}_\sigma$, and let $T_0 \in \mathbb{R}$ be such that s is constant in $\{t \leq T_0\}$. Then for any $T_1 \leq T_0$, we can paste together microlocal energy estimates for $\widetilde{P}_\sigma$ near $\overline{C_-}$ and standard energy estimates for the wave equation in $\{t \leq T_1\}$ away from Y_- as in the derivation of [114, Equation (3.29)], and thereby obtain

$$\|u\|_{H^1(\{t\leq T_1\})} \lesssim \|\widetilde{P}_\sigma u\|_{H^0(\{t\leq T_1\})}; \tag{5.4.2}$$

thus, for $f \in \mathcal{C}^\infty(\widetilde{X})$, $\operatorname{supp} f \subset \{t \geq T_1\}$ implies $\operatorname{supp} \widetilde{P}_\sigma^{-1} f \subset \{t \geq T_1\}$. Choosing $\phi \in \mathcal{C}_c^\infty(X)$ with support in $\{t \geq T_1\}$ and $\psi \in \mathcal{C}^\infty(\widetilde{X})$ with support in $\{t \leq T_1\}$, we therefore obtain $\psi \widetilde{P}_\sigma^{-1}\phi = 0$. Since $\widetilde{P}_\sigma^{-1}$ is meromorphic, this continues to hold for all $\sigma \in \mathbb{C}$ such that $\operatorname{Im}\sigma > 1/2 - s$. Since $T_1 \leq T_0$ is arbitrary, this, together with standard energy estimates on the asymptotically de Sitter space X, proves that P_σ^{-1} propagates supports forward, provided P_σ is invertible. Moreover, elements of $\ker \widetilde{P}_\sigma$ are supported in $\overline{C_+}$.

This (and the corresponding statement for the adjoint P_σ^*) implies the invertibility of P_σ on the spaces in the statement in the theorem; this follows from [8, Lemma 8.3], see also Footnote 15 there. Concretely, let $E\colon H^{s-1}(X)^{\bullet,-} \to H^{s-1}(\widetilde{X})$ be a continuous extension operator that extends by 0 in $\overline{C_-}$ and $R\colon H^s(\widetilde{X}) \to H^s(X)^{-,-}$ the restriction, then $R \circ \widetilde{P}_\sigma^{-1} \circ E$ does not have poles. Since

$$\bigcup_{T_1 \leq T_0} H^s(\{t > T_1\})^{\bullet,-} \subset H^s(X)^{\bullet,-},$$

where ($\bullet$) denotes supported distributions at $\{t = T_1\}$, resp. Y_-, is dense, $R \circ \widetilde{P}_\sigma^{-1} \circ E$ in fact maps into $H^s(X)^{\bullet,-}$, thus $P_\sigma^{-1} = R \circ \widetilde{P}_\sigma^{-1} \circ E$ indeed exists and has the claimed properties. □

In our quest for finding forward solutions of semilinear equations, we restrict ourselves to a submanifold with boundary $\Omega \subset \overline{X}$ containing and localized near future infinity, so that we can work in fixed order Sobolev spaces; moreover, it will be useful to measure the conormal regularity of solutions to the linear equation at the conormal bundle of the boundary of X at future infinity more precisely. So let $H^{s,k}(\widetilde{X}, Y_+)$ be the subspace of

$H^s(\widetilde{X})$ with k-fold regularity with respect to the $\Psi^0(\widetilde{X})$-module

$$\mathcal{M} = \{X \in \Psi^1(\widetilde{X}) \colon \sigma_1(X)|_{N^*Y_+} = 0\} \tag{5.4.3}$$

of first order ps.d.o.s with principal symbol vanishing on N^*Y_+. By [59, Theorem 6.3], with $s_0 = 1/2 - \operatorname{Im}\sigma$ in our case, shows that $f \in H^{s-1,k}(\widetilde{X}, Y_+)$, $\widetilde{P}_\sigma u = f$, u a distribution, in fact imply that $u \in H^{s,k}(\widetilde{X}, Y_+)$. So if we let $H^{s,k}(\Omega)^{\bullet,-}$ denote the space of all $u \in H^s(X)^{\bullet,-}$ which are restrictions to Ω of functions in $H^{s,k}(\widetilde{X}, Y_+)$, supported in $\Omega \cup \overline{C_+}$, the argument of Theorem 5.4.1 shows that we have a forward solution operator

$$S_\sigma \colon H^{s-1,k}(\Omega)^{\bullet,-} \to H^{s,k}(\Omega)^{\bullet,-},$$

provided

$$s < 1/2 - \operatorname{Im}\sigma. \tag{5.4.4}$$

The analysis of semilinear equations thus requires the study of algebra properties of the spaces $H^{s,k}(\Omega)^{\bullet,-}$, which will be the subject of §5.4.2.

The backward problem

Another problem that we will briefly consider below is the backward problem, i.e. where one solves the equation on X backward from Y_+, which is the same, up to relabelling, as solving the equation forward from Y_-. Thus, we have a backward solution operator $S_\sigma^- \colon H^{s-1,k}(\Omega)^{-,\bullet} \to H^{s,k}(\Omega)^{-,\bullet}$ (where Ω is chosen as above so that we can use constant order Sobolev spaces), provided $s > 1/2 - \operatorname{Im}\sigma$. Similarly to the above, $(-)$ denotes extendible distributions at $\partial\Omega \cap X^\circ$ and $(\bullet)$ supported distributions at Y_+; the module regularity is measured at Y_+.

5.4.2 Algebra properties of Sobolev spaces with module regularity

We now study spaces like $H^{s,k}(\Omega)-,\bullet$ in a slightly more general setting.

Let us call a polynomially bounded measurable function $w \colon \mathbb{R}^n \to (0,\infty)$ a *weight function*. For such a weight function w, we define

$$H^{(w)}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) \colon w\hat{u} \in L^2(\mathbb{R}^n)\}.$$

The following lemma is similar in spirit to, but different from, Strichartz' result on Sobolev algebras [104]; it is the basis for the multiplicative properties of the more delicate spaces considered below.

Lemma 5.4.2. *Let w_1, w_2, w be weight functions such that one of the quantities*

$$\begin{aligned} M_+ &:= \sup_{\xi\in\mathbb{R}^n} \int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\right)^2 d\eta \\ M_- &:= \sup_{\eta\in\mathbb{R}^n} \int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\right)^2 d\xi \end{aligned} \tag{5.4.5}$$

is finite. Then $H^{(w_1)}(\mathbb{R}^n) \cdot H^{(w_2)}(\mathbb{R}^n) \subset H^{(w)}(\mathbb{R}^n)$.

Proof. For $u, v \in \mathscr{S}(\mathbb{R}^n)$, we use Cauchy-Schwarz to estimate

$$\begin{aligned} \|uv\|^2_{H^{(w)}} &= \int w(\xi)^2 |\widehat{uv}(\xi)|^2 \, d\xi \\ &= \int w(\xi)^2 \left(\int w_1(\eta)|\widehat{u}(\eta)| w_2(\xi-\eta)|\widehat{v}(\xi-\eta)| w_1(\eta)^{-1} w_2(\xi-\eta)^{-1} \, d\eta\right)^2 d\xi \\ &\le \int \left(\int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\right)^2 d\eta\right) \\ &\qquad\qquad \times \left(\int w_1(\eta)^2 |\widehat{u}(\eta)|^2 w_2(\xi-\eta)^2 |\widehat{v}(\xi-\eta)|^2 \, d\eta\right) d\xi \\ &\le M_+ \|u\|^2_{H^{(w_1)}} \|v\|^2_{H^{(w_2)}} \end{aligned}$$

as well as

$$\begin{aligned} \|uv\|^2_{H^{(w)}} &\le \int \left(\int w_2(\xi-\eta)^2 |\widehat{v}(\xi-\eta)|^2 \, d\eta\right) \\ &\qquad \times \left(\int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\right)^2 w_1(\eta)^2 |\widehat{u}(\eta)|^2 \, d\eta\right) d\xi \\ &= \|v\|^2_{H^{(w_2)}} \int w_1(\eta)^2 |\widehat{u}(\eta)|^2 \left(\int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\right)^2 d\xi\right) d\eta \\ &\le M_- \|u\|^2_{H^{(w_1)}} \|v\|^2_{H^{(w_2)}}. \end{aligned}$$

Since $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^{(w_1)}(\mathbb{R}^n)$ and $H^{(w_2)}(\mathbb{R}^n)$, the lemma follows. ☐

In particular, if

$$\left\| \frac{w(\xi)}{w(\eta)w(\xi-\eta)} \right\|_{L^\infty_\xi L^2_\eta} < \infty, \tag{5.4.6}$$

then $H^{(w)}$ is an algebra.

For example, the weight function $w(\xi) = \langle\xi\rangle^s$ for $s > n/2$ satisfies (5.4.6) as we will check below, which implies that $H^s(\mathbb{R}^n)$ is an algebra for $s > n/2$; this is the special case $k = 0$ of Lemma 5.4.4 below, and is well-known, see e.g. [108, Chapter 13.3]. Also, product-type weight functions $w_d(\xi) = \langle\xi'\rangle^s\langle\xi''\rangle^k$ (where $\xi = (\xi',\xi'') \in \mathbb{R}^{d+(n-d)}$ for $s > d/2, k > (n-d)/2$ satisfy (5.4.6).

The following lemma, together with the triangle inequality $\langle\xi\rangle^\alpha \lesssim \langle\eta\rangle^\alpha + \langle\xi-\eta\rangle^\alpha$ for $\alpha \geq 0$, will often be used to check conditions like (5.4.5).

Lemma 5.4.3. *Suppose $\alpha, \beta \geq 0$ are such that $\alpha + \beta > n$. Then*

$$\int_{\mathbb{R}^n} \frac{d\eta}{\langle\eta\rangle^\alpha\langle\xi-\eta\rangle^\beta} \in L^\infty(\mathbb{R}^n_\xi).$$

Proof. Splitting the domain of integration into the two regions $\{\langle\eta\rangle < \langle\xi-\eta\rangle\}$ and $\{\langle\eta\rangle \geq \langle\xi-\eta\rangle\}$, we obtain the bound

$$\int_{\mathbb{R}^n} \frac{d\eta}{\langle\eta\rangle^\alpha\langle\xi-\eta\rangle^\beta} \leq 2\int_{\mathbb{R}^n} \frac{d\eta}{\langle\eta\rangle^{\alpha+\beta}},$$

which is finite in view of $\alpha + \beta > n$. □

Another important consequence of Lemma 5.4.2 is that $H^{s'}(\mathbb{R}^n)$ is an $H^s(\mathbb{R}^n)$-module provided $|s'| \leq s$, $s > n/2$, which follows for $s' \geq 0$ from $M_+ < \infty$, and for $s' < 0$ either by duality or from $M_- < \infty$ (with $M_\pm$ as in the statement of the lemma, with the corresponding weight functions).

Lemma 5.4.4. *Write $x \in \mathbb{R}^n$ as $x = (x', x'') \in \mathbb{R}^{d+(n-d)}$. For $s \in \mathbb{R}, k \in \mathbb{N}_0$, let*

$$\mathcal{Y}^{s,k}_d(\mathbb{R}^n) = \{u \in H^s(\mathbb{R}^n) \colon D^k_{x''}u \in H^s(\mathbb{R}^n)\}.$$

Then for $s > d/2, s + k > n/2$, $\mathcal{Y}^{s,k}_d(\mathbb{R}^n)$ is an algebra.

Proof. Using the Leibniz rule, we see that it suffices to show that if $u, v \in \mathcal{Y}^{s,k}_d$, then $D^\alpha_{x''}uD^\beta_{x''}v \in H^s$, provided $|\alpha| + |\beta| \leq k$. Since $D^\alpha_{x''}u \in \mathcal{Y}^{s,k-|\alpha|}_d$ and $D^\beta_{x''}v \in \mathcal{Y}^{s,k-|\beta|}_d$, this amounts to showing that

$$\mathcal{Y}_d^{s,a} \cdot \mathcal{Y}_d^{s,b} \subset H^s \text{ if } a + b \geq k. \tag{5.4.7}$$

Using the characterization $\mathcal{Y}_d^{s,a} = H^{(w)}$ for $w(\xi) = \langle\xi\rangle^s \langle\xi''\rangle^k$, Lemma 5.4.2 in turn reduces this to the estimate

$$\begin{aligned}
&\int \frac{\langle\xi\rangle^{2s}}{\langle\eta\rangle^{2s}\langle\eta''\rangle^{2a}\langle\xi-\eta\rangle^{2s}\langle\xi''-\eta''\rangle^{2b}}\,d\eta \\
&\quad\lesssim \int \frac{d\eta}{\langle\eta''\rangle^{2a}\langle\xi-\eta\rangle^{2s}\langle\xi''-\eta''\rangle^{2b}} + \int \frac{d\eta}{\langle\eta\rangle^{2s}\langle\eta''\rangle^{2a}\langle\xi''-\eta''\rangle^{2b}} \\
&\quad\leq \int \frac{d\eta'}{\langle\xi'-\eta'\rangle^{2s'}} \int \frac{d\eta''}{\langle\eta''\rangle^{2a}\langle\xi''-\eta''\rangle^{2b+2(s-s')}} \\
&\qquad + \int \frac{d\eta'}{\langle\eta'\rangle^{2s'}} \int \frac{d\eta''}{\langle\eta''\rangle^{2a+2(s-s')}\langle\xi''-\eta''\rangle^{2b}},
\end{aligned}$$

where we choose $d/2 < s' < s$ such that $a + b + s - s' > (n - d)/2$, which holds if $k + s > (n - d)/2 + s'$, which is possible by our assumptions on s and k. The integrals are uniformly bounded in ξ: For the η'-integrals, this follows from $s' > d/2$; for the η''-integrals, we use Lemma 5.4.3. □

We shall now use this (non-invariant) result to prove algebra properties for spaces with iterated module regularity: Consider a compact manifold without boundary X and a submanifold Y. Let $\mathcal{M} \supset \Psi^0(X)$ be the $\Psi^0(X)$-module of first order ps.d.o.s whose principal symbol vanishes on N^*Y. For $s \in \mathbb{R}, k \in \mathbb{N}_0$, define

$$H^{s,k}(X,Y) = \{u \in H^s(X) \colon \mathcal{M}^k u \in H^s(X)\}.$$

Proposition 5.4.5. *Suppose* $\dim(X) = n$ *and* $\operatorname{codim}(Y) = d$. *Assume that* $s > d/2$ *and* $s + k > n/2$. *Then* $H^{s,k}(X,Y)$ *is an algebra.*

Proof. Away from Y, $H^{s,k}(X,Y)$ is just $H^{s+k}(X)$, which is an algebra since $s + k > \dim(X)/2$. Thus, since the statement is local, we may assume that we have a product decomposition near Y, namely $X = \mathbb{R}^d_{x'} \times \mathbb{R}^{n-d}_{x''}$, $Y = \{x' = 0\}$, and that we are given arbitrary $u, v \in H^{s,k}(X,Y)$ with compact support close to $(0,0)$ for which we have to show $uv \in H^{s,k}(X,Y)$. Notice that for $f \in H^s(X)$ with such small support, $f \in H^{s,k}(X,Y)$ is equivalent to $\mathcal{M}'^k f \in H^s(X)$, where $\mathcal{M}'$ is the $C^\infty(M)$-module of differential operators generated by $\mathrm{Id}, \partial_{x''_i}, x'_j\partial_{x'_k}$, where $1 \leq i \leq n - d, 1 \leq j, k \leq d$.

Thus the proposition follows from the following statement: For s, k as in the statement of the proposition,

$$H^{s,k}(\mathbb{R}^n, \mathbb{R}^{n-d}) := \{u \in H^s(\mathbb{R}^n) \colon (x')^{\widetilde{\alpha}} D_{x'}^{\alpha} D_{x''}^{\beta} u \in H^s(\mathbb{R}^n), |\widetilde{\alpha}| = |\alpha|, |\alpha| + |\beta| \leq k\}$$

is an algebra. Using the Leibniz rule, we thus have to show that

$$((x')^{\widetilde{\alpha}} D_{x'}^{\alpha} D_{x''}^{\beta} u)((x')^{\widetilde{\gamma}} D_{x'}^{\gamma} D_{x''}^{\delta} v) \in H^s, \tag{5.4.8}$$

provided $|\widetilde{\alpha}| = |\alpha|, |\widetilde{\gamma}| = |\gamma|, |\alpha| + |\beta| + |\gamma| + |\delta| \leq k$. Since the two factors in (5.4.8) lie in $H^{s,k-|\alpha|-|\beta|}$ and $H^{s,k-|\gamma|-|\delta|}$, respectively, this amounts to showing that $H^{s,a} \cdot H^{s,b} \subset H^s$ for $a + b \geq k$. This however is easy to see, since $H^{s,c} \subset \mathcal{Y}_d^{s,c}$ for all $c \in \mathbb{N}_0$, and $\mathcal{Y}_d^{s,a} \cdot \mathcal{Y}_d^{s,b} \subset H^s$ was proved in (5.4.7). □

In order to be able to obtain sharper results for particular nonlinear equations in §5.4.3, we will now prove further results in the case $\operatorname{codim}(Y) = 1$, which we will assume to hold from now on; also, we fix $n = \dim(X)$.

Proposition 5.4.6. *Assume that $s > 1/2$ and $k > (n-1)/2$. Then*

$$H^{s,k}(X, Y) \cdot H^{s-1,k}(X, Y) \subset H^{s-1,k}(X, Y).$$

Proof. Using the Leibniz rule, this follows from $\mathcal{Y}_1^{s,a} \cdot \mathcal{Y}_1^{s-1,b} \subset H^{s-1}$ for $a + b \geq k$. This, as before, can be reduced to the local statement on $\mathbb{R}^n = \mathbb{R}_{x_1} \times \mathbb{R}_{x'}^{n-1}$ with $Y = \{x_1 = 0\}$. We write $\xi = (\xi_1, \xi') \in \mathbb{R}^{1+(n-1)}$ and $\eta = (\eta_1, \eta') \in \mathbb{R}^{1+(n-1)}$. By Lemma 5.4.2, the case $s \geq 1$ follows from the estimate

$$\begin{aligned}
&\int \frac{\langle\xi\rangle^{2(s-1)}}{\langle\eta\rangle^{2s}\langle\eta'\rangle^{2a}\langle\xi-\eta\rangle^{2(s-1)}\langle\xi'-\eta'\rangle^{2b}}\,d\eta \\
&\quad \lesssim \int \frac{d\eta}{\langle\eta\rangle^{2}\langle\eta'\rangle^{2a}\langle\xi-\eta\rangle^{2(s-1)}\langle\xi'-\eta'\rangle^{2b}} + \int \frac{d\eta}{\langle\eta\rangle^{2s}\langle\eta'\rangle^{2a}\langle\xi'-\eta'\rangle^{2b}} \\
&\quad \leq 2 \int \frac{d\eta_1}{\langle\eta_1\rangle^{2s}} \int \frac{d\eta'}{\langle\eta'\rangle^{2a}\langle\xi'-\eta'\rangle^{2b}} \in L_\xi^\infty
\end{aligned}$$

by Lemma 5.4.3.

If $1/2 < s \leq 1$, then ξ_1 and ξ' play different roles in the following sense: The background regularity to be proved is H^{s-1}, $s - 1 \leq 0$, thus the continuity of multiplication in the conormal direction to Y is proved by 'duality' (i.e. using Lemma 5.4.2 with $M_- < \infty$), whereas the continuity in the tangential (to Y) directions, where both factors have $k > (n-1)/2$ derivatives, is proved directly (i.e. using Lemma 5.4.2 with $M_+ < \infty$). Concretely then, let $u \in \mathcal{Y}_1^{s,a}$, $v \in \mathcal{Y}_1^{s-1,b}$, and put

$$u_0(\xi) = \langle\xi\rangle^s \langle\xi'\rangle^a u(\xi) \in L^2(\mathbb{R}^n), \quad v_0(\xi) = \langle\xi\rangle^{s-1}\langle\xi'\rangle^b v(\xi) \in L^2(\mathbb{R}^n).$$

Then

$$\langle\xi\rangle^{s-1}\widehat{uv}(\xi) = \int \frac{\langle\eta\rangle^{1-s}}{\langle\xi\rangle^{1-s}\langle\eta'\rangle^b\langle\xi-\eta\rangle^s\langle\xi'-\eta'\rangle^a} u_0(\xi-\eta)v_0(\eta)\,d\eta,$$

hence by Cauchy-Schwarz and Lemma 5.4.3

$$\begin{aligned}
&\int \langle\xi\rangle^{2(s-1)}|\widehat{uv}(\xi)|^2\,d\xi \\
&\quad \leq \int \left(\int \frac{d\eta'}{\langle\eta'\rangle^{2b}\langle\xi'-\eta'\rangle^{2a}}\right)\left(\int \left|\int \frac{\langle\eta\rangle^{1-s}}{\langle\xi\rangle^{1-s}\langle\xi-\eta\rangle^s} u_0(\xi-\eta)v_0(\eta)\,d\eta_1\right|^2 d\eta'\right) d\xi \\
&\quad \lesssim \iint \left(\int |u_0(\xi-\eta)|^2\,d\eta_1\right)\left(\int \frac{\langle\eta\rangle^{2(1-s)}}{\langle\xi\rangle^{2(1-s)}\langle\xi-\eta\rangle^{2s}}|v_0(\eta)|^2\,d\eta_1\right) d\eta'\,d\xi \\
&\quad \lesssim \iint \|u_0(\cdot,\xi'-\eta')\|_{L^2}^2 |v_0(\eta)|^2 \\
&\qquad\qquad \times \left(\int \frac{1}{\langle\xi-\eta\rangle^{2s}} + \frac{1}{\langle\xi\rangle^{2(1-s)}\langle\xi-\eta\rangle^{2(2s-1)}}\,d\xi_1\right) d\xi'\,d\eta \\
&\quad \lesssim \|u\|_{\mathcal{Y}_1^{s,a}}^2 \|v\|_{\mathcal{Y}_1^{s-1,b}}^2,
\end{aligned}$$

since $1/2 < s \leq 1$, thus $1-s \geq 0$ and $2s-1 > 0$, and the ξ_1-integral is thus bounded from above by

$$\int \frac{1}{\langle\xi_1-\eta_1\rangle^{2s}} + \frac{1}{\langle\xi_1\rangle^{2(1-s)}\langle\xi_1-\eta_1\rangle^{2(2s-1)}}\,d\xi_1 \in L^\infty_{\eta_1}.$$

The proof is complete. $\square$

For semilinear equations whose non-linearity does not involve any derivatives, one can afford to lose derivatives in multiplication statements. We give two useful results in this context, the first being a consequence of Proposition 5.4.6.

Corollary 5.4.7. *Let $\mu \in \mathcal{C}^\infty(X)$ be a defining function for Y, i.e. $\mu|_Y \equiv 0$, $d\mu \neq 0$ on Y, and μ vanishes on Y only. Suppose $s > 1/2$ and $\ell \in \mathbb{C}$ are such that $\operatorname{Re}\ell + 3/2 > s$. Then multiplication by μ_+^ℓ defines a continuous map $H^{s,k}(X,Y) \to H^{s-1,k}(X,Y)$ for all $k \in \mathbb{N}_0$.*

Proof. By the Leibniz rule, it suffices to prove the statement for $k = 0$. We have $\mu_+^\ell \in H^{\operatorname{Re}\ell+1/2-\epsilon;\infty}(X,Y)$ for all $\epsilon > 0$: Indeed, the Fourier transform of $\chi(x)x_+^\ell$ on $\mathbb{R}$, with $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$, is bounded by a constant multiple of $\langle\xi\rangle^{-\operatorname{Re}\ell-1}$, which is an element of $\langle\xi\rangle^{-r}L^2_\xi$ if and only if $r - \operatorname{Re}\ell - 1 < -1/2$, i.e. if $\operatorname{Re}\ell + 1/2 > r$. Hence the corollary follows from Proposition 5.4.6, since one has $\operatorname{Re}\ell + 1/2 - \epsilon \geq s - 1$ for some $\epsilon > 0$ provided $\operatorname{Re}\ell + 3/2 > s$. □

Proposition 5.4.8. *Let $0 \leq s', s_1, s_2 < 1/2$ be such that $s' < s_1 + s_2 - 1/2$, and let $k > (n-1)/2$. Then $H^{s_1,k}(X,Y) \cdot H^{s_2,k}(X,Y) \subset H^{s',k}(X,Y)$.*

Proof. Using the Leibniz rule, this reduces to the statement that $\mathcal{Y}_1^{s_1,a} \cdot \mathcal{Y}_1^{s_2,b} \subset H^{s'}$ if $a + b \geq k$. Splitting variables $\xi = (\xi_1, \xi')$, $\eta = (\eta_1, \eta')$, Lemma 5.4.2 in turn reduces this to the observation that

$$\begin{aligned}\int & \frac{\langle\xi\rangle^{2s'}}{\langle\eta\rangle^{2s_1}\langle\eta'\rangle^{2a}\langle\xi-\eta\rangle^{2s_2}\langle\xi'-\eta'\rangle^{2b}}\,d\eta \\ &\lesssim \left(\int \frac{d\eta_1}{\langle\eta_1\rangle^{2(s_1-s')}\langle\xi_1-\eta_1\rangle^{2s_2}} + \int \frac{d\eta_1}{\langle\eta_1\rangle^{2s_1}\langle\xi_1-\eta_1\rangle^{2(s_2-s')}}\right) \\ &\qquad \times \int \frac{d\eta'}{\langle\eta'\rangle^{2a}\langle\xi'-\eta'\rangle^{2b}}\end{aligned}$$

is uniformly bounded in ξ by Lemma 5.4.3 in view of $s' < s_1 + s_2 - 1/2 < \min\{s_1, s_2\}$, thus $s_1 - s' > 0$ and $s_2 - s' > 0$, and $s_1 + s_2 - s' > 1/2$, as well as $a + b > (n-1)/2$. □

Corollary 5.4.9. *Let $p \in \mathbb{N}$ and $s = 1/2 - \epsilon$ with $0 \leq \epsilon < 1/2p$, and let $k > (n-1)/2$. Then $u \in H^{s,k}(X,Y) \Rightarrow u^p \in H^{0,k}(X,Y)$.*

Proof. Proposition 5.4.8 gives $u^2 \in H^{1/2-2\epsilon-\epsilon_2',k}$ for all $\epsilon_2' > 0$, thus $u^3 \in H^{1/2-3\epsilon-\epsilon_3',k}$ for all $\epsilon_3' > 0$, since $\epsilon_2' > 0$ is arbitrary; continuing in this way gives $u^p \in H^{1/2-p\epsilon-\epsilon_p',k}$ for all $\epsilon_p' > 0$, and the claim follows. □

5.4.3 A class of semilinear equations

Recall that we have a forward solution operator $S_\sigma\colon H^{s-1,k}(\Omega)^{\bullet,-} \to H^{s,k}(\Omega)^{\bullet,-}$ of P_σ, defined in (5.4.1), provided $s < 1/2 - \operatorname{Im}\sigma$. Let us fix such $s \in \mathbb{R}$ and $\sigma \in \mathbb{C}$. Undoing the conjugation, we obtain a forward solution operator

$$S = \mu^{-1/2}\mu^{-i\sigma/2+(n+1)/4}S_\sigma\mu^{i\sigma/2-(n+1)/4}\mu^{-1/2},$$

with the mapping property

$$S\colon \mu^{(n+3)/4+\operatorname{Im}\sigma/2}H^{s-1,k}(\Omega)^{\bullet,-} \to \mu^{(n-1)/4+\operatorname{Im}\sigma/2}H^{s,k}(\Omega)^{\bullet,-},$$

of $\Box_g - (n-1)^2/4 - \sigma^2$. Since g is a 0-metric, the natural vector fields to appear in a nonlinear equation are 0-vector fields; see §5.4.5 for a brief discussion of these concepts. However, since the analysis is based on ordinary Sobolev spaces relative to which one has b-regularity (regularity with respect to the module $\mathcal{M}$), we consider b-vector fields in the non-linearities. In case one does use 0-vector fields, the solvability conditions can be relaxed; see §5.4.4.

Theorem 5.4.10. *Suppose $s < 1/2 - \operatorname{Im}\sigma$. Let*

$$\begin{aligned} q\colon \mu^{(n-1)/4+\operatorname{Im}\sigma/2}H^{s,k}(\Omega)^{\bullet,-} &\times \mu^{(n-1)/4+\operatorname{Im}\sigma/2}H^{s,k-1}(\Omega;{}^{\mathrm{b}}T^*_\Omega M)^{\bullet,-} \\ &\to \mu^{(n+3)/4+\operatorname{Im}\sigma/2}H^{s-1,k}(\Omega)^{\bullet,-} \end{aligned}$$

be a continuous function with $q(0,0) = 0$ such that there exists a continuous non-decreasing function $L\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying

$$\|q(u, {}^{\mathrm{b}}du) - q(v, {}^{\mathrm{b}}dv)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R.$$

Then there is a constant $C_L > 0$ so that the following holds: If $L(0) < C_L$, then for small $R > 0$, there exists $C > 0$ such that for all $f \in \mu^{(n+3)/4+\operatorname{Im}\sigma/2}H^{s-1,k}(\Omega)^{\bullet,-}$ with norm $\leq C$, the equation

$$\left(\Box_g - \left(\frac{n-1}{2}\right)^2 - \sigma^2\right)u = f + q(u, {}^{\mathrm{b}}du)$$

has a unique solution $u \in \mu^{(n-1)/4+\operatorname{Im}\sigma/2}H^{s,k}(\Omega)^{\bullet,-}$, with norm $\leq R$, that depends continuously on f.

Proof. Use the Banach fixed point theorem as in the proof of Theorem 5.2.6. □

Remark 5.4.11. As in Theorem 5.2.6, we can also allow non-linearities $q(u, {}^{\mathrm{b}}du, \Box_g u)$, provided

$$\begin{aligned} q\colon \mu^{(n-1)/4+\operatorname{Im}\sigma/2}H^{s,k}(\Omega)^{\bullet,-} &\times \mu^{(n-1)/4+\operatorname{Im}\sigma/2}H^{s-1,k}(\Omega;{}^{\mathrm{b}}T^*_\Omega M)^{\bullet,-} \\ &\times \mu^{(n+3)/4+\operatorname{Im}\sigma/2}H^{s-1,k}(\Omega)^{\bullet,-} \end{aligned}$$

$$\to \mu^{(n+3)/4+\operatorname{Im}\sigma/2} H^{s-1,k}(\Omega)^{\bullet,-}$$

is continuous, $q(0,0,0)=0$ and q has a small Lipschitz constant near 0.

5.4.4 Semilinear equations with polynomial non-linearities

Next, we want to find a forward solution of the semilinear PDE

$$\Big(\Box_g - \Big(\frac{n-1}{2}\Big)^2 - \sigma^2\Big)u = f + c\mu^A u^p X(u) \tag{5.4.9}$$

where $c \in C^\infty(\widetilde{X})$, and $X(u) = \prod_{j=1}^q X_j u$ is a q-fold product of derivatives of u along vector fields $X_j \in \mathcal{M}$; recall the definition (5.4.3) of the module $\mathcal{M}$. What follows is a computation in the course of which we will obtain conditions on A, p, q which guarantee that the map $u \mapsto c\mu^A u^p X(u)$ satisfies the conditions of the map q in Theorem 5.4.10. Note that the derivatives in the non-linearity lie in the module $\mathcal{M}$ (in coordinates: $\mu\partial_\mu$, ∂_y), whereas, as mentioned above, the natural vector fields are 0-derivatives (in coordinates: $x\partial_x = 2\mu\partial_\mu$ and $x\partial_y = \mu^{1/2}\partial_y$), but since it does not make the computation more difficult, we consider module instead of 0-derivatives and compensate this by allowing any weight μ^A in front of the non-linearity.

Rephrasing the equation (5.4.9) in terms of P_σ using $\widetilde{u} = \mu^{i\sigma/2-(n+1)/4+1/2}u$ and $\widetilde{f} = \mu^{-1/2+i\sigma/2-(n+1)/4}f$, we obtain

$$\begin{aligned} P_\sigma \widetilde{u} &= \widetilde{f} + c\mu^A \mu^{-1/2+i\sigma/2-(n+1)/4}\mu^{(p+q)(-i\sigma/2+(n-1)/4)}\widetilde{u}^p \prod_{j=1}^q (f_j + X_j\widetilde{u}) \\ &= \widetilde{f} + c\mu^\ell \widetilde{u}^p \prod_{j=1}^q (f_j + X_j \widetilde{u}), \end{aligned} \tag{5.4.10}$$

where $f_j \in \mathcal{C}^\infty(\widetilde{X})$ and

$$\ell = A + (p+q-1)(-i\sigma/2 + (n-1)/4) - 1. \tag{5.4.11}$$

Therefore, if $\widetilde{u} \in H^{s,k}(\Omega)^{\bullet,-}$, we obtain that the right hand side of the equation lies in $H^{s,k-1}(\Omega)^{\bullet,-}$ if $\widetilde{f} \in H^{s,k-1}(\Omega)^{\bullet,-}$, $s > 1/2, k > (n+1)/2$, which by Proposition 5.4.5

implies that $H^{s,k-1}(\Omega)^{\bullet,-}$ is an algebra, and if

$$\operatorname{Re}\ell + 1/2 = A + (p+q-1)(\operatorname{Im}\sigma/2 + (n-1)/4) - 1/2 > s, \tag{5.4.12}$$

since this condition ensures that $\mu^\ell \in H^{s,\infty}(X)$, which implies that multiplication by μ^ℓ is a bounded map $H^{s,k-1}(\Omega)^{\bullet,-} \to H^{s,k-1}(\Omega)^{\bullet,-}$.[20] Given the restriction (5.4.4) on s and $\operatorname{Im}\sigma$, we see that by choosing $s > 1/2$ close to $1/2$, $\operatorname{Im}\sigma < 0$ close to 0, we obtain the condition

$$p + q > 1 + \frac{4(1-A)}{n-1}. \tag{5.4.13}$$

If these conditions are satisfied, the right hand side of equation (5.4.10) is an element of $H^{s,k-1}(\Omega)^{\bullet,-} \subset H^{s-1,k}(\Omega)^{\bullet,-}$, so Theorem 5.4.10 is applicable, and thus (5.4.9) is well-posed in these spaces. For instance, from (5.4.13) with $A = 0$, we see that quadratic non-linearities are fine for $n \geq 6$, cubic ones for $n \geq 4$.

To sum this up, we revert back to $u = \mu^{(n-1)/4 - i\sigma/2}\widetilde{u}$ and $f = \mu^{(n+3)/4 - i\sigma/2}\widetilde{f}$:

Theorem 5.4.12. *Let $s > 1/2, k > (n+1)/2$, and assume $A \in \mathbb{R}$ and $p, q \in \mathbb{N}_0$, $p + q \geq 2$ satisfy condition (5.4.12). Moreover, suppose $\sigma \in \mathbb{C}$ satisfies (5.4.4), i.e. $\operatorname{Im}\sigma < 1/2 - s$. Finally, let $c \in \mathcal{C}^\infty(\widetilde{M})$ and $X(u) = \prod_{j=1}^q X_j u$, where X_j are vector fields in $\mathcal{M}$. Then for small enough $R > 0$, there exists a constant $C > 0$ such that for all $f \in \mu^{(n+3)/4 + \operatorname{Im}\sigma/2} H^{s,k}(\Omega)^{\bullet,-}$ with norm $\leq C$, the PDE*

$$\left(\Box_g - \left(\frac{n-1}{2}\right)^2 - \sigma^2\right)u = f + c\mu^A u^p X(u)$$

has a unique solution $u \in \mu^{(n-1)/4 + \operatorname{Im}\sigma/2} H^{s,k}(\Omega)^{\bullet,-}$, with norm $\leq R$, that depends continuously on f.

The same conclusion holds if the non-linearity is a finite sum of terms of the form $c\mu^A u^p X(u)$, provided each such term separately satisfies (5.4.4).

Proof. Reformulating the PDE in terms of $\widetilde{u}$ and $\widetilde{f}$ as above, this follows from an application

[20] If one works in higher regularity spaces, $s \geq 3/2$, we in fact only need $\operatorname{Re}\ell + 3/2 > s$, since then multiplication by μ^ℓ is a bounded map $H^{s,k-1}(\Omega)^{\bullet,-} \subset H^{s-1,k}(\Omega)^{\bullet,-} \to H^{s-1,k}(\Omega)^{\bullet,-}$. However, the solvability criterion (5.4.13) would be weaker, namely the role of the dimension n shifts by 2, since in order to use $s \geq 3/2$, we need $\operatorname{Im}\sigma < -1$.

of the Banach fixed point theorem to the map

$$H^{s,k}(\Omega)^{\bullet,-} \ni \widetilde{u} \mapsto S_\sigma\Big(\widetilde{f} + \mu^\ell \widetilde{w}^p \prod_{j=1}^q (f_j + X_j\widetilde{u})\Big) \in H^{s,k}(\Omega)^{\bullet,-}$$

with ℓ given by (5.4.11) and $f_j \in \mathcal{C}^\infty(\widetilde{X})$. Here, $p+q \geq 2$ and the smallness of R ensure that this map is a contraction on the metric ball of radius R in $H^{s,k}(\Omega)^{\bullet,-}$. □

Remark 5.4.13. Even though the above conditions force $\operatorname{Im}\sigma < 0$, let us remark that the conditions of the theorem, most importantly (5.4.12), can be satisfied if $m^2 = (n-1)^2/4 + \sigma^2 > 0$ is real, which thus means that we are in fact considering a nonlinear equation involving the Klein-Gordon operator $\Box_g - m^2$. Indeed, let $\sigma = i\widetilde{\sigma}$ with $\widetilde{\sigma} < 0$, then condition (5.4.12) with $A = 0, p+q = 2$, becomes $\widetilde{\sigma} > 2-(n-1)/2$ (where we accordingly have to choose $s > 1/2$ close, depending on $\widetilde{\sigma}$, to $1/2$), and the requirement $\widetilde{\sigma} < 0$ forces $n \geq 6$. On the other hand, we want $(n-1)^2/4 - \widetilde{\sigma}^2 = m^2 > 0$; we thus obtain the condition

$$0 < m^2 < \Big(\frac{n-1}{2}\Big)^2 - \Big(2 - \frac{n-1}{2}\Big)^2$$

for masses m that Theorem 5.4.12 can handle, which does give a non-trivial range of allowed m for $n \geq 6$.

Remark 5.4.14. Let us compare the numerology in Theorem 5.4.12 with the numerology for the static model of an asymptotically de Sitter space in §5.2: First, we can solve fewer equations globally on asymptotically de Sitter spaces, and second, we need stronger regularity assumptions in order to make an iterative argument work: In the static model, we needed to be in a b-Sobolev space of order $> (n+2)/2$, which in the non-blown-up picture corresponds to 0-regularity of order $> (n+2)/2$, see §5.4.5, whereas in the global version, we need a background Sobolev regularity $> 1/2$, relative to which we have 'b-regularity' (i.e. regularity with respect to the module $\mathcal{M}$) of order $> (n+1)/2$. This comparison is of course only a qualitative one, though, since the underlying geometries in the two cases are different.

Using Proposition 5.4.6 and Corollary 5.4.7, one can often improve this result. Thus, let us consider the most natural case of equation (5.4.9) in which we use 0-derivatives X_j, corresponding to the 0-structure on the *not* even-ified manifold X, and no additional weight. The only difference this makes is if there are tangential 0-derivatives (in coordinates:

$\mu^{1/2}\partial_y$). For simplicity of notation, let us therefore assume that $X_j = \mu^{1/2}\widetilde{X}_j$, $1 \leq j \leq \alpha$, and $X_j = \widetilde{X}_j$, $\alpha < j \leq q$, where the $\widetilde{X}_j$ are merely vector fields in $\mathcal{M}$. Then the PDE (5.4.9), rewritten in terms of P_σ, $\widetilde{u}$ and $\widetilde{f}$, becomes

$$P_\sigma \widetilde{u} = \widetilde{f} + c\mu^\ell \widetilde{u}^p \prod_{j=1}^{q} (\widetilde{f}_j + \widetilde{X}_j \widetilde{u}) \tag{5.4.14}$$

with $\widetilde{f}_j \in \mathcal{C}^\infty(\widetilde{X})$, where

$$\ell = \alpha/2 + (p+q-1)(-i\sigma/2 + (n-1)/4) - 1.$$

First, suppose that there are no derivatives in the non-linearity so that $p \geq 2$, $q = \alpha = 0$. Then $\mu^\ell \widetilde{u}^p \in H^{s-1,k}(\Omega)^{\bullet,-}$ provided $\operatorname{Re}\ell + 3/2 > s > 1/2$ by Corollary 5.4.7; choosing s arbitrarily close to $1/2$, this is equivalent to

$$\operatorname{Im}\sigma/2 + (n-1)/4 > 0. \tag{5.4.15}$$

This is a very natural condition: The solution operator for the linear wave equation produces solutions with asymptotics $\mu^{(n-1)/4 \pm i\sigma/2}$, given the numerology (5.2.9) and recalling that we are working with the even-ified manifold with boundary defining function $\mu = x^2$; the nonlinear equation (5.4.9) should therefore only be well-behaved if solutions to the linear equation decay at infinity, i.e. if $\pm \operatorname{Im}\sigma + (n-1)/4 \geq 0$. Since we need $\operatorname{Im}\sigma < 0$ to be allowed to take $s > 1/2$, condition (5.4.15) is equivalent to the (small) decay of solutions to the linear equation at infinity (where $\mu = 0$).

Next, let us assume that $q > 0$. Then the nonlinear term in equation (5.4.14) is an element of

$$\mu^\ell H^{s,k}(\Omega)^{\bullet,-} \cdot H^{s,k-1}(\Omega)^{\bullet,-} \subset H^{s,k-1}(\Omega)^{\bullet,-}$$

by Proposition 5.4.6, provided $\operatorname{Re}\ell + 1/2 > s > 1/2$, which gives the condition

$$\operatorname{Im}\sigma/2 + (n-1)/4 > 1 - \alpha/2$$

where we again choose $s > 1/2$ arbitrarily close to $1/2$, i.e. for $\alpha = 2$, we again get condition (5.4.15), and for $\alpha > 2$, we get an even weaker one.

Finally, let us discuss a nonlinear term of the form $c\mu^A u^p$, $p \geq 2$, in the setting of even lower regularity $0 \leq s < 1/2$, the technical tool here being Corollary 5.4.9: Rewriting the PDE (5.4.9) with this non-linearity in terms of P_σ, $\widetilde{u}$ and $\widetilde{f}$, we get

$$P_\sigma \widetilde{u} = \widetilde{f} + c\mu^\ell \widetilde{u}^p, \quad \ell = A + (p-1)(-i\sigma/2 + (n-1)/4) - 1.$$

Let $s = 1/2 - \epsilon$ with $0 \leq \epsilon < 1/2p$. Then if $\widetilde{u} \in H^{1/2-\epsilon,k}(\Omega)^{\bullet,-}$ with $k > (n-1)/2$, Corollary 5.4.9 yields $\widetilde{u}^p \in H^{0,k}(\Omega)^{\bullet,-}$, thus

$$\mu^\ell \widetilde{u}^p \in H^{0,k}(\Omega)^{\bullet,-} \subset H^{s-1,k}(\Omega)^{\bullet,-}$$

provided $\operatorname{Re} \ell \geq 0$, i.e.

$$n > 1 + \frac{4(1-A)}{p-1} - 2 \operatorname{Im} \sigma, \tag{5.4.16}$$

where we still require $\operatorname{Im} \sigma < 1/2 - s = \epsilon$, which in particular allows σ to be real if $\epsilon > 0$.

In summary:

Theorem 5.4.15. *Let $p \geq 2$ be an integer, $1/2 - 1/2p < s \leq 1/2, k > (n-1)/2$, and suppose $\sigma \in \mathbb{C}$ is such that $\operatorname{Im} \sigma < 1/2 - s$. Moreover, assume $A \in \mathbb{R}$ and the dimension n satisfy condition (5.4.16). Then for small enough $R > 0$, there exists a constant $C > 0$ such that for all $f \in \mu^{(n+3)/4 + \operatorname{Im} \sigma/2} H^{s,k}(\Omega)^{\bullet,-}$ with norm $\leq C$, the PDE*

$$\left(\Box_g - \left(\frac{n-1}{2}\right)^2 - \sigma^2\right) u = f + c\mu^A u^p$$

has a unique solution $u \in \mu^{(n-1)/4 + \operatorname{Im} \sigma/2} H^{s,k}(\Omega)^{\bullet,-}$, with norm $\leq R$, that depends continuously on f.

In particular, if $1/4 < s < 1/2$, $0 < \operatorname{Im} \sigma < 1/2 - s$ and $A = 0$, then quadratic non-linearities are fine for $n \geq 5$; if $\operatorname{Im} \sigma = 0$ and $A = 0$, then they work for $n \geq 6$.

Backward solutions to semilinear equations with polynomial non-linearities

Let us briefly turn to the backward problem for (5.4.9), which we rephrase in terms of P_σ as above. For simplicity, let us only consider the 'least sophisticated' conditions, namely $s > 1/2, k > (n+1)/2$,

$$A + (p+q-1)(\operatorname{Im} \sigma/2 + (n-1)/4) - 1/2 > s, \tag{5.4.17}$$

and the important change compared to the forward problem is that now $s > 1/2 - \operatorname{Im}\sigma$, which guarantees the existence of the backward solution operator S^-_σ. Thus, if $\operatorname{Im}\sigma > 0$ is large enough and $s > 1/2$ satisfies (5.4.17), then (5.4.9) is solvable in any dimension.

In the special case that we only consider 0-derivatives and no extra weight, which corresponds to putting $A = q + \alpha/2$, we obtain the condition

$$\operatorname{Im}\sigma > \frac{4(1 - q - \alpha/2) - (p + q - 1)(n - 1)}{2(p + q + 1)}$$

if we choose $s > 1/2 - \operatorname{Im}\sigma$ close to $1/2$, which in particular allows $\operatorname{Im}\sigma \geq 0$, and thus σ^2 *arbitrary*, if $p > 1 + \frac{4}{n-1}$ (so $p \geq 2$ is acceptable if $n \geq 6$) or $q + \alpha/2 \geq 1$.

5.4.5 From static parts to global asymptotically de Sitter spaces

Let us consider the equation

$$(\Box_g - m^2)u = f + q(u, {}^0du), \tag{5.4.18}$$

where the reason for using the 0-differential ${}^0d$, see below, will be given momentarily. The idea is that every point in X lies in the interior of the backward light cone from some point p at future infinity Y_+, denoted S_p; that is, the blow-up of $\overline{X}$ at p contains the static part S_p of an asymptotically de Sitter space where the solvability statements have been explained in §5.2. Consider a suitable neighborhood $\Omega_p \subset [\overline{X}; p]$ of the static patch as in §5.2, so the boundary of Ω_p is the union of ∂S_p and an 'artificial' spacelike boundary, which on the non-blown-up space $\overline{X}$ all meet at the point p. In fact, we may choose the Ω_p in a fashion that is uniform in p, using the construction in §2.2.2, see in particular (2.2.16). We then solve equation (5.4.18) on Ω_p, thereby obtaining a forward solution u_p, and by local uniqueness for $\Box_g - m^2$ in X, all such solutions agree on their overlap, i.e. $u_p \equiv u_q$ on $\Omega_p \cap \Omega_q$. Therefore, we can define a function u by setting $u = u_p$ on Ω_p, $p \in Y_+$, which then is a solution of (5.4.18) on X. To make this precise, we need to analyze the relationships between the function spaces on the Ω_p, $p \in Y_+$, and X. As we will see in Lemma 5.4.16 below, b-Sobolev spaces on the blow-ups Ω_p of $\overline{X}$ at boundary points are closely related to 0-Sobolev spaces on X.

Recall the definition of 0-Sobolev spaces on a manifold with boundary M (for us, $M = \overline{X}$) with a 0-metric, i.e. a metric of the form $x^{-2}\widehat{g}$ with x a boundary defining function, where $\widehat{g}$ extends non-degenerately to the boundary; 0-geometry was introduced in [81] to analyze the resolvent on asymptotically hyperbolic spaces: If $\mathcal{V}_0(M) = x\mathcal{V}(M)$ denotes the Lie algebra of 0-vector fields, where $\mathcal{V}(M)$ are smooth vector fields on M, and $\mathrm{Diff}_0^*(M)$ the enveloping algebra of 0-differential operators, then

$$H_0^k(M) = \{u \in L^2(M, d\mathrm{vol})\colon Pu \in L^2(M, d\mathrm{vol}), P \in \mathrm{Diff}_0^k(M)\},$$

and $H_0^{k,\ell}(M) = x^\ell H_0^k(M)$ when x is a boundary defining function. For clarity, we shall write $L_0^2(M) = L^2(M, d\mathrm{vol})$. We also recall the definition of the 0-(co)tangent spaces: If $\mathcal{I}_p$ denotes the ideal of $\mathcal{C}^\infty(M)$ functions vanishing at $p \in M$, then the 0-tangent space at p is defined as ${}^0T_pM = \mathcal{V}_0(M)/\mathcal{I}_p \cdot \mathcal{V}_0(M)$, and the 0-cotangent space at p, ${}^0T_p^*M$, as the dual of ${}^0T_pM$. In local coordinates $(x, y) \in \mathbb{R}_x \times \mathbb{R}_y^{n-1}$ near the boundary of M, we have $d\mathrm{vol} = f(x, y)\frac{dx}{x}\frac{dy}{x^{n-1}}$ with f smooth and non-vanishing, and $\mathcal{V}_0(M)$ is spanned by $x\partial_x$ and $x\partial_y$; also $x\partial_x$ and $x\partial_{y_j}$, $j = 2, \ldots, n$, form a basis of ${}^0T_pM$ (for $p \in \partial M$, which is the only place where 0-spaces differ from the standard spaces), and $\frac{dx}{x}$, $\frac{dy_j}{x}$, $j = 2, \ldots, n$, form a basis of ${}^0T_p^*M$. The exterior derivative d induces the first order 0-differential operator ${}^0d$ on sections of Λ^0TM; this follows from

$$df = (\partial_x f)\,dx + (\partial_y f)\,dy = (x\partial_x f)\,\frac{dx}{x} + (x\partial_y f)\,\frac{dy}{x}.$$

Now, let $\Omega \subset \overline{X}$ be a domain as in §5.4.1. Moreover, let $\beta_p\colon \Omega_p \to X$ be the blow-down map. We then have:

Lemma 5.4.16. *Let $k \in \mathbb{N}_0$, $\ell \in \mathbb{R}$. Then there are constants $C > 0$ and $C_\delta > 0$ such that for all $\delta > 0$,*

$$\|f\|_{H_0^{k,\ell-(n-1)/2-\delta}(\Omega)^\bullet} \leq C_\delta \sup_{p \in Y_+} \|\beta_p^* f\|_{H_{\mathrm{b}}^{k,\ell}(\Omega_p)^{\bullet,-}} \leq C C_\delta \|f\|_{H_0^{k,\ell}(\Omega)^\bullet}. \tag{5.4.19}$$

Here, ($\bullet$) indicates supported distributions at the 'artificial' boundary and ($-$) extendible distributions at all other boundary hypersurfaces.

Proof. Let us work locally near a point $p \in Y_+$; since $Y_+ \cong \mathbb{S}^{n-1}$ is compact, all constructions are uniform in p. The only possible issues are near the boundary $Y_+ = \{x = 0\}$, with x a boundary defining function; hence, let us work in a product neighborhood $Y_+ \times [0, 2\epsilon)_x$, $\epsilon > 0$, of Y_+, and let us assume u is supported is $Y_+ \times [0, \epsilon]$.

We use coordinates $x, y_2, \ldots, y_n$ such that $y_j = 0$ at p. Coordinates on Ω_p are then $x, Y_2, \ldots, Y_n$ with $Y_j = y_j/x$, i.e. $\beta_p(x,Y) = (x, xY)$, with the restriction $\sum_{j=2}^n |Y_j|^2 \leq 1$. Therefore,

$$\begin{aligned}\|\beta_p^* f\|_{L^2_{\mathrm{b}}}^2 &\approx \int_{\Omega_p} |\beta_p^* f(x,Y)|^2 \,\frac{dx}{x}\,dY = \int_{\beta_p(\Omega_p)} |f(x,xY)|^2 \,\frac{dx}{x}\,dY \\ &\leq \int |f(x,y)|^2 \,\frac{dx}{x}\,\frac{dy}{x^{n-1}} \approx \|f\|_{L^2_0}^2.\end{aligned}$$

Adding weights to this estimate is straightforward. Next, we observe

$$\begin{aligned}x\partial_x(\beta_p^* f)(x,Y) &= x\partial_x f(x,xY) + Yx\partial_y f(x,xY) \\ \partial_Y(\beta_p^* f)(x,Y) &= x\partial_y f(x,xY),\end{aligned} \tag{5.4.20}$$

and since $|Y|$ is bounded on Ω_p, we conclude that $\beta_p^* f \in H^1_{\mathrm{b}}(\Omega_p)$ is equivalent to f, $x\partial_x f$, $x\partial_y f \in L^2_0(\beta_p(\Omega_p))$, which proves the second inequality in (5.4.19) in the case $k = 1$; the general case is similar.

For the first inequality in (5.4.19), we first note that the additional weight comes from the number of static parts, i.e. interiors of backward light cones from points in Y_+, that one needs to cover any fixed half space $\{x \geq x_0\}$: Namely, for $0 < x_0 \leq \epsilon$, let $\mathscr{B}(x_0) \subset Y_+$ be a set of points such that every point in $\{x \geq x_0\}$ lies in Ω_p for some $p \in \mathscr{B}(x_0)$; then we can choose $\mathscr{B}(x_0)$ such that $|\mathscr{B}(x_0)| \leq C x_0^{-(n-1)}$, where $|\cdot|$ denotes the number of elements in a set. This follows from the observation that the area of the slice $x = x_0$ of Ω_p within $Y_+ \cong \mathbb{S}^{n-1}$ is bounded from below by $c x_0^{n-1}$ for some p-independent constant $c > 0$, where we fix a volume element on $\mathbb{S}^{n-1}$. Indeed, note that null-geodesics of the 0-metric g are, up to reparametrization, the same as null-geodesics of the conformally related metric $x^2 g$, which is a non-degenerate Lorentzian metric up to Y_+. See also Figure 5.3 below.

Thus, putting $\alpha = (n-1)/2 + \delta$, $\delta > 0$, we estimate

$$\begin{aligned}\int_{x\leq\epsilon} |x^\alpha f(x,y)| \,\frac{dx}{x}\,\frac{dy}{x^{n-1}} &= \sum_{j=0}^\infty \int_{2^{-j-1}\epsilon < x \leq 2^{-j}\epsilon} |x^\alpha f(x,y)|^2 \,\frac{dx}{x}\,\frac{dy}{x^{n-1}} \\ &\lesssim \sum_{j=0}^\infty 2^{-2\alpha j} \sum_{p\in\mathscr{B}(2^{-j-1}\epsilon)} \|\beta_p^* f\|_{L^2_{\mathrm{b}}}^2 \lesssim \sum_{j=0}^\infty 2^{-2\alpha j} (2^{-j-1}\epsilon)^{-(n-1)} \sup_{p\in Y_+} \|\beta_p^* f\|_{L^2_{\mathrm{b}}}^2 \\ &\lesssim \sum_{j=0}^\infty 2^{-j(2\alpha-n+1)} \sup_{p\in Y_+} \|\beta_p^* f\|_{L^2_{\mathrm{b}}}^2,\end{aligned}$$

with the sum converging since $2\alpha - n + 1 = 2\delta > 0$. Weights and higher order Sobolev spaces are handled similarly, using (5.4.20). □

In particular, this explains why in equation (5.4.18) we take $d = {}^0 d\colon H_0^{k,\ell}(X) \to H_0^{k-1,\ell}(X; {}^0T^*X)$, namely this is necessary in order to make the global equation interact well with the static patches.

Since we want to consider local problems to solve the global one, the non-linearity q must be local in the sense that $q(u, {}^0 du)(p)$ for $p \in X$ only depends on p and its arguments evaluated at p; let us for simplicity assume that q is in fact a polynomial as in (5.2.15). Using Corollary 5.2.9, we then obtain:

Theorem 5.4.17. *Let $0 \leq \epsilon < \epsilon_0$ with ϵ_0 as in §5.2.3, and $s > \max(3/2 + \epsilon, n/2 + 1)$, $s \in \mathbb{N}$. Let*

$$q(u, {}^0 du) = \sum_{2 \leq j + |\alpha| \leq d} q_{j\alpha} u^j \prod_{k \leq |\alpha|} X_{\alpha,k} u,$$

$q_{j,\alpha} \in \mathbb{C} + H_0^s(\overline{X})$, $X_{\alpha,k} \in \mathcal{V}_0(M)$. Then there exists $C > 0$ such that for all $f \in H_0^{s-1,\epsilon}(\Omega)^\bullet$ with norm $\leq C$, the equation

$$(\Box_g - m^2) u = f + q(u, {}^0 du)$$

has a unique solution $u \in \bigcap_{\delta > 0} H_0^{s, \epsilon - (n-1)/2 - \delta}(\Omega)^\bullet$ that depends continuously on f. Here, we allow $m = 0$ if every summand of q contains at least one 0-derivative, and require $m > 0$ if this is not the case, e.g. if $q = q(u)$ is simply the sum of (multiples of) power of u.

The analogous conclusion also holds for $\Box_g u = f + q({}^0 du)$ provided $\epsilon > 0$, with the solution u being in $\bigcap_{\delta > 0} H_0^{s, -(n-1)/2 - \delta}(\Omega)^\bullet$. Moreover, for all $p \in Y_+$, the limit $u_\partial(p) := \lim_{p' \to p, p' \in X} u(p')$ exists, $u_\partial \in C^{0,\epsilon}(Y_+)$, and $u - u_\partial(\phi \circ \mu) \in x^\epsilon C^0(\overline{X})$, where $\phi \circ \mu$ is identically 1 near Y_+ and vanishes near the 'artificial' boundary of Ω.

Proof. We start by proving the first part: If $f \in H_0^{s-1,\epsilon}(\Omega)^\bullet$, then $f_p = \beta_p^* f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega_p)$ is a uniformly bounded family in the respective norms by Lemma 5.4.16. We can then use Corollary 5.2.9 to solve

$$(\Box_g - m^2) u_p = f_p + q(u_p, {}^{\mathrm{b}} du_p)$$

in the static part Ω_p, where we use that q is a polynomial and the fact that ${}^{\mathrm{b}}T^*_{p'}\Omega_p$ naturally injects into ${}^0T^*_{\beta_p(p')}\Omega$ for $p' \in \Omega_p$ to make sense of the non-linearity; we thus obtain a uniformly bounded family $u_p = \widetilde{u}_p|_{\Omega_p} \in H_{\mathrm{b}}^{s,\epsilon}(\Omega_p)^{\bullet,-}$. By local uniqueness and since f

vanishes near Y_-, we see that the function u, defined by $u(\beta_p(p')) = u_p(p')$ for $p \in Y_+$, $p' \in \Omega_p$, is well-defined, and by Lemma 5.4.16, we indeed have $u \in H_0^{s,\epsilon-(n-1)/2-\delta}(\Omega)^\bullet$ for all $\delta > 0$.

For the second part, we follow the same strategy, obtaining solutions $u_p = c_p(\phi \circ \mu) + u_p'$ of

$$\Box_g u_p = f_p + q({}^{\mathrm{b}}du_p),$$

where $c_p \in \mathbb{C}$ and $u_p' \in H_{\mathrm{b}}^{s,\epsilon}(\Omega_p)^{\bullet,-}$ are uniformly bounded, thus u_p is uniformly bounded in $H_{\mathrm{b}}^{s,-\delta}(\Omega)^\bullet$ for every fixed $\delta > 0$, and therefore the existence of a unique solution u follows as before. Put $u_\partial(p) := c_p$, then $u_\partial(p) = \lim_{p' \to p, p' \in \Omega_p} u(p')$, since $u_p' \in x^\epsilon C^0(\Omega_p)$ by the Sobolev embedding theorem. We first prove that u_∂ so defined is ϵ-Hölder continuous. Let us work in local coordinates (x, y) near a point $(0, y_0)$ in Y_+. Now, u_p' is uniformly bounded in $x^\epsilon C^0(\Omega_p)$, and since for $x_0 > 0$ arbitrary, we have $c_{p_1} + u_{p_1}'(x_0, y_*) = c_{p_2} + u_{p_2}'(x_0, y_*)$ for all $p_1, p_2 \in Y_+$, provided $|p_1 - p_2| \leq cx_0$ for some constant $c > 0$, which ensures that $\Omega_{p_1} \cap \Omega_{p_2} \cap \{x = x_0\}$ is non-empty and thus contains a point (x_0, y_*) (see Figure 5.3), we obtain

$$|c_{p_1} - c_{p_2}| = |u_{p_1}'(x_0, y_*) - u_{p_2}'(x_0, y_*)| \leq Cx_0^\epsilon, \quad |p_1 - p_2| \leq cx_0$$

for all x_0, thus

$$\frac{|u_\partial(p_1) - u_\partial(p_2)|}{|p_1 - p_2|^\epsilon} \leq C, \quad p_1, p_2 \in Y_+.$$

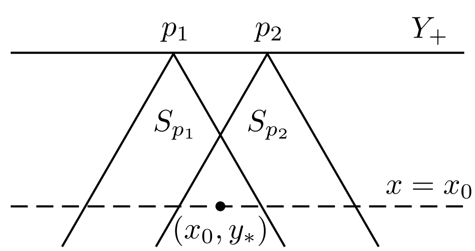

Figure 5.3: Setup for the proof of $u_\partial \in C^{0,\epsilon}(Y_+)$: Shown are the backward light cones from two nearby points $p_1, p_2 \in Y_+$ that intersect within the slice $\{x = x_0\}$ at a point (x_0, y_*).

This in particular implies that

$$\begin{aligned} |u(x,y) - u_\partial(0, y_0)| &\leq |u(x,y) - u_\partial(0,y)| + |u_\partial(0,y) - u_\partial(0,y_0)| \\ &\leq C(|y - y_0|^\epsilon + x^\epsilon) \xrightarrow{x \to 0, y \to y_0} 0, \end{aligned} \tag{5.4.21}$$

hence we in fact have $u_\partial(p) = \lim_{p'\to p, p'\in X} u(p')$. Finally, putting $y = y_0$ in (5.4.21) proves that $u - u_\partial(\phi\circ\mu) \in x^\epsilon C^0(\overline{X})$. □

The major lossy part of the argument is the conversion from f to the family $\beta_p^* f$: Even though the second inequality in Lemma 5.4.16 is optimal (e.g., for functions which are supported in a single static patch), one loses $(n-1)/2$ orders of decay relative to the gluing estimate, i.e. the first inequality in Lemma 5.4.16, which is used to pass from the family u_p to u. Observe on the other hand that the decay properties of u, without regard to those of f, in the first part of the theorem are very natural, since the constant function 1 is an element of $\bigcap_{\delta>0} H_0^{\infty,-(n-1)/2-\delta}(X)$, thus u has an additional decay of ϵ relative to constants.

Remark 5.4.18. For the proof of Theorem 5.4.17, it is irrelevant whether certain 0-Sobolev spaces are algebras, since the main analysis, Corollary 5.2.9, is carried out on b-Sobolev spaces.

5.5 Lorentzian scattering spaces

5.5.1 The linear Fredholm framework

We now consider n-dimensional non-trapping asymptotically Minkowski spacetimes (M, g), a notion which includes the radial compactification of Minkowski spacetime. This notion was briefly recalled in §5.1; here we restate this in the notation of [8, §3] where this notion was introduced.

Thus, M is compact with smooth boundary, with a boundary defining function ρ (we switch the notation from τ mainly to emphasize that ρ is not everywhere timelike), and scattering vector fields $V \in \mathcal{V}_{\mathrm{sc}}(M)$, introduced in [83], are smooth vector fields of the form $\rho V'$, $V' \in \mathcal{V}_{\mathrm{b}}(M)$. Hence, if the z_j are local coordinates on ∂M extended to a neighborhood in M, then a local basis of these vector fields over $\mathcal{C}^\infty(M)$ is $\rho^2\partial_\rho, \rho\partial_{z_j}$. Correspondingly, $\mathcal{V}_{\mathrm{sc}}(M)$ is the set of smooth sections of a vector bundle ${}^{\mathrm{sc}}TM$, which is therefore, roughly speaking, $\rho\,{}^{\mathrm{b}}TM$. The dual bundle, called the scattering cotangent bundle, is denoted by ${}^{\mathrm{sc}}T^*M$. If M is the radial compactification of $\mathbb{R}^n$, then $\mathcal{V}_{\mathrm{sc}}(M)$ is spanned by (the lifts of) the translation invariant vector fields over $\mathcal{C}^\infty(M)$. (Recall from §2.1.3 that the radial compactification means gluing a sphere at infinity to $\mathbb{R}^n$ via the reciprocal polar coordinate map $(r,\omega) \mapsto (r^{-1},\omega) \in (0,1)_\rho \times \mathbb{S}^{n-1}_\omega$, i.e. adding $\rho = 0$ to the right hand side, corresponding to '$r = \infty$'.) The vector field $\rho^2\partial_\rho$ is well-defined up to a positive factor at $\rho = 0$, and is called the scattering normal vector field.

Definition 5.5.1. A *Lorentzian scattering metric* g is a Lorentzian signature, taken to be $(1, n-1)$, metric on ${}^{\mathrm{sc}}TM$, i.e. a smooth symmetric section of ${}^{\mathrm{sc}}T^*M \otimes {}^{\mathrm{sc}}T^*M$ with this signature with the following additional properties:

(1) There is a real $\mathcal{C}^\infty$ function v defined on M with dv, $d\rho$ linearly independent at 'the light cone at infinity,' $S = \{v = 0,\ \rho = 0\}$,

(2) $g(\rho^2\partial_\rho, \rho^2\partial_\rho)$ has the same sign as v at $\rho = 0$, thus $\rho^2\partial_\rho$ is timelike in $v > 0$, spacelike in $v < 0$;

(3) near S,

$$g = v\frac{d\rho^2}{\rho^4} - \Big(\frac{d\rho}{\rho^2} \otimes \frac{\alpha}{\rho} + \frac{\alpha}{\rho} \otimes \frac{d\rho}{\rho^2}\Big) - \frac{\widetilde{h}}{\rho^2},$$

where α is a smooth one-form on M,

$$\alpha = \frac{1}{2}\,dv + \mathcal{O}(v) + \mathcal{O}(\rho),$$

$\widetilde{h}$ is a smooth 2-cotensor on M, which is positive definite on the (codimension two) annihilator of $d\rho$ and dv.

A Lorentzian scattering metric is *non-trapping* if

(1) $S = S_+ \cup S_-$ (each a disjoint union of connected components), in $X = \partial M$ the open set $\{v > 0\} \cap X$ decomposes as $C_+ \cup C_-$ (disjoint union), with $\partial C_+ = S_+$, $\partial C_- = S_-$; we write $C_0 = \{v < 0\} \cap X$,

(2) the projections of all null-bicharacteristics in ${}^{\mathrm{sc}}T^*M \setminus o$ to M tend to $S_\pm$ as their parameter tends to $\pm\infty$ or vice versa.

Since a conformal factor only reparameterizes bicharacteristics, this means that with $\widehat{g} = \rho^2 g$, which is a b-metric on M, the projections of all null-bicharacteristics of $\widehat{g}$ in ${}^{\mathrm{b}}T^*M \setminus o$ tend to $S_\pm$. As already pointed out in §5.1, the difference between the de Sitter-type and Minkowski settings is that at the spherical conormal bundle ${}^{\mathrm{b}}SN^*S$ of S, the nature of the radial points is source/sink rather than a saddle point of the flow at $L_\pm$ discussed in §§3.3.1 and 5.2.

We first state solvability properties, namely we show that under the assumptions of [8, §3], the problem of finding a tempered solution to $\Box_g w = f$ is a Fredholm problem in suitable weighted Sobolev spaces. In particular, there is only a finite dimensional obstruction to existence. Then we strengthen the assumptions somewhat and show actual solvability in the strong sense that in these spaces the solution w satisfies that if f is vanishing to infinite order near $\overline{C_-}$, then so does w.

Let

$$L = \rho^{-(n-2)/2}\rho^{-2}\Box_g \rho^{(n-2)/2} \in \mathrm{Diff}^2_{\mathrm{b}}(M) \tag{5.5.1}$$

be the 'conjugated' b-wave operator (as in [8, §4]), which is formally self-adjoint with respect to the density of the Lorentzian b-metric

$$\widehat{g} = \rho^2 g, \tag{5.5.2}$$

further

$$L = \Box_{\widehat{g}} - \gamma,$$

where $\gamma \in \mathcal{C}^\infty(M)$ is real-valued. Let

$$\begin{gathered} m \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M) \text{ a variable (Sobolev) order function, decreasing along} \\ \text{the direction of the Hamilton flow oriented to the future, i.e. towards } S_+. \end{gathered} \tag{5.5.3}$$

Remark 5.5.2. In the actual application of asymptotically Minkowski spaces, one can take m to be a function on M rather than ${}^{\mathrm{b}}S^*M$ by making it take constant values near $\overline{C_+}$, resp. $\overline{C_-}$, corresponding to the requirements at $\mathcal{R}_+$, resp. $\mathcal{R}_-$ below, and transitioning in between using a time function $\widetilde{\mathfrak{t}}$ as introduce in the discussion preceding Theorem 5.5.4, i.e. making m of the form $F \circ \widetilde{\mathfrak{t}}$ for appropriate F. Since this simplifies some arguments below, we assume this whenever it is convenient.

With

$$\mathcal{R}_+ = {}^{\mathrm{b}}SN^*S_+, \text{ resp. } \mathcal{R}_- = {}^{\mathrm{b}}SN^*S_-,$$

the future, resp. past, radial sets in ${}^{\mathrm{b}}S^*M$, see [8, §3.6], and with

$$m + l < 1/2 \text{ at } \mathcal{R}_+,\ m + l > 1/2 \text{ at } \mathcal{R}_-,$$

m constant near $\mathcal{R}_+ \cup \mathcal{R}_-$, one has an estimate

$$\|u\|_{H_b^{m,l}} \leq C\|Lu\|_{H_b^{m-1,l}} + C\|u\|_{H_b^{m',l}}, \tag{5.5.4}$$

provided one assumes $m' < m$,

$$m' + l > 1/2 \text{ at } \mathcal{R}_-, \;\; u \in H_b^{m',l}.$$

To see this, we recall and record a slight improvement of [8, Proposition 4.4]:

Proposition 5.5.3. *Suppose L is as above.*

If $m + l < 1/2$, and if $u \in H_b^{-\infty,l}(M)$ then $\mathcal{R}_\pm$ (and thus a neighborhood of $\mathcal{R}_\pm$) is disjoint from $\mathrm{WF}_b^{m,l}(u)$ provided $\mathcal{R}_\pm \cap \mathrm{WF}_b^{m-1,l}(Lu) = \emptyset$ and a punctured neighborhood of $\mathcal{R}_\pm$, with $\mathcal{R}_\pm$ removed, in $\Sigma \cap {}^{\mathrm{b}}S^*M$ is disjoint from $\mathrm{WF}_b^{m,l}(u)$.

On the other hand, if $m' + l > 1/2$, $m \geq m'$, $u \in H_b^{-\infty,l}(M)$ and if $\mathrm{WF}_b^{m',l}(u) \cap \mathcal{R}_\pm = \emptyset$ then $\mathcal{R}_\pm$ (and thus a neighborhood of $\mathcal{R}_\pm$) is disjoint from $\mathrm{WF}_b^{m,l}(u)$ provided $\mathcal{R}_\pm \cap \mathrm{WF}_b^{m-1,l}(Lu) = \emptyset$.

Proof. The first statement is proved in [8, Proposition 4.4]. The second statement follows the same way, but in that case the product of the required powers of the boundary defining functions, $\rho^{-2l}\widehat{\rho}^{-2m+1}$, with $\widehat{\rho}$ the defining function of fiber infinity[21] as in §3.3.1, in the commutant of [8, Proposition 4.4] provides a favorable sign, thus [8, Equation (4.1)] holds without the E term. However, when regularizing, the regularizer contributes a term with the opposite sign, exactly as in [114, Proof of Propositions 2.3-2.4]; this forces the requirement on the a priori regularity, namely $\mathrm{WF}_b^{m',l}(u) \cap \mathcal{R}_\pm = \emptyset$, exactly as in the referred results of [114]; see also Proposition 3.3.8 above. □

Indeed, due to the closed graph theorem, (5.5.4) follows immediately from the b-radial point regularity statements of Proposition 5.5.3 for sources/sinks, and the propagation of b-singularities for variable order Sobolev spaces, which is not proved in [8], but whose analogue in standard Sobolev spaces is proved there in [8, Proposition A.1] (with additional references given to related results in the literature), and as it is a purely symbolic argument, the extension to the b-setting is straightforward, cf. Proposition 3.3.8 here and [8, Proposition 4.4] extending the radial point results, Propositions 2.3-2.4, of [114], from the boundaryless setting to the b-setting.

[21] This defining function is denoted by ν in [8].

One also has a similar estimate for L when one replaces m by a weight $\widetilde{m}$ which is increasing along the direction of the Hamilton flow oriented towards the past,

$$\widetilde{m}+\widetilde{l}>1/2 \text{ at } \mathcal{R}_+, \quad \widetilde{m}+\widetilde{l}<1/2 \text{ at } \mathcal{R}_-,$$

provided one assumes $\widetilde{m}'<\widetilde{m}$,

$$\widetilde{m}'+\widetilde{l}>1/2 \text{ at } \mathcal{R}_+, \quad u\in H_{\mathrm{b}}^{\widetilde{m}',\widetilde{l}}.$$

Further L can be replaced by L^*. Thus,

$$\|u\|_{H_{\mathrm{b}}^{\widetilde{m},\widetilde{l}}} \le C\|L^*u\|_{H_{\mathrm{b}}^{\widetilde{m}-1,\widetilde{l}}} + C\|u\|_{H_{\mathrm{b}}^{\widetilde{m}',\widetilde{l}}}. \tag{5.5.5}$$

Just as in the asymptotically de Sitter/Kerr-de Sitter settings, one wants to improve these estimates so that the space $H_{\mathrm{b}}^{m,l}$, resp. $H_{\mathrm{b}}^{\widetilde{m},\widetilde{l}}$, on the left hand side includes compactly into the error term on the right hand side. This argument is completely analogous to §5.2.1 using the Mellin transformed normal operator estimates obtained in [8, §5]. We thus further assume that there are no poles of the Mellin conjugate $\widehat{L}(\sigma)$ on the line $\operatorname{Im}\sigma=-l$. Then using the Mellin transform and the estimates for $\widehat{L}(\sigma)$ (including the high energy estimates, which imply that for all but a discrete set of l the aforementioned lines do not contain such poles), as in §5.2.1, we obtain that on $\mathbb{R}_+\times\partial M$

$$\|v\|_{H_{\mathrm{b}}^{\widehat{m},l}} \le C\|N(L)v\|_{H_{\mathrm{b}}^{\widehat{m}-1,l}} \tag{5.5.6}$$

when $\widehat{m}\in\mathcal{C}^\infty(S^*\partial M)$ is a variable order function decreasing along the direction of the Hamilton flow oriented to the future, Λ_+, resp. Λ_-, the future, resp. past, radial sets in $S^*\partial M$, and with

$$\widehat{m}+l<1/2 \text{ at } \Lambda_+,\ \widehat{m}+l>1/2 \text{ at } \Lambda_-.$$

One can take

$$\widehat{m}=m|_{T^*\partial M},$$

for instance, under the identification of $T^*\partial M$ as a subspace of ${}^{\mathrm{b}}T^*_{\partial M}M$ by means of the boundary defining function ρ (see §2.1.1), taking into account that homogeneous degree zero functions on $T^*\partial M\setminus o$ are exactly functions on $S^*\partial M$, and analogously on ${}^{\mathrm{b}}T^*_{\partial M}M$.

However, in the limit $\sigma \to \infty$, one should use norms depending on σ reflecting the dependence of the semiclassical norm on h. We recall from Remark 5.5.2 that in the main case of interest one can take m to be a pullback from M, and thus the Mellin transformed operator norms are independent of σ. In either case, we simply write m in place of $\widehat{m}$.

Again, we have an analogous estimate for $N(L^*)$:

$$\|v\|_{H_{\mathrm{b}}^{\widetilde{m},\widetilde{l}}} \leq C\|N(L^*)v\|_{H_{\mathrm{b}}^{\widetilde{m}-1,\widetilde{l}}}, \tag{5.5.7}$$

provided $-\widetilde{l}$ is not the imaginary part of a pole of $\widehat{L^*}$, and provided $\widetilde{m}$ satisfies the requirements above. As $\widehat{L^*}(\sigma) = (\widehat{L})^*(\overline{\sigma})$, the requirement on $-\widetilde{l}$ is the same as $\widetilde{l}$ not being the imaginary part of a pole of $\widehat{L}$.

At this point the argument in §5.2.1, which in turn followed §3.2.1, can be repeated verbatim to yield that for m with $m + l > 3/2$ at $\mathcal{R}_-$ (with the stronger restriction coming from the requirements on m' at $\mathcal{R}_-$, $\widetilde{m}'$ at $\mathcal{R}_+$, and $m' < m-1$, $\widetilde{m}' < \widetilde{m}-1$; recall that one needs to estimate the normal operator on these primed spaces), and $m + l < 1/2$ at $\mathcal{R}_+$,

$$\|u\|_{H_{\mathrm{b}}^{m,l}} \leq C\|Lu\|_{H_{\mathrm{b}}^{m-1,l}} + C\|u\|_{H_{\mathrm{b}}^{m'+1,l-1}}, \tag{5.5.8}$$

where now the inclusion $H_{\mathrm{b}}^{m,l} \to H_{\mathrm{b}}^{m'+1,l-1}$ is compact (as we choose $m' < m - 1$); this argument required m, l, m' satisfied the requirements preceding (5.5.4), and that $-l$ is not the imaginary part of any pole of $\widehat{L}$.

Analogous estimates hold for L^*:

$$\|u\|_{H_{\mathrm{b}}^{\widetilde{m},\widetilde{l}}} \leq C\|L^*u\|_{H_{\mathrm{b}}^{\widetilde{m}-1,\widetilde{l}}} + C\|u\|_{H_{\mathrm{b}}^{m'+1,\widetilde{l}-1}}, \tag{5.5.9}$$

provided $\widetilde{m}$, $\widetilde{l}$, $\widetilde{m}'$ satisfy the requirements stated before (5.5.5), $\widetilde{m}' < \widetilde{m} - 1$, and provided $-\widetilde{l}$ is not the imaginary part of a pole of $\widehat{L^*}$ (i.e. $\widetilde{l}$ of $\widehat{L}$).

Via the same functional analytic argument as in §5.2.1 we thus obtain Fredholm properties of L, in particular solvability, modulo a (possible) finite dimensional obstruction, in $H_{\mathrm{b}}^{m,l}$ if

$$m + l > 3/2 \text{ at } \mathcal{R}_-,\ m + l < -1/2 \text{ at } \mathcal{R}_+. \tag{5.5.10}$$

More precisely, we take $\widetilde{m} = 1 - m$, $\widetilde{l} = -l$, so $m + l < -1/2$ at $\mathcal{R}_+$ means $\widetilde{m} + \widetilde{l} = 1 - (m + l) > 3/2$, so the space on the left hand side of (5.5.8) is dual to that in the first term on the right hand side of (5.5.9), and the same for the equations interchanged. Then the Fredholm statement holds for

$$L\colon \mathcal{X}^{m,l} \to \mathcal{Y}^{m-1,l},$$

where

$$\mathcal{X}^{s,r} = \{u \in H_{\mathrm{b}}^{s,r} \colon Lu \in H_{\mathrm{b}}^{s-1,r}\}, \quad \mathcal{Y}^{s,r} = H_{\mathrm{b}}^{s,r}.$$

Note that, by propagation of singularities, i.e. most importantly using Proposition 5.5.3, with $\operatorname{Ker} L \subset H_{\mathrm{b}}^{m,l}$, $\operatorname{Ker} L^* \subset H_{\mathrm{b}}^{1-m,-l}$ a priori,

$$\operatorname{Ker} L \subset H_{\mathrm{b}}^{m^\flat,l},\ \operatorname{Ker} L^* \subset H_{\mathrm{b}}^{1-m^\flat,-l} \tag{5.5.11}$$

if $m^\flat + l > 1/2$ at $\mathcal{R}_-$, $m^\flat + l < 1/2$ at $\mathcal{R}_+$.

Using the same argument, we can thus improve (5.5.10) using the propagation of singularities. Namely, suppose one merely has

$$m + l > 3/2 \text{ at } \mathcal{R}_-,\ m + l < 1/2 \text{ at } \mathcal{R}_+, \tag{5.5.12}$$

so the requirement at $\mathcal{R}_+$ is weakened. Then let $m^\sharp = m - 1$ near $\mathcal{R}_+$, $m^\sharp \leq m$ everywhere, but still satisfying the requirements for the order function along the Hamilton flow, so the Fredholm result is applicable with $m^\sharp$ in place of m. Now, if $u \in \mathcal{X}^{m^\sharp,l}$, $Lu = f$, $f \in \mathcal{Y}^{m-1,l} \subset \mathcal{Y}^{m^\sharp-1,l}$, then Proposition 5.5.3 gives $u \in \mathcal{X}^{m,l}$. Further, if $\operatorname{Ker} L$ and $\operatorname{Ker} L^*$ are trivial, this gives that for m, l as in (5.5.12), satisfying also the conditions along the Hamilton flow, $L\colon \mathcal{X}^{m,l} \to \mathcal{Y}^{m-1,l}$ is invertible.

Now, as invertibility (the absence of kernel and cokernel) is preserved under sufficiently small perturbations, it holds in particular for perturbations of the Minkowski metric which are Lorentzian scattering metrics in our sense, with closeness measured in smooth sections of the second symmetric power of ${}^{\mathrm{b}}T^*M$. (Note that non-trapping is also preserved under such perturbations.)

For more general asymptotically Minkowski metrics we note that, due to Theorem 5.2.3 (which does not have any requirements for the timelike nature of the boundary defining function, and which works locally near $\overline{C_-}$ either by working on (extendible) function spaces or by using the localization given by wave propagation as in §3.3 of [114] or §5.4.1 here) elements of $\operatorname{Ker} L$ on $H_{\mathrm{b}}^{m,l}$, with m, l as above, lie in $\dot{\mathcal{C}}^\infty(M)$ locally near $\overline{C_-}$ provided all resonances, i.e. poles of $\widehat{L}(\sigma)$, in $\operatorname{Im}\sigma < -l$ have polar parts (coefficients of the Laurent series) that map into distributions supported on $\overline{C_+}$. As shown in [117, Remark 4.17] when $\widehat{L}(\sigma)$ arises from a Lorentzian conic metric as in[22] [117, Equation (3.5)], but with the arguments applicable without significant changes in our more general case, see also [8, §7] for our general setting, and [114, Remark 4.6] for a related discussion with complex absorption, the resonances of $\widehat{L}(\sigma)$ consist of the resonances of the asymptotically hyperbolic resolvents on the caps, namely $\mathcal{R}_{C_+}(\sigma)$, $\mathcal{R}_{C_-}(-\sigma)$, as well as possibly imaginary integers, $\sigma \in i\mathbb{Z} \setminus \{0\}$, with resonant states when $\operatorname{Im}\sigma < 0$ being differentiated delta distributions at $S_+ = \partial C_+$ while the dual states are differentiated delta distributions at $S_- = \partial C_-$ when $\operatorname{Im}\sigma > 0$; the latter arise, e.g. as poles on even dimensional Minkowski space. More generally, when composed with extension of $\mathcal{C}^\infty_{\mathrm{c}}(\overline{C_-} \cup C_0)$ by zero to $\mathcal{C}^\infty(X)$ from the right and with restriction to $\overline{C_-} \cup C_0$ from the left, the only poles of $\widehat{L}(\sigma)$ are those of $\mathcal{R}_{C_-}(-\sigma)$ as well as the possible $\sigma \in i\mathbb{N}_+$. Thus, fixing $l > -1$, one can conclude that elements of $\operatorname{Ker} L$ are in $\dot{\mathcal{C}}^\infty(M)$ locally near $\overline{C_-}$ provided $\mathcal{R}_{C_-}(\widetilde{\sigma})$ has no poles in $\operatorname{Im}\widetilde{\sigma} > l$. (The only change for $l \leq -1$ is that one needs to exclude the potential pure imaginary integer poles as well.) The analogous statement for $\operatorname{Ker} L^*$ on $H_{\mathrm{b}}^{\widetilde{m},\widetilde{l}}$ is that fixing $\widetilde{l} > -1$, elements are in $\dot{\mathcal{C}}^\infty(M)$ near $\overline{C_+}$ provided $\mathcal{R}_{C_+}(\widetilde{\sigma})$ has no poles in $\operatorname{Im}\widetilde{\sigma} > \widetilde{l}$. As $\widetilde{l} = -l$ for our duality arguments, the weakest symmetric assumption (in terms of strength at C_+ and C_-) is that $\mathcal{R}_{C_\pm}$ do not have any poles in the closed upper half plane; here the closure is added to make sure L is actually Fredholm on $H_{\mathrm{b}}^{m,l}$ with $l = 0$. In general, if one wants to use other values of l, one needs to assume the absence of poles in $\operatorname{Im}\sigma \geq -|l|$ (if one wants to keep the hypotheses symmetric).

Note that assuming $\frac{d\rho}{\rho}$ is timelike (with respect to $\widehat{g}$, defined in (5.5.2)) near $\overline{C_-}$, one *automatically* has the absence of poles of $\mathcal{R}_{C_-}$ in an upper half plane, and the finiteness (with multiplicity) of the number of poles in any upper half plane, by the semiclassical estimates of [114, §§3.2, 7.2] (one can ignore the complex absorption discussion there), so in this case the issue is that of a possible finite number of resonances. There is an analogous statement if $\frac{d\rho}{\rho}$ is timelike near $\overline{C_+}$ for $\mathcal{R}_{C_+}$.

Now, assuming still that $\frac{d\rho}{\rho}$ is timelike at, hence near $\overline{C_-}$, it is easy to construct a

[22] In [117], the boundary defining function used to define the Mellin transform is replaced by its reciprocal, which effectively switches the sign of σ in the operator, but also the backward propagator is considered (propagating toward the past light cone), which reverses the role of σ and $-\sigma$ again, so in fact, the signs in [117] and [8] agree for the formulae connecting the asymptotically hyperbolic resolvents and the global operator, $\widehat{L}(\sigma)$.

function $\mathfrak{t}$ which has a timelike differential near $\overline{C_-}$, and appropriate sublevel sets are small neighborhoods of $\overline{C_-}$. Once one has such a function $\mathfrak{t}$, energy estimates can be used to conclude that rapidly vanishing, in such a neighborhood, solutions of $Lu = 0$ actually vanish in this neighborhood, so elements of $\operatorname{Ker} L$ have support disjoint from $\overline{C_-}$; similarly elements of $\operatorname{Ker} L^*$ have support disjoint from $\overline{C_+}$.

Concretely, with $\widehat{G}$ the dual b-metric of $\widehat{g}$, let U_- be a neighborhood of $\overline{C_-}$, and let $0 < \epsilon_0 < \epsilon_1$, $\widetilde{\epsilon} > 0$, $\delta > 0$ be such that $\{\rho \leq \widetilde{\epsilon},\ v \geq -\epsilon_1\} \cap U_-$ is a compact subset of U_-, and on U_-

$$\rho < \widetilde{\epsilon},\ v > -\epsilon_1 \Rightarrow \widehat{G}\Big(\frac{d\rho}{\rho}, \frac{d\rho}{\rho}\Big) > \delta,$$
$$\rho < \widetilde{\epsilon},\ -\epsilon_1 < v < -\epsilon_0 \Rightarrow \widehat{G}\Big(\frac{d\rho}{\rho}, dv\Big) < 0,\ \widehat{G}(dv, dv) > 0.$$

Such U_- and constants indeed exist. First, there is U_- and $\widetilde{\epsilon}' > 0$, $\epsilon_1' > 0$ such that $\{\rho \leq \widetilde{\epsilon}',\ v \geq -\epsilon_1'\} \cap U_-$ is a compact subset of U_- since $\overline{C_-}$ is defined by $\{\rho = 0,\ v \geq 0\}$ in a neighborhood of $\overline{C_-}$ with $d\rho \neq 0$ there and $dv \neq 0$ near $v = 0$; we then consider $\widetilde{\epsilon} < \widetilde{\epsilon}'$, $\epsilon_1 < \epsilon_1'$ below. Next, since $\widehat{G}(\frac{d\rho}{\rho}, \frac{d\rho}{\rho})$ is positive on a neighborhood of $\overline{C_-}$ by assumption (thus for any sufficiently small $\epsilon_1, \widetilde{\epsilon}$ there is a desired δ so that the first inequality is satisfied) and $\widehat{G}(\frac{d\rho}{\rho}, dv)|_{S_-} = -2$, any sufficiently small ϵ_1 and $\widetilde{\epsilon}$ give $\widehat{G}(\frac{d\rho}{\rho}, dv) < 0$ in the desired region, and finally $\widehat{G}(dv, dv) > 0$ on C_0 near S_- (as $\widehat{G}(dv, dv) = -4v + \mathcal{O}(v^2)$ there), so choosing ϵ_1 sufficiently small, $\epsilon_0 < \epsilon_1$, and then $\widetilde{\epsilon}$ sufficiently small satisfies all criteria.

Now let ϵ_-, ϵ_+ be such that $0 < \epsilon_- < \epsilon_+ < \widetilde{\epsilon}$, and let $\phi \in \mathcal{C}^\infty(\mathbb{R})$ have $\phi' \leq 0$, $\phi = 0$ near $[-\epsilon_0, \infty)$, $\phi > \widetilde{\epsilon}$ near $(-\infty, -\epsilon_1]$, $\phi' < 0$ when ϕ takes values in $[\epsilon_-, \epsilon_+]$. Then $\mathfrak{t} = \rho + \phi(v)$ has the property that on U_-

$$\mathfrak{t} \leq \epsilon_+ \Rightarrow \rho, \phi(v) \leq \epsilon_+ \Rightarrow \rho < \widetilde{\epsilon}, v > -\epsilon_1,$$

and

$$v \geq -\epsilon_0 \Rightarrow \mathfrak{t} = \rho.$$

Thus, on U_- if $v \geq -\epsilon_0$ and $\mathfrak{t} \leq \epsilon_+$ then $d\mathfrak{t}$ is timelike as $d\rho$ is such, while if $v < -\epsilon_0$, $\mathfrak{t} \leq \epsilon_+$ then

$$\widehat{G}(d\mathfrak{t}, d\mathfrak{t}) = \rho^2 \widehat{G}\Big(\frac{d\rho}{\rho}, \frac{d\rho}{\rho}\Big) + 2\phi'(v)\rho\widehat{G}\Big(\frac{d\rho}{\rho}, dv\Big) + (\phi'(v))^2\widehat{G}(dv, dv)$$

and all terms are ≥ 0 in view of $-\epsilon_1 < v < -\epsilon_0$, $\rho \leq \widetilde{\epsilon}$, with the inequality being strict when

$\mathfrak{t} \in [\epsilon_-, \epsilon_+]$ (as well as in $M^\circ \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+]))$. Thus, near $\mathfrak{t}^{-1}([\epsilon_-, \epsilon_+]) \cap U_-$, $\mathfrak{t}$ is a timelike function; the same is true on $M^\circ \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+]) \cap U_-$. Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ with $\chi' \leq 0$, $\chi = 1$ near $(-\infty, \epsilon_-]$, $\chi = 0$ near $[\epsilon_+, \infty)$, and let $\chi \circ \mathfrak{t}$, defined by this formula in U_-, be extended to M as 0 outside U_-; since $\mathfrak{t}^{-1}((-\infty, \epsilon_+]) \cap U_-$ is a compact subset of U_-, this gives a $\mathcal{C}^\infty$ function. Further, ρ is also timelike, with $\frac{d\rho}{\rho}$ and $d\mathfrak{t}$ in the same component of the timelike cone; see Figure 5.4. Correspondingly, one can apply energy estimates using the timelike vector field $V = (\chi \circ \mathfrak{t})\rho^{-\ell}\widehat{G}(\frac{d\rho}{\rho}, .)$, cf. [114, §3.3] leading up to Equation (3.24) and the subsequent discussion, which in turn is based on [113, §§3-4]. Here one needs to make both $-\chi'$ large relative to χ and $\ell > 0$ large (making the b-derivative of $\rho^{-\ell}$ large relative to $\rho^{-\ell}$), as discussed in the Mellin transformed setting in [114, §3.3], in [113, §§3-4], as well as §4.1 here (with τ in place of ρ, but with the sign of ℓ reversed due to the difference between b-saddle points and b-sinks/sources). Notice that taking ℓ large is exactly where the rapid decay near $\overline{C_-}$ is used.

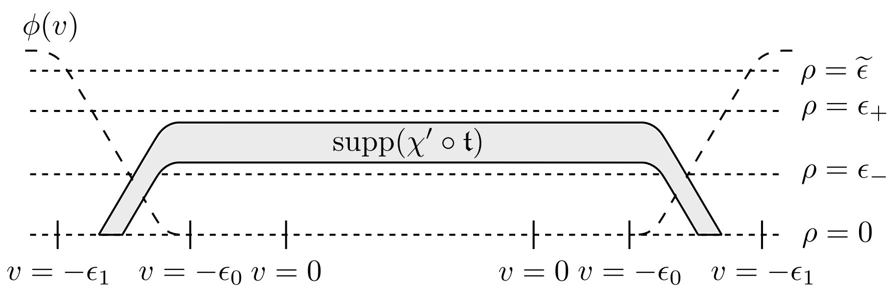

Figure 5.4: Setup for energy estimates near $\overline{C_-}$: The shaded region is the support of $\chi' \circ \mathfrak{t}$, where $-\chi'$ is used to dominate χ to give positivity in the energy estimate; near $\rho = 0$ and on supp$(\chi \circ \mathfrak{t})$, i.e. in the region between $\rho = 0$ and the shaded region, a sufficiently large weight $\rho^{-\ell}$ gives positivity.

We have seen that the existence of appropriate timelike functions, such as $\mathfrak{t}$, in a neighborhood of $\overline{C_+}$ and $\overline{C_-}$ is automatic (in a slightly degenerate sense at $\overline{C_\pm}$ themselves) when $\frac{d\rho}{\rho}$ is timelike in these regions; indeed these functions could be extended to a neighborhood of C_0 if v is appropriately chosen. In order to conclude that elements of $\operatorname{Ker} L$ and $\operatorname{Ker} L^*$ vanish globally, however, we need to control *all* of the interior of M. This can be accomplished by showing global hyperbolicity[23] of M°, which in turn can be seen by applying a result due to Geroch [53]. Namely, by [53, Theorem 11] it suffices to show that a suitable $\mathcal{S}$ is a Cauchy surface, which by [53, Property 6] follows if we show that $\mathcal{S}$ is achronal, closed,

[23] In Geroch's notation, our M° is M.

and every null-geodesic intersects and then re-emerges from $\mathcal{S}$. In order to define $\mathcal{S}$, it is useful to define $\widehat{\mathfrak{t}} = \psi \circ \mathfrak{t}$ in U_-, where $\psi \in \mathcal{C}^\infty(\mathbb{R})$, $\psi' \geq 0$, $\psi(t) = t$ near $t \leq \epsilon_-$, $\psi'(t) > 0$ for $t < \epsilon_+$, $\psi'(t) = 0$ for $t \geq \epsilon_+$; let $T = \psi(\epsilon_+) > \epsilon_-$. Further, extend $\widehat{\mathfrak{t}}$ to M as $= T$ outside U_-; since $U_- \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+])$ is compact, this gives a $\mathcal{C}^\infty$ function on M. Thus, $\widehat{\mathfrak{t}} \in \mathcal{C}^\infty(M)$ is a globally weakly time-like function in that $\widehat{G}(d\widehat{\mathfrak{t}}, d\widehat{\mathfrak{t}}) \geq 0$, and it is strictly time-like in $M^\circ \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+))$. In particular, it is monotone along all null-geodesics. Further, $\widehat{\mathfrak{t}} = 0$ at S_- and $\widehat{\mathfrak{t}} = T > 0$ at S_+, indeed near S_+. Then we claim that $\mathcal{S} = \widehat{\mathfrak{t}}^{-1}(\epsilon_-) \cap M^\circ$ is a Cauchy surface.

Now, $\mathcal{S}$ is closed in M° since $\overline{\mathcal{S}}$ is closed in M; indeed it is a closed embedded submanifold. By our non-trapping assumption, every null-geodesic in M° tends to S_+ in one direction and S_- in the other direction, so on future oriented null-geodesics (ones tending to S_+), $\widehat{\mathfrak{t}}$ is monotone increasing, attaining all values in $(0, T]$. Since at the ϵ_- level set of $\mathfrak{t}$, hence of $\widehat{\mathfrak{t}}$, $d\widehat{\mathfrak{t}}$ is strictly time-like, the value ϵ_- is attained exactly once for $\widehat{\mathfrak{t}}$ along null-geodesics. Thus, every null-geodesic intersects $\mathcal{S}$ and then re-emerges from it. Finally, $\mathcal{S}$ is achronal, i.e. there exist no time-like curves connecting two points on $\mathcal{S}$: any future oriented time-like curve (meaning with tangent vector in the time-like cone whose boundary is the future light cone) in $M^\circ \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+))$ has $\widehat{\mathfrak{t}}$ monotone increasing, with the increase being strict near $\mathcal{S}$, so again the value ϵ_- can be attained at most once on such a curve. In summary, this proves that M° is globally hyperbolic, so every solution of $Lu = 0$ with vanishing Cauchy data on $\mathcal{S}$ vanishes identically, in particular by what we have observed, $\operatorname{Ker} L$ and $\operatorname{Ker} L^*$ are trivial on the indicated spaces.

In summary:

Theorem 5.5.4. *If (M, g) is a non-trapping Lorentzian scattering metric in the sense of [8], $|l| < 1$, and*

(1) The induced asymptotically hyperbolic resolvents $\mathcal{R}_{C_\pm}$ have no poles in $\operatorname{Im}\sigma \geq -|l|$,

(2) $\frac{d\rho}{\rho}$ is timelike near $\overline{C_+} \cup \overline{C_-}$,

then for order functions $m \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$ satisfying (5.5.3) and (5.5.12), the forward problem for the conjugated wave operator L, see (5.5.1), i.e. with L considered as a map

$$L \colon \mathcal{X}^{m,l} \to \mathcal{Y}^{m-1,l},$$

is invertible.

Extending the notation of [8], especially [8, §4], we denote by $H_{\mathrm{b}}^{m,l,k}(M)$, where $m, l \in \mathbb{R}, k \in \mathbb{N}_0$, the space of all $u \in H_{\mathrm{b}}^{m,l}(M)$ (i.e. $u \in \rho^l H_{\mathrm{b}}^m(M)$, where ρ is the boundary defining function of M) such that $\mathcal{M}^j u \in H_{\mathrm{b}}^{m,l}(M)$ for all $0 \leq j \leq k$. Here, $\mathcal{M} \subset \Psi_{\mathrm{b}}^1(M)$ is the $\Psi_{\mathrm{b}}^0(M)$-module of pseudodifferential operators with principal symbol vanishing on the radial set $\mathcal{R}_+$ of the operator $L = \rho^{-(n-2)/2}\rho^{-2}\Box_g \rho^{(n-2)/2}$; in the coordinates ρ, v, y as in [8] (ρ being as above, v a defining function of the light cone at infinity within ∂M, y coordinates within in the light cone at infinity), $\mathcal{M}$ has local generators $\rho\partial_\rho, \rho\partial_v, v\partial_v, \partial_y$. Then the results of [8], concretely Proposition 4.4, extend our theorem to the spaces with module regularity. Namely the reference guarantees the module regularity $u \in H_{\mathrm{b}}^{m,l,k}(M)$ of a solution u of $Lu = f$ if f has matching module regularity[24] $f \in H_{\mathrm{b}}^{m-1,l,k}(M)$ and if u is in $H_{\mathrm{b}}^{m+k,l}(M)$ near $\overline{C_-}$. If $f \in H_{\mathrm{b}}^{m-1,l,k}(M)$, then in particular f is locally in $H_{\mathrm{b}}^{m+k-1,l}$ near $\overline{C_-}$, thus, taking into account that $m + l > 1/2$ already there, u is in $H_{\mathrm{b}}^{m+k,l}$ in that region by Proposition 5.5.3 (by the first case there, i.e. in the high regularity regime). Thus, an application of the closed graph theorem gives the following boundedness result:

Theorem 5.5.5. *Under the assumptions of Theorem 5.5.4, L^{-1} has the property that it restricts to*

$$L^{-1} : H_{\mathrm{b}}^{m-1,l,k} \to H_{\mathrm{b}}^{m,l,k},\ k \geq 0,$$

as a bounded map.

In particular, letting $\Omega = \{\widetilde{\mathfrak{t}} \geq 0\}$, where $\widetilde{\mathfrak{t}} = \widehat{\mathfrak{t}} - \epsilon_-$ so that it attains the value 0 within $M \setminus (\overline{C_+} \cup \overline{C_-})$, we have a forward solution operator S of L which maps $H_{\mathrm{b}}^{m-1,l,k}(\Omega)^\bullet$ into $H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$, given that $m+l < 1/2$; let us assume that m is constant in Ω. Here, $H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$ consists of supported distributions at $\partial\Omega \cap C_0^\circ = \{\widetilde{\mathfrak{t}} = 0\}$.

Remark 5.5.6. Using the arguments leading to Theorem 5.5.4 in the current, forward problem, setting, but now also using standard energy estimates near the artificial boundary $\widetilde{\mathfrak{t}} = 0$ of Ω, we see that if suffices to control the resonances of the asymptotically hyperbolic resolvent in the upper cap C_+ in order to ensure the invertibility of the forward problem.

[24] This Proposition in [8] is stated making the stronger assumption, $f \in H_{\mathrm{b}}^{m-1+k,l}(M)$. However, the proof goes through for just $f \in H_{\mathrm{b}}^{m-1,l,k}(M)$ in a completely analogous manner to the result of Haber and Vasy [59, Theorem 6.3], where (in the boundaryless setting, for a Lagrangian radial set) the result is stated in this generality.

5.5.2 Algebra properties of b-Sobolev spaces with module regularity

In order to discuss nonlinear wave equations on an asymptotically Minkowski space, we need to discuss the algebra properties of the spaces $H_{\mathrm{b}}^{m,l,k}$. Even though we are only interested in $H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$, we consider $H_{\mathrm{b}}^{m,l,k}(M)$, where m is constant on M for notational simplicity, and the results we prove below are valid for $H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$ by the same proofs.

We start with the following lemma:

Lemma 5.5.7. *Let $l_1, l_2 \in \mathbb{R}$, $k > n/2$. Then $H_{\mathrm{b}}^{0,l_1,k} \cdot H_{\mathrm{b}}^{0,l_2,k} \subset H_{\mathrm{b}}^{0,l_1+l_2-1/2,k}$.*

Proof. The generators $\rho\partial_\rho, \rho\partial_v, v\partial_v, \partial_y$ of $\mathcal{M}$ take on a simpler form if we blow up the point $(\rho, v) = (0, 0)$. It is most convenient to use projective coordinates on the blown-up space, namely:

(1) Near the interior of the front face, we use the coordinates $\widetilde{\rho} = \rho \geq 0$ and $s = v/\rho \in \mathbb{R}$. We compute $\rho\partial_\rho = \widetilde{\rho}\partial_{\widetilde{\rho}} - s\partial_s$, $v\partial_v = s\partial_s$, $\rho\partial_v = \partial_s$; and since $\frac{d\rho}{\rho}\,dv\,dy = d\widetilde{\rho}\,ds\,dy$ (this is the b-density from $H_{\mathrm{b}}^{0,l,k}$), the space $H_{\mathrm{b}}^{0,l,k}$ becomes

$$A^{l,k} := \{u \in \widetilde{\rho}^l L^2(d\widetilde{\rho}\,ds\,dy) \colon \mathcal{A}^j u \in \widetilde{\rho}^l L^2(d\widetilde{\rho}\,ds\,dy), 0 \leq j \leq k\},$$

where $\mathcal{A}$ is the C^∞-module of differential operators generated by $\partial_s, \widetilde{\rho}\partial_{\widetilde{\rho}}, \partial_y$.

Now, observe that $\widetilde{\rho}^l L^2(d\widetilde{\rho}\,ds\,dy) = \widetilde{\rho}^{l-1/2} L^2(\frac{d\widetilde{\rho}}{\rho}\,ds\,dy)$; therefore, we can rewrite

$$\begin{aligned} A^{l,k} &= \{u \in \widetilde{\rho}^{l-1/2} L^2(\tfrac{d\widetilde{\rho}}{\rho}\,ds\,dy) \colon \mathcal{A}^j u \in \widetilde{\rho}^{l-1/2} L^2(\tfrac{d\widetilde{\rho}}{\rho}\,ds\,dy), 0 \leq j \leq k\} \\ &= \widetilde{\rho}^{l-1/2} H_{\mathrm{b}}^k(\tfrac{d\widetilde{\rho}}{\rho}\,ds\,dy). \end{aligned}$$

In particular, by the Sobolev algebra property, Lemma 5.2.7, and the locality of the multiplication, choosing $k > n/2$ ensures that $\widetilde{\rho}^{l_1-1/2} H_{\mathrm{b}}^k \cdot \widetilde{\rho}^{l_2-1/2} H_{\mathrm{b}}^k \subset \widetilde{\rho}^{l_1+l_2-1} H_{\mathrm{b}}^k$, which is to say $A^{l_1,k} \cdot A^{l_2,k} \subset A^{l_1+l_2-1/2,k}$.

(2) Near either corner of the blown-up space, we use $\widetilde{v} = v$ and $t = \rho/v$ (say, $\widetilde{v} \geq 0, t \geq 0$). We compute $\rho\partial_\rho = t\partial_t$, $v\partial_v = \widetilde{v}\partial_{\widetilde{v}} - t\partial_t$, $\rho\partial_v = t\widetilde{v}\partial_{\widetilde{v}} - t^2\partial_t$; and since $\frac{d\rho}{\rho}\,dv\,dy = \frac{dt}{t}\,d\widetilde{v}\,dy$, the space $H_{\mathrm{b}}^{0,l,k}$ becomes

$$B^{l,k} := \{u \in (t\widetilde{v})^l L^2(\tfrac{dt}{t}\,d\widetilde{v}\,dy) \colon \mathcal{B}^j u \in (t\widetilde{v})^l L^2(\tfrac{dt}{t}\,d\widetilde{v}\,dy), 0 \leq j \leq k\},$$

where $\mathcal{B}$ is the C^∞-module of differential operators generated by $t\partial t, \widetilde{v}\partial_{\widetilde{v}}, \partial_y$. Again, we can rewrite this as

$$B^{l,k} = t^l \widetilde{v}^{l-1/2} H_{\mathrm{b}}^k(\tfrac{dt}{t}\,\tfrac{d\widetilde{v}}{\widetilde{v}}\,dy),$$

which implies that for $k > n/2$,

$$B^{l_1,k} \cdot B^{l_2,k} \subset t^{l_1+l_2} v^{l_1+l_2-1} H_{\mathrm{b}}^k(\tfrac{dt}{t}\,\tfrac{d\widetilde{v}}{\widetilde{v}}\,dy) \subset B^{l_1+l_2-1/2,k}.$$

To relate these two statements to the statement of the lemma, we use cutoff functions χ_A, χ_B to localize within the two coordinate systems. More precisely, choose a cutoff function $\chi \in C_c^\infty(\mathbb{R}_s)$ such that $\chi(s) \equiv 1$ near $s = 0$, $\chi(s) = 0$ for $|s| \geq 2$, and $\chi^{1/2} \in C_c^\infty(\mathbb{R}_s)$. Then multiplication with $\chi_A(\rho, v) := \chi(v/\rho)$ is a continuous map $H_{\mathrm{b}}^{0,l,k} \to A^{l,k}$. Indeed, to check this, one simply observes that $\mathcal{M}^j \chi_A \in L^\infty$ for all $j \in \mathbb{N}_0$. Similarly, letting $\chi_B(\rho, v) := 1 - \chi_A(\rho, v)$, multiplication with χ_B is a continuous map $H_{\mathrm{b}}^{0,l,k} \to B^{l,k}$. Finally, note that we have $A^{l,k}, B^{l,k} \subset H_{\mathrm{b}}^{0,l,k}$.

To put everything together, take $u_j \in H_{\mathrm{b}}^{0,l_j,k}$ $(j = 1, 2)$, then

$$u_1 u_2 = (\chi_A u_1)(\chi_A u_2) + (\chi_B u_1)(\chi_B u_2) + (\chi_A u_1)(\chi_B u_2) + (\chi_B u_1)(\chi_A u_2).$$

The first two terms then lie in $H_{\mathrm{b}}^{0,l_1+l_2-1/2,k}$. To deal with the third term, write

$$(\chi_A u_1)(\chi_B u_2) = (\chi_A^{1/2} u_1)(\chi_A^{1/2} \chi_B u_2) \in A^{l_1,k} \cdot A^{l_2,k} \subset H_{\mathrm{b}}^{0,l_1+l_2-1/2,k};$$

likewise for the fourth term. Thus, $u_1 u_2 \in H_{\mathrm{b}}^{0,l_1+l_2-1/2,k}$, as claimed. □

Remark 5.5.8. The proof actually shows more, namely that

$$H_{\mathrm{b}}^{0,l,k} H_{\mathrm{b}}^{0,l',k} \subset \rho_{\mathrm{ff}}^{-1/2} H_{\mathrm{b}}^{0,l+l',k}, \tag{5.5.13}$$

where ρ_{ff} is the defining function of the front face $\rho = v = 0$, e.g. $\rho_{\mathrm{ff}} = (\rho^2 + v^2)^{1/2}$. The reason for (5.5.13) to be a natural statement is that module- and b-derivatives are the same away from $\rho = v = 0$, hence regularity with respect to the module $\mathcal{M}$ is, up to a weight, which is a power of ρ_{ff}, the same as b-regularity.

More abstractly speaking, the above proof shows the following: If ρ_b denotes a boundary defining function of the other boundary hypersurface of $[M;S_+]$, i.e. $\partial[M;S_+]\setminus\mathrm{ff}$, then

$$H_{\mathrm{b}}^{0,l,k}\cong\rho_{\mathrm{ff}}^{-1/2}(\rho_{\mathrm{ff}}\rho_b)^l H_{\mathrm{b}}^k([M;S_+]).$$

Note that one can also show this in one step, introducing the coordinates $\rho_{\mathrm{ff}}\geq 0$ and $s=v/(\rho+\rho_{\mathrm{ff}})\in[-1,1]$ on $[M;S_+]$ in a neighborhood of ff, and mimicking the above proof, which however is computationally less convenient.

Remark 5.5.9. We can extend the lemma to $H_{\mathrm{b}}^{m,l,k}H_{\mathrm{b}}^{m,l',k}\subset H_{\mathrm{b}}^{m,l+l'-1/2,k}$ for $m\in\mathbb{N}_0$ using the Leibniz rule to distribute the m b-derivatives among the two factors, and then using the lemma for the case $m=0$.

The following corollary improves Lemma 5.5.7 if we have higher b-regularity; it will play an important role in §5.5.5.

Corollary 5.5.10. *Let $k>n/2$, $0\leq\delta<1/n$ and $l,l'\in\mathbb{R}$. Then*

(1) $H_{\mathrm{b}}^{1,l,k}H_{\mathrm{b}}^{0,l',k}\subset H_{\mathrm{b}}^{0,l+l'-1/2+\delta,k}$.

(2) $H_{\mathrm{b}}^{1,l,k}H_{\mathrm{b}}^{1,l',k}\subset H_{\mathrm{b}}^{1,l+l'-1/2+\delta,k}$.

Proof. Take $s=1/(2\delta)>n/2$, then

$$H_{\mathrm{b}}^{s,l,k}H_{\mathrm{b}}^{0,l',k}\subset H_{\mathrm{b}}^{0,l+l',k}; \tag{5.5.14}$$

indeed, using the Leibniz rule to distribute the k module derivatives among the two factors and cancelling the weights, this amounts to showing that $H_{\mathrm{b}}^{s,0,k_1}H_{\mathrm{b}}^{0,0,k_2}\subset H_{\mathrm{b}}^{0,0,0}$ for $k_1+k_2\geq k$; but this is true even for $k_1=k_2=0$, since H_{b}^s is a multiplier on H_{b}^0 provided $s>n/2$.

The lemma on the other hand gives

$$H_{\mathrm{b}}^{0,l,k}H_{\mathrm{b}}^{0,l',k}\subset\rho^{-1/2}H_{\mathrm{b}}^{0,l+l',k}. \tag{5.5.15}$$

Interpolating in the first factor between (5.5.14) and (5.5.15) thus gives the first statement.

For the second statement, use the Leibniz rule to distribute the one b-derivative to either factor; then, one has to show $H_{\mathrm{b}}^{1,l,k}H_{\mathrm{b}}^{0,l',k}\subset H_{\mathrm{b}}^{0,l+l'-1/2+\delta,k}$, and the same inclusion with l and l' switched, which is what we just proved. □

Lemma 5.5.7 and Remark 5.5.8 imply that for $u\in H_{\mathrm{b}}^{m,l,k}$, $p\geq 1$, with $m\geq 0, k>n/2$, we have $u^p\in H_{\mathrm{b}}^{m,pl-(p-1)/2,k}$; in fact, $u^p\in\rho_{\mathrm{ff}}^{-(p-1)/2}H_{\mathrm{b}}^{m,pl,k}$. Using Corollary 5.5.10, we can improve this to the statement $u \in H_{\mathrm{b}}^{m,l,k} \Rightarrow u^p \in H_{\mathrm{b}}^{m,pl-(p-1)/2+(p-1)\delta,k}$ for $m \geq 1$.

For non-linearities that only involve powers u^p, we can afford to lose differentiability, as at the end of §5.4.2, and gain decay in return, as the following lemma shows.

Lemma 5.5.11. *Let $\alpha > 1/2$, $l \in \mathbb{R}$, $k \in \mathbb{N}_0$. Then $\rho_{\mathrm{ff}}^{-\alpha} H_{\mathrm{b}}^{0,l,k} \subset \rho^{1/2-\alpha} H_{\mathrm{b}}^{-1,l,k}$, where $\rho_{\mathrm{ff}} = (\rho^2 + v^2)^{1/2}$.*

Proof. We may assume $l = 0$, and that u is supported in $|v| < 1$, $\rho < 1$. First, consider the case $k = 0$. Let $u \in \rho_{\mathrm{ff}}^{-\alpha} H_{\mathrm{b}}^0$, and put

$$\widetilde{u}(\rho, v, y) = \int_{-\infty}^{v} u(\rho, w, y)\, dw,$$

so $\partial_v \widetilde{u} = u$. We have to prove $\chi\widetilde{u} \in \rho^{1/2-\alpha} H_{\mathrm{b}}^0$ if $\chi \equiv 1$ near $\operatorname{supp} u$, which implies $u \in H_{\mathrm{b}}^{-1}$, as $\partial_v : H_{\mathrm{b}}^0 \to H_{\mathrm{b}}^{-1}$, and the b-Sobolev space are local spaces. But

$$|\widetilde{u}(\rho, v, y)|^2 \leq \left(\int_{-1}^{1} \rho_{\mathrm{ff}}(\rho, w)^{2\alpha} |u(\rho, w, y)|^2\, dw \right) \int_{-1}^{1} \rho_{\mathrm{ff}}(\rho, w)^{-2\alpha}\, dw; \tag{5.5.16}$$

now,

$$\int_{-1}^{1} \rho_{\mathrm{ff}}^{-2\alpha}\, dw = \rho^{1-2\alpha} \int_{-1/\rho}^{1/\rho} \frac{dz}{(1 + |z|^2)^\alpha} \lesssim \rho^{1-2\alpha}$$

for $\alpha > 1/2$, therefore, with the v integral considered on a fixed interval, say $|v| < 2$ (notice that the right hand side in (5.5.16) is independent of v!),

$$\iiint \rho^{2\alpha-1} |\widetilde{u}(\rho, v, y)|^2 \frac{d\rho}{\rho}\, dv\, dy \lesssim \iiint \rho_{\mathrm{ff}}^{2\alpha} |u(\rho, w, y)|^2 \frac{d\rho}{\rho}\, dw\, dy,$$

proving the claim for $k = 0$. Now, $\rho\partial_\rho$ and ∂_y just commute with this calculation, so the corresponding derivatives are certainly well-behaved. On the other hand, $\partial_v \widetilde{u} = u$, so the estimates involving at least one v-derivative are just those for u itself. □

Corollary 5.5.12. *Let $k, p \in \mathbb{N}$ be such that $k > n/2$, $p \geq 2$. Let $l \in \mathbb{R}$, $u \in H_{\mathrm{b}}^{0,l,k}$. Then $u^p \in H_{\mathrm{b}}^{-1,lp-(p-1)/2+1/2-\delta,k}$ with $\delta = 0$ if $p \geq 3$ and $\delta > 0$ if $p = 2$.*

Proof. This follows from $u^p \in \rho_{\mathrm{ff}}^{-(p-1)/2-\delta} H_{\mathrm{b}}^{0,lp,k}$ and the previous lemma, using that $(p-1)/2 + \delta > 1/2$ with δ as stated. □

In other words, we gain the decay $\rho^{1/2-\delta}$ if we give up one derivative.

5.5.3 A class of semilinear equations

We are now set to discuss solutions to nonlinear wave equations on an asymptotically Minkowski space. Under the assumptions of Theorem 5.5.4, we obtain a forward solution operator $S\colon H_{\mathrm{b}}^{m-1,l,k}(\Omega)^\bullet \to H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$ of $P = \rho^{-(n-2)/2}\rho^{-2}\Box_g\rho^{(n-2)/2}$ provided $|l| < 1$, $m + l < 1/2$ and $k \geq 0$.

Undoing the conjugation, we obtain a forward solution operator of $\Box_g$,

$$\widetilde{S} = \rho^{(n-2)/2} S \rho^{-2} \rho^{-(n-2)/2},$$

which is a bounded operator

$$\widetilde{S}\colon H_{\mathrm{b}}^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet \to H_{\mathrm{b}}^{m,l+(n-2)/2,k}(\Omega)^\bullet.$$

Since g is a Lorentzian scattering metric, the natural vector fields to appear in a nonlinear equation are scattering vector fields. Since the wave equation (as opposed to the Klein-Gordon equation with non-zero mass) can be recast as a non-degenerate b-equation, we in fact allow b-vector fields:

Theorem 5.5.13. *Let*

$$q\colon H_{\mathrm{b}}^{m,l+(n-2)/2,k}(\Omega)^\bullet \times H_{\mathrm{b}}^{m-1,l+(n-2)/2,k}(\Omega; {}^{\mathrm{b}}T^*_\Omega M)^\bullet \to H_{\mathrm{b}}^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet$$

be a continuous function with $q(0,0) = 0$ such that there exists a continuous non-decreasing function $L\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying

$$\|q(u, {}^{\mathrm{b}}du) - q(v, {}^{\mathrm{b}}dv)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R.$$

Then there is a constant $C_L > 0$ so that the following holds: If $L(0) < C_L$, then for small $R > 0$, there exists $C > 0$ such that for all $f \in H_{\mathrm{b}}^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet$ with norm $\leq C$, the equation

$$\Box_g u = f + q(u, {}^{\mathrm{b}}du)$$

has a unique solution $u \in H_{\mathrm{b}}^{m,l+(n-2)/2,k}(\Omega)^\bullet$, with norm $\leq R$, that depends continuously on f.

Proof. Use the Banach fixed point theorem as in the proof of Theorem 5.2.6. $\Box$

Remark 5.5.14. Here, just as in Theorem 5.4.10, we can also allow q to depend on $\Box_g u$ as well.

5.5.4 Semilinear equations with polynomial non-linearities

Next, we want to find a forward solution of the semilinear PDE

$$\Box_g u = f + cu^p X(u),$$

where $c \in \mathcal{C}^\infty(M)$, $p \in \mathbb{N}_0$, and $X(u) = \prod_{j=1}^q \rho V_j(u)$ is a q-fold product of derivatives of u along scattering vector fields; here, V_j are b-vector fields. Let us assume $p + q \geq 2$ in order for the equation to be genuinely nonlinear. We rewrite the PDE as

$$\begin{aligned} L(\rho^{-(n-2)/2}u) = \rho^{-(n-2)/2-2} f + c\rho^{-2}\rho^{(p-1)(n-2)/2}(\rho^{-(n-2)/2}u)^p \\ \times \prod_{j=1}^q \rho V_j(\rho^{(n-2)/2}\rho^{-(n-2)/2}u). \end{aligned}$$

Introducing $\widetilde{u} = \rho^{-(n-2)/2}u$ and $\widetilde{f} = \rho^{-(n-2)/2-2}f$ yields the equation

$$\begin{aligned} L\widetilde{u} &= \widetilde{f} + c\rho^{(p-1)(n-2)/2-2}\widetilde{u}^p \prod_{j=1}^q \rho^{n/2}(f_j\widetilde{u} + V_j\widetilde{u}) \\ &= \widetilde{f} + c\rho^{(p-1)(n-2)/2+qn/2-2}\widetilde{u}^p \prod_{j=1}^q (f_j\widetilde{u} + V_j\widetilde{u}), \end{aligned} \tag{5.5.17}$$

where the f_j are smooth functions. Now suppose that $\widetilde{u} \in H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$ with $m + l < 1/2$, $m \geq 1$ and $k > n/2$, so $H_{\mathrm{b}}^{m-1,-\infty,k}(\Omega)^\bullet$ is an algebra. Then the second summand of the right hand side of (5.5.17) lies in $H_{\mathrm{b}}^{m-1,\ell,k}(\Omega)^\bullet$, where

$$\ell = (p-1)(n-2)/2 + qn/2 - 2 + pl - (p-1)/2 + ql - (q-1)/2 - 1/2.$$

For this space to lie in $H_{\mathrm{b}}^{m-1,l,k}(\Omega)^\bullet$ (which we want in order to be able to apply the solution operator S and land in $H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$ so that a fixed point argument as in §5.2 can be applied), we thus need $\ell \geq l$, which can be rewritten as

$$(p-1)(l + (n-3)/2) + q(l + (n-1)/2) \geq 2. \tag{5.5.18}$$

With the amount $m = 1$ of b-regularity and correspondingly weight $l < 1/2 - m$ less than, but close to $-1/2$, we thus get the condition

$$(p-1)(n-4) + q(n-2) > 4.$$

If there are only non-linearities involving derivatives of u, i.e. $p = 0$, we get the condition $q > 1 + 2/(n-2)$, i.e. quadratic non-linearities are fine for $n \geq 5$, cubic ones for $n \geq 4$.

Note that if $q = 0$, we can actually choose $m = 0$ and $l < 1/2$ close to $1/2$, and we have Corollary 5.5.12 at hand. Thus we can improve (5.5.18) to $(p-1)(1/2+(n-3)/2) > 2-1/2$, i.e. $p > 1 + 3/(n-2)$, hence quadratic non-linearities can be dealt with if $n \geq 6$, whereas cubic non-linearities are fine as long as $n \geq 4$. Observe that this condition on p always implies $p > 1$, which is a natural condition, since $p = 1$ would amount to changing $\Box_g$ into $\Box_g - m^2$ (if one chooses the sign appropriately). However, the Klein-Gordon operator naturally fits into a scattering framework rather than the b-framework discussed here, i.e. requires a different analysis; we will not pursue this further here.

To summarize the general case, we undo the conjugation used to define L in terms of $\Box_g$: Note that $\widetilde{u} \in H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$ is equivalent to $u \in H_{\mathrm{b}}^{m,l+(n-2)/2,k}(\Omega)^\bullet$, and $\widetilde{f} \in H_{\mathrm{b}}^{m-1,l,k}(\Omega)^\bullet$ to $f \in H_{\mathrm{b}}^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet$. Thus:

Theorem 5.5.15. *Let $|l| < 1, m + l < 1/2, k > n/2$, and assume that $p, q \in \mathbb{N}_0$, $p + q \geq 2$, satisfy condition (5.5.18) or the weaker conditions given above in the cases where $p = 0$ or $q = 0$; let $m \geq 0$ if $q = 0$, otherwise let $m \geq 1$. Moreover, let $c \in \mathcal{C}^\infty(M)$ and $X(u) = \prod_{j=1}^q X_j u$, where X_j is a scattering vector field on M. Then for small enough $R > 0$, there exists a constant $C > 0$ such that for all $f \in H_{\mathrm{b}}^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet$ with norm $\leq C$, the equation*

$$\Box_g u = f + c u^p X(u)$$

has a unique solution $u \in H_{\mathrm{b}}^{m,l+(n-2)/2,k}(\Omega)^\bullet$, with norm $\leq R$, that depends continuously on f.

The same conclusion holds if the non-linearity is a finite sum of terms of the form $cu^p X(u)$, provided each such term separately satisfies (5.5.18).

Proof. Reformulating the PDE in terms of $\widetilde{u}$ and $\widetilde{f}$ as above, this follows from an application of the Banach fixed point theorem to the map

$$H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet \ni \widetilde{u} \mapsto S\Big(\widetilde{f} + c\rho^{(p-1)(n-2)/2+qn/2-2}\widetilde{u}^p \prod_{j=1}^{q}(f_j\widetilde{u} + V_j\widetilde{u})\Big) \in H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$$

with m, l, k as in the statement of the theorem. Here, $p + q \geq 2$ and the smallness of R ensure that this map is a contraction on the ball of radius R in $H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$. □

Remark 5.5.16. If the derivatives in the non-linearity only involve module derivatives, we get a slightly better result since we can work with $\widetilde{u} \in H_{\mathrm{b}}^{0,l,k}(\Omega)^\bullet$: Indeed, a module derivative falling on $\widetilde{u}$ gives an element of $H_{\mathrm{b}}^{0,l,k-1}(\Omega)^\bullet$, applied to which the forward solution operator produces an element of $H_{\mathrm{b}}^{1,l,k-1}(\Omega)^\bullet \subset H_{\mathrm{b}}^{0,l,k}(\Omega)^\bullet$.

The numerology works out as follows: In condition (5.5.18), we now take $l < 1/2$ close to $1/2$, thus obtaining

$$(p-1)(n-2) + qn > 4.$$

Thus, in the case that there are only derivatives in the non-linearity, i.e. $p = 0$, we get $q > 1 + 2/n$, which allows for quadratic non-linearities provided $n \geq 3$.

Remark 5.5.17. We can further improve (5.5.18) in the case $p \geq 1$, $q \geq 1$, $m \geq 1$ by using the δ-improvement from Corollary 5.5.10, namely, the right hand side of (5.5.17) actually lies in $H_{\mathrm{b}}^{m-1,\ell,k}(\Omega)^\bullet$, where now

$$\ell = (p-1)(n-2)/2 + qn/2 - 2 + pl - (p-1)/2 + (p-1)\delta + ql - (q-1)/2 - 1/2 + \delta,$$

which satisfies $\ell \geq l$ if

$$(p-1)(l + (n-3)/2 + \delta) + q(l + (n-1)/2) + \delta \geq 2,$$

which for $l < -1/2$ close to $-1/2$ means: $(p-1)(n-4+2\delta) + q(n-2) + 2\delta > 4$, where $0 < \delta < 1/n$.

Remark 5.5.18. Let us compare the above result with Christodoulou's [19]. A special case of his theorem states that the Cauchy problem for the wave equation on Minkowski space with small initial data in[25] $H_{k,k-1}(\mathbb{R}^{n-1})$ admits a global solution $u \in H_{\mathrm{loc}}^k(\mathbb{R}^n)$ with decay $|u(x)| \lesssim (1 + (v/\rho)^2)^{-(n-2)/2}$; here, $k = n/2 + 2$, and n is assumed to ≥ 4 and even; in

[25] Note that n is the dimension of Minkowski space here, whereas Christodoulou uses $n + 1$.

case $n = 4$, the non-linearity is moreover assumed to satisfy the null condition. The only polynomial non-linearity that we cannot deal with using the above argument is thus the null-form non-linearity in 4 dimensions.

To make a further comparison possible, we express $H_{k,\delta}(\mathbb{R}^{n-1})$ as a b-Sobolev space on the radial compactification of $\mathbb{R}^{n-1}$: Note that $u \in H_{k,\delta}(\mathbb{R}^{n-1})$ is equivalent to $(\langle x\rangle D_x)^\alpha u \in \langle x\rangle^{-\delta} L^2(\mathbb{R}^{n-1})$, $|\alpha| \leq k$. In terms of the boundary defining function ρ of $\partial\overline{\mathbb{R}^{n-1}}$ and the standard measure $d\omega$ on the unit sphere $\mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$, we have $L^2(\mathbb{R}^{n-1}) = L^2(\frac{d\rho}{\rho^2}\frac{dy}{\rho^{n-2}}) = \rho^{(n-1)/2} L^2(\frac{d\rho}{\rho}\,dy)$, and thus $H_{k,\delta}(\mathbb{R}^{n-1}) = \rho^{(n-1)/2+\delta} H_{\mathrm{b}}^k(\tilde{\mathfrak{t}} = 0)$. Therefore, converting the Cauchy problem into a forward problem, the forcing lies in $H_{\mathrm{b}}^{k,(n-1)/2+k-1,0}(\Omega)^\bullet = H_{\mathrm{b}}^{n/2+2,n+1/2,0}(\Omega)^\bullet$. Comparing this with the space $H_{\mathrm{b}}^{0,l+(n-2)/2+2,n/2+1}$ (with $l < 1/2$) needed for our argument, we see that Christodoulou's result applies to a regime of fast decay which is disjoint from our slow decay (or even mild growth) regime.

Remark 5.5.19. In the case of non-linearities u^p, the result of Christodoulou [19] implies the existence of global solutions to $\Box_g u = f + u^p$ if the spacetime dimension n is *even* and $n \geq 4$ if $p \geq 3$; in even dimensions $n \geq 6$, $p \geq 2$ suffices; the above result extends this to all dimensions satisfying the respective inequalities. In a somewhat similar context, see the work of Chruściel and Łęski [21], it has been proved that $p \geq 2$ in fact works in all dimensions $n \geq 5$.

5.5.5 Semilinear equations with null condition

With g the Lorentzian scattering metric on an asymptotically Minkowski space satisfying the assumptions of Theorem 5.5.4 as before, define the null form $Q({}^{\mathrm{sc}}du, {}^{\mathrm{sc}}dv) := g^{jk}\partial_j u \partial_k v$, where ${}^{\mathrm{sc}}d\colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M; {}^{\mathrm{sc}}T^*M)$ is the scattering differential, defined analogously to ${}^{\mathrm{b}}d$ and ${}^0 d$. For brevity, let us write $Q({}^{\mathrm{sc}}du)$ for $Q({}^{\mathrm{sc}}du, {}^{\mathrm{sc}}du)$. We are interested in solving the PDE

$$\Box_g u = Q({}^{\mathrm{sc}}du) + f.$$

The previous discussion solves this for $n \geq 5$; thus, let us from now on assume $n = 4$. To make the computations more transparent, we will keep the n in the notation and only substitute $n = 4$ when needed. Rewriting the PDE in terms of the operator $L = \rho^{-2}\rho^{-(n-2)/2}\Box_g \rho^{(n-2)/2}$ as above, we get

$$L\tilde{u} = \tilde{f} + \rho^{-(n-2)/2-2} Q({}^{\mathrm{sc}}d(\rho^{(n-2)/2}\tilde{u})),$$

where $\widetilde{u} = \rho^{-(n-2)/2}u$ and $\widetilde{f} = \rho^{-(n-2)/2-2}f$. We can write $Q({}^{\mathrm{sc}}du) = \frac{1}{2}\Box_g(u^2) - u\Box_g u$, thus the PDE becomes

$$\begin{aligned} L\widetilde{u} &= \widetilde{f} + \rho^{-(n-2)/2-2}\big(\tfrac{1}{2}\Box_g(\rho^{n-2}\widetilde{u}^2) - \rho^{(n-2)/2}\widetilde{u}\Box_g(\rho^{(n-2)/2}\widetilde{u})\big) \\ &= \widetilde{f} + \tfrac{1}{2}L(\rho^{(n-2)/2}\widetilde{u}^2) - \rho^{(n-2)/2}\widetilde{u}L\widetilde{u}. \end{aligned}$$

Since the results of §5.5.2 give small improvements on the decay of products of $H_{\mathrm{b}}^{1,*,*}$ functions with $H_{\mathrm{b}}^{m,*,*}$ functions ($m \geq 0$), one wants to solve this PDE on a function space that keeps track of these small improvements.

Definition 5.5.20. For $l \in \mathbb{R}, k \in \mathbb{N}_0$ and $\alpha \geq 0$, define the space $\mathcal{X}^{l,k,\alpha} := \{v \in H_{\mathrm{b}}^{1,l+\alpha,k}(\Omega)^\bullet \colon Lv \in H_{\mathrm{b}}^{0,l,k}(\Omega)^\bullet\}$ with norm

$$\|v\|_{\mathcal{X}^{l,k,\alpha}} = \|v\|_{H_{\mathrm{b}}^{1,l+\alpha,k}(\Omega)^\bullet} + \|Lv\|_{H_{\mathrm{b}}^{0,l,k}(\Omega)^\bullet}. \tag{5.5.19}$$

By an argument similar to the one used in the proof of Theorem 5.2.6, we see that $\mathcal{X}^{l,k,\alpha}$ is a Banach space. On $\mathcal{X}^{l,k,\alpha}$, which $\alpha > 0$ chosen below, we want to run an iteration argument: Start by defining the operator $T \colon \mathcal{X}^{l,k,\alpha} \to H_{\mathrm{b}}^{1,-\infty,k}(\Omega)^\bullet$ by

$$T \colon \widetilde{u} \mapsto S\big(\widetilde{f} - \rho^{(n-2)/2}\widetilde{u}L\widetilde{u}\big) + \tfrac{1}{2}\rho^{(n-2)/2}\widetilde{u}^2.$$

Note that $\widetilde{u} \in \mathcal{X}^{l,k,\alpha}$ implies, using Corollary 5.5.10 with $\delta < 1/n$,

$$\begin{aligned} \rho^{(n-2)/2}\widetilde{u}^2 \in \rho^{(n-2)/2}H_{\mathrm{b}}^{1,2(l+\alpha)-1/2+\delta,k}(\Omega)^\bullet &= H_{\mathrm{b}}^{1,2l+\alpha+(n-3)/2+\delta+\alpha,k}(\Omega)^\bullet, \\ \rho^{(n-2)/2}\widetilde{u}L\widetilde{u} &\in H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet, \\ S(\rho^{(n-2)/2}\widetilde{u}L\widetilde{u}) &\in H_{\mathrm{b}}^{1,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet, \end{aligned} \tag{5.5.20}$$

where in the last inclusion, we need to require $1 + (2l + \alpha + (n-3)/2 + \delta) < 1/2$, which for $n = 4$ means

$$l < -1/2 - (\alpha + \delta)/2; \tag{5.5.21}$$

let us assume from now on that this condition holds. Furthermore, (5.5.20) implies $T\widetilde{u} \in H_{\mathrm{b}}^{1,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet$. Finally, we analyze

$$L(T\widetilde{u}) \in H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet + \frac{1}{2}L(\rho^{(n-2)/2}\widetilde{u}^2).$$

Using that L is a second-order b-differential operator, we have

$$\begin{aligned}\rho^{(n-2)/2}L(\widetilde{u}^2) &\in 2\rho^{(n-2)/2}\widetilde{u}L\widetilde{u} + \rho^{(n-2)/2}H_{\mathrm{b}}^{0,l+\alpha,k}(\Omega)^\bullet H_{\mathrm{b}}^{0,l+\alpha,k}(\Omega)^\bullet \\ &\subset H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet + H_{\mathrm{b}}^{0,2(l+\alpha)+(n-3)/2,k}(\Omega)^\bullet \\ &= H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^\bullet,\end{aligned}$$

which gives

$$\begin{aligned}L(\rho^{(n-2)/2}\widetilde{u}^2) &\in L(\rho^{(n-2)/2})\widetilde{u}^2 + \rho^{(n-2)/2}L(\widetilde{u}^2) \\ &\qquad + \rho^{(n-2)/2}H_{\mathrm{b}}^{1,l+\alpha,k}(\Omega)^\bullet H_{\mathrm{b}}^{0,l+\alpha,k}(\Omega)^\bullet \\ &\subset H_{\mathrm{b}}^{1,2l+\alpha+(n-3)/2+\delta+\alpha,k}(\Omega)^\bullet + H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^\bullet \\ &\qquad + H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\delta+\alpha}(\Omega)^\bullet \\ &= H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^\bullet.\end{aligned}$$

Hence, putting everything together,

$$L(T\widetilde{u}) \in H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^\bullet.$$

Therefore, we have $T\widetilde{u} \in \mathcal{X}^{l,k,\alpha}$ provided

$$\begin{aligned}2l+\alpha+(n-3)/2+\delta &\geq l+\alpha \\ 2l+\alpha+(n-3)/2+\min\{\alpha,\delta\} &\geq l,\end{aligned}$$

which for $0<\alpha<\delta$ and $n=4$ is equivalent to

$$l \geq -1/2-\delta, \quad l \geq -1/2-2\alpha. \tag{5.5.22}$$

This is consistent with condition (5.5.21) if $-1/2-(\alpha+\delta)/2 > -1/2-2\alpha$, i.e. if $\alpha > \delta/3$.

Finally, for the map T to be well-defined, we need $S\widetilde{f} \in \mathcal{X}^{l,k,\alpha}$, hence $\widetilde{f} \in \mathrm{Ran}_{\mathcal{X}^{l,k,\alpha}} L$, which is in particular satisfied if $\widetilde{f} \in H_{\mathrm{b}}^{0,l+\alpha,k}(\Omega)^\bullet$. Indeed, since $1+l+\alpha < 1-1/2-(\delta-\alpha)/2 < 1/2$ by condition (5.5.21), the element $S\widetilde{f} \in H_{\mathrm{b}}^{1,l+\alpha,k}(\Omega)^\bullet$ is well-defined.

We have proved:

Theorem 5.5.21. *Let $c \in \mathbb{C}$, $0<\delta<1/4$, $\delta/3<\alpha<\delta$, and let $-1/2-2\alpha \leq l <$*

$-1/2 - (\alpha + \delta)/2$. *Then for small enough* $R > 0$*, there exists a constant* $C > 0$ *such that for all* $f \in H_{\mathrm{b}}^{0,l+3+\alpha,k}(\Omega)^{\bullet}$ *with norm* $\leq C$*, the equation*

$$\Box_g u = f + cQ({}^{\mathrm{sc}}du)$$

has a unique solution $u \in \mathcal{X}^{l+1,k,\alpha}$*, with norm* $\leq R$*, that depends continuously on* f*.*

Chapter 6

Resonance expansions for tensor-valued waves

6.1 Introduction

We study linear tensor-valued wave equations on perturbations of Schwarzschild-de Sitter spaces (thus including Kerr-de Sitter spaces) with spacetime dimension $n \geq 4$; in particular, this includes wave equations for differential forms and symmetric 2-tensors. (We mention symmetric 2-tensors here explicitly because of their role in the study of Einstein's field equations, which, as stated in Chapter 1, is one of the main motivations for large parts of this thesis.) As mentioned in Remark 5.3.5, the additional complications of working on sections of non-trivial bundles rather than on scalar functions are twofold: One needs to prove high energy estimates, in a strip below the real line in order to obtain exponential decay up to a finite-dimensional space of resonances, and one needs to understand this latter space in case one wants to study nonlinear equations. In this chapter, we tackle the first complication; in Chapter 7, the second, in the case that the bundle is the differential form bundle and the operator is the Hodge d'Alembertian.

In the form that is easiest to state, we will prove:

Theorem 6.1.1. *Let (M, g) denote a Kerr-de Sitter spacetime in $n \geq 4$ spacetime dimensions, with small angular momentum. Let $\mathcal{E} \subset \mathcal{T}_k$ be a subbundle of the bundle $\mathcal{T}_k$ of (covariant) rank k tensors on M, so that the tensor wave operator $\Box_g = -\operatorname{tr} \nabla^2$ acts on sections of $\mathcal{E}$; for instance, one can take $\mathcal{E}$ to be equal to $\mathcal{T}_k$, symmetric rank k-tensors or* *differential forms of degree k. Let Ω denote a small neighborhood of the domain of outer communications, bounded beyond but close to the cosmological and the black hole horizons by spacelike boundaries as in §§2.3 and 2.4, and let t_* be a smooth time coordinate on Ω, given by (2.4.2). See Figure 2.5 for the setup.*

Then for any $f \in \mathcal{C}_c^\infty(\Omega, \mathcal{E})$, the wave equation $\Box_g u = f$ has a unique global forward solution (supported in the causal future of $\operatorname{supp} f$) $u \in \mathcal{C}^\infty(\Omega, \mathcal{E})$, and u has an asymptotic expansion

$$u = \sum_{j=1}^{N} \sum_{m=0}^{m_j-1} \sum_{\ell=1}^{d_j} e^{-it_*\sigma_j} t_*^m u_{jm\ell} a_{jm\ell}(x) + u',$$

where $u_{jm\ell} \in \mathbb{C}$, the resonant states $a_{jm\ell}$, only depending on $\Box_g$, are smooth functions of the spatial coordinates and $\sigma_j \in \mathbb{C}$ are resonances with $\operatorname{Im} \sigma_j > -\delta$ (whose multiplicity is $m_j \geq 1$ and for which the space of resonant states has dimension d_j), while $u' \in e^{-\delta t_*} L^\infty(\Omega, \mathcal{E})$ is exponentially decaying, for $\delta > 0$ small; we measure the size of sections of $\mathcal{E}$ by means of a t_*-independent positive definite inner product.

The same result holds true if we add any stationary 0-th order term to $\Box$, and one can also add stationary first order terms which are either small or subject to a natural, but somewhat technical condition, which we explain in Remark 6.4.9. In fact, we can even work on spacetimes which merely *approach a stationary perturbation of Schwarzschild-de Sitter space exponentially fast*. See §6.2 for the form of the Schwarzschild-de Sitter metric and the precise assumptions on regularity and asymptotics of perturbations, for details on the setup, and Theorem 6.2.1 for the full statement of Theorem 6.1.1.

The resonances and resonant states depend strongly on the precise form of the operator and which bundle one is working on. In the case of the trivial bundle, thus considering scalar waves, they were computed in the Kerr-de Sitter setting by Dyatlov [40], following work by Sá Barreto and Zworski [5] as well as Bony and Häfner [13]. In Chapter 7, we will compute the resonances for the Hodge d'Alembertian on differential forms, which equals the tensor wave operator plus a zeroth order curvature term: We show that there is only one resonance $\sigma_1 = 0$ in $\operatorname{Im} \sigma \geq 0$, of order $m_1 = 1$, and we canonically identify the 0-resonant states with cohomological information of the underlying spacetime. Note however that we will deal with a very general class of warped product type spacetimes with asymptotically hyperbolic ends, while the present chapter is only concerned with (perturbations of) Schwarzschild-de Sitter spacetimes. We remark that in general one expects that $\Box_g = -\operatorname{tr} \nabla^2$ on a bundle $\mathcal{E}$ as in

Theorem 6.1.1 has resonances in $\operatorname{Im}\sigma > 0$, thus causing linear waves to grow exponentially in time.

We point out that if there are no resonances for $\Box_g$ (plus lower order terms) in $\operatorname{Im}\sigma \geq 0$, thus solutions decay exponentially, we can combine Theorem 6.1.1 with the framework for quasilinear wave-type equations developed in Chapters 8 and 9 and immediately obtain the *global solvability of quasilinear equations*. This also works if there is merely a simple resonance at $\sigma = 0$ which is annihilated by the nonlinearity. See Remark 7.5.3 for an example for differential forms.

The proof of Theorem 6.1.1 is essentially the same as the proof of the analogous Theorems 5.2.3 and 5.3.1. In the context of scalar waves, more general and precise versions of Theorem 6.1.1 are known, see the references in §5.1.1. Thus, the main advance is that we give a conceptually transparent framework that allows us to deal with tensor-valued waves on black hole spacetimes, where the natural inner product on the tensor bundle induced by the spacetime metric is not positive definite. Notice that in order to obtain energy estimates for waves, one needs to work with positive inner products on the tensor bundle, relative to which however $\Box$ is in general not well-behaved: Most severely, it is in general far from being symmetric at the trapped set, which prevents the use of semiclassical estimates at normally hyperbolic trapping; see the statement of Theorem 3.3.14 for the role of symmetry for the normally hyperbolic b-estimate. On a pragmatic level, we show that one can conjugate $\Box$ by a suitable 0-th order pseudodifferential operator so as to make the conjugated operator (almost) symmetric at the trapped set with respect to a positive definite inner product, and one can then directly apply Dyatlov's methods [42] to obtain a spectral gap. In other words, we reduce the high frequency analysis of tensor-valued waves to an essentially scalar problem. The conceptually correct point of view to accomplish this conjugation is that of *pseudodifferential inner products*, which we introduce in §6.3.

Roughly speaking, pseudodifferential inner products on $\mathcal{E} \to M$ (with M a closed manifold now for simplicity) replace ordinary inner products $\int \langle B_0(u), v\rangle\,|dg|$, where B_0 is an inner product on the fibers of $\mathcal{E}$, mapping $\mathcal{E}$ into its anti-dual $\overline{\mathcal{E}}^*$, by 'inner products' of the form $\int \langle B(x,D)u, v\rangle\,|dg|$, where $B \in \Psi^0$ is a zeroth order pseudodifferential operator mapping sections of $\mathcal{E}$ into sections of $\overline{\mathcal{E}}^*$. Thus, we gain a significant amount of flexibility, since we can allow the inner product to *depend on the position in phase space*, rather than merely on the position in the base: Indeed, the principal symbol $b = \sigma_0(B)$ is an inner product on the vector bundle $\pi^*\mathcal{E}$ over $T^*M \setminus o$, where $\pi\colon T^*M \setminus o \to M$ is the projection.

One can define adjoints of operators $P \in \Psi^m(M, \mathcal{E})$ (e.g. $P = \Box_g$), acting on sections of $\mathcal{E}$, relative to a pseudodifferential inner product B, denoted P^{*B}, which are well-defined modulo smoothing operators. Moreover, there is an invariant symbolic calculus involving the *subprincipal operator* $S_{\mathrm{sub}}(P)$, which is a first order differential operator on $T^*M \setminus o$ acting on sections of $\pi^*\mathcal{E}$ that invariantly encodes the subprincipal part of P, for computing principal symbols of commutators and imaginary parts of such operators. In the case that P is principally scalar and real, the principal symbol of $P - P^{*B} \in \Psi^{m-1}(M, \mathcal{E})$ then vanishes in some conic subset of phase space $T^*M \setminus o$ if and only if $S_{\mathrm{sub}}(P) - S_{\mathrm{sub}}(P)^{*b}$ does, which in turn can be reinterpreted as saying that the principal symbol of $QPQ^{-1} - (QPQ^{-1})^{*B_0}$ vanishes there, where B_0 is an ordinary inner product on $\mathcal{E}$, and $Q \in \Psi^0(M, \mathcal{E})$ is a suitably chosen elliptic operator. In the case considered in Theorem 6.1.1 then, it turns out that the subprincipal operator of $\Box_g$ on tensors, decomposed into parts acting on tangential and normal tensors according to the product decompositions $M = \mathbb{R}_t \times X_x$ and $X = (r_-, r_+) \times \mathbb{S}^{n-2}$, at the trapped set equals the derivative along the Hamilton vector field H_G, G the dual metric function, plus a *nilpotent* zeroth order term. This then enables one to choose a positive definite inner product b on $\pi^*\mathcal{E}$ relative to which $S_{\mathrm{sub}}(\Box_g)$ is arbitrarily close to being symmetric at the trapped set; see §6.3.5 for the argument in a toy example. Thus with $B = b(x, D)$, the operator $\Box_g$ is arbitrarily close to being symmetric with respect to the pseudodifferential inner product B. Hence, one can indeed appeal to Dyatlov's results on spectral gaps by considering a conjugate of $\Box_g$, which is the central ingredient in the proof of Theorem 6.1.1.

We point out that refined microlocal propagation results, in the sense of polarization sets, for systems were proved by Dencker [34], and in fact the subprincipal operator we define here is very closely related to the partial connection along the Hamilton flow defined in [34]; see also Remark 6.3.10.

6.1.1 Previous and related work

The study of non-scalar waves on black hole backgrounds has focused primarily on Maxwell's equations, which describe the electromagnetic field on Lorentzian spacetimes: Sterbenz and Tataru [103] showed local energy decay for Maxwell's equations on a class of spherically symmetric asymptotically flat spacetimes including Schwarzschild. Blue [11] established conformal energy and pointwise decay estimates in the exterior of the Schwarzschild black hole; Andersson and Blue [3] proved similar estimates on slowly rotating Kerr spacetimes.

These followed earlier results for Schwarzschild by Inglese and Nicolo [65] on energy and pointwise bounds for integer spin fields in the far exterior of the Schwarzschild black hole, and by Bachelot [4], who proved scattering for electromagnetic perturbations. Finster, Kamran, Smoller and Yau [47] proved local pointwise decay for Dirac waves on Kerr. There are further works which in particular establish bounds for certain components of the Maxwell field, see Donninger, Schlag and Soffer [37] and Whiting [120]. Dafermos [22, 23] studied the nonlinear Einstein-Maxwell-scalar field system under the assumption of spherical symmetry. See §5.1.1 for further references.

We moreover point out that Vasy [112] proved the meromorphic continuation of the resolvent of the Laplacian on differential forms on asymptotically hyperbolic spaces (following earlier works by Mazzeo and Melrose [81] and Guillarmou [57] in the scalar setting and Mazzeo [79], Carron and Pedon [15] and Guillarmou, Moroianu and Park [58] for forms and spinors; see also the work of Dyatlov, Faure and Guillarmou [45], which in particular involves a discussion of Laplacians on compact hyperbolic manifolds acting on symmetric tensors). The fact that the analysis presented in [114], which underlies [112], works on sections of vector bundles just as it does on functions is crucial for us here.

6.2 Detailed setup and proof of the main theorem

We denote by Ω the domain (2.3.9) inside the extension M of Schwarzschild-de Sitter space in $n \geq 4$ spacetime dimensions, and equip M with the Schwarzschild-de Sitter metric g_0, which is a Lorentzian b-metric. Suppose g is a Lorentzian b-metric such that for some smooth Lorentzian b-metric g', we have $g - g' \in H_{\mathrm{b}}^{\infty,r}(\Omega, S^{2\mathrm{b}}T^*M)$ for some $r > 0$ as in (5.2.11). Changing g' so as to make it invariant under time translations does not affect this condition, so let us assume g' is t_*-invariant. We consider the wave operator $\Box_g$ acting on sections of the bundle $\mathcal{T}_k$ of covariant tensors of rank k over Ω. We assume that g' and g_0 are close (in the C^k sense for sufficiently high k), so that the dynamical and geometric structure of g is close to that of g_0; in other words, the metric g is exponentially approaching a stationary metric close to the Schwarzschild-de Sitter metric, so for instance perturbations (within this setting) of Kerr-de Sitter spaces are allowed. Most importantly, the nature of the trapping for g' (and thus for g) is still normally hyperbolic, and the subprincipal operator (see §6.3.3) of $\Box_g$ at the trapped set, while not necessarily having the nilpotent structure alluded to in the introduction and explained in §6.4.2, has small imaginary part relative to (the symbol of) a pseudodifferential inner product on $\mathcal{T}_k$. Recall that the trapping for Schwarzschild-de Sitter space is r-normally hyperbolic for every r, and r-normal hyperbolicity (for large, but finite r) is structurally stable under perturbations of the metric, so this perturbation framework is indeed quite flexible. In the language of Definition 2.5.1, our setup amounts to allowing non-trapping spacetimes with normally hyperbolic trapping which are close to Schwarzschild-de Sitter space within this class of spacetimes.

We then have:

Theorem 6.2.1. *In the above notation, if g' is sufficiently close to the Schwarzschild-de Sitter metric g_0, then there exist $s_0 \in \mathbb{R}$ and $\delta > 0$ as well as a finite set $\{\sigma_j : j = 1, \ldots, N\} \subset \mathbb{C}$, $\operatorname{Im}\sigma_j > -\delta$, integers $m_j \geq 1$ and $d_j \geq 1$, and smooth functions $a_{jm\ell} \in \mathcal{C}^\infty(\partial_\infty\Omega)$, $1 \leq j \leq N$, $0 \leq m \leq m_j - 1$, $1 \leq \ell \leq d_j$, such that the following holds: The equation*

$$\Box_g u = f, \quad f \in H_{\mathrm{b}}^{s,\delta}(\Omega, \mathcal{T}_k)^{\bullet,-}, \quad s \geq s_0, \tag{6.2.1}$$

has a unique solution $u \in H_{\mathrm{b}}^{-\infty,-\infty}(\Omega, \mathcal{T}_k)^{\bullet,-}$, which has an asymptotic expansion

$$u = \chi(\tau) \sum_{j=1}^{N} \sum_{m=0}^{m_j-1} \sum_{\ell=1}^{d_j} \tau^{i\sigma_j} |\log\tau|^m u_{jm\ell} a_{jm\ell} + u',$$

where χ is a cutoff function, i.e. $\chi(\tau) \equiv 1$ near $\tau = 0$ and $\chi(\tau) \equiv 0$ near the Cauchy surface H_1, and $u_{jm\ell} \in \mathbb{C}$, while the remainder term is $u' \in H_{\mathrm{b}}^{s,\delta}(\Omega, \mathcal{T}_k)^{\bullet,-}$.

The same result holds true if we restrict to a subbundle of $\mathcal{T}_k$ which is preserved by the action of $\Box$, for instance the degree k form bundle, or the symmetric rank k tensor bundle.

If $V \in \mathcal{C}^\infty(M, \operatorname{End}(\mathcal{T}_k)) + H_{\mathrm{b}}^{\infty,r}(\Omega, \operatorname{End}(\mathcal{T}_k))$, $r > 0$, is a smooth (conormal) $\operatorname{End}(\mathcal{T}_k)$-valued potential (without restriction on its size), the analogous result holds for $\Box_g$ replaced by $\Box_g + V$. We may even change $\Box_g$ by adding a first order b-differential operator L acting on $\mathcal{T}_k$ with coefficients which are elements of $\mathcal{C}^\infty + H_{\mathrm{b}}^{\infty,r}$, provided either the coefficients of L are small, or the subprincipal operator of $\Box_g + L$ is sufficiently close to being symmetric with respect to a pseudodifferential inner product on $\mathcal{T}_k$, see Remark 6.4.9.

The numbers σ_j are of course the resonances, and the functions $a_{jm\ell}$ the resonant states. They have been computed in various special cases; see the discussion in the introduction for references. The threshold regularity s_0 is related to the dynamics of the flow of the

Hamiltonian vector field H_G of the dual metric function G (i.e. $G(x,\xi) = |\xi|^2_{G(x)}$, with G the dual metric of g) near the horizons which are generalized radial sets, see §3.3.1. Thus, s_0 can easily be made explicit, but this is not the point of the present chapter.

The proof of Theorem 6.2.1 proceeds in the same way as the proofs of Theorems 5.2.3 and 5.3.1 using a contour deformation argument, regaining derivatives lost in view of treating $\Box_g - N(\Box_g)$ as a perturbation by appealing to the b-radial point and b-normally hyperbolic trapping estimates from §§3.3.1 and 3.3.2. The main issue is to show high energy estimates for $\widehat{\Box}(\sigma)^{-1}$, see below. The fact that the remainder term u' has the same regularity as the forcing term f, thus u' loses 2 derivatives relative to the elliptic gain of 2 derivatives, comes from the high energy estimate losing a power of 2, which in turn is caused by the same loss for high energy estimates at normally hyperbolic trapping, see [42, Theorem 1], and 9.2.5 for a microlocalized version of Dyatlov's estimate, as well as Theorem 9.2.9 for the global estimate (in the more general setting of non-trapping spacetimes with normally hyperbolic trapping).

Thus, the crucial point is to obtain high energy estimates at the trapped set for the operator $\Box$ acting on $\mathcal{T}_k$ in $\operatorname{Im}\sigma > -\delta$. Dyatlov's result [42, Theorem 1] (see also the discussion preceding Theorem 9.2.5) shows that a sufficient condition for these to hold is

$$|\sigma|^{-1}\sigma_{\mathrm{b},1}\Big(\frac{1}{2i}(\Box - \Box^*)\Big) < \nu_{\min}/2 \tag{6.2.2}$$

at the trapped set Γ, where $\nu_{\min}$ is the minimal normal expansion rate of the Hamilton flow at the trapping, see [42] and the computation in §2.3, in particular (2.3.11). Here, the adjoint is taken with respect to a *positive definite inner product* on $\mathcal{T}_k$; note that the inner product induced by g, with respect to which $\Box$ is of course symmetric, is not positive definite, except when $k = 0$, i.e. for the scalar wave equation. Since g is close to the Schwarzschild-de Sitter metric, it suffices (by the dynamical stability of the trapping) to obtain such a bound for the Schwarzschild-de Sitter metric g_0. While this bound is impossible to obtain directly for the full range of Schwarzschild-de Sitter spacetimes, we show in §6.4.2 how it can be obtained if we use pseudodifferential products. Prosaically, this means that we consider a conjugated operator $P := Q\Box Q^-$, where $Q \in \Psi^0_{\mathrm{b}}(M,\mathcal{T}_k)$ is elliptic with parametrix Q^-, and for any $\epsilon > 0$, we can arrange $|\sigma|^{-1}\sigma_{\mathrm{b},1}(\frac{1}{2i}(P-P^*)) < \epsilon$ (with the adjoint taken relative to an ordinary positive definite inner product on $\mathcal{T}_k$), thus (6.2.2) holds for $\Box$ replaced by P; we will prove this in Theorem 6.4.8. Hence [42, Theorem 1] applies to P, establishing a spectral gap; indeed, by the remark following [42, Theorem 1], Dyatlov's result applies for operators on bundles as well, *as soon as one establishes* (6.2.2). Arranging (6.2.2) in a natural fashion lies at the heart of §§6.3 and 6.4.

For later reference, we recall from (2.3.12) that the spacetime trapped set, i.e. the set of points in phase space that never escape through either horizon along the Hamilton flow, not restricted to future infinity, is given by

$$\Gamma = \{(t, r = r_p, \omega; \sigma, \xi = 0, \eta) \colon \sigma^2 = \Psi^2 |\eta|^2\}, \tag{6.2.3}$$

where $\Psi = \alpha r^{-1}$, $\Psi'(r_p) = 0$. *We thus change the notation from* $\widetilde{\Gamma}$ *in* (2.3.12) *to* Γ *here* to make the notation less cumbersome.

6.3 Pseudodifferential inner products

We now develop a general theory of pseudodifferential inner products, which we apply to the setting of Theorem 6.2.1 in §6.4.

We work on a complex rank N vector bundle $\mathcal{E}$ over the smooth compact n-dimensional manifold X without boundary. We will define *pseudodifferential inner products* on $\mathcal{E}$, which are inner products depending on the position in phase space T^*X, rather than merely the position in the base X. As indicated in the introduction, we achieve this by replacing ordinary inner products by pseudodifferential operators whose symbols are inner products on the bundle $\pi^*\mathcal{E} \to T^*X \setminus o$, where $\pi \colon T^*X \setminus o \to X$ is the projection.

6.3.1 Notation

Let $\mathcal{V}$ be a complex N-dimensional vector space. We denote by $\overline{\mathcal{V}}$ the complex conjugate of $\mathcal{V}$, i.e. $\overline{\mathcal{V}} = \mathcal{V}$ as sets, and the identity map $\iota \colon \mathcal{V} \to \overline{\mathcal{V}}$ is antilinear, so $\iota(\lambda v) = \overline{\lambda}\iota(v)$ for $v \in \mathcal{V}$, $\lambda \in \mathbb{C}$, which defines the linear structure on $\overline{\mathcal{V}}$. (We prefer to write $\iota(v)$ rather than $\overline{v}$ to prevent possible confusion with taking complex conjugates in complexifications of real vector spaces.) A Hermitian inner product H on $\mathcal{V}$ is thus a linear map $H \colon \mathcal{V} \otimes \overline{\mathcal{V}} \to \mathbb{C}$ such that $H(u, \iota(v)) = \overline{H(v, \iota(u))}$ for $u, v \in \mathcal{V}$, and $H(u, \iota(u)) > 0$ for all non-zero $u \in \mathcal{V}$. This can be rephrased this in terms of the linear map $B \colon \mathcal{V} \to \overline{\mathcal{V}}^*$ defined by $B(u) = H(u, \cdot)$ and the natural dual pairing of $\overline{\mathcal{V}}^*$ with $\overline{\mathcal{V}}$, namely $\langle Bu, \iota(v)\rangle = \overline{\langle Bv, \iota(u)\rangle}$, and $\langle Bu, \iota(u)\rangle > 0$ for $u \in \mathcal{V}$ non-zero.

A map $A\colon \mathcal{V} \to \overline{\mathcal{V}}^*$ has a transpose $A^T\colon \overline{\mathcal{V}} \to \mathcal{V}^*$, which satisfies $\langle Au, \iota(v)\rangle = \langle u, A^T\iota(v)\rangle$ for all $u, v \in \mathcal{V}$, and an adjoint $A^*\colon \mathcal{V} \to \overline{\mathcal{V}}^*$ satisfying $\langle Au, \iota(v)\rangle = \overline{\langle A^*v, \iota(u)\rangle}$. Concretely, defining the antilinear map

$$j\colon \mathcal{V}^* \to \overline{\mathcal{V}}^*, \quad \langle j(\ell), \iota(v)\rangle = \overline{\langle \ell, v\rangle},$$

we have $A^* = jA^T\iota$. The symmetry of a Hermitian inner product B as above is simply expressed by $B = B^*$. Similarly, a map $P\colon \mathcal{V} \to \mathcal{V}$ has a transpose $P^T\colon \mathcal{V}^* \to \mathcal{V}^*$ and an adjoint $P^*\colon \overline{\mathcal{V}}^* \to \overline{\mathcal{V}}^*$ defined by $\langle \bar\ell, \iota(Pv)\rangle = \langle P^*\bar\ell, \iota(v)\rangle$ for $\bar\ell \in \overline{\mathcal{V}}^*$ and $v \in \mathcal{V}$, and one easily finds $P^* = jP^Tj^{-1}$. We point out that the definitions of adjoints of maps $A\colon \mathcal{V} \to \overline{\mathcal{V}}^*$ and $P\colon \mathcal{V} \to \mathcal{V}$ are compatible in the sense that $(AP)^* = P^*A^*$. Furthermore, if $B\colon \mathcal{V} \to \overline{\mathcal{V}}^*$ is a Hermitian inner product and $Q\colon \mathcal{V} \to \mathcal{V}$ is invertible, then $B_1 = Q^*BQ$ defines another Hermitian inner product, $\langle B_1u, \iota(v)\rangle = \langle BQu, \iota(Qv)\rangle$.

Now, given an inner product B on $\mathcal{V}$ and any map $P\colon \mathcal{V} \to \mathcal{V}$, the adjoint P^{*B} of P with respect to B is the unique map $P^{*B}\colon \mathcal{V} \to \mathcal{V}$ such that $\langle BPu, \iota(v)\rangle = \langle Bu, \iota(P^{*B}v)\rangle$ for all $u, v \in \mathcal{V}$. We find a formula for P^{*B} by computing

$$\langle BPu, \iota(v)\rangle = \overline{\langle B^*(B^*)^{-1}P^*B^*v, \iota(u)\rangle} = \langle Bu, \iota((BPB^{-1})^*v)\rangle,$$

i.e. $P^{*B} = (BPB^{-1})^* = B^{-1}P^*B$. The self-adjointness of P with respect to B is thus expressed by the equality $P = B^{-1}P^*B$.

If $\mathcal{E}$ is a complex rank N vector bundle, we can similarly define the complex conjugate bundle $\overline{\mathcal{E}}$ as well as adjoints of vector bundle maps $\mathcal{E} \to \mathcal{E}$ and $\mathcal{E} \to \overline{\mathcal{E}}^*$. We can also define adjoints of pseudodifferential operators mapping between these bundles: For convenience, we remove the dependence of adjoints on a volume density on X by tensoring all bundles with the half-density bundle $\Omega^{\frac{1}{2}}$ over X, and we have a natural pairing

$$(\mathcal{E}^* \otimes \Omega^{\frac{1}{2}})_x \times (\mathcal{E} \otimes \Omega^{\frac{1}{2}})_x \ni (\bar\ell, \iota(v)) \mapsto \langle \bar\ell, \iota(v)\rangle \in \Omega^1_x, \quad x \in X,$$

likewise for the complex conjugate of $\mathcal{E}$. Thus, an operator $A \in \Psi^m(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}}, \overline{\mathcal{E}}^* \otimes \Omega^{\frac{1}{2}})$ has an adjoint $A^* \in \Psi^m(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}}, \overline{\mathcal{E}}^* \otimes \Omega^{\frac{1}{2}})$ defined by

$$\int_X \langle A^*u, \iota(v)\rangle = \int_X \overline{\langle Av, \iota(u)\rangle},$$

with principal symbol $\sigma_m(A^*) = \sigma_m(A)^* \in S^m(T^*X \setminus o, \pi^* \operatorname{Hom}(\mathcal{E}, \overline{\mathcal{E}}^*))$, and likewise $P \in \Psi^m(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$ has an adjoint $P^* \in \Psi^m(X, \overline{\mathcal{E}}^* \otimes \Omega^{\frac{1}{2}})$ with $\sigma_m(P^*) = \sigma_m(P)^*$.

6.3.2 Definition of pseudodifferential inner products; adjoints

We work with classical, i.e. one-step polyhomogeneous, symbols and operators, and denote by $S^m_{\text{hom}}(T^*X \setminus o)$ symbols which are homogeneous of degree m with respect to dilations in the fibers of $T^*X \setminus o$.

Definition 6.3.1. A *pseudodifferential inner product* (or Ψ-inner product) *on the vector bundle* $\mathcal{E} \to X$ is a pseudodifferential operator $B \in \Psi^0(X; \mathcal{E} \otimes \Omega^{\frac{1}{2}}, \overline{\mathcal{E}}^* \otimes \Omega^{\frac{1}{2}})$ satisfying $B = B^*$, and such that moreover the principal symbol $\sigma^0(B) = b \in S^0_{\text{hom}}(T^*X \setminus o; \pi^* \operatorname{Hom}(\mathcal{E}, \overline{\mathcal{E}}^*))$ of B satisfies

$$\langle b(x,\xi)u, \iota(u)\rangle > 0 \tag{6.3.1}$$

for all non-zero $u \in \mathcal{E}_x$, where $x \in X$, $\xi \in T^*_x X \setminus o$. If the context is clear, we will also call the sesquilinear pairing

$$\mathcal{C}^\infty(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}}) \times \mathcal{C}^\infty(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}}) \ni (u, v) \mapsto \int_X \langle B(x,D)u, \iota(v)\rangle$$

the pseudodifferential inner product associated with B.

In particular, the principal symbol b of B is a Hermitian inner product on $\pi^*\mathcal{E}$. Conversely, for any $b \in S^0_{\text{hom}}(T^*X \setminus o; \pi^* \operatorname{Hom}(\mathcal{E}, \overline{\mathcal{E}}^*))$ satisfying $b = b^*$ and (6.3.1), there exists a Ψ-inner product B with $\sigma^0(B) = b$; indeed, simply take $\widetilde{B}$ to be any quantization of b and put $B = \frac{1}{2}(\widetilde{B} + \widetilde{B}^*)$.

Remark 6.3.2. While we will develop the theory of Ψ-inner products only in the standard calculus on a closed manifold, everything works *mutatis mutandis* in other settings as well. Thus, in the b-calculus, see §3.3, Ψ_{b}-inner products on a manifold with boundary are defined similarly to Ψ-inner products, except that adjoints are defined on the space $\dot{\mathcal{C}}^\infty$ of functions vanishing to infinite order at the boundary, and the space of 'trivial,' smoothing operators is now $\Psi_{\mathrm{b}}^{-\infty}$, likewise for the scattering calculus [84], replacing 'b' by 'sc.' In the semiclassical calculus on a closed manifold, adjoints are again defined on $\mathcal{C}^\infty$, but the space of 'trivial' operators is now $h^\infty \Psi_\hbar^{-\infty}$, and suitable factors of h need to be put in for computations involving subprincipal symbols.

We next discuss adjoints of ps.d.o.s relative to Ψ-inner products.

Definition 6.3.3. Let B be a Ψ-inner product, and let $P \in \Psi^m(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$, then $P^{*B} \in \Psi^m(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$ is called an *adjoint of P with respect to B* if there exists an operator $R \in \Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}}, \overline{\mathcal{E}}^* \otimes \Omega^{\frac{1}{2}})$ such that

$$\int \langle BPu, \iota(v)\rangle = \int \langle Bu, \iota(P^{*B}v)\rangle + \int \langle Ru, \iota(v)\rangle \tag{6.3.2}$$

for all $u, v \in \mathcal{C}^\infty(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$.

Remark 6.3.4. This definition and the following lemma have straightforward generalizations to the case that P maps sections of $\mathcal{E}$ into sections of another vector bundle $\mathcal{F}$, provided a (Ψ-)inner product on $\mathcal{F}$ is given.

Lemma 6.3.5. *In the notation of Definition 6.3.3, the adjoint of P with respect to B exists and is uniquely determined modulo $\Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$. In fact, $P = (BPB^-)^*$, where B^- is a parametrix for B. Moreover, $(P^{*B})^{*B} = P$ modulo $\Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$. In particular, $\operatorname{Im}^B P = \frac{1}{2i}(P - P^{*B})$ is self-adjoint with respect to B (i.e. its own adjoint modulo $\Psi^{-\infty}$).*

Proof. Let B^- be a parametrix of B and put $R_L = I - B^-B \in \Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$. Then

$$\int \langle BPu, \iota(v)\rangle = \int \langle BPB^-Bu, \iota(v)\rangle + \langle BPR_Lu, \iota(v)\rangle,$$

hence (6.3.2) holds with $P^{*B} = (BPB^-)^*$ and $R = BPR_L$. To show the uniqueness of P^{*B} modulo smoothing operators, suppose that $\widetilde{P}$ is another adjoint of P with respect to B, with error term $\widetilde{R}$ (i.e. (6.3.2) holds with P^{*B} and R replaced by $\widetilde{P}$ and $\widetilde{R}$). Then

$$\begin{aligned}\int \overline{\langle B(P^{*B} - \widetilde{P})v, \iota(u)\rangle} &= \int \langle Bu, \iota((P^{*B} - \widetilde{P})v)\rangle = \int \langle (\widetilde{R} - R)u, \iota(v)\rangle \\ &= \int \overline{\langle (\widetilde{R} - R)^*v, \iota(u)\rangle}\end{aligned}$$

for $u, v \in \mathcal{C}^\infty(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$, so $B(P^{*B} - \widetilde{P}) = (\widetilde{R} - R)^* \in \Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}}, \overline{\mathcal{E}}^* \otimes \Omega^{\frac{1}{2}})$, and the ellipticity of B implies $P^{*B} - \widetilde{P} \in \Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$, as claimed.

Since B is self-adjoint, we can assume that B^- is self-adjoint by replacing it by $\frac{1}{2}(B^- + (B^-)^*)$ (which changes B^- by an operator in $\Psi^{-\infty}$). Then the second claim follows from

$$(P^{*B})^{*B} = (BP^{*B}B^-)^* = B^-BPB^-B = P$$

modulo $\Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$. ∎

Lemma 6.3.6. *Suppose $P \in \Psi^m(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$ is self-adjoint with respect to B. Then its principal symbol p is self-adjoint with respect to $b = \sigma^0(B)$, i.e.*

$$\langle b(x,\xi)p(x,\xi)u, \iota(v)\rangle = \langle b(x,\xi)u, \iota(p(x,\xi)v)\rangle, \quad x \in X, \xi \in T_xX, u, v \in \mathcal{E}_x.$$

Proof. The hypothesis on P means $(BPB^-)^* = P$ modulo $\Psi^{-\infty}$, thus on the level of principal symbols, $p = b^{-1}p^*b = p^{*b}$, which proves the claim. □

We now specialize to the case that $P \in \Psi^m(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$ has a real, scalar principal symbol. Fix a coordinate system of X and a local trivialization of $\mathcal{E}$, then the full symbol of P is a sum of homogeneous symbols $p \sim p_m + p_{m-1} + \ldots$, with p_j homogeneous of degree j and valued in complex $N \times N$ matrices. Recall from [64, §18] that the subprincipal symbol

$$\sigma_{\mathrm{sub}}(P) = p_{m-1}(x,\xi) - \frac{1}{2i}\sum_j \partial_{x_j\xi_j} p_m(x,\xi) \in S^{m-1}_{\mathrm{hom}}(T^*X \setminus o, \mathbb{C}^{N\times N}) \tag{6.3.3}$$

is well-defined under changes of coordinates; however, it does depend on the choice of local trivialization of $\mathcal{E}$. We compute the principal symbol of

$$\mathrm{Im}^B P := \frac{1}{2i}(P - P^{*B})$$

for such P in a local trivialization of $\mathcal{E}$; we will give an invariant formulation in Proposition 6.3.11 below.

Lemma 6.3.7. *Let $P \in \Psi^m(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$ be a principally real and scalar, and let $B = b(x, D)$ be a Ψ-inner product on $\mathcal{E}$. Then $\mathrm{Im}^B P \in \Psi^{m-1}(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$ has the principal symbol*

$$\sigma^{m-1}(\mathrm{Im}^B P) = \mathrm{Im}^b \sigma_{\mathrm{sub}}(P) + \frac{1}{2}b^{-1}H_p(b), \tag{6.3.4}$$

where $\mathrm{Im}^b \sigma_{\mathrm{sub}}(P) = \frac{1}{2i}\big(\sigma_{\mathrm{sub}}(P) - \sigma_{\mathrm{sub}}(P)^{*b}\big)$. Here, we interpret b and $\sigma_{\mathrm{sub}}(P)$ as $N \times N$ matrices of scalar-valued symbols using a local frame of $\mathcal{E}$ and the corresponding dual frame of $\overline{\mathcal{E}}^*$, and the action of H_p is component-wise.

Proof. We compute in a local coordinate system over which $\mathcal{E}$ and $\overline{\mathcal{E}}$ are trivialized by a choice of N linearly independent sections $e_1, \ldots, e_N$, and $\mathcal{E}^*$ and $\overline{\mathcal{E}}^*$ are trivialized by the dual sections $e_1^*, \ldots, e_N^* \in \mathcal{E}^*$ satisfying $e_i^*(e_j) = \delta_{ij}$, extended linearly as linear functionals on $\mathcal{E}$, resp. on $\overline{\mathcal{E}}$, in the case of $\mathcal{E}^*$, resp. $\overline{\mathcal{E}}^*$. We trivialize $\Omega^{\frac{1}{2}}$ using the section $|dx|^{\frac{1}{2}}$. Let

$b_{ij}(x,\xi) = \langle b(x,\xi)e_j, \iota(e_i)\rangle$, then $b(x,\xi) = (b_{ij}(x,\xi))_{i,j=1,\ldots,N}$, a linear map from the fibers of $\mathcal{E}$ to the fibers of $\overline{\mathcal{E}}^*$, is the symbol of B in local coordinates: If $u = \sum_j u_j e_j |dx|^{\frac{1}{2}}$ and $v = \sum_j v_j e_j |dx|^{\frac{1}{2}}$, we have

$$\langle b(x,\xi)u, \iota(v)\rangle = \sum_{ij} b_{ij}(x,\xi)u_j \cdot \overline{v_i}|dx|,$$

thus

$$\int \langle Bu, \iota(v)\rangle = \sum_{ij} \int (b_{ij}(x,D)u_j) \cdot \overline{v_j}\, dx.$$

Note that $b(x,\xi)$ is a Hermitian matrix, i.e. $b_{ij}(x,\xi) = \overline{b_{ji}(x,\xi)}$, and in fact $B = b(x,D)$ is self-adjoint (with respect to the standard Hermitian inner product on $\mathbb{C}^N$). The adjoint of $P = p(x,D)$, which in local coordinates is simply an $N \times N$ matrix of scalar ps.d.o.s, with respect to B is the operator $\widetilde{P} = \widetilde{p}(x,D)$ such that

$$\int b(x,D)p(x,D)u \cdot \overline{v}\, dx = \int b(x,D)u \cdot \overline{\widetilde{p}(x,D)v}\, dx + \int Ru \cdot \overline{v}\, dx, \quad R \in \Psi^{-\infty}.$$

Let $B^- := b^-(x,D)$ be a parametrix for $b(x,D)$, in particular $b^-(x,\xi) = b(x,\xi)^{-1}$ modulo S^{-1}; we may assume $B^-(x,D)^* = B^-(x,D)$. We then have

$$\widetilde{p}(x,D) = b^-(x,D)p(x,D)^*b(x,D)$$

by Lemma 6.3.5. Write $p(x,\xi) = p_m(x,\xi) + p_{m-1}(x,\xi) + \ldots$, then the full symbol of $P - \widetilde{P} = B^-(BP - P^*B)$ (where P^* is the adjoint of P with respect to the standard Hermitian inner product on $\mathbb{C}^N$) is given, modulo S^{m-2}, by

$$\begin{aligned} b^{-1}\Big(&bp_m + \frac{1}{i}\sum_j \partial_{\xi_j} b \partial_{x_j} p_m + bp_{m-1} \\ &- p_m^* b - \frac{1}{i}\sum_j (\partial_{x_j\xi_j} p_m^*)b - \frac{1}{i}\sum_j \partial_{\xi_j} p_m^* \partial_{x_j} b - p_{m-1}^* b\Big) \\ = \Big(&p_{m-1} - \frac{1}{2i}\sum_j \partial_{x_j\xi_j} p_m\Big) - b^{-1}\Big(p_{m-1} - \frac{1}{2i}\sum_j \partial_{x_j\xi_j} p_m\Big)^* b + ib^{-1}H_{p_m}(b), \end{aligned}$$

where we used that p_m is scalar and real. The claim follows. □

6.3.3 Invariant formalism for subprincipal symbols of operators acting on bundles

We continue to denote by $P \in \Psi^m(X, \mathcal{E}\otimes\Omega^{\frac{1}{2}})$ a principally scalar ps.d.o. acting on the vector bundle $\mathcal{E}$, with principal symbol p. (The discussion until Proposition 6.3.8 in fact works for principally non-scalar operators as well with mostly notational changes.) We will show how to modify the definition (6.3.3) of the subprincipal symbol of P, expressed in terms of a local trivialization of $\mathcal{E}$, in an invariant fashion, i.e. in a way that is both independent of the choice of local trivialization and of local coordinates on X. This provides a completely invariant formulation of Lemma 6.3.7.

Let $U \subset X$ be an open subset over which $\mathcal{E}$ is trivial, and pick a frame $e(x) = \{e_1(x), \ldots, e_N(x)\}$ trivializing $\mathcal{E}$ over U. Let us write P^e for P in the frame e, i.e. $P^e = (P^e_{jk})_{j,k=1,\ldots,N}$ is the $N \times N$ matrix of operators $P^e_{jk} \in \Psi^m(U, \Omega^{\frac{1}{2}})$ defined by

$$P(\sum_k u_k(x)e_k(x)) = \sum_{jk} P^e_{jk}(u_k)e_j(x), \quad u_k \in \mathcal{C}^\infty(U, \Omega^{\frac{1}{2}}).$$

Then $\sigma^e_{\mathrm{sub}}(P)$ as defined in (6.3.3), with the superscript making the choice of frame explicit, is simply an $N \times N$ matrix of scalar symbols:

$$\sigma^e_{\mathrm{sub}}(P) = (\sigma_{\mathrm{sub}}(P^e_{jk}))_{j,k=1,\ldots,N}.$$

We will consider the effect of a change of frame on the subprincipal symbol (6.3.3). Thus, let $C \in \mathcal{C}^\infty(U, \mathrm{End}(\mathcal{E}))$ be a change of frame, i.e. $C(x)$ is invertible for all $x \in X$. Then $e_j(x) = C(x)e'_j(x)$ defines another frame $e'(x) = \{e'_1(x), \ldots, e'_N(x)\}$ of $\mathcal{E}$ over U. One easily computes

$$\sigma^{e'}_{\mathrm{sub}}(C^{-1}PC) = (C^{e'})^{-1}\sigma^{e'}_{\mathrm{sub}}(P)C^{e'} - i(C^{e'})^{-1}H_p(C^{e'}),$$

with H_p interpreted as the diagonal $N \times N$ matrix $1_{N\times N}H_p$ of first order differential operators, and $C^{e'}$ is the matrix of C in the frame e'. Now note that $(C^{-1}PC)^{e'} = P^e$ and $(C^{e'})^{-1}H_p(C^{e'}) = (C^{e'})^{-1}H_pC^{e'} - H_p$; thus, we obtain

$$\sigma^e_{\mathrm{sub}}(P) - iH_p = (C^{e'})^{-1}\big(\sigma^{e'}_{\mathrm{sub}}(P) - iH_p\big)C^{e'} \tag{6.3.5}$$

Thus, viewing $\sigma^{e'}_{\mathrm{sub}}(P) - iH_p$ as the $N \times N$ matrix (in the frame e') of a differential operator acting on $\mathcal{C}^\infty(T^*X \setminus o, \pi^*\mathcal{E})$, the right hand side of (6.3.5) is the matrix of the same differential operator, but expressed in the frame e. Notice that the principal symbol p of P as a scalar, i.e. diagonal, $N \times N$ matrix of symbols, is well-defined independently of the choice of frame. To summarize:

Definition 6.3.8. For $P \in \Psi^m(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$ with scalar principal symbol p, there is a well-defined *subprincipal operator* $S_{\mathrm{sub}}(P) \in \mathrm{Diff}^1(T^*X \setminus o, \pi^*\mathcal{E})$, homogeneous of degree $m-1$ with respect to dilations in the fibers of $T^*X \setminus o$, defined as follows: If $\{e_1(x), \ldots, e_N(x)\}$ is a local frame of $\mathcal{E}$, define the operators $P_{jk} \in \Psi^m(X, \Omega^{\frac{1}{2}})$ by $P(\sum_k u_k(x)e_k(x)) = \sum_{jk} P_{jk}(u_k)e_j(x)$, $u_k \in \mathcal{C}^\infty(X, \Omega^{\frac{1}{2}})$. Then

$$S_{\mathrm{sub}}(P)\Big(\sum_k q_k(x,\xi)e_k(x)\Big) := \sum_{jk}(\sigma_{\mathrm{sub}}(P_{jk})q_k)e_j - i\sum_k (H_p q_k)e_k.$$

In shorthand notation, $S_{\mathrm{sub}}(P) = \sigma_{\mathrm{sub}}(P) - iH_p$, understood in a local frame as a matrix of first order differential operators. We emphasize the dependence on the order of the operator by writing $S_{\mathrm{sub},m}(P)$, so that for $P \in \Psi^m(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$, we have $S_{\mathrm{sub},m+1}(P) = \sigma_m(P)$.

We shall compute the subprincipal operator of the Laplace-Beltrami operator acting on sections of the tensor bundle in §6.4.

Remark 6.3.9. For Ψ_{b}-inner products, the subprincipal operator of $P \in \Psi_{\mathrm{b}}^m(X, \mathcal{E} \otimes \Omega_{\mathrm{b}}^{\frac{1}{2}})$ acting on $\mathcal{E}$-valued b-half-densities is an element of $\mathrm{Diff}_{\mathrm{b}}^1({}^{\mathrm{b}}T^*X \setminus o, \pi_{\mathrm{b}}^*\mathcal{E})$, where $\pi_{\mathrm{b}} \colon {}^{\mathrm{b}}T^*X \setminus o \to X$ is the projection. In the semiclassical setting, $P \in \Psi_\hbar^m(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$, we have $S_{\mathrm{sub}}(P) \in \mathrm{Diff}^1(T^*X, \pi^*\mathcal{E})$.

Remark 6.3.10. Dencker [34] proved that polarization sets propagate along so-called Hamilton orbits, which are line subbundles of the pullback of $\pi^*\mathcal{E}$ to null-bicharacteristics, and which are spanned by sections of this bundle which are parallel with respect to a partial connection D_P. In the case of interest for us, when P is principally scalar, his definition [34, Equation (4.6)] (taking $\widetilde{p} = \mathrm{id}$) agrees with our definition of $S_{\mathrm{sub}}(P)$ up to a factor of i.

We can now express the symbols of commutators and imaginary parts in a completely invariant fashion:

Proposition 6.3.11. *Let $P \in \Psi^m(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$ be a ps.d.o. with scalar principal symbol p.*

(1) Suppose $Q \in \Psi^{m'}(X, \mathcal{E} \otimes \Omega^{\frac{1}{2}})$ is an operator acting on $\mathcal{E}$-valued half-densities, with

principal symbol q. (We do not assume Q is principally scalar.) Then

$$\sigma^{m+m'-1}([P,Q]) = [S_{\mathrm{sub}}(P), q].$$

If Q is elliptic with parametrix Q^-, then

$$S_{\mathrm{sub}}(QPQ^-) = qS_{\mathrm{sub}}(P)q^{-1}. \tag{6.3.6}$$

(2) *Suppose in addition that p is real. Let B be a Ψ-inner product on $\mathcal{E}$ with principal symbol b, then*

$$\sigma^{m-1}(\operatorname{Im}^B P) = \operatorname{Im}^b S_{\mathrm{sub}}(P), \tag{6.3.7}$$

where $\operatorname{Im}^b S_{\mathrm{sub}}(P) = \frac{1}{2i}\big(S_{\mathrm{sub}}(P) - S_{\mathrm{sub}}(P)^{*b}\big)$; we take the adjoint of the differential operator $S_{\mathrm{sub}}(P)$ with respect to the inner product b on $\pi^*\mathcal{E}$ and the symplectic volume density on T^*X.

Proof. We verify this in a local frame $e(x) = \{e_1(x), \dots, e_N(x)\}$ of $\mathcal{E}$. We compute

$$\begin{aligned} &S_{\mathrm{sub}}(P)\Big(\sum_{jk} q_{jk}(x,\xi)u_k(x,\xi)e_j(x)\Big) \\ &\qquad = \sum_{j\ell}\Big(\sum_k \sigma_{\mathrm{sub}}(P)_{jk}q_{k\ell} - iH_p(q_{j\ell})\Big)u_\ell e_j - iq_{j\ell}H_p(u_\ell)e_j - iq_{j\ell}u_\ell e_j H_p, \end{aligned}$$

while

$$\begin{aligned} &qS_{\mathrm{sub}}(P)\Big(\sum_{\ell} u_\ell(x,\xi)e_\ell(x)\Big) \\ &\qquad = \sum_{j\ell}\Big(\sum_k q_{jk}\sigma_{\mathrm{sub}}(P)_{k\ell}\Big)u_\ell e_j - iq_{j\ell}H_p(u_\ell)e_j - iq_{j\ell}u_\ell e_j H_p, \end{aligned}$$

hence $S_{\mathrm{sub}}(P)q - qS_{\mathrm{sub}}(P) = [\sigma_{\mathrm{sub}}(P), q] - iH_p(q)$ as an endomorphism (a zeroth order differential operator acting on sections of $\mathcal{E}$) of $\mathcal{E}$ in the frame e, which equals $\sigma^{m+m'-1}([P,Q])$ according to the usual (full) symbolic calculus.

Furthermore,

$$\begin{aligned} S_{\mathrm{sub},m}(QPQ^-) &= S_{\mathrm{sub},m}(P) + S_{\mathrm{sub},m}(Q[P,Q^-]) \\ &= S_{\mathrm{sub},m}(P) + q\sigma_{m+m'-1}([P,Q^-]) = S_{\mathrm{sub},m}(P) + q[S_{\mathrm{sub},m}(P), q^{-1}] \end{aligned}$$

$$= qS_{\mathrm{sub},m}(P)q^{-1},$$

noting that $Q[P, Q^-]$ is of order $m-1$.

For the second part, we have $S_{\mathrm{sub}}(P)^{*b} = \sigma_{\mathrm{sub}}(P)^{*b} - (iH_p)^{*b} = b^{-1}\sigma_{\mathrm{sub}}(P)^*b + ib^{-1}(H_p)^*b$, where $(H_p)^*$ is the adjoint of H_p as an operator acting on $\mathcal{C}^\infty_{\mathrm{c}}(T^*X \setminus o)$, and we equip T^*X with the natural symplectic volume density $|dx\,d\xi|$. We have $(H_p)^* = -H_{\bar{p}} = -H_p$ since p is real. Therefore,

$$\begin{aligned} S_{\mathrm{sub}}(P) - S_{\mathrm{sub}}(P)^{*b} &= \sigma_{\mathrm{sub}}(P) - \sigma_{\mathrm{sub}}(P)^{*b} - iH_p + ib^{-1}H_p b \\ &= \sigma_{\mathrm{sub}}(P) - \sigma_{\mathrm{sub}}(P)^{*b} + ib^{-1}H_p(b), \end{aligned}$$

which indeed gives (6.3.4) upon division by $2i$. □

In particular, (6.3.7) provides a very elegant point of view for understanding the imaginary part of a principally scalar and real (pseudo)differential operator with respect to a Ψ-inner product B, as already indicated in the introduction: For instance, the principal symbol of the imaginary part $\mathrm{Im}^B P$ vanishes (or is small relative to $b = \sigma^0(B)$) in a subset of phase space if and only if the imaginary part of the first order differential operator $S_{\mathrm{sub}}(P)$ on $T^*X \setminus o$ has vanishing (or small with respect to the fiber inner product b of $\pi^*\mathcal{E}$) coefficients in this subset.

6.3.4 Interpretation of pseudodifferential inner products in traditional terms

We now show how to interpret the imaginary part $\mathrm{Im}^B P$ of an operator P with respect to a Ψ-inner product B in terms of the imaginary part of a conjugated version of P with respect to a standard inner product:

Proposition 6.3.12. *Let B be a Ψ-inner product on $\mathcal{E}$. Then for any positive definite Hermitian inner product $B_0 \in \mathcal{C}^\infty(X, \mathrm{Hom}(\mathcal{E}\otimes\Omega^{\frac12}, \overline{\mathcal{E}}^*\otimes\Omega^{\frac12}))$ on $\mathcal{E}$, there exists an elliptic operator $Q \in \Psi^0(X, \mathrm{End}(\mathcal{E}\otimes\Omega^{\frac12}))$ such that $B - Q^*B_0Q \in \Psi^{-\infty}(X, \mathrm{Hom}(\mathcal{E}\otimes\Omega^{\frac12}, \overline{\mathcal{E}}^*\otimes\Omega^{\frac12}))$.*

In particular, denoting by $Q^- \in \Psi^0(X, \mathrm{End}(\mathcal{E}\otimes\Omega^{\frac12}))$ a parametrix of Q, we have for any $P \in \Psi^m(X, \mathcal{E}\otimes\Omega^{\frac12})$ with real and scalar principal symbol:

$$Q(\mathrm{Im}^B P)Q^- = \mathrm{Im}^{B_0}(QPQ^-), \tag{6.3.8}$$

and $\sigma^{m-1}(\operatorname{Im}^B P)$ *and* $\sigma^{m-1}(\operatorname{Im}^{B_0}(QPQ^-))$ *(which are self-adjoint with respect to* $\sigma^0(B)$ *and* B_0*, respectively, hence diagonalizable) have the same eigenvalues.*

On a symbolic level, equation 6.3.8 is the same as equation (6.3.6).

Proof of Proposition 6.3.12. In order to shorten the notation, fix a global trivialization of $\Omega^{\frac{1}{2}}$ over X and use it to identify $\mathcal{E}\otimes\Omega^{\frac{1}{2}}$ with $\mathcal{E}$, likewise for all other half-density bundles appearing in the statement. Denote the principal symbol of B by $b\in S^0_{\mathrm{hom}}(T^*X\setminus o,\pi^*\operatorname{Hom}(\mathcal{E},\overline{\mathcal{E}}^*))$. We similarly put $b_0:=B_0$, which is an inner product on $\pi^*\mathcal{E}$ that only depends on the base point.

We start with on the symbolic level by constructing an elliptic symbol $q_1\in S^0_{\mathrm{hom}}(T^*X\setminus o,\pi^*\operatorname{End}(\mathcal{E}))$ such that $b=q_1^*b_0q_1$; recall that $q_1^*\in S^0_{\mathrm{hom}}(T^*X\setminus o,\pi^*\operatorname{End}(\overline{\mathcal{E}}^*))$. For $t\in[0,1]$, define the Hermitian inner product $b_t:=(1-t)b_0+tb$. We will construct a differentiable family q_t of symbols such that $b_t=q_t^*b_0q_t$ for $t\in[0,1]$. Observe that for any such family, we have $\partial_tb_t=b-b_0=(\partial_tq_t)^*b_0q_t+q_t^*b_0\partial_tq_t$, which suggests requiring $\partial_tq_t=\frac{1}{2}b_0^{-1}(q_t^*)^{-1}(b-b_0)$, which we can write as a linear expression in q_t by noting that $(q_t^*)^{-1}=b_0q_tb_t^{-1}$. Moreover, $q_0=\mathrm{id}$ is a valid choice for q_t at $t=0$. Thus, we are led to define q_t, $t\in[0,1]$, as the solution of the ODE

$$\partial_tq_t=\frac{1}{2}q_tb_t^{-1}(b-b_0),\quad q_0=\mathrm{id}\,.$$

Reversing these arguments, for the solution q_t we then have $q_t^*b_0q_t=b_t$ for $t=0$, and both $q_t^*b_0q_t$ and b_t are solutions of the same ODE, namely

$$\partial_t\widetilde{b}_t=\frac{1}{2}\big((b-b_0)b_t^{-1}\widetilde{b}_t+\widetilde{b}_tb_t^{-1}(b-b_0)\big),\quad \widetilde{b}_0=b_0,$$

hence $q_t^*b_0q_t=b_t$ for all $t\in[0,1]$.

Let $Q_1\in\Psi^0(X,\operatorname{End}(\mathcal{E}))$ be a quantization of q_1, then we conclude that $B-Q_1^*B_0Q_1\in\Psi^{-1}$. We iteratively remove this error to obtain a smoothing error: Suppose the operator $Q_k\in\Psi^0(X,\operatorname{End}(\mathcal{E}))$ is such that $B-Q_k^*B_0Q_k\in\Psi^{-k}$ for some $k\geq 1$. We will find $D_k\in\Psi^{-k}$, a quantization of $d_k\in S^{-k}_{\mathrm{hom}}(T^*X\setminus o,\pi^*\mathcal{E})$, such that $Q_{k+1}:=Q_k+D_k$ satisfies $B-Q_{k+1}^*B_0Q_{k+1}\in\Psi^{-k-1}$. This is equivalent to the equality of symbols

$$r_k:=\sigma^{-k}(B-Q_k^*B_0Q_k)=\sigma^{-k}(D_k^*B_0Q_k+Q_k^*B_0D_k)=d_k^*b_0q_1+(b_0q_1)^*d_k,$$

which in view of $r_k^*=r_k$ is satisfied for $d_k=\frac{1}{2}((b_0q_1)^*)^{-1}r_k$. We define $Q\in\Psi^0(X,\operatorname{End}(\mathcal{E}))$ to be the asymptotic limit of the Q_k as $k\to\infty$, i.e. $Q\sim Q_1+\sum_{k=1}^{\infty}D_k$, which thus satisfies

$B - Q^* B_0 Q \in \Psi^{-\infty}$. This proves the first part of the proposition.

For the second part, denote parametrices of B and Q by B^- and Q^-, respectively. Then, modulo operators in $\Psi^{-\infty}$, we have

$$P^{*B} = (BPB^-)^* = (Q^* B_0 Q P Q^- B_0^{-1} (Q^-)^*)^* = Q^-(QPQ^-)^{*B_0} Q,$$

hence

$$Q(P - P^{*B})Q^- = (QPQ^-) - (QPQ^-)^{*B_0}$$

modulo $\Psi^{-\infty}$. □

6.3.5 A simple example

On $\mathbb{R}^n_x = \mathbb{R}_{x_1} \times \mathbb{R}^{n-1}_{x'}$, we consider the operator $P = D_{x_1} + A \in \Psi^1(\mathbb{R}^n, \mathbb{C}^N)$, where $A = A(x, D) \in \Psi^0(\mathbb{R}^n, \mathbb{C}^N)$ is independent of x_1. Trivializing the half-density bundle over $\mathbb{R}^n$ via $|dx|^{\frac{1}{2}}$, we can consider P as an operator in $\Psi^1(\mathbb{R}^n, \mathbb{C}^N \otimes \Omega^{\frac{1}{2}})$. Its principal symbol is $\sigma_1(P)(x, \xi) = \xi_1$, where we use the standard coordinates on $T^*\mathbb{R}^n$, i.e. writing covectors as $\xi\, dx$, so the Hamilton vector field is $H_{\sigma_1(P)} = \partial_{x_1}$; moreover, in the trivialization of $\mathbb{C}^N$ by means of its standard basis, $\sigma_{\mathrm{sub}}(P)(x, \xi) = A(x, \xi)$. Thus, the subprincipal operator of P is

$$S_{\mathrm{sub}}(P)(x, \xi) = A(x, \xi) - i\partial_{x_1} \in \mathrm{Diff}^1(T^*\mathbb{R}^n \setminus o, \pi^*\mathbb{C}^N),$$

with A homogeneous of degree 0 in the fiber variables. Suppose we are interested in bounding $\frac{1}{2i}(P - P^*)$ on $Z := T^*_{\{x'=0\}}\mathbb{R}^n \setminus o$ relative to a suitably chosen inner product. Let us assume that $A(0, \xi)$ is nilpotent for all $|\xi| = 1$, and that in fact at $x = 0$ and $|\xi| = 1$, we can choose a smooth frame $e_1(\xi), \ldots, e_N(\xi)$ of the bundle $\pi^*\mathbb{C}^N \to T^*\mathbb{R}^n \setminus o$ so that $A(0, \xi)$, written in the basis $e_1(\xi), \ldots, e_N(\xi)$, is a single Jordan block with zeros on the diagonal and ones directly above. Extend the e_j by homogeneity (of degree 0) in the fiber variables, and define them to be constant in the x_1-direction along Z, i.e. $e_j(x_1, 0; \xi) = e_j(0, 0; \xi)$, and extend them in an arbitrary manner to a neighborhood of Z.

Now, on Z we have $Ae_j = e_{j-1}$, writing $e_0 := 0$. Introduce a new frame $e'_j := \epsilon^j e_j$ with $\epsilon > 0$ fixed, then $Ae'_j = \epsilon e'_{j-1}$. Define the inner product b on $\pi^*\mathbb{C}^N$ by

$$\langle b(x, \xi)(e'_i(x, \xi)), \iota(e'_j(x, \xi))\rangle = \delta_{ij},$$

that is, $\{e'_1, \ldots, e'_N\}$ is an orthonormal frame for b. Then on Z, we find that $\mathrm{Im}^b\, S_{\mathrm{sub}}(P)$

(which is of order 0) in the frame $\{e'_1, \ldots, e'_N\}$ is given by the matrix which is zero apart from entries $\epsilon/2i$ directly above and $-\epsilon/2i$ directly below the diagonal. Thus, defining the Ψ-inner product $B = b(x, D)$, we have arranged that $\|\sigma_0(\operatorname{Im}^B P)(x, \xi)\|_b \leq \epsilon$ on Z. Since $\sigma_0(\operatorname{Im}^B P)$ is self-adjoint with respect to b, this is really the statement that its eigenvalues are bounded from above and below by ϵ and $-\epsilon$, respectively.

Using Proposition 6.3.12, we can rephrase this as follows: If v_j denotes the standard basis of $\mathbb{C}^N$ and $\langle B_0(v_i), \iota(v_j)\rangle = \delta_{ij}$ the standard inner product on $\mathbb{C}^N$ (the particular choice of an ordinary inner product being irrelevant, see the statement of Proposition 6.3.12), define the map $q(x, \xi) \in S^0_{\mathrm{hom}}(T^*\mathbb{R}^n \setminus o, \pi^*\mathbb{C}^N)$ by $q(x, \xi)e'_j(x, \xi) = v_j$. Let $Q = q(x, D)$ and denote by Q^- a parametrix of Q, then we find that $QPQ^- \in \Psi^1(\mathbb{R}^n, \mathbb{C}^N)$ satisfies $\|\sigma_0(\operatorname{Im}^{B_0} QPQ^-)\|_{B_0} \leq \epsilon$.

If A has several Jordan blocks not all of which are nilpotent, one can (under the assumption of the existence of a smooth family of Jordan bases) similarly construct a Ψ-inner product so that the imaginary part of A relative to it is bounded by the maximal imaginary part of the eigenvalues of A (plus ϵ) from above, and by the minimal imaginary part (minus ϵ) from below.

6.4 Subprincipal operators of tensor Laplacians

Let (M, g) be a smooth manifold equipped with a metric tensor g of arbitrary signature. Denote by $\mathcal{T}_k M = \bigotimes^k T^*M$, $k \geq 1$, the bundle of (covariant) tensors of rank k on M. The metric g induces a metric (which we also call g) on $\mathcal{T}_k M$. We study the symbolic properties of $\Delta_k = -\operatorname{tr}\nabla^2 \in \mathrm{Diff}^2(M, \mathcal{T}_k M)$, the Laplace-Beltrami operator on M acting on the bundle $\mathcal{T}_k M$. Denote by $G \in \mathcal{C}^\infty(T^*M)$ the metric function, i.e. $G(x, \xi) = |\xi|^2_{G(x)}$, where G is the dual metric of g.

Proposition 6.4.1. *The subprincipal operator of Δ_k is*

$$S_{\mathrm{sub}}(\Delta_k)(x, \xi) = -i\nabla^{\pi^*\mathcal{T}_k M}_{H_G} \in \mathrm{Diff}^1(T^*M \setminus o, \pi^*\mathcal{T}_k M), \tag{6.4.1}$$

where $\nabla^{\pi^*\mathcal{T}_k M}$ is the pullback connection, with $\pi\colon T^*M \setminus o \to M$ being the projection.

Proof. Since both sides of (6.4.1) are invariantly defined, it suffices to prove the equality in an arbitrary local coordinate system. At a fixed point $x_0 \in M$, introduce normal coordinates so that $\partial_k g_{ij} = 0$ at x_0. Then we schematically have

$$\begin{aligned}(\Delta_k u)_{i_1\dots i_k} &= -g^{jk} u_{i_1\dots i_k,jk} = -g^{jk}(\partial_k u_{i_1\dots i_k,j} + \Gamma\cdot\partial u)\\ &= -g^{jk}\partial_{jk} u_{i_1\dots i_k} + \partial(\Gamma\cdot u) + \Gamma\cdot\partial u\\ &= -g^{jk}\partial_{jk} u_{i_1\dots i_k} + \Gamma\cdot\partial u + \partial\Gamma\cdot u,\end{aligned}$$

with Γ denoting Christoffel symbols. This suffices to see that the full symbol of Δ_k in the local coordinate system is given by

$$\sigma(\Delta_k)(x,\xi) = g^{jk}(x)\xi_j\xi_k + (x^j - x_0^j)\ell_j(x,\xi) + e(x),$$

where $\ell_j(x,\xi)$ is a linear map in ξ with values in $\mathrm{End}((\mathcal{T}_k M)_x)$, and $e(x)$ is an endomorphism of $(\mathcal{T}_k M)_x$. Therefore, $\sigma_{\mathrm{sub}}(\Delta_k)(x_0,\xi) = 0$, since $\partial_i g^{jk}(x_0) = 0$. Thus,

$$S_{\mathrm{sub}}(\Delta_k)(x_0,\xi) = -iH_{|\xi|_g^2} = -2ig^{jk}\xi_k\partial_{x^j}. \tag{6.4.2}$$

We now compute the right hand side of (6.4.1). First, writing $dx^I = dx^{i_1}\otimes\cdots\otimes dx^{i_k}$ for multiindices $I = (i_1,\dots,i_k)$, we note that sections of $\pi^*\mathcal{T}_k M$ are of the form $u_I(x,\xi)\,dx^I$, while pullbacks (under π) of sections of $\mathcal{T}_k M$ are of the form $u_I(x)\,dx^I$. By definition, the pullback connection $\nabla^{\pi^*\mathcal{T}_k M}$ is given by

$$\nabla^{\pi^*\mathcal{T}_k M}_{\partial_{x^j}}(u_I(x)\,dx^I) = \nabla^{\mathcal{T}_k M}_{\partial_{x^j}}(u_I(x)\,dx^I), \quad \nabla^{\pi^*\mathcal{T}_k M}_{\partial_{\xi_k}}(u_I(x)\,dx^I) = 0$$

on pulled back sections and extended to sections of the pullback bundle using the Leibniz rule; thus,

$$\begin{aligned}\nabla^{\pi^*\mathcal{T}_k M}_{\partial_{x^j}}(u_I(x,\xi)\,dx^I) &= \nabla^{\mathcal{T}_k M}_{\partial_{x^j}}(u_I(\cdot,\xi)\,dx^I)(x),\\ \nabla^{\pi^*\mathcal{T}_k M}_{\partial_{\xi_k}}(u_I(x,\xi)\,dx^I) &= \partial_{\xi_k} u_I(x,\xi)\,dx^I.\end{aligned}$$

Thus, in normal coordinates at $x_0 \in M$, we simply have $\nabla^{\pi^*\mathcal{T}_k M}_{\partial_{x^j}} = \partial_{x^j}$ and $\nabla^{\pi^*\mathcal{T}_k M}_{\partial_{\xi_k}} = \partial_{\xi_k}$, therefore

$$\nabla^{\pi^*\mathcal{T}_k M}_{H_{|\xi|_g^2}} = 2g^{jk}\xi_k\partial_{x^j}$$

at x_0, which verifies (6.4.1) in view of (6.4.2). $\square$

To simplify the study of the pullback connection on $\pi^*\mathcal{T}_kM$ for general k, we observe that there is a canonical bundle isomorphism $\pi^*\mathcal{T}_kM \cong \bigotimes^k \pi^*T^*M$; hence the connection $\nabla^{\pi^*\mathcal{T}_kM}$ is simply the product connection on $\bigotimes^k \pi^*T^*M$. Therefore, if we understand certain properties of $S_{\mathrm{sub}}(\Delta_1)$, we can easily deduce them for $S_{\mathrm{sub}}(\Delta_k)$ for any k. In our application, we will need to choose a *positive definite* pseudodifferential inner product $B_k = b_k(x, D)$ on the bundle $\mathcal{T}_kM$ with respect to which Δ_k is arbitrarily close to being symmetric in certain subsets of phase space. Concretely, this means that we want the operator $S_{\mathrm{sub}}(\Delta_k)$ to be (almost) symmetric with respect to the inner product b_k on $\pi^*\mathcal{T}_kM$. The following lemma shows that it suffices to accomplish this for $k = 1$:

Lemma 6.4.2. *Let $U \subset T^*M\backslash o$ be open, and let $f \in \mathcal{C}^\infty(U)$ be real-valued. Fix a Hermitian inner product b (antilinear in the second slot) on π^*T^*M, and define $R \in \mathrm{End}(\pi^*T^*M)$ by requiring that*

$$\int_U \langle i\nabla^{\pi^*T^*M}_{H_f} u, v\rangle_b \, d\sigma - \int_U \langle u, i\nabla^{\pi^*T^*M}_{H_f} v\rangle_b \, d\sigma = \int_U \langle u, Rv\rangle_b \, d\sigma$$

for all $u, v \in \mathcal{C}^\infty_{\mathrm{c}}(U, \pi^*T^*M)$, where $d\sigma$ is the natural symplectic volume density on T^*M. There exists a constant $C_k > 0$, independent of U, f and b, such that the following holds: If $\sup_U \|R\|_b \leq \epsilon$ (using b to measure the operator norm of R acting on each fiber) for some $\epsilon > 0$, then the inner product $b_k = \bigotimes^k b$ induced by b on $\bigotimes^k \pi^*T^*M \cong \pi^*\mathcal{T}_kM$ satisfies

$$\int_U \langle i\nabla^{\pi^*\mathcal{T}_kM}_{H_f} u, v\rangle_{b_k} \, d\sigma - \int_U \langle u, i\nabla^{\pi^*\mathcal{T}_kM}_{H_f} v\rangle_{b_k} \, d\sigma = \int_U \langle u, R_k v\rangle_{b_k} \, d\sigma,$$

$u, v \in \mathcal{C}^\infty_{\mathrm{c}}(U, \pi^*\mathcal{T}_kM)$, for $R_k \in \mathrm{End}(\pi^*\mathcal{T}_kM)$ satisfying $\sup_U \|R_k\|_{b_k} \leq k\epsilon$.

Proof. We show this for $k = 2$, the proof for general k being entirely analogous. Denote $S = i\nabla^{\pi^*T^*M}_{H_f}$, then $S_2 = i\nabla^{\pi^*\mathcal{T}_2M}_{H_f}$ acts by $S_2(u_1 \otimes u_2) = Su_1 \otimes u_2 + u_1 \otimes Su_2$. Hence using $S(au) = aSu + iH_f(a)u$ for sections u of π^*T^*M and functions a on U, we calculate

$$\begin{aligned}
\int_U \langle S_2(u_1 \otimes u_2), v_1 \otimes v_2\rangle_{b_2} \, d\sigma &= \int_U \langle Su_1, v_1\rangle_b \langle u_2, v_2\rangle_b + \langle u_1, v_1\rangle_b \langle Su_2, v_2\rangle_b \, d\sigma \\
&= \int_U \Big\langle u_1, S\big(v_1\langle u_2, v_2\rangle_b\big)\Big\rangle_b + \int_U \Big\langle u_2, S\big(v_2\langle u_1, v_1\rangle_b\big)\Big\rangle_b \, d\sigma \\
&\quad + \int_U \langle u_1 \otimes u_2, (R \otimes \mathrm{id} + \mathrm{id} \otimes R)(v_1 \otimes v_2)\rangle_{b_2} \, d\sigma \\
&= \int_U \langle u_1 \otimes u_2, S_2(v_1 \otimes v_2)\rangle_{b_2} \, d\sigma - i\int_U H_f(\langle u_1, v_1\rangle_b \langle u_2, v_2\rangle_b) \, d\sigma
\end{aligned}$$

$$
\begin{aligned}
&+ \int_U \langle u_1 \otimes u_2, R_2(v_1 \otimes v_2)\rangle_{b_2}\, d\sigma \\
&= \int_U \langle u_1 \otimes u_2, S_2(v_1 \otimes v_2)\rangle_{b_2}\, d\sigma + \int_U \langle u_1 \otimes u_2, R_2(v_1 \otimes v_2)\rangle_{b_2}\, d\sigma
\end{aligned}
$$

with $R_2 = R \otimes \mathrm{id} + \mathrm{id} \otimes R$, where we used that $\int_U H_f u\, d\sigma = -\int_U u H_f 1\, d\sigma = 0$ for $u \in \mathcal{C}^\infty_c(U)$. From the explicit form of R_2, we see that $\|R_2\|_{b_2} \leq 2\epsilon$ indeed. □

6.4.1 Warped product spacetimes

Let X be an $(n-1)$-dimensional manifold equipped with a smooth Riemannian metric $h = h(x, dx)$, and let $\alpha \in \mathcal{C}^\infty(X)$ be a positive function. We consider the manifold $M = \mathbb{R}_t \times X$, equipped with the Lorentzian metric

$$g = \alpha^2\, dt^2 - h. \tag{6.4.3}$$

On such a spacetime, we have a natural splitting of 1-forms into their tangential and normal part relative to $\alpha\, dt$, i.e.

$$u = u_T + u_N \alpha\, dt. \tag{6.4.4}$$

In this section, we will compute the form of $\nabla^{\pi^* T^* M}_{H_G}$ as a 2×2 matrix of differential operators with respect to this decomposition. For brevity, we will use the notation $\widetilde{\nabla}^M := \nabla^{\pi^* T^* M}$, similarly $\widetilde{\nabla}^X := \nabla^{\pi^* T^* X}$, and we will moreover use the abstract index notation, fixing $x^0 = t$, and $x' = (x^1, \ldots, x^{n-1})$ are coordinates on X (independent of t). We let Greek indices $\mu, \nu, \lambda, \ldots$ run from 0 to $n-1$, Latin indices $i, j, k, \ldots$ from 1 to $n-1$. Moreover, the canonical dual variables[26] $\xi_0 =: \sigma$ and $\xi' = (\xi_1, \ldots, \xi_{n-1})$ on the fibers of T^*M are indexed by decorated Greek indices $\widetilde{\mu}$ (running from 0 to $n-1$) and Latin indices $\widetilde{i}, \widetilde{j}, \ldots$ (running from 1 to $n-1$). If an index appears both with and without tilde in one expression, it is summed accordingly, for instance $a_j b_{\widetilde{j}} = \sum_{j=1}^n a_j b_{\widetilde{j}}$. Thus, for a section u of $\pi^* T^* M$, we have

$$\widetilde{\nabla}^M_\mu u_\nu = \nabla^M_\mu u_\nu, \quad \widetilde{\nabla}^M_{\widetilde{\mu}} u_\nu = \partial_{\widetilde{\mu}} u_\nu,$$

where we interpret ∇^M_μ as acting on u for fixed values of the fiber variables, i.e. viewing u as a family of sections of T^*M depending on the fiber variables. As before, we denote by G the

[26] Thus, once we discuss Schwarzschild-de Sitter space in the next section, in the region where $t_* = t$ (which we can in particular arrange near the trapped set), σ in the present notation is equal to $-\sigma$ in the notation of §6.2.

metric function on T^*M, and we let H denote the metric function on T^*X, interpreted as a (t,σ)-independent function on T^*M. Lastly, we denote the Christoffel symbols of (M,g) by ${}^M\Gamma^\kappa_{\mu\nu}$, and those of (X,h) by ${}^X\Gamma^k_{ij}$.

Lemma 6.4.3. *The Christoffel symbols of M are given by:*

$$\begin{aligned} {}^M\Gamma^0_{00} = 0, \quad {}^M\Gamma^0_{i0} = \alpha^{-1}\alpha_i, \quad {}^M\Gamma^0_{ij} = 0, \\ {}^M\Gamma^k_{00} = \alpha h^{k\ell}\alpha_\ell, \quad {}^M\Gamma^k_{i0} = 0, \quad {}^M\Gamma^k_{ij} = {}^X\Gamma^k_{ij}. \end{aligned} \tag{6.4.5}$$

Proof. We have $g_{00} = \alpha^2$, $g_{0i} = g_{i0} = 0$ and $g_{ij} = -h_{ij}$, and g is t-independent, thus $\partial_0 g_{\mu\nu} = 0$. Using ${}^M\Gamma_{\kappa\mu\nu} = \frac{1}{2}(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu})$, we then compute

$$\begin{aligned} {}^M\Gamma_{000} = 0, \quad {}^M\Gamma_{0i0} = \alpha\alpha_i, \quad {}^M\Gamma_{0ij} = 0, \\ {}^M\Gamma_{k00} = -\alpha\alpha_k, \quad {}^M\Gamma_{ki0} = 0, \quad {}^M\Gamma_{kij} = -{}^X\Gamma_{kij}, \end{aligned}$$

which immediately gives (6.4.5). □

Proposition 6.4.4. *For the metric g as in (6.4.3), the subprincipal operator of $\Box_1$ (the tensor wave operator acting on 1-forms on M) in the decomposition (6.4.4) of 1-forms is given by*

$$\begin{aligned} & iS_{\mathrm{sub}}(\Box_1)(t,x',\sigma,\xi') \\ &= \begin{pmatrix} 2\alpha^{-2}\sigma\partial_t + \sigma^2\widetilde{\nabla}^X_{H_{\alpha^{-2}}} - \widetilde{\nabla}^X_{H_H} & -2\alpha^{-2}\sigma\, d\alpha \\ -2\alpha^{-2}\sigma i_{\nabla^X\alpha} & 2\alpha^{-2}\sigma\partial_t + \sigma^2 H_{\alpha^{-2}} - H_H \end{pmatrix}. \end{aligned}$$

Proof. We start by computing the form of $\widetilde{\nabla}^M_\mu u_\nu$ and $\widetilde{\nabla}^M_{\widetilde{\mu}} u_\nu$ for tangential and normal 1-forms. For tangential forms $u = u_\mu\, dx^\mu$ with $u_0 = 0$, we have

$$\begin{aligned} \widetilde{\nabla}^M_0 u_0 = -{}^M\Gamma^\lambda_{00}u_\lambda = -\alpha\langle d\alpha, u\rangle_H, \quad \widetilde{\nabla}^M_0 u_i = \partial_0 u_i, \\ \widetilde{\nabla}^M_j u_0 = 0, \quad \widetilde{\nabla}^M_j u_i = \nabla^X_j u_i, \quad \widetilde{\nabla}^M_{\widetilde{\mu}} u_0 = 0, \quad \widetilde{\nabla}^M_{\widetilde{\mu}} u_i = \partial_{\widetilde{\mu}} u_i, \end{aligned}$$

while for normal forms $u = u_\mu\, dx^\mu$ with $u_i = 0$ and $u_0 = \alpha v$, we compute

$$\begin{aligned} \widetilde{\nabla}^M_0 u_0 = \alpha\partial_t v, \quad \widetilde{\nabla}^M_0 u_i = -\alpha_i v, \\ \widetilde{\nabla}^M_j u_0 = \partial_j(\alpha v) - \alpha_j v = \alpha\partial_j v, \quad \widetilde{\nabla}^M_j u_i = 0, \quad \widetilde{\nabla}^M_{\widetilde{\mu}} u_0 = \alpha\partial_{\widetilde{\mu}} v, \quad \widetilde{\nabla}^M_{\widetilde{\mu}} u_i = 0. \end{aligned}$$

Since $G = \alpha^{-2}\sigma^2 - H$, we find $H_G = 2\alpha^{-2}\sigma\partial_t + \sigma^{-2}H_{\alpha^{-2}} - H_H$. Using $\langle d\alpha, \cdot\rangle_H = i_{\nabla^X\alpha}$, we obtain

$$\widetilde{\nabla}^M_{\partial_t} = \begin{pmatrix} \partial_t & -d\alpha \\ -i_{\nabla^X\alpha} & \partial_t \end{pmatrix}.$$

Moreover, for any $f \in \mathcal{C}^\infty(T^*X)$ (we will take $f = \alpha^{-2}$ and $f = H$), viewed as a (t,σ)-independent function on T^*M, we have $H_f = f_{\tilde{j}}\partial_j - f_j\partial_{\tilde{j}}$. Hence on tangential forms,

$$\widetilde{\nabla}^M_{H_f}u_0 = 0, \quad \widetilde{\nabla}^M_{H_f}u_i = f_{\tilde{j}}\nabla^X_j u_i - f_j\partial_{\tilde{j}}u_i = \widetilde{\nabla}^X_{H_f}u_i,$$

while on normal forms as above,

$$\widetilde{\nabla}^M_{H_f}u_0 = \alpha f_{\tilde{j}}\partial_j v - \alpha f_j\partial_{\tilde{j}}v = \alpha H_f v, \quad \widetilde{\nabla}^M_{H_f}u_i = 0.$$

Thus,

$$\widetilde{\nabla}^M_{H_f} = \begin{pmatrix} \widetilde{\nabla}^X_{H_f} & 0 \\ 0 & H_f \end{pmatrix}.$$

The claim follows. □

6.4.2 Schwarzschild-de Sitter space

We stay in the setting of the previous section, and now the spatial metric h has a decomposition

$$h = \alpha^{-2}\,dr^2 + r^2\,d\omega^2,$$

where $d\omega^2$ is the round metric on the unit sphere $Y = \mathbb{S}^{n-2}$, with dual metric denoted Ω; see (2.3.1). Thus, writing ξ, resp. η, for the dual variables of r, resp. $\omega \in \mathbb{S}^{n-2}$, we have $H = \alpha^2\xi^2 + r^{-2}|\eta|^2_\Omega$. Write 1-forms on X as

$$u = u_T + u_N\alpha^{-1}\,dr. \tag{6.4.6}$$

Abbreviate the derivative of a function f with respect to r by f'. Since $d\alpha = \alpha'\,dr$ and $\nabla^X\alpha = \alpha^2\alpha'\partial_r$, we have, in the decomposition (6.4.6),

$$d\alpha = \begin{pmatrix} 0 \\ \alpha\alpha' \end{pmatrix}, \quad i_{\nabla^X\alpha} = \begin{pmatrix} 0 & \alpha\alpha' \end{pmatrix}.$$

We will need the Christoffel symbols of h. We continue using the notation to the previous section, except now $x^1 = r$ and $\xi_1 = \xi$, while $x^2, \ldots, x^n$ are r-independent coordinates on $\mathbb{S}^{n-2}$, and moreover the lower bound for Greek indices is 1, and 2 for Latin indices.

Lemma 6.4.5. *The Christoffel symbols of X are given by:*

$$\begin{gathered} {}^X\Gamma_{11}^1 = -\alpha^{-1}\alpha', \quad {}^X\Gamma_{i1}^1 = 0, \quad {}^X\Gamma_{ij}^1 = -r\alpha^2(d\omega^2)_{ij}, \\ {}^X\Gamma_{11}^k = 0, \quad {}^X\Gamma_{i1}^k = r^{-1}\delta_i^k, \quad {}^X\Gamma_{ij}^k = {}^Y\Gamma_{ij}^k. \end{gathered} \tag{6.4.7}$$

Proof. We have $h_{11} = \alpha^{-2}$, $h_{1i} = h_{i1} = 0$ and $h_{ij} = r^2(d\omega^2)_{ij}$, and $(d\omega^2)_{ij}$ is r-independent. We then compute

$$\begin{gathered} {}^X\Gamma_{111} = -\alpha^{-3}\alpha', \quad {}^X\Gamma_{1i1} = 0, \quad {}^X\Gamma_{1ij} = -r(d\omega^2)_{ij}, \\ {}^X\Gamma_{k11} = 0, \quad {}^X\Gamma_{ki1} = r(d\omega^2)_{ki}, \quad {}^X\Gamma_{kij} = r^{2Y}\Gamma_{kij}, \end{gathered}$$

which immediately gives (6.4.7). □

We are only interested in the subprincipal operator of $\Box_1$ at the trapped set, which we recall from (6.2.3) to be the set

$$\Gamma = \{r = r_p, \xi = 0, \sigma^2 = \Psi^2|\eta|^2\}, \quad \text{where } \Psi = \alpha r^{-1}, \Psi'(r_p) = 0. \tag{6.4.8}$$

Thus, at Γ, we have

$$H_H = 2\alpha^2\xi\partial_r - 2\alpha\alpha'\xi^2\partial_\xi + 2r^{-3}|\eta|^2\partial_\xi + r^{-2}H_{|\eta|^2} = 2r^{-3}|\eta|^2\partial_\xi + r^{-2}H_{|\eta|^2},$$

while $\sigma^2 H_{\alpha^{-2}} = 2\sigma^2\alpha^{-3}\alpha'\partial_\xi$. Now $\alpha^{-1}\alpha' = (r\Psi)^{-1}(r\Psi)' = r^{-1}$ at $r = r_p$, therefore $\sigma^2\alpha^{-3}\alpha' = r^{-3}|\eta|^2$, and we thus obtain

$$\sigma^2 H_{\alpha^{-2}} - H_H = -r^{-2}H_{|\eta|^2} \text{ at } \Gamma. \tag{6.4.9}$$

Notice that $|\eta|^2 \in \mathcal{C}^\infty(T^*Y)$ is independent of (r, ξ).

Lemma 6.4.6. *For a function $f \in \mathcal{C}^\infty(T^*Y)$, viewed as an (r,ξ)-independent function on X, we have*

$$\widetilde{\nabla}_{H_f}^X = \begin{pmatrix} \widetilde{\nabla}_{H_f}^Y & \alpha r(i_{H_f}d\omega^2) \\ -\alpha r^{-1}i_{H_f} & H_f \end{pmatrix}.$$

in the decomposition (6.4.6) *of 1-forms on* X*.*

Proof. On tangential forms u, i.e. $u_1 = 0$, we have

$$\widetilde{\nabla}^X_j u_1 = -r^{-1}u_j, \quad \widetilde{\nabla}^X_j u_i = \nabla^Y_j u_i, \quad \widetilde{\nabla}^X_{\tilde{j}} u_1 = 0, \quad \widetilde{\nabla}^X_{\tilde{j}} u_i = \partial_{\tilde{j}} u_i,$$

thus using $H_f = f_{\tilde{j}}\partial_j - f_j\partial_{\tilde{j}}$, we get, using that π^*T^*X can be canonically identified with the horizontal subbundle of $T^*(T^*X)$:

$$\widetilde{\nabla}^X_{H_f} u_1 = -r^{-1}f_{\tilde{j}}u_j = -r^{-1}u(H_f) = -r^{-1}i_{H_f}u, \quad \widetilde{\nabla}^X_{H_f} u_i = \widetilde{\nabla}^Y_{H_f} u_i.$$

On normal forms u, i.e. $u_1 = \alpha^{-1}v$, $u_i = 0$, we compute

$$\widetilde{\nabla}^X_j u_1 = \alpha^{-1}\partial_j v, \quad \widetilde{\nabla}^X_j u_i = r\alpha(d\omega^2)_{ij}v, \quad \widetilde{\nabla}^X_{\tilde{j}} u_1 = \alpha^{-1}\partial_{\tilde{j}} v, \quad \widetilde{\nabla}^X_{\tilde{j}} u_i = 0,$$

hence

$$\begin{aligned}\widetilde{\nabla}^X_{H_f} u_1 &= \alpha^{-1}f_{\tilde{j}}\partial_j v - \alpha^{-1}f_j\partial_{\tilde{j}} v = \alpha^{-1}H_f v,\\ \widetilde{\nabla}^X_{H_f} u_i &= f_{\tilde{j}}r\alpha(d\omega^2)_{ij}v = \alpha r(i_{H_f}d\omega^2)v.\end{aligned}$$

The claim follows immediately. □

Combining Proposition 6.4.4 and Lemma 6.4.6, we can thus compute the subprincipal operator of $\Box_1$ acting on 1-forms (sections of the pullback of T^*M to $T^*M \setminus o$) decomposed as

$$u = u_{TT} + u_{TN}\alpha^{-1}\,dr + u_N\alpha\,dt. \tag{6.4.10}$$

In view of (6.4.9), we merely need to apply Lemma 6.4.6 to $f = |\eta|^2$, in which case $H_f = 2\Omega^{jk}\eta_j\partial_k - \partial_\ell\Omega^{jk}\eta_j\eta_k\partial_{\tilde{\ell}}$, so $i_{H_f} = 2i_\eta$ on 1-forms (identifying the 1-form η with a tangent vector using the metric $d\omega^2$), while $i_{H_f}d\omega^2 = 2\eta$. Thus, we obtain:

Proposition 6.4.7. *In the decomposition* (6.4.10)*, the subprincipal operator of* $\Box_1$ *on*

Schwarzschild-de Sitter space at the trapped set Γ *is given by*

$$iS_{\mathrm{sub}}(\Box_1) = \begin{pmatrix} 2\alpha^{-2}\sigma\partial_t - r^{-2}\widetilde{\nabla}^Y_{H_{|\eta|^2}} & -2\alpha r^{-1}\eta & 0 \\ 2\alpha r^{-3} i_\eta & 2\alpha^{-2}\sigma\partial_t - r^{-2}H_{|\eta|^2} & -2r^{-1}\sigma \\ 0 & -2r^{-1}\sigma & 2\alpha^{-2}\sigma\partial_t - r^{-2}H_{|\eta|^2} \end{pmatrix}. \tag{6.4.11}$$

Since $\Box_1$ is symmetric with respect to the natural inner product G on the 1-form bundle, which in the decomposition (6.4.10) is an orthogonal direct sum of inner products, $G = (-r^{-2}\Omega) \oplus (-1) \oplus 1$, the operator $S_{\mathrm{sub}}(\Box_1)$ is a symmetric operator acting on sections of π^*T^*M over $T^*M \setminus o$ if we equip π^*T^*M with the fiber inner product G and use the symplectic volume density on $T^*M \setminus o$.

The matrix $-2r^{-2}s$, with

$$s = \begin{pmatrix} 0 & \Psi r^2 \eta & 0 \\ -\Psi i_\eta & 0 & r\sigma \\ 0 & r\sigma & 0 \end{pmatrix},$$

of 0-th order terms of $S_{\mathrm{sub}}(\Box_1)$ is nilpotent, which suggests in analogy to the discussion in §6.3.5 that the imaginary part of $S_{\mathrm{sub}}(\Box_1)$ with respect to a *Riemannian* fiber inner product can be made arbitrarily small. Indeed, for any fixed $\epsilon > 0$, define the 'change of basis matrix'

$$q = \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \epsilon^{-1}\Psi r^2 & 0 \\ -\epsilon^{-2}|\eta|^{-1}\Psi^2 r^2 i_\eta & 0 & \epsilon^{-2}|\eta|^{-1}\Psi r^3\sigma \end{pmatrix},$$

then

$$qsq^{-1} = \begin{pmatrix} 0 & \epsilon\eta & 0 \\ 0 & 0 & \epsilon|\eta| \\ 0 & 0 & 0 \end{pmatrix}.$$

In order to compute $qS_{\mathrm{sub}}(\Box_1)q^{-1}$, we note that the diagonal matrix of t-derivatives in (6.4.11) commutes with q, and it remains to study the derivatives along $H_{|\eta|^2}$; more specifically, q has a block structure, with the columns and rows $1,3$ being the first block and the $(2,2)$ entry the second, and the $(2,2)$ block is an η-independent multiple of the identity, hence commutes with the relevant $(2,2)$ entry $ir^{-2}H_{|\eta|^2}$ of $S_{\mathrm{sub}}(\Box_1)$. For the $1,3$ block, we compute

$$\begin{aligned}&\left[\begin{pmatrix}\widetilde{\nabla}^Y_{H_{|\eta|^2}} & 0\\ 0 & H_{|\eta|^2}\end{pmatrix}, \begin{pmatrix}\mathrm{id} & 0\\ -\epsilon^{-2}|\eta|^{-1}\Psi^2 r^2 i_\eta & \epsilon^{-2}|\eta|^{-1}\Psi r^3\sigma\end{pmatrix}\right]\\ &\qquad = \epsilon^{-2}\Psi^2 r^2|\eta|^{-1}\begin{pmatrix}0 & 0\\ i_\eta\widetilde{\nabla}^Y_{H_{|\eta|^2}} - H_{|\eta|^2} i_\eta & 0\end{pmatrix}.\end{aligned} \tag{6.4.12}$$

Now $\widetilde{\nabla}^Y_{H_{|\eta|^2}}$ and $H_{|\eta|^2}$ are the restrictions of the pullback connection $\nabla^{\pi^*\Lambda\mathbb{S}^{n-2}}_{H_{|\eta|^2}}$ of the full form bundle to 1-forms and functions, respectively, and the latter commutes with i_η, since by Proposition 6.3.11,

$$0 = S_{\mathrm{sub}}([\Box,\delta]) = -i[S_{\mathrm{sub}}(\Box), i_\eta] = -\big[\nabla^{\pi^*\Lambda\mathbb{S}^{n-2}}_{H_{|\eta|^2}}, i_\eta\big],$$

where $\Box$ denotes the Hodge d'Alembertian on the form bundle and δ is the codifferential. Thus, (6.4.12) in fact vanishes, and therefore

$$\begin{aligned}&qS_{\mathrm{sub}}(\Box_1)q^{-1}\\ &\quad = -i\begin{pmatrix}2\alpha^{-2}\sigma\partial_t - r^{-2}\widetilde{\nabla}^Y_{H_{|\eta|^2}} & -2r^2\epsilon\eta & 0\\ 0 & 2\alpha^{-2}\sigma\partial_t - r^{-2}H_{|\eta|^2} & -2r^2\epsilon|\eta|\\ 0 & 0 & 2\alpha^{-2}\sigma\partial_t - r^{-2}H_{|\eta|^2}\end{pmatrix}.\end{aligned}$$

Equip the 1-form bundle over M in the decomposition (6.4.10) with the Hermitian inner product

$$B_0 = \Omega\oplus 1\oplus 1, \tag{6.4.13}$$

then $qS_{\mathrm{sub}}(\Box_1)q^{-1}$ has imaginary part (with respect to B_0) of size $\mathcal{O}(\epsilon)$. Put differently, $S_{\mathrm{sub}}(\Box_1)$ has imaginary part of size $\mathcal{O}(\epsilon)$ relative to the Hermitian inner product $b := B_0(q\cdot, q\cdot)$, which is the symbol of a pseudodifferential inner product on π^*T^*M. We can now invoke Lemma 6.4.2 on a neighborhood of $\Gamma\cap\{|\sigma|=1\}$ and use the homogeneity of q, b and $S_{\mathrm{sub}}(\Box_1)$ to obtain:

Theorem 6.4.8. *For any $\epsilon>0$, there exists a (positive definite) t_*-independent pseudodifferential inner product $B = b(x,D)$ on $\mathcal{T}_kM$ (thus, b is an inner product on $\pi^*\mathcal{T}_kM$,* *homogeneous of degree* 0 *with respect to dilations in the base* $T^*M \setminus o$*), such that*

$$\sup_\Gamma |\sigma|^{-1} \left\| \frac{1}{2i}(S_{\mathrm{sub}}(\Box_k) - S_{\mathrm{sub}}(\Box_k)^{*b}) \right\|_b \leq \epsilon,$$

where Γ *is the trapped set* (6.4.8)*. Put differently, there is an elliptic ps.d.o.* Q*, invariant under* t_**-translations, acting on sections of* $\mathcal{T}_k M$*, with parametrix* Q^-*, such that relative to the ordinary positive definite inner product* (6.4.13)*, we have*

$$\sup_\Gamma |\sigma|^{-1} \left\| \sigma_1 \left(\frac{1}{2i}(Q\Box_k Q^- - (Q\Box_k Q^-)^{*B_0}) \right) \right\|_{B_0} \leq \epsilon.$$

By restriction, the analogous statements are true for $\Box$ *acting on subbundles of the tensor bundle on* M*, for instance differential forms of all degrees and symmetric 2-tensors.*

By the t_*-translation invariance of the involved symbols, inner products and operators, this is really a statement about Ψ_{b}-inner products, and Q is a b-pseudodifferential operator; see the discussion preceding Theorem 6.2.1 for the relationship of the stationary and the b-picture.

Remark 6.4.9. Adding a 0-th order term to $\Box$ does not change $\Box$ or its imaginary part at the principal symbol level, thus does not affect the subprincipal operator of $\Box$ either; therefore, Theorem 6.4.8 holds in this case as well.

Adding a first order operator L (acting on sections of $\mathcal{T}_k M$), which we assume to be t-independent for simplicity, does affect the subprincipal operator, more specifically its 0-th order part, since $S_{\mathrm{sub}}(\Box + L) = S_{\mathrm{sub}}(\Box) + \sigma_1(L)$. Thus, if $\sigma_1(L)$ is small at Γ, we can use the same Ψ-inner product as for $\Box$ and obtain a bound on $\mathrm{Im}^b\, S_{\mathrm{sub}}(\Box + L)$ which is small, but no longer arbitrarily small. However, the bound merely needs to be smaller than $\nu_{\mathrm{min}}/2$, see (6.2.2), which does hold for small L.

If we do not restrict the size of L, we can still obtain a spectral gap, provided one can choose a Ψ-inner product as in Theorem 6.4.8, again with $\epsilon > 0$ sufficiently (but not necessarily arbitrarily) small. This is the case if the 0-th order part of $S_{\mathrm{sub}}(\Box + L)$ is nilpotent (or has small eigenvalues) and can be conjugated in a t-independent manner to an operator which is sufficiently close to being symmetric, in the sense that it satisfies the bound (6.2.2) with $\Box$ replaced by $\Box + L$.

We remark that the subprincipal operator $iS_{\mathrm{sub}}(\Box) = H_G + i\sigma_{\mathrm{sub}}(G)$ induces a notion of parallel transport on $\pi^*\mathcal{T}_k M$ along the Hamilton flow of H_G. As a consequence of the nilpotent structure of $S_{\mathrm{sub}}(\Box)$ at the trapped set, parallel sections along the trapped set grow only polynomially in size (with respect to a fixed t-invariant positive definite inner product), rather than exponentially. Parallel sections as induced by $S_{\mathrm{sub}}(\Box + L)$, with L as in Remark 6.4.9, may grow exponentially, with their size bounded by $Ce^{\kappa|\sigma|t}$ for some constants $C > 0$ and κ, where the additional factor of $|\sigma|$ in the exponent accounts for the homogeneity of the parallel transport. If such a bound does not hold for any $\kappa < \nu_{\min}/2$, the dispersion of waves concentrated at the trapped set caused by the normally hyperbolic nature of the trapping is expected to be too weak to counteract the exponential growth caused by the subprincipal part of $\Box + L$, and correspondingly one does not expect a spectral gap. Notice that the growth of parallel sections is an averaged condition in that it involves the behavior of the parallel transport for large times, while the choice of Ψ-inner products as explained above is a local condition and depends on the pointwise structure of $S_{\mathrm{sub}}(\Box)$.

Chapter 7

Resonances for differential forms

7.1 Introduction

Maxwell's equations describe the dynamics of the electromagnetic field on a 4-dimensional spacetime (M, g). Writing them in the form $(d + \delta_g)F = 0$, where δ_g is the codifferential, for the electromagnetic field F (a 2-form) suggests studying the operator $d + \delta_g$, whose square $\Box_g = (d + \delta_g)^2$ is the Hodge d'Alembertian, i.e. the wave operator on differential forms. It is then very natural to study solutions of $(d + \delta_g)u = 0$ or $\Box_g u = 0$ without restrictions on the form degree. Important examples of spacetimes that fit into the class of spacetimes studied in the present chapter are Schwarzschild-de Sitter spacetimes with spacetime dimension $n \geq 4$, and by very simple perturbation arguments, we can readily analyze waves on perturbations of these, in particular on Kerr-de Sitter spaces. Concretely, a special case of our general results is:

Theorem 7.1.1. *Let (M, g_a) denote a non-degenerate Kerr-de Sitter space with black hole mass $M_\bullet > 0$, cosmological constant $\Lambda > 0$ and angular momentum a, see §2.4, more precisely a suitable neighborhood Ω of the domain of outer communications as in (2.3.9), and denote by t_* a smooth time coordinate. Suppose $u \in \mathcal{C}^\infty(M; \Lambda M)$ is a solution of the equation*

$$(d + \delta_{g_a})u = 0,$$

with smooth initial data, and denote by u_j the form degree j part of u, $j = 0, \ldots, 4$. Then u_2 decays exponentially in t_* to a stationary state, which is a linear combination of the t_*-independent 2-forms $u_{a,1}, u_{a,2}$. In the standard (Boyer-Lindquist) local coordinate system *on Kerr-de Sitter space, $u_{a,1}$ and $u_{a,2}$ have explicit closed form expressions; in particular, on Schwarzschild-de Sitter space, $u_{0,1} = r^{-2}\,dt \wedge dr$, and $u_{0,2} = \omega$ is the volume element of the round unit 2-sphere. Moreover, u_1 and u_3 decay exponentially to 0, while u_0 decays exponentially to a constant, and u_4 to a constant multiple of the volume form.*

Suppose now $u \in \mathcal{C}^\infty(M; \Lambda M)$ instead solves the wave equation

$$\Box_{g_a} u = 0$$

with smooth initial data, then the same decay as before holds for u_0, u_2 and u_4, while u_1 decays exponentially to a member of a 2-dimensional family of stationary states, likewise for u_3.

The Schwarzschild-de Sitter case of this theorem, i.e. the special case $a = 0$, will be proved in §7.4.2, and we give explicit expressions for all stationary states, see Theorems 7.4.3 and 7.4.5, and §7.5 provides the perturbation arguments, see in particular Theorem 7.5.1. For the explicit form of $u_{a,1}$ and $u_{a,2}$, see Remark 7.5.4. Notice that asymptotics and decay of differential form solutions to the wave equation are much stronger statements than corresponding statements for Maxwell's equations or for the Hodge-de Rham equation.

We stress that the main feature of the spacetimes (M, g) considered in this chapter is a warped product type structure of the metric, whereas *we do not make any symmetry assumptions on M*. From a geometric point of view then, the main result of this chapter is a general cohomological interpretation of stationary states, which in the above theorem are merely explicitly given. On a technical level, we show how to explicitly analyze quasinormal modes (or resonances) for equations on vector bundles whose natural inner product is not positive definite, which is somewhat complementary to the high frequency analysis in Chapter 6. To stress the generality of the method, we point out that symmetries only become relevant in explicit calculations for specific examples such as Schwarzschild-de Sitter and Kerr-de Sitter spaces. Even then, the perturbation analysis around Schwarzschild-de Sitter space works without restrictions on the perturbation; only for the explicit form of the space $\langle u_{a,1}, u_{a,2}\rangle$ of stationary states do we need the very specific form of the Kerr-de Sitter metric. Thus, combining the perturbation analysis with the nonlinear framework developed in §9, we can immediately solve suitable *quasilinear wave equations on differential forms* on Kerr-de Sitter spacetimes; see Remark 7.5.3. To put this into context, part of the motivation for the present chapter again is the black hole stability problem, and we expect that the approach taken here will facilitate the linear part of the stability analysis, which, when accomplished, rather directly gives the nonlinear result when combined with the nonlinear analysis presented in Chapters 8 and 9.

7.1.1 Outline of the general result

Going back to the linear problem studied here, we proceed to explain the general setup in more detail; one should keep Schwarzschild-de Sitter space, as presented in §2.3, as the main example in mind.

Remark 7.1.2. Notationally, M_S°, X_S° in §2.3 correspond to M and X in the present chapter, whereas the extended manifolds are called $\widetilde{M}$ and $\widetilde{X}$ here: This is in order to emphasize the role of M and X (and of the warped product metric on them), while the analysis on the extended spaces $\widetilde{M}$ and $\widetilde{X}$, even though it plays a central role in the setup, is somewhat secondary for our analysis here.

Thus, let $\overline{X}$ be a connected, compact, orientable $(n-1)$-dimensional manifold with non-empty boundary $Y = \partial\overline{X} \neq \emptyset$ and interior $X = \overline{X}^\circ$, and let $M = \mathbb{R}_t \times X$, which is thus n-dimensional. Denote the connected components of Y, which are of dimension $(n-2)$, by Y_i, for i in a finite index set I. We assume that M is equipped with the metric

$$g = \alpha(x)^2\, dt^2 - h(x, dx), \tag{7.1.1}$$

where h is a smooth Riemannian metric on $\overline{X}$ (in particular, incomplete) and α is a boundary defining function of X, i.e. $\alpha \in \mathcal{C}^\infty(\overline{X})$, $\alpha = 0$ on Y, $\alpha > 0$ in X and $d\alpha|_Y \neq 0$. We moreover assume that every connected component Y_i of Y, $i \in I$, has a collar neighborhood $[0, \epsilon_i)_\alpha \times (Y_i)_y$ in which h takes the form

$$h = \widetilde{\beta}_i(\alpha^2, y)\, d\alpha^2 + k_i(\alpha^2, y, dy) \tag{7.1.2}$$

with $\widetilde{\beta}_i(0, y) \equiv \beta_i > 0$ constant along Y_i. In particular, h is an *even* metric in the sense of Guillarmou [57]. Thus, de Sitter and Schwarzschild-de Sitter spaces fit into this framework, whereas asymptotically flat spacetimes like Schwarzschild (or Kerr) do not. We change the smooth structure on $\overline{X}$ to only include even functions of α, and show how one can then extend the metric g to a stationary metric (denoted $\widetilde{g}$, but dropped from the notation in the sequel) on a bigger spacetime $\widetilde{M} = \mathbb{R}_{t_*} \times \widetilde{X}$, where t_* is a shifted time coordinate. Since the operator $d+\delta$ commutes with time translations, it is natural to consider the normal operator family

$$\widetilde{\delta}(\sigma)+\widetilde{\delta}(\sigma)=e^{it_*\sigma}(d+\delta)e^{-it_*\sigma}$$

acting on differential forms (valued in the form bundle of M) on a slice of constant t_*, identified with $\widetilde{X}$; the normal operator family $\widetilde{\Box}(\sigma)$ of $\Box$ is defined completely analogously. As discussed before, see in particular §3.3.3, the proper way to view the normal operator family is as a family of operators on the boundary at infinity of a bordified version of $\widetilde{M}$, where one introduces $\tau=e^{-t_*}$ and adds $\tau=0$, i.e. future infinity, to the manifold $\widetilde{M}$.

Since the Hodge d'Alembertian (and hence the normal operator family $\widetilde{\Box}(\sigma)$) has a scalar principal symbol, it can easily be shown to fit into the microlocal framework developed by Vasy [114]; we prove this in §7.2. In particular, $\widetilde{\Box}(\sigma)^{-1}$ is a meromorphic family of operators in $\sigma\in\mathbb{C}$, and under the assumption that the inverse family $\widetilde{\Box}(\sigma)^{-1}$ verifies suitable high energy bounds as $|\operatorname{Re}\sigma|\to\infty$ and $\operatorname{Im}\sigma>-C$ (for $C>0$ small), one can deduce exponential decay of solutions to $\Box u=0$, up to contributions from a finite dimensional space of resonances, as in Theorems 5.2.3 and 5.3.1. Thus again, proving wave decay and asymptotics is reduced to studying high energy estimates, which for the problem at hand depend purely on geometric properties of the spacetime and will be further discussed below, and the location of resonances as well as the spaces of resonant states. Our main theorem is then:

Theorem 7.1.3. *The only resonance of $d+\delta$ in $\operatorname{Im}\sigma\geq 0$ is $\sigma=0$, and 0 is a simple resonance. Zero resonant states are smooth, and the space $\widetilde{\mathcal{H}}$ of these resonant states is equal to $\ker\widetilde{d}(0)\cap\ker\widetilde{\delta}(0)$. (In other words, resonant states, viewed as t_*-independent differential forms on $\widetilde{M}$, are annihilated by d and δ.) Using the grading $\widetilde{\mathcal{H}}=\bigoplus_{k=0}^n\widetilde{\mathcal{H}}^k$ of $\widetilde{\mathcal{H}}$ by form degrees, there is a canonical exact sequence*

$$0\to H^k(\overline{X})\oplus H^{k-1}(\overline{X},\partial\overline{X})\to\widetilde{\mathcal{H}}^k\to H^{k-1}(\partial\overline{X}). \tag{7.1.3}$$

Furthermore, the only resonance of $\Box$ in $\operatorname{Im}\sigma\geq 0$ is $\sigma=0$. Zero resonant states are smooth, and the space $\widetilde{\mathcal{K}}=\bigoplus_{k=0}^n\widetilde{\mathcal{K}}^k$ of these resonant states, graded by form degree and satisfying $\widetilde{\mathcal{K}}^k\supset\widetilde{\mathcal{H}}^k$, fits into the short exact sequence

$$0\to H^k(\overline{X})\oplus H^{k-1}(\overline{X},\partial\overline{X})\to\widetilde{\mathcal{K}}^k\to H^{k-1}(\partial\overline{X})\to 0. \tag{7.1.4}$$

Lastly, the Hodge star operator on $\widetilde{M}$ induces natural isomorphisms $\star\colon \widetilde{\mathcal{H}}^k \xrightarrow{\cong} \widetilde{\mathcal{H}}^{n-k}$ and $\star\colon \widetilde{\mathcal{K}}^k \xrightarrow{\cong} \widetilde{\mathcal{K}}^{n-k}$, $k = 0, \ldots, n$.

See Theorem 7.3.20 for the full statement, including the precise definitions of the maps in the exact sequences. In fact, the various cohomology groups in (7.1.3) and (7.1.4) correspond to various types of resonant differential forms, namely forms which are square integrable on X with respect to a natural Riemannian inner product on forms on M (obtained by switching the sign in (7.1.1)), as well as 'tangential' and 'normal' forms in a decomposition $u = u_T + \alpha^{-1}\,dt \wedge u_N$ of the form bundle corresponding to the warped product structure of the metric. Roughly speaking, (7.1.4) encodes the fact that resonant states for which a certain boundary component vanishes are square integrable with respect to the natural Riemannian inner product on X and can be shown to canonically represent absolute (for tangential forms) or relative (for normal forms) de Rham cohomology of $\overline{X}$, while the aforementioned boundary component is a harmonic form on Y and can be specified freely for resonant states of $\Box$. (Notice by contrast that the last map in the exact sequence (7.1.3) for $d + \delta$ is not necessarily surjective.)

The proof of Theorem 7.1.3 proceeds in several steps. First, we exclude resonances in $\operatorname{Im}\sigma > 0$ in §7.3.1; the idea here is to relate the normal operator family of $d + \delta$ (a family of operators on the extended space $\widetilde{X}$) to another normal operator family $\widehat{d}(\sigma) + \widehat{\delta}(\sigma) = e^{it\sigma}(d + \delta)e^{-it\sigma}$, which is a family of operators on X that degenerates at $\partial\overline{X}$, but has the advantage of having a simple form in view of the warped product type structure (7.1.1) of the metric: Since one formally obtains $\widehat{d}(\sigma) + \widehat{\delta}(\sigma)$ by replacing each ∂_t in the expression for $d + \delta$ by $-i\sigma$, we see that on a formal level $\widehat{d}(\sigma) + \widehat{\delta}(\sigma)$ for purely imaginary σ resembles the normal operator family of the Hodge-de Rham operator of the Riemannian metric on M mentioned above; then one can show the triviality of $\ker(\widehat{d}(\sigma) + \widehat{\delta}(\sigma))$ in a way that is very similar to how one would show the triviality of $\ker(A + \sigma)$ for self-adjoint A and $\operatorname{Im}\sigma > 0$. For not purely imaginary σ, but still with $\operatorname{Im}\sigma > 0$, one can change the tangential part of the metric on M in (7.1.1) by a complex phase and then run a similar argument, using that the resulting 'inner product,' while complex, still has some positivity properties. Next, in §7.3.2, we exclude non-zero real resonances by means of a boundary pairing argument, which is a standard technique in scattering theory [84]. Finally, the analysis of the zero resonance in §7.3.3 relies on a boundary pairing type argument, and we again use the Riemannian inner product on forms on M. The fact that this Riemannian inner product is singular at $\partial\overline{X}$ implies that resonant states are not necessarily square integrable, and whether or not a state is square integrable is determined by the absence of a certain boundary component of the state. This is a crucial element of the cohomological interpretation of resonant states in §7.3.4.

As already alluded to, deducing wave expansions and decay from Theorem 7.1.3 requires high energy estimates for the normal operator family. These are easy to obtain if the metric h on X is non-trapping, i.e. all geodesics escape to $\partial\overline{X}$, as is the case for the static patch of de Sitter space, discussed in the present chapter in §7.4.1 and in the scalar setting in §5.2. Another instance in which suitable estimates hold is when the only trapping within X is normally hyperbolic, as is the case for Kerr-de Sitter spaces with parameters in a certain range. As discussed in Chapter 6, such estimates are now widely available in the scalar setting [42, 124]; the proof of exponential decay then relies on high energy estimates in a strip below the real line. For $\Box$ acting on differential forms, obtaining high energy estimates requires a smallness assumption on the imaginary part of the subprincipal symbol of $\Box$ relative to a *positive definite* inner product on the form bundle, and we showed how to tackle this issue by means of pseudodifferential inner products in Chapter 6 for $\Box$ on tensors of arbitrary rank on perturbations of Schwarzschild-de Sitter space.

This chapter gives the first proof of asymptotics for differential forms solving the wave or Hodge-de Rham equation in all form degrees and in this generality, and also the first to demonstrate the forward solvability of non-scalar quasilinear wave equations on black hole spacetimes; however, we point out that for applications in general relativity, our results require the cosmological constant to be positive, as discussed in Chapter 1 and §5.1, whereas previous works on Maxwell's equations deal with asymptotically flat spacetimes; see §6.1.1 for references.

We moreover remark that Vasy's proof of the meromorphy of the (modified) resolvent of the Laplacian on differential forms on asymptotically hyperbolic spaces [112] makes use of the same microlocal framework as the present chapter, and it also shows how to link the 'intrinsic' structure of the asymptotically hyperbolic space and the form of the Hodge-Laplacian with a 'non-degenerately extended' space and operator.

7.2 Analytic setup

Recall that we are working on a spacetime $M = \mathbb{R}_t \times X$, equipped with a metric g as in (7.1.1)-(7.1.2), where X is the interior of a connected, compact, orientable manifold $\overline{X}$ with non-empty boundary $Y = \partial\overline{X} \neq \emptyset$ and boundary defining function $\alpha \in \mathcal{C}^\infty(\overline{X})$. Fixing a collar neighborhood of Y identified with $[0,\epsilon)_\alpha \times Y$, denote by $\overline{X}_{\mathrm{even}}$ the manifold $\overline{X}$ with the smooth structure changed so that only even functions in α are smooth, i.e. smooth functions are precisely those for which all odd terms in the Taylor expansion at all boundary components vanish. For brevity, we assume from now on that Y is connected,

$$h = \widetilde{\beta}(\alpha^2, y)^2\,d\alpha^2 + k(\alpha^2, y, dy) \tag{7.2.1}$$

in a collar neighborhood of Y, and thus $\widetilde{\beta}(0,y) \equiv \beta$ is a single constant, but all of our arguments readily go through in the case of multiple boundary components. The main examples of spaces which directly fit into this setup are the static patch of de Sitter space (with 1 boundary component) and Schwarzschild-de Sitter space (with 2 boundary components); see §7.4 for details.

On M, we consider the Hodge-de Rham operator $d+\delta$, acting on differential forms. We put its square, the Hodge d'Alembertian $\Box = (d+\delta)^2$, which is principally scalar, into the microlocal framework developed in [114]. First, we resolve the coordinate singularity at $\alpha = 0$; proceeding as in §2.3, see in particular (2.3.5), we renormalize the time coordinate t in the collar neighborhood of Y by writing

$$t = t_* + F(\alpha), \quad \partial_\alpha F(\alpha) = -\frac{\widetilde{\beta}}{\alpha} - 2\alpha c(\alpha^2, y) \tag{7.2.2}$$

with c smooth, hence $F(\alpha) \in -\beta\log\alpha + \mathcal{C}^\infty(\overline{X}_{\mathrm{even}})$; notice that the above requirement on F only makes sense near Y. We introduce the boundary defining function $\mu = \alpha^2$ of $\overline{X}_{\mathrm{even}}$; then one computes

$$g = \mu\,dt_*^2 - (\widetilde{\beta} + 2\mu c)\,dt_*\,d\mu + (\mu c^2 + \widetilde{\beta}c)\,d\mu^2 - k(\mu, y, dy). \tag{7.2.3}$$

In particular, the determinant of g in these coordinates equals $-\frac{\widetilde{\beta}^2}{4}\det(k)$, hence g is non-degenerate up to Y. Furthermore, we claim that we can choose $c(\mu, y)$ such that dt_* is timelike on $\mathbb{R}_{t_*} \times \overline{X}_{\mathrm{even}}$; this requirement is explained below and in [114, §7], as well as in §4.2. That is, we want to arrange, with G denoting the dual metric to g, that

$$G(dt_*, dt_*) = -4\widetilde{\beta}^{-2}(\mu c^2 + \widetilde{\beta}c) > 0. \tag{7.2.4}$$

This is trivially satisfied if $c = -\widetilde{\beta}/2\mu$, which corresponds to undoing the change of coordinates in (7.2.2), however we want c to be smooth at $\mu = 0$. But for $\mu \geq 0$, (7.2.4) holds provided $-\widetilde{\beta}/\mu < c < 0$; hence, we can choose a smooth c verifying (7.2.4) in $\mu \geq 0$ and such that moreover $c = -\widetilde{\beta}/2\mu$ in $\mu \geq \mu_1$ (intersected with the collar neighborhood of Y) for any fixed $\mu_1 > 0$. Thus, we can choose F as in (7.2.2) with $F = 0$ in $\alpha^2 \geq \mu_1$ (in particular, F is defined globally on X) such that (7.2.4) holds.

Since the metric g in (7.2.3) is stationary (t_*-independent) and non-degenerate on $\overline{X}_{\text{even}}$, it can be extended to a stationary Lorentzian metric on an extension $\widetilde{X}_\delta$ into which $\overline{X}_{\text{even}}$ embeds. Concretely, one defines $\widetilde{X}_\delta = (\overline{X}_{\text{even}} \sqcup ([-\delta, \epsilon)_\mu \times Y_y))/\sim$ with the obvious smooth structure, where $\sim$ identifies elements of $[0, \epsilon)_\mu \times Y_y$ with points in $\overline{X}_{\text{even}}$ by means of the collar neighborhood of Y. Then, extending $\widetilde{\beta}$ and k, and thus g, in an arbitrary t_*-independent manner to $\widetilde{X}_\delta$, the extended metric, which we denote by $\widetilde{g}$, is non-degenerate on $\widetilde{X}_\delta$ for sufficiently small $\delta > 0$, and ∂_{t_*} remains timelike uniformly on $\mathbb{R}_{t_*} \times \widetilde{X}_\delta$: Indeed, in $\mu < 0$, (7.2.4) (with the dual metric $\widetilde{G}$ of $\widetilde{g}$ in place of G) holds for any negative function c as long as $\widetilde{\beta}$ remains positive on $\widetilde{X}_\delta$. Reducing $\delta > 0$ further if necessary (to enforce the relevant structure of the null-geodesic flow near Y within $\widetilde{X}_\delta \setminus \overline{X}_{\text{even}}$, see [114, §2]), we let $\widetilde{X}$ be the double space of $\widetilde{X}_\delta$, which is thus a compact manifold without boundary, and denote by $\widetilde{g}$ the extended metric on $\widetilde{X}$, slightly modified near $\partial\widetilde{X}_\delta$ to ensure the smoothness of $\widetilde{g}$ on the double space $\widetilde{X}$.

The operator $d + \delta_g$ on M now extends to an operator $d + \delta_{\widetilde{g}}$ on $\widetilde{M} = \mathbb{R}_{t_*} \times \widetilde{X}$. Correspondingly, the wave operator $\Box_g$ on M extends to the wave operator $\Box_{\widetilde{g}}$ on $\widetilde{M}$. Denote the normal operator family of $\Box_{\widetilde{g}}$ by $\widetilde{\Box}_{\widetilde{g}}(\sigma)$, that is to say (using e^{-t_*} as the Mellin transform variable, and dropping the subscript $\widetilde{g}$ for brevity)

$$\widetilde{\Box}(\sigma) = e^{it_*\sigma}\Box e^{-it_*\sigma};$$

since $\Box$ is invariant under translations in t_*, this amounts to replacing each ∂_{t_*} in the expression for $\Box$ by $-i\sigma$. The operator $\widetilde{\Box}(\sigma)$ acts on sections of the pullback $\Lambda_{\widetilde{X}}\widetilde{M}$ of the form bundle $\Lambda\widetilde{M}$ under the map $\widetilde{X} \to \widetilde{M}$, $\widetilde{x} \mapsto (0, \widetilde{x})$, and writing differential forms $\widetilde{u}$ on $\widetilde{M}$ as

$$\widetilde{u} = \widetilde{u}_T + dt_* \wedge \widetilde{u}_N \tag{7.2.5}$$

with $\widetilde{u}_T$ and $\widetilde{u}_N$ valued in forms on $\widetilde{X}$, we can identify $\Lambda_{\widetilde{X}}\widetilde{M}$ with $\Lambda\widetilde{X} \oplus \Lambda\widetilde{X}$.

The last step required to show that $\Box$, more precisely $\widetilde{\Box}(\sigma)$, fits into the framework described in [114] is classical non-trapping for the bicharacteristic flow of $\widetilde{\Box}(\sigma)$; complex absorption can be dealt with by the arguments of [114, §§3-4]. But in fact, we even have ellipticity in X: Indeed, on X, we have

$$\widetilde{\Box}(\sigma) = e^{-iF\sigma}e^{it\sigma}\Box e^{-it\sigma}e^{iF\sigma} = e^{-iF\sigma}\widehat{\Box}(\sigma)e^{iF\sigma}, \tag{7.2.6}$$

where $\widehat{\Box}(\sigma) = e^{it\sigma}\Box e^{-it\sigma}$ is the conjugation of $\Box$ by the Fourier transform in $-t$, and F is as in (7.2.2); here, we view $\widehat{\Box}(\sigma)$ as an operator acting on sections of $\Lambda_{\widetilde{X}}\widetilde{M}|_X$. Now, the latter bundle can by identified with $\Lambda X \oplus \Lambda X$ by writing differential forms as $u = u_T + dt \wedge u_N$, with u_T and u_N valued in forms on X, and switching between this identification and (7.2.5) amounts to conjugating $\widehat{\Box}(\sigma)$ by a bundle isomorphism on $\Lambda X \oplus \Lambda X$, which preserves ellipticity. The standard principal symbol of $\widehat{\Box}(\sigma)$ as a second order operator acting on sections of $\Lambda X \oplus \Lambda X$ is given by $(-H) \oplus (-H)$, where H is the dual metric to h, here identified with the dual metric function on T^*X; this follows from the calculations in the next section. Since H is Riemannian, this implies that $\widehat{\Box}(\sigma)$, hence $\widetilde{\Box}(\sigma)$, is classically elliptic in X, which trivially implies the non-trapping property.

Hence by [114, Theorem 7.3], $\widetilde{\Box}(\sigma)$ is an analytic family of Fredholm operators on suitable function spaces, and the inverse family $\widetilde{\Box}(\sigma)^{-1} \colon \mathcal{C}^\infty(\widetilde{X}; \Lambda\widetilde{X} \oplus \Lambda\widetilde{X}) \to \mathcal{C}^{-\infty}(\widetilde{X}; \Lambda\widetilde{X} \oplus \Lambda\widetilde{X})$ (where we use the identification (7.2.5)) admits a meromorphic continuation from $\operatorname{Im}\sigma \gg 0$ to the complex plane; note however that without further assumptions on the geodesic flow (for instance, semiclassical non-trapping or normally hyperbolic trapping), we do not obtain any high energy bounds. Moreover (see [114, Lemma 3.5]), the Laurent coefficient at the poles are finite rank operators mapping sufficiently regular distributions to elements of $\mathcal{C}^\infty(\widetilde{X}; \Lambda\widetilde{X} \oplus \Lambda\widetilde{X})$.

For present purposes, it is actually more convenient to replace complex absorption by Cauchy hypersurfaces outside of $\overline{X}_{\text{even}}$ as in Chapter 5, for instance §5.2; the above properties on $\widetilde{\Box}(\sigma)^{-1}$ hold true in this setting as well. We then deduce:

Lemma 7.2.1. *A complex number $\sigma \in \mathbb{C}$ is a resonance of $\Box$, i.e. $\widetilde{\Box}(\sigma)^{-1}$ has a pole at σ, if and only if there exists a non-zero $u \in \alpha^{-i\beta\sigma}\mathcal{C}^\infty(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}} \oplus \Lambda\overline{X}_{\text{even}})$ (using the identification (7.2.5)) such that $\widehat{\Box}(\sigma)u = 0$.*

Proof. If $\sigma \in \mathbb{C}$ is a resonance, then there exists a non-zero $\widetilde{u} \in \mathcal{C}^\infty(\widetilde{X}; \Lambda\widetilde{X} \oplus \Lambda\widetilde{X})$ with $\widetilde{\Box}(\sigma)\widetilde{u} = 0$. Restricting to $\overline{X}$, this implies by (7.2.6) and (7.2.2) that $\widehat{\Box}(\sigma)u = 0$ for $u = e^{iF\sigma}\widetilde{u}|_{\overline{X}} \in \alpha^{-i\beta\sigma}\mathcal{C}^\infty(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}} \oplus \Lambda\overline{X}_{\text{even}})$. If $u = 0$, then $\widetilde{u}$ vanishes to infinite order at Y, and since $\widetilde{\Box}(\sigma)$ is a conjugate of a wave or Klein-Gordon operator on an asymptotically de Sitter space, see [117], unique continuation at infinity on the de Sitter side as in [111, Proposition 5.3] (which is in the scalar setting, but works similarly in the present context since it relies on a semiclassical argument in which only the principal symbol of the wave operator matters, and this is the same in our setting) shows that $\widetilde{u} \equiv 0$ on $\widetilde{X}$; this is the place where we use that we capped off $\widetilde{X}$ outside of $\overline{X}_{\mathrm{even}}$ by a Cauchy hypersurface. Hence, $u \neq 0$, as desired.

Conversely, given a $u \in \alpha^{-i\beta\sigma}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda\overline{X}_{\mathrm{even}} \oplus \Lambda\overline{X}_{\mathrm{even}})$ with $\widehat{\Box}(\sigma)u = 0$, we define $\widetilde{u}' \in \mathcal{C}^\infty(\widetilde{X}; \Lambda\widetilde{X} \oplus \Lambda\widetilde{X})$ to be any smooth extension of $e^{-iF\sigma}u$ from $\overline{X}_{\mathrm{even}}$ to $\widetilde{X}$. Then $\widetilde{\Box}(\sigma)\widetilde{u}'$ is identically zero in X and thus vanishes to infinite order at Y; hence, we can solve

$$\widetilde{\Box}(\sigma)\widetilde{v} = -\widetilde{\Box}(\sigma)\widetilde{u}'$$

in $\widetilde{X} \setminus X$ with $\widetilde{v}$ vanishing to infinite order at Y; thus, extending $\widetilde{v}$ by 0 to X, we find that $\widetilde{u} = \widetilde{u}' + \widetilde{v}$ is a non-zero solution to $\widetilde{\Box}(\sigma)\widetilde{u} = 0$ on $\widetilde{X}$. □

Since $\Box = (d + \delta)^2$, we readily obtain the following analogue of Lemma 7.2.1 for $d + \delta$, dropping the bundles from the notation for simplicity:

Lemma 7.2.2. *The map* $\ker_{\mathcal{C}^\infty(\widetilde{X})}(\widetilde{d}(\sigma) + \widetilde{\delta}(\sigma)) \to \ker_{\alpha^{-i\beta\sigma}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}})}(\widehat{d}(\sigma) + \widehat{\delta}(\sigma))$, $\widetilde{u} \mapsto e^{iF\sigma}\widetilde{u}|_X$, *is an isomorphism.*

Proof. Since $\widetilde{u} \in \ker(\widetilde{d}(\sigma) + \widetilde{\delta}(\sigma))$ implies $\widetilde{u} \in \ker\widetilde{\Box}(\sigma)$, injectivity follows from the proof of Lemma 7.2.1. To show surjectivity, take $u \in e^{iF\sigma}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}})$ with $(\widehat{d}(\sigma) + \widehat{\delta}(\sigma))u = 0$ and choose any smooth extension $\widetilde{u}'$ of $e^{-iF\sigma}u$ to $\widetilde{X}$. Solving $\widetilde{\Box}(\sigma)\widetilde{v}' = -(\widetilde{d}(\sigma) + \widetilde{\delta}(\sigma))\widetilde{u}'$ with $\operatorname{supp}\widetilde{v}' \subset \widetilde{X} \setminus X$ and then defining $\widetilde{v} = (\widetilde{d}(\sigma) + \widetilde{\delta}(\sigma))\widetilde{v}'$, we see that $\widetilde{u} = \widetilde{u}' + \widetilde{v}$ extends $\widetilde{u}'$ to $\widetilde{X}$ and is annihilated by $\widetilde{d}(\sigma) + \widetilde{\delta}(\sigma)$. □

Thus, when studying the location and structure of resonances, we already have very precise information about regularity and asymptotics (on X) of potential resonant states.

Lastly, we remark that since $\widetilde{\Box}(\sigma) = (\widetilde{d}(\sigma) + \widetilde{\delta}(\sigma))^2$ is an analytic family of Fredholm operators with meromorphic inverse, the same holds for $\widetilde{d}(\sigma) + \widetilde{\delta}(\sigma)$. More precisely, $\widetilde{\Box}(\sigma)\colon \mathcal{X}^s \to \mathcal{Y}^{s-1}$ with $\mathcal{X}^s = \{u \in H^s(\widetilde{X}; \Lambda\widetilde{X} \oplus \Lambda\widetilde{X})^- \colon \widetilde{\Box}(\sigma)u \in H^{s-1}(\widetilde{X}; \Lambda\widetilde{X} \oplus \Lambda\widetilde{X})\}$ and $\mathcal{Y}^{s-1} = H^{s-1}(\widetilde{X}; \Lambda\widetilde{X} \oplus \Lambda\widetilde{X})^-$, with $(-)$ denoting extendible distributions [64, Appendix B], is Fredholm provided s is large enough (depending on $\operatorname{Im}\sigma$), and thus $\widetilde{d}(\sigma) + \widetilde{\delta}(\sigma)\colon \mathcal{Z}^s \to \mathcal{Y}^s$ with $\mathcal{Y}^s = \{u \in H^s(\widetilde{X}; \Lambda\widetilde{X} \oplus \Lambda\widetilde{X})^- \colon (\widetilde{d}(\sigma) + \widetilde{\delta}(\sigma))u \in H^s(\widetilde{X}; \Lambda\widetilde{X} \oplus \Lambda\widetilde{X})\}$ is Fredholm for the same s. In addition, $\widetilde{d}(\sigma)+\widetilde{\delta}(\sigma)$ acting on these spaces is invertible if and only if its square $\widetilde{\Box}(\sigma)$ is. In particular, $\widetilde{d}(\sigma)+\widetilde{\delta}(\sigma)$ has index 0, being an analytic family of Fredholm operators which is invertible for $\operatorname{Im}\sigma \gg 0$.

7.3 Resonances in the closed upper half plane

Using Lemma 7.2.2, we now study the resonances of in $\operatorname{Im}\sigma \geq 0$ by analyzing the operator $\widehat{d}(\sigma)+\widehat{\delta}(\sigma)$ (and related operators) on $\overline{X}_{\mathrm{even}}$. Recall that a resonance at $\sigma \in \mathbb{C}$ and a corresponding resonant state $\widetilde{u}$ yield a solution $(d+\delta)(e^{-it_*\sigma}\widetilde{u})=0$, hence $\operatorname{Im}\sigma>0$ implies in view of $|e^{-it_*\sigma}|=e^{t_*\operatorname{Im}\sigma}$ that $e^{-it_*\sigma}\widetilde{u}$ grows exponentially in t_*, whereas resonances with $\operatorname{Im}\sigma=0$ yield solutions which at most grow polynomially in t_* (and do not decay). *We will continue to drop the metric g or $\widetilde{g}$ from the notation for brevity.*

In order to keep track of fiber inner products and volume densities, we will use the following notation.

Definition 7.3.1. For a density μ on X and a complex vector bundle $\mathcal{E}\to X$ equipped with a positive definite Hermitian form B, let $L^2(X,\mu;\mathcal{E},B)$ be the space of all sections u of $\mathcal{E}$ for which $\|u\|^2_{\mu,B}:=\int_X B(u,u)\,d\mu<\infty$.

If B is merely assumed to be sesquilinear (but not necessarily positive definite), we define the pairing

$$\langle u,v\rangle_{\mu,B}:=\int_X B(u,v)\,d\mu$$

for all sections u,v of $\mathcal{E}$ for which $B(u,v)\in L^1(X,\mu)$. If the choice of the density μ or inner product B is clear from the context, it will be dropped from the notation.

Remark 7.3.2. It will always be clear what bundle $\mathcal{E}$ we are using at a given time, so $\mathcal{E}$ will from now on be dropped from the notation; also, X will mostly be suppressed.

Since the metric g in (7.1.1) has a warped product structure and $\alpha\,dt$ has unit squared norm, it is natural to write differential forms on $M=\mathbb{R}_t\times X_x$ as

$$u(t,x)=u_T(t,x)+\alpha\,dt\wedge u_N(t,x), \tag{7.3.1}$$

where the tangential and normal forms u_T and u_N are t-dependent forms on X, and we will often write this as

$$u(t,x)=\begin{pmatrix}u_T(t,x)\\ u_N(t,x)\end{pmatrix}.$$

Thus, the differential d on M is given in terms of the differential d_X on X by

$$d = \begin{pmatrix} d_X & 0 \\ \alpha^{-1}\partial_t & -\alpha^{-1}d_X\alpha \end{pmatrix}. \tag{7.3.2}$$

Since the dual metric is given by $G = \alpha^{-2}\partial_t^2 - H$, the fiber inner product G_k on k-forms is given by

$$G_k = \begin{pmatrix} (-1)^k H_k & 0 \\ 0 & (-1)^{k-1}H_{k-1} \end{pmatrix}, \tag{7.3.3}$$

where H_q denotes the fiber inner product on q-forms on X. Furthermore, the volume density on M is $|dg| = \alpha|dt\,dh|$, and we therefore compute the $L^2(M,|dg|)$-adjoint of d to be

$$\delta = \begin{pmatrix} -\alpha^{-1}\delta_X\alpha & -\alpha^{-1}\partial_t \\ 0 & \delta_X \end{pmatrix}, \tag{7.3.4}$$

where δ_X is the $L^2(X,|dh|;\Lambda X,H)$-adjoint of d_X. Thus,

$$\widehat{d}(\sigma) = \begin{pmatrix} d_X & 0 \\ -i\sigma\alpha^{-1} & -\alpha^{-1}d_X\alpha \end{pmatrix}, \quad \widehat{\delta}(\sigma) = \begin{pmatrix} -\alpha^{-1}\delta_X\alpha & i\sigma\alpha^{-1} \\ 0 & \delta_X \end{pmatrix}. \tag{7.3.5}$$

In the course of our arguments we will need to justify various integrations by parts and boundary pairing arguments. This requires a precise understanding of the asymptotics of u_T and u_N for potential resonant states u at $Y = \partial\overline{X}_{\mathrm{even}}$. To this end, we further decompose the bundle $\Lambda X \oplus \Lambda X$ near Y by writing u_T as

$$u_T = u_{TT} + d\alpha \wedge u_{TN} \tag{7.3.6}$$

and similarly for u_N, hence

$$u = u_{TT} + d\alpha \wedge u_{TN} + \alpha\,dt \wedge u_{NT} + \alpha\,dt \wedge d\alpha \wedge u_{NN}, \tag{7.3.7}$$

where the $u_{\bullet\bullet}$ are forms on X valued in ΛY. Now for a resonant state u, we have

$$u = \alpha^{-i\beta\sigma}(\widetilde{u}'_{TT} + d(\alpha^2) \wedge \widetilde{u}'_{TN} + dt_* \wedge \widetilde{u}'_{NT} + dt_* \wedge d(\alpha^2) \wedge \widetilde{u}'_{NN}) \tag{7.3.8}$$

near Y with $\widetilde{u}'_{\bullet\bullet} \in \mathcal{C}^\infty(\overline{X}_{\mathrm{even}};\Lambda Y)$, which we rewrite in terms of the decomposition (7.3.7) using (7.2.2), obtaining

$$u = \alpha^{-i\beta\sigma}\big(\widetilde{u}'_{TT} + d\alpha \wedge (2\alpha\widetilde{u}'_{TN} - F'(\alpha)\widetilde{u}'_{NT}) \\ + \alpha\, dt \wedge \alpha^{-1}\widetilde{u}'_{NT} + 2\alpha\, dt \wedge d\alpha \wedge \widetilde{u}'_{NN}\big);$$

hence introducing the 'change of basis' matrix

$$\mathscr{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & \beta\alpha^{-1} & 0 \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and defining the space

$$\mathcal{C}^\infty_{(\sigma)} := \mathscr{C}\alpha^{-i\beta\sigma} \begin{pmatrix} \mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \\ \mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \\ \mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \\ \mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \end{pmatrix} \subset \begin{pmatrix} \alpha^{-i\beta\sigma}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \\ \alpha^{-i\beta\sigma-1}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \\ \alpha^{-i\beta\sigma-1}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \\ \alpha^{-i\beta\sigma}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \end{pmatrix}, \tag{7.3.9}$$

we obtain

$$\begin{pmatrix} u_{TT} \\ u_{TN} \\ u_{NT} \\ u_{NN} \end{pmatrix} = \mathscr{C}\alpha^{-i\beta\sigma} \begin{pmatrix} \widetilde{u}_{TT} \\ \widetilde{u}_{TN} \\ \widetilde{u}_{NT} \\ \widetilde{u}_{NN} \end{pmatrix} \in \mathcal{C}^\infty_{(\sigma)} \tag{7.3.10}$$

with $\widetilde{u}_{\bullet\bullet} \in \mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y)$, where the $u_{\bullet\bullet}$ are the components of u in the decomposition (7.3.7).

We will also need the precise form of $\widehat{d}(\sigma)$ and $\widehat{\delta}(\sigma)$ near Y. Since in the decomposition (7.3.6), the fiber inner product on ΛX-valued forms is $H = K \oplus \widetilde{\beta}^{-2}K$ in view of (7.2.1), we have

$$d_X = \begin{pmatrix} d_Y & 0 \\ \partial_\alpha & -d_Y \end{pmatrix} \quad \text{and} \quad \delta_X = \begin{pmatrix} \delta_Y & \partial_\alpha^* \\ 0 & -\delta_Y \end{pmatrix}, \tag{7.3.11}$$

where d_Y is the differential on Y and ∂_α^* is the formal adjoint of $\partial_\alpha \colon \mathcal{C}^\infty(X; \Lambda Y) \subset L^2(X, |dh|; \Lambda Y, K) \to L^2(X, |dh|; \Lambda Y, \widetilde{\beta}^{-2}K)$. Thus, if $\widetilde{\beta}$ and k are independent of α near

Y, we simply have

$$\partial_\alpha^* = -\beta^{-2}\partial_\alpha,$$

and in general, $\partial_\alpha^* = -\beta^{-2}\partial_\alpha + \alpha^2 p_1\partial_\alpha + \alpha p_2$, where $p_1, p_2 \in \mathcal{C}^\infty(\overline{X}_{\mathrm{even}})$.

Finally, we compute the form of $\widehat{d}(\sigma)$ near Y acting on forms as in (7.3.10):

$$\widehat{d}(\sigma)\mathscr{C} = \begin{pmatrix} d_Y & 0 & 0 & 0 \\ \partial_\alpha & -\alpha d_Y & -\beta\alpha^{-1}d_Y & 0 \\ -i\sigma\alpha^{-1} & 0 & -\alpha^{-1}d_Y & 0 \\ 0 & -i\sigma & -i\sigma\beta\alpha^{-2} - \alpha^{-1}\partial_\alpha & d_Y \end{pmatrix}. \tag{7.3.12}$$

Thus, applying $\widehat{d}(\sigma)$ to $u \in \mathcal{C}^\infty_{(\sigma)}$ yields an element

$$\widehat{d}(\sigma)u \in \begin{pmatrix} \alpha^{-i\beta\sigma}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \\ \alpha^{-i\beta\sigma-1}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \\ \alpha^{-i\beta\sigma-1}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \\ \alpha^{-i\beta\sigma}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \end{pmatrix},$$

where we use that there is a cancellation in the $(4,3)$ entry of $\widehat{d}(\sigma)\mathscr{C}$ in view of $(i\sigma\beta\alpha^{-2} + \alpha^{-1}\partial_\alpha)\alpha^{-i\beta\sigma} = 0$; without this cancellation, the fourth component of $\widehat{d}(\sigma)u$ would only lie in $\alpha^{-i\beta\sigma-2}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y)$. Similarly, we compute

$$\widehat{\delta}(\sigma)\mathscr{C} = \begin{pmatrix} -\delta_Y & -\alpha^{-1}\partial_\alpha^*\alpha^2 & -\beta\alpha^{-1}\partial_\alpha^* + i\sigma\alpha^{-2} & 0 \\ 0 & \alpha\delta_Y & \beta\alpha^{-1}\delta_Y & i\sigma\alpha^{-1} \\ 0 & 0 & \alpha^{-1}\delta_Y & \partial_\alpha^* \\ 0 & 0 & 0 & -\delta_Y \end{pmatrix}, \tag{7.3.13}$$

thus applying $\widehat{\delta}(\sigma)$ to $u \in \mathcal{C}^\infty_{(\sigma)}$ also gives an element

$$\widehat{\delta}(\sigma)u \in \begin{pmatrix} \alpha^{-i\beta\sigma}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \\ \alpha^{-i\beta\sigma-1}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \\ \alpha^{-i\beta\sigma-1}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \\ \alpha^{-i\beta\sigma}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y) \end{pmatrix},$$

where there is again a cancellation in the $(1,3)$ entry of $\widehat{\delta}(\sigma)\mathscr{C}$; without this cancellation, the first component of $\widehat{d}(\sigma)u$ would only lie in $\alpha^{-i\beta\sigma-2}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}};\Lambda Y)$.

In fact, a bit more is true: Namely, one checks that the operators $\alpha^{i\beta\sigma}\mathscr{C}^{-1}\widehat{d}(\sigma)\mathscr{C}\alpha^{-i\beta\sigma}$ and $\alpha^{i\beta\sigma}\mathscr{C}^{-1}\widehat{\delta}(\sigma)\mathscr{C}\alpha^{-i\beta\sigma}$ preserve the space $\mathcal{C}^\infty(\overline{X}_{\mathrm{even}};\Lambda Y)^4$ (in the decomposition (7.3.8)), hence if $u\in\mathcal{C}^\infty_{(\sigma)}$, then also $\widehat{d}(\sigma)u,\widehat{\delta}(\sigma)u\in\mathcal{C}^\infty_{(\sigma)}$. Indeed, this follows either by a direct computation, or one notes that these operators are equal (up to a smooth phase factor) to the matrices of the Fourier transforms in t_* of d and δ with respect to the form decomposition (7.3.8), which are smooth on the extended manifold $\widetilde{X}$. Since it will be useful later, we check this explicitly for $\sigma=0$ by computing

$$\mathscr{C}^{-1}\widehat{d}(0)\mathscr{C}=\begin{pmatrix} d_Y & 0 & 0 & 0\\ \alpha^{-1}\partial_\alpha & -d_Y & 0 & 0\\ 0 & 0 & -d_Y & 0\\ 0 & 0 & -\alpha^{-1}\partial_\alpha & d_Y\end{pmatrix} \tag{7.3.14}$$

and

$$\mathscr{C}^{-1}\widehat{\delta}(0)\mathscr{C}=\begin{pmatrix} -\delta_Y & -\alpha^{-1}\partial_\alpha^*\alpha^2 & -\alpha^{-1}\partial_\alpha^*\beta & 0\\ 0 & \delta_Y & 0 & -\beta\alpha^{-1}\partial_\alpha^*\\ 0 & 0 & \delta_Y & \alpha\partial_\alpha^*\\ 0 & 0 & 0 & -\delta_Y\end{pmatrix}. \tag{7.3.15}$$

7.3.1 Absence of resonances in the upper half plane

The fiber inner product on the form bundle is not positive definite, thus we cannot use standard arguments for (formally) self-adjoint operators to exclude a non-trivial kernel of $\widehat{d}(\sigma)+\widehat{\delta}(\sigma)$. We therefore introduce a different inner product (by which we mean here a non-degenerate sesquilinear form), related to the natural inner product induced by the metric, which does have some positivity properties. Concretely, for $\theta\in(-\pi/2,\pi/2)$, we use the inner product $H\oplus e^{-2i\theta}H$, i.e. on pure degree k-forms on M, the fiber inner product is given by $H_k\oplus e^{-2i\theta}H_{k-1}$ in the decomposition into tangential and normal components as in (7.3.1).

Lemma 7.3.3. *Let $\theta\in(-\pi/2,\pi/2)$. Suppose that $u\in L^2(\alpha|dh|;H\oplus H)$ is such that $\langle u,u\rangle_{H\oplus e^{-2i\theta}H}=0$. Then $u=0$.*

Proof. With $u = u_T + \alpha\, dt \wedge u_N$, we have $\|u_T\|^2_{L^2(\alpha|dh|;H)} + e^{-2i\theta}\|u_N\|^2_{L^2(\alpha|dh|;H)} = 0$. Multiplying this equation by $e^{i\theta}$ and taking real parts gives

$$\cos(\theta)\|u\|^2_{L^2(\alpha|dh|;H\oplus H)} = 0,$$

hence $u = 0$, since $\cos\theta > 0$ for θ in the given range. □

Using the volume density $\alpha|dh|$ to compute adjoints, we have

$$\langle \widehat{d}(\sigma)u, v\rangle_{H\oplus e^{-2i\theta}H} = \langle u, \widehat{\delta}_\theta(\sigma)v\rangle_{H\oplus e^{-2i\theta}H}, \quad u, v \in \mathcal{C}^\infty_c(X; \Lambda X \oplus \Lambda X)$$

for the operator

$$\widehat{\delta}_\theta(\sigma) = \begin{pmatrix} \alpha^{-1}\delta_X\alpha & ie^{2i\theta}\bar{\sigma}\alpha^{-1} \\ 0 & -\delta_X \end{pmatrix},$$

which equals $-\widehat{\delta}(\sigma)$ provided $e^{2i\theta}\bar{\sigma} = -\sigma$, i.e. $\sigma \in e^{i\theta}\cdot i(0,\infty)$.

Remark 7.3.4. Since the inner product $H \oplus e^{-2i\theta}H$ is not Hermitian, we do *not* have $\langle \widehat{\delta}_\theta(\sigma)u, v\rangle_{H\oplus e^{-2i\theta}H} = \langle u, \widehat{d}(\sigma)v\rangle_{H\oplus e^{-2i\theta}H}$ in general. Rather, one computes

$$\begin{aligned}\langle \widehat{\delta}_\theta(\sigma)u, v\rangle_{H\oplus e^{2i\theta}H} &= \overline{\langle v, \widehat{\delta}_\theta(\sigma)u\rangle_{H\oplus e^{-2i\theta}H}} \\ &= \overline{\langle \widehat{d}(\sigma)v, u\rangle_{H\oplus e^{-2i\theta}H}} = \langle u, \widehat{d}(\sigma)v\rangle_{H\oplus e^{2i\theta}H}.\end{aligned} \tag{7.3.16}$$

Now suppose $u \in \mathcal{C}^\infty_{(\sigma)}$ is a solution, with $\operatorname{Im}\sigma > 0$, of

$$(\widehat{d}(\sigma) + \widehat{\delta}(\sigma))u = 0. \tag{7.3.17}$$

We claim that every such u must vanish. To show this, we apply $\widehat{d}(\sigma)$ to (7.3.17) and pair the result with u; this gives

$$\begin{aligned}0 &= \langle \widehat{d}(\sigma)\widehat{\delta}(\sigma)u, u\rangle_{H\oplus e^{-2i\theta}H} = \langle \widehat{\delta}(\sigma)u, \widehat{\delta}_\theta(\sigma)u\rangle_{H\oplus e^{-2i\theta}H} \\ &= -\langle \widehat{\delta}(\sigma)u, \widehat{\delta}(\sigma)u\rangle_{H\oplus e^{-2i\theta}H},\end{aligned} \tag{7.3.18}$$

where we choose $\theta \in (-\pi/2, \pi/2)$ so that $\sigma \in e^{i\theta}\cdot i(0,\infty)$; the integration by parts will be justified momentarily. By Lemma 7.3.3, this implies $\widehat{\delta}(\sigma)u = 0$. On the other hand, applying $\widehat{\delta}(\sigma)$ to (7.3.17) and using (7.3.16), we get, for $\sigma \in e^{i\theta} \cdot i(0,\infty)$,

$$\begin{aligned} 0 &= \langle \widehat{\delta}(\sigma)\widehat{d}(\sigma)u, u\rangle_{H\oplus e^{2i\theta}H} = -\langle \widehat{\delta}_\theta(\sigma)\widehat{d}(\sigma)u, u\rangle_{H\oplus e^{2i\theta}H} \\ &= -\langle \widehat{d}(\sigma)u, \widehat{d}(\sigma)u\rangle_{H\oplus e^{2i\theta}H}, \end{aligned} \tag{7.3.19}$$

hence $\widehat{d}(\sigma)u = 0$ by Lemma 7.3.3, again modulo justifying the integration by parts.

Using the splitting (7.3.1) and the form (7.3.5) of $\widehat{d}(\sigma)$, the second component of the equation $\widehat{d}(\sigma)u = 0$ gives $i\sigma u_T + d_X \alpha u_N = 0$. Taking the $L^2(\alpha|dh|; H)$-pairing of this with u_T gives (the integration by parts to be justified below)

$$0 = i\sigma\|u_T\|^2 + \langle d_X\alpha u_N, u_T\rangle = i\sigma\|u_T\|^2 + \langle u_N, \delta_X \alpha u_T\rangle, \tag{7.3.20}$$

and then the first component of $\widehat{\delta}(\sigma)u = 0$, i.e. $\delta_X\alpha u_T = i\sigma u_N$, can be used to rewrite the pairing on the right hand side; we obtain $0 = i(\sigma\|u_T\|^2 - \bar{\sigma}\|u_N\|^2)$. Writing $\sigma = ie^{i\theta}\widetilde{\sigma}$ with $\widetilde{\sigma} > 0$ real, this becomes

$$0 = \widetilde{\sigma}(e^{i\theta}\|u_T\|^2 + e^{-i\theta}\|u_N\|^2), \tag{7.3.21}$$

and taking the real part of this equation gives $u_T = 0 = u_N$, hence $u = 0$.

We now justify the integrations by parts used in (7.3.18) and (7.3.19), which is only an issue at Y. First of all, since $u \in \mathcal{C}^\infty_{(\sigma)}$ and $\operatorname{Im}\sigma > 0$, the pairings are well-defined in the strong sense that all functions which appear in the pairings are elements of $L^2(\alpha|dh|; H\oplus H)$; in fact, all functions in these pairings lie in $\mathcal{C}^\infty_{(\sigma)}$. In view of the block structure $H\oplus e^{-2i\theta}H = K\oplus\widetilde{\beta}^{-2}K\oplus e^{-2i\theta}K\oplus\widetilde{\beta}^{-2}e^{-2i\theta}K$ of the inner product, the only potentially troublesome term for the integration by parts there is the pairing of the first components, since this is where we need the cancellation mentioned after (7.3.13) to ensure that $\widehat{\delta}(\sigma)u \in L^2$. However, if we only use the cancellation in one of the terms, we pair $\alpha^{-i\beta\sigma}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y)$ against $\alpha^{-i\beta\sigma-2}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y)$ in the first component, thus this pairing is still absolutely integrable and one can integrate by parts. Likewise, the integration by parts used in (7.3.19) only has potential issues in the pairing of the fourth components, since we need the cancellation mentioned after (7.3.12) to ensure that $\widehat{d}(\sigma)u \in L^2$. But again, if we only use this cancellation in one of the terms, we pair $\alpha^{-i\beta\sigma}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y)$ against $\alpha^{-i\beta\sigma-2}\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y)$, which is absolutely integrable.

In order to justify (7.3.20), we observe using (7.3.11) that near Y,

$$u_T, d_X \alpha u_N \in \begin{pmatrix} \alpha^{-i\beta\sigma}\mathcal{C}^\infty \\ \alpha^{-i\beta\sigma-1}\mathcal{C}^\infty \end{pmatrix}, \quad u_N, \delta_X \alpha u_T \in \begin{pmatrix} \alpha^{-i\beta\sigma-1}\mathcal{C}^\infty \\ \alpha^{-i\beta\sigma}\mathcal{C}^\infty \end{pmatrix},$$

where we write $\mathcal{C}^\infty = \mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y)$. These membership statements do not rely on any cancellations, and since all these functions are in $L^2(\alpha|dh|; \Lambda Y, K)$ near Y, the integration by parts in (7.3.20) is justified.

We summarize the above discussion and extend it to a quantitative version:

Proposition 7.3.5. *There exists a constant $C > 0$ such that for all $\sigma \in \mathbb{C}$ with $\operatorname{Im}\sigma > 0$, we have the following estimate for $u \in \mathcal{C}^\infty_{(\sigma)}$:*

$$\|u\|_{L^2(\alpha|dh|; H\oplus H)} \leq C\frac{|\sigma|}{|\operatorname{Im}\sigma|^2}\|(\widehat{d}(\sigma) + \widehat{\delta}(\sigma))u\|_{L^2(\alpha|dh|;H\oplus H)}. \tag{7.3.22}$$

Proof. Write $\sigma = ie^{i\theta}\widetilde{\sigma}$, $\theta \in (-\pi/2, \pi/2)$, $\widetilde{\sigma} > 0$, as before. Let $f = (\widehat{d}(\sigma) + \widehat{\delta}(\sigma))u$; in particular $f \in \mathcal{C}^\infty_{(\sigma)}$. Then $\widehat{d}(\sigma)\widehat{\delta}(\sigma)u = \widehat{d}(\sigma)f$, so

$$\langle\widehat{\delta}(\sigma)u, \widehat{\delta}(\sigma)u\rangle_{H\oplus e^{-2i\theta}H} = -\langle\widehat{d}(\sigma)\widehat{\delta}(\sigma)u, u\rangle_{H\oplus e^{-2i\theta}H} = \langle f, \widehat{\delta}(\sigma)u\rangle_{H\oplus e^{-2i\theta}H}, \tag{7.3.23}$$

and similarly

$$\langle\widehat{d}(\sigma)u, \widehat{d}(\sigma)u\rangle_{H\oplus e^{2i\theta}H} = \langle f, \widehat{d}(\sigma)u\rangle_{H\oplus e^{2i\theta}H}. \tag{7.3.24}$$

Multiply (7.3.23) by $e^{i\theta}$, (7.3.24) by $e^{-i\theta}$ and take the sum of both equations to get

$$\begin{aligned} e^{i\theta}(\|(\widehat{\delta}(\sigma)u)_T\|^2 + \|(\widehat{d}(\sigma)u)_N\|^2) + e^{-i\theta}(\|(\widehat{\delta}(\sigma)u)_N\|^2 + \|(\widehat{d}(\sigma)u)_T\|^2) \\ = e^{i\theta}\langle f, \widehat{\delta}(\sigma)u\rangle_{H\oplus e^{-2i\theta}H} + e^{-i\theta}\langle f, \widehat{d}(\sigma)u\rangle_{H\oplus e^{2i\theta}H}. \end{aligned}$$

Here, the norms without subscript are $L^2(\alpha|dh|; H \oplus H)$-norms as usual. Taking the real part and applying Cauchy-Schwarz to the right hand side produces the estimate

$$\|\widehat{d}(\sigma)u\| + \|\widehat{\delta}(\sigma)u\| \leq \frac{4}{\cos\theta}\|f\| = \frac{4|\sigma|}{\operatorname{Im}\sigma}\|f\|. \tag{7.3.25}$$

We estimate u in terms of the left hand side of (7.3.25) by following the arguments leading to (7.3.21): Put $v = \widehat{d}(\sigma)u$ and $w = \widehat{\delta}(\sigma)u$. Then $i\sigma u_T + d_X\alpha u_N = -\alpha v_N$; we pair this with u_T in $L^2(\alpha|dh|;H)$ and obtain

$$i\sigma\|u_T\|^2 + \langle u_N, \delta_X \alpha u_T\rangle = -\langle \alpha v_N, u_T\rangle.$$

Using $-\delta_X\alpha u_T + i\sigma u_N = \alpha w_T$, this implies

$$i\sigma\|u_T\|^2 - i\bar{\sigma}\|u_N\|^2 = -\langle \alpha v_N, u_T\rangle + \langle u_N, \alpha w_T\rangle,$$

thus

$$\widetilde{\sigma}(e^{i\theta}\|u_T\|^2 + e^{-i\theta}\|u_N\|^2) = \langle \alpha v_N, u_T\rangle - \langle u_N, \alpha w_T\rangle.$$

Taking the real part and applying Cauchy-Schwarz, we get

$$(\cos\theta)\|u\| \leq |\sigma|^{-1}(\|\alpha v\| + \|\alpha w\|) \lesssim |\sigma|^{-1}(\|v\| + \|w\|).$$

In combination with (7.3.25), this yields (7.3.22). □

7.3.2 Boundary pairing and absence of non-zero real resonances

We proceed to exclude non-zero real resonances for $d+\delta$ by means of a boundary pairing argument similar to [84, §2.3].

Proposition 7.3.6. *Suppose $\sigma \in \mathbb{R}$, $\sigma \neq 0$. If $u \in \mathcal{C}^\infty_{(\sigma)}$ solves $(\widehat{d}(\sigma) + \widehat{\delta}(\sigma))u = 0$, then $u = 0$.*

Proof. Writing $u = u_T + \alpha\, dt \wedge u_N$ as usual, we can expand $(\widehat{d}(\sigma) + \widehat{\delta}(\sigma))u = 0$ as

$$\begin{aligned}(\alpha d_X - \delta_X\alpha)u_T + i\sigma u_N &= 0 \\ -i\sigma u_T + (-d_X\alpha + \alpha\delta_X)u_N &= 0.\end{aligned} \tag{7.3.26}$$

Applying $(-d_X\alpha + \alpha\delta_X)$ to the first equation and using the second equation to simplify the resulting expression produces a second order equation for u_T,

$$(d_X\alpha\delta_X\alpha + \alpha\delta_X\alpha d_X - d_X\alpha^2 d_X - \sigma^2)u_T = 0. \tag{7.3.27}$$

Writing $u_T = u_{TT} + d\alpha \wedge u_{TN}$ as in (7.3.6), we see from the definition of the space $\mathcal{C}^\infty_{(\sigma)}$ that

$$u_T \in \mathcal{C}^\infty_{(\sigma),T} := \alpha^{-i\beta\sigma}\mathcal{C}^\infty(\overline{X}_{\text{even}};\Lambda Y) \oplus \alpha^{-i\beta\sigma-1}\mathcal{C}^\infty(\overline{X}_{\text{even}};\Lambda Y)$$

near Y. Notice that the space $\mathcal{C}^\infty_{(\sigma),T}$ barely fails to be contained in $L^2(\alpha|dh|)$.

We will deduce from (7.3.27) that $u_T = 0$; equation (7.3.26) then gives $u_N = 0$, as $\sigma \neq 0$. Now, the $L^2(\alpha|dh|;H)$-adjoint of $d_X\alpha$ is $\delta_X\alpha$, hence even ignoring the term $d_X\alpha^2 d_X$, the operator in (7.3.27) is not symmetric. However, we can obtain a simpler equation from (7.3.27) by applying d_X to it; write $v_T = d_X u_T \in \mathcal{C}^\infty_{(\sigma),T}$, and near Y,

$$v_T = \begin{pmatrix} \alpha^{-i\beta\sigma}\widetilde{v}_{TT} \\ \alpha^{-i\beta\sigma-1}\widetilde{v}_{TN} \end{pmatrix}, \quad \widetilde{v}_{TT}, \widetilde{v}_{TN} \in \mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y).$$

Then v_T satisfies the equation

$$(d_X\alpha\delta_X\alpha - \sigma^2)v_T = 0,$$

and $d_X\alpha\delta_X\alpha$ *is* symmetric with respect to the $L^2(\alpha|dh|;H)$-inner product. We now compute the boundary pairing formula (using the same inner product); to this end, pick a cutoff function $\chi \in \mathcal{C}^\infty(\overline{X})$ such that in a collar neighborhood $[0,\delta)_\alpha \times Y_y$ of Y in $\overline{X}$, $\chi = \chi(\alpha)$ is identically 0 near $\alpha = 0$ and identically 1 in $\alpha \geq \delta/2$, and extend χ by 1 to all of $\overline{X}$. Define $\chi_\epsilon(\alpha) = \chi(\alpha/\epsilon)$ and $\chi'_\epsilon(\alpha) = \chi'(\alpha/\epsilon)$. Then

$$\begin{aligned} 0 &= \lim_{\epsilon\to 0}(\langle (d_X\alpha\delta_X\alpha - \sigma^2)v_T, \chi_\epsilon v_T\rangle - \langle v_T, \chi_\epsilon(d_X\alpha\delta_X\alpha - \sigma^2)v_T\rangle) \\ &= \lim_{\epsilon\to 0}\langle v_T, [d_X\alpha\delta_X\alpha, \chi_\epsilon]v_T\rangle. \end{aligned} \tag{7.3.28}$$

The coefficients of the commutator are supported near Y, hence we use (7.3.11) to compute its form as

$$\begin{aligned} [d_X\alpha\delta_X\alpha, \chi_\epsilon] &= \left[\begin{pmatrix} d_Y\alpha\delta_Y\alpha & d_Y\alpha\partial^*_\alpha\alpha \\ \partial_\alpha\alpha\delta_Y\alpha & \partial_\alpha\alpha\partial^*_\alpha\alpha + d_Y\alpha\delta_Y\alpha \end{pmatrix}, \chi_\epsilon\right] \\ &= \begin{pmatrix} 0 & d_Y\alpha[\partial^*_\alpha, \chi_\epsilon]\alpha \\ [\partial_\alpha, \chi_\epsilon]\alpha\delta_Y\alpha & [\partial_\alpha\alpha\partial^*_\alpha\alpha, \chi_\epsilon] \end{pmatrix} \\ &= \epsilon^{-1}\begin{pmatrix} 0 & -\beta^{-2}(\alpha^2 + \mathcal{O}(\alpha^4))\chi'_\epsilon d_Y \\ \chi'_\epsilon\alpha\delta_Y\alpha & \chi'_\epsilon\alpha\partial^*_\alpha\alpha - \partial_\alpha(\alpha^2 + \mathcal{O}(\alpha^4))\beta^{-2}\chi'_\epsilon \end{pmatrix}. \end{aligned}$$

In (7.3.28), the off-diagonal terms of this give terms of the form

$$\int_Y \int \alpha^{\mp i\beta\sigma} \alpha^{\pm i\beta\sigma - 1} \epsilon^{-1} \alpha^2 \chi'_\epsilon \widetilde{v}\, d\alpha\, |dk| \tag{7.3.29}$$

with $\widetilde{v} \in \mathcal{C}^\infty(\overline{X}_{\text{even}})$, and are easily seen to vanish in the limit $\epsilon \to 0$. For the non-zero diagonal term, recall that the volume density is given by $\alpha|dh| = \alpha\beta\, d\alpha|dk|$, and the fiber inner product in the (TN)-component is $\beta^{-2}K$, so

$$\begin{aligned} &\epsilon^{-1}\langle \alpha^{-i\beta\sigma-1}\widetilde{v}_{TN}, (\chi'_\epsilon \alpha\partial^*_\alpha \alpha - \partial_\alpha \alpha^2\beta^{-2}\chi'_\epsilon)\alpha^{-i\beta\sigma-1}\widetilde{v}_{TN}\rangle_{L^2(X;\alpha\beta\, d\alpha|dk|;\Lambda Y;\beta^{-2}K)} \\ &\quad = 2\int_Y \int \langle \widetilde{v}_{TN}, i\beta^{-2}\sigma\widetilde{v}_{TN}\rangle_K \epsilon^{-1}\chi'_\epsilon\, d\alpha|dk| + o(1) \\ &\quad \xrightarrow{\epsilon\to 0} -2i\beta^{-2}\sigma\|\widetilde{v}_{TN}|_Y\|^2_{L^2(Y,|dk|;K)}; \end{aligned}$$

here, both summands in the pairing yield the same result, as is most easily seen by integrating by parts in α, hence the factor of 2, and the $o(1)$-term comes from differentiating $\widetilde{v}_{TN}$, which produces a term of the form (7.3.29). We thus arrive at

$$0 = \langle (d_X\alpha\delta_X\alpha - \sigma^2)v_T, v_T\rangle - \langle v_T, (d_X\alpha\delta_X\alpha - \sigma^2)v_T\rangle = -2i\beta^{-2}\sigma\|\widetilde{v}_{TN}|_Y\|^2,$$

whence $\widetilde{v}_{TN}|_Y = 0$ in view of $\sigma \neq 0$, so we in fact have

$$v_T = \begin{pmatrix} \alpha^{-i\beta\sigma}\widetilde{v}_{TT} \\ \alpha^{-i\beta\sigma}\widetilde{v}'_{TN} \end{pmatrix}, \quad \widetilde{v}'_{TN} \in \mathcal{C}^\infty(\overline{X}_{\text{even}}; \Lambda Y). \tag{7.3.30}$$

For the next step, we need the language of 0-differential operators, explained briefly in §5.4.5. Often, as in our case, one is considering solutions of 0-differential equations with additional properties, such as having an expansion in powers of α (and perhaps $\log\alpha$) with smooth coefficients, i.e. polyhomogeneous functions. In these cases $\alpha\mathrm{Diff}_{\mathrm{b}}(\overline{X}) \subset \mathrm{Diff}_0(\overline{X})$ acts 'trivially' on an expansion in that it maps each term to one with an additional order of vanishing, so in particular, one can analyze the asymptotic expansion of solutions of 0-differential equations in this restrictive class by ignoring the $\alpha\mathrm{Diff}_{\mathrm{b}}(\overline{X})$ terms. Notice that $\alpha\partial_{y_j} \in \alpha\mathrm{Diff}_{\mathrm{b}}(\overline{X})$ in particular, so the tangential 0-derivatives can be dropped for this purpose. The indicial equation is then obtained by freezing the coefficients of $A \in \mathrm{Diff}_0(\overline{X})$ at ∂X, i.e. writing it as $\sum_{k,\beta} a_{k,\beta}(\alpha, y)(\alpha\partial_\alpha)^k(\alpha\partial_y)^\beta$, where $a_{k,\beta}$ are bundle endomorphism valued, and restricting α to 0, and dropping all terms with a positive power of $\alpha\partial_y$, to obtain

$\sum_k a_{k,0}(0,y)(\alpha\partial_\alpha)^k$. This can be thought of as a regular-singular ODE in α for each y; its indicial roots are called the indicial roots of the original 0-operator, and they determine the asymptotics of solutions of the homogeneous PDE with this a priori form.

Now $d_X\alpha\delta_X\alpha-\sigma^2\in\mathrm{Diff}_0^2(\overline{X})$ is a 0-differential operator which equals

$$d_X\alpha\delta_X\alpha-\sigma^2=\begin{pmatrix}-\sigma^2 & 0\\ 0 & -\beta^{-2}\partial_\alpha\alpha\partial_\alpha\alpha-\sigma^2\end{pmatrix}$$

modulo $\alpha\mathrm{Diff}_{\mathrm{b}}^2(\overline{X})$; hence its indicial roots are $\pm i\beta\sigma-1$. In particular, $-i\beta\sigma+j$, $j\in\mathbb{N}_0$, is not an indicial root. Thus, a standard inductive argument starting with (7.3.30) shows that $v_T\in\dot{\mathcal{C}}^\infty(\overline{X};\Lambda X)$.

Next, we note that v_T lies in the kernel of the operator

$$d_X\alpha\delta_X\alpha+\alpha^{-1}\delta_X\alpha^3 d_X-\sigma^2\in\mathrm{Diff}_0^2(\overline{X};{}^0\Lambda\overline{X}),$$

which has the same principal part as $\alpha^2\Delta_X$, hence is principally a 0-Laplacian; thus, we can apply Mazzeo's result on unique continuation at infinity [80] to conclude that the rapidly vanishing v_T must in fact vanish identically.

We thus have proved $d_X u_T=0$. Since u_T satisfies (7.3.27), we deduce that u_T itself satisfies

$$(d_X\alpha\delta_X\alpha-\sigma^2)u_T=0,$$

thus repeating the above argument shows that this implies $u_T=0$, hence $u=0$, and the proof is complete. □

7.3.3 Analysis of the zero resonance

We have shown now that the only potential resonance for $d+\delta$ in $\operatorname{Im}\sigma\geq 0$ is $\sigma=0$, and we proceed to study the zero resonance in detail, in particular giving a cohomological interpretation of it in §7.3.4.

We begin by establishing the order of the pole of $(\widetilde{d}(\sigma)+\widetilde{\delta}(\sigma))^{-1}$:

Lemma 7.3.7. *$(\widetilde{d}(\sigma)+\widetilde{\delta}(\sigma))^{-1}$ has a pole of order 1 at $\sigma=0$.*

Proof. Since $\widetilde{d}(0)+\widetilde{\delta}(0)$ annihilates constant functions (which are indeed elements of $\mathcal{C}^\infty_{(0)}$), $(\widetilde{d}(\sigma)+\widetilde{\delta}(\sigma))^{-1}$ does have a pole at 0. Denote the order of the pole by N. Then there is a holomorphic family $\widetilde{u}(\sigma)\in\mathcal{C}^\infty(\widetilde{X})$ with $\widetilde{u}(0)\neq 0$ such that $(\widetilde{d}(\sigma)+\widetilde{\delta}(\sigma))\widetilde{u}(\sigma)=\sigma^N\widetilde{v}$, where $\widetilde{v} \in \mathcal{C}^\infty(\widetilde{X})$. Define $u(\sigma) = e^{iF\sigma}\widetilde{u}(\sigma)|_X \in \mathcal{C}^\infty_{(\sigma)}$ and $v(\sigma) = e^{iF\sigma}\widetilde{v}|_X \in \mathcal{C}^\infty_{(\sigma)}$, then $(\widehat{d}(0) + \widehat{\delta}(0))u(\sigma) = \sigma^N v(\sigma)$. Moreover, since $(\widetilde{d}(0) + \widetilde{\delta}(0))\widetilde{u}(0) = 0$ and $\widetilde{u}(0)$ is non-zero, Lemma 7.2.2 shows that $u(0) \neq 0$.

Let us assume now that $N \geq 2$. For $\sigma \in i(0,\infty)$ close to 0, the quantitative estimate in Proposition 7.3.5 now gives

$$\|u(\sigma)\| \lesssim |\sigma|^{-1+N}\|v(\sigma)\| \leq |\sigma|\|v(\sigma)\|, \tag{7.3.31}$$

where we use the norm of $L^2(\alpha|dh|; H \oplus H)$; observe that in the notation of §7.3.1, we have $\widehat{\delta_0}(0) = -\widehat{\delta}(0)$, hence using the Riemannian fiber inner product $H \oplus H$ is indeed natural when studying the zero resonance. Notice that (7.3.31) does not immediately give $u(0) = 0$ since $v(0) \notin L^2(\alpha|dh|; H \oplus H)$. However, we can quantify the degeneration of the L^2-norm of $v(\sigma)$ as $\sigma \to 0$. To see this, we first observe that the L^2-norm of $v(\sigma)$ restricted to the complement of any fixed neighborhood of Y does stay bounded, so it remains to analyze the L^2-norms of the four components of $v(\sigma)$ near Y in the notation of (7.3.7); denote these components by $\alpha^{-i\beta\sigma}\widetilde{v}_{TT}(\sigma), \alpha^{-i\beta\sigma-1}\widetilde{v}_{TN}(\sigma), \alpha^{-i\beta\sigma-1}\widetilde{v}_{NT}(\sigma)$ and $\alpha^{-i\beta\sigma}\widetilde{v}_{NN}(\sigma)$, so that the $\widetilde{v}_{\bullet\bullet}(\sigma) \in \mathcal{C}^\infty(\overline{X}_{\text{even}}; \Lambda Y)$ uniformly. Since the fiber metric in this basis has a block diagonal form and any $\mathcal{C}^\infty(\overline{X}_{\text{even}})$-multiple of $\alpha^{-i\beta\sigma}$ is uniformly square-integrable with respect to the volume density $\alpha|dh|$, the degeneration of the L^2-norm of v is caused by the (TN) and (NT) components. For these, we compute, with $\widetilde{w}(\sigma) \in \mathcal{C}^\infty(\overline{X}_{\text{even}}; \Lambda Y)$ denoting any continuous family supported near Y,

$$\begin{aligned}\int_Y \int \alpha^{2(-i\beta\sigma-1)}\|\widetilde{w}\|_K^2\, \alpha\, d\alpha|dk| \\ = \|\widetilde{w}(0)\|^2_{L^2(Y,|dk|;K)} \int \alpha^{-2i\beta\sigma-1}\chi(\alpha)\, d\alpha + \mathcal{O}(1),\end{aligned}$$

where $\chi \in \mathcal{C}^\infty(\overline{X})$ is a cutoff, equal to 1 near $\alpha = 0$. We can rewrite the integral using an integration by parts, which yields

$$\int \alpha^{-2i\beta\sigma-1}\chi(\alpha)\, d\alpha = \frac{1}{2i\beta\sigma}\int \alpha^{-2i\beta\sigma}\chi'(\alpha)\, d\alpha = \mathcal{O}(|\sigma|^{-1}).$$

Therefore, we obtain the bound $\|v(\sigma)\| = \mathcal{O}(|\sigma|^{-1/2})$. Plugging this into (7.3.31), we conclude using Fatou's Lemma that $u(0) = 0$, which contradicts our assumption that $u(0) \neq 0$. Hence, the order of the pole is $N \leq 1$, but since it is at least 1, it must be equal to 1. □

Next, we identify the resonant states. *For brevity, we will write* $\widehat{d} = \widehat{d}(0)$, $\widehat{\delta} = \widehat{\delta}(0)$ *and* $\widehat{\Box} = \widehat{\Box_g}(0)$.

Proposition 7.3.8. $\ker_{\mathcal{C}^\infty_{(0)}}(\widehat{d} + \widehat{\delta})$ *is equal to the space*

$$\mathcal{H} = \{u \in \mathcal{C}^\infty_{(0)} \colon \widehat{d}u = 0, \widehat{\delta}u = 0\}. \tag{7.3.32}$$

Proof. Given $u \in \mathcal{C}^\infty_{(0)}$ with $(\widehat{d} + \widehat{\delta})u = 0$, we conclude that $\widehat{\Box}u = 0$, and since $\widehat{\Box}$ is symmetric on $L^2(\alpha|dh|; H \oplus H)$, we can obtain information about u by a boundary pairing type argument: Concretely, for a cutoff $\chi \in \mathcal{C}^\infty(\overline{X})$ as in the proof of Proposition 7.3.6, identically 0 near Y, identically 1 outside a neighborhood of Y and a function of α in a collar neighborhood of Y, and with $\chi_\epsilon(\alpha) = \chi(\alpha/\epsilon)$, $\chi'_\epsilon(\alpha) = \chi'(\alpha/\epsilon)$, we have

$$\begin{aligned} 0 &= -\lim_{\epsilon\to 0}\langle \chi_\epsilon(\widehat{d}\,\widehat{\delta} + \widehat{\delta}\,\widehat{d})u, u\rangle = \lim_{\epsilon\to 0}(\langle \widehat{\delta}u, \widehat{\delta}\chi_\epsilon u\rangle + \langle \widehat{d}u, \widehat{d}\chi_\epsilon u\rangle) \\ &= \lim_{\epsilon\to 0}(\|\chi_\epsilon^{1/2}\widehat{\delta}u\|^2 + \|\chi_\epsilon^{1/2}\widehat{d}u\|^2) + \lim_{\epsilon\to 0}(\langle \widehat{\delta}u, [\widehat{\delta}, \chi_\epsilon]u\rangle + \langle \widehat{d}u, [\widehat{d}, \chi_\epsilon]u\rangle). \end{aligned} \tag{7.3.33}$$

Since the commutators are supported near Y, we can compute them in the basis (7.3.7). Let us write $u = \mathscr{C}\widetilde{u}$ as in (7.3.10) with $\sigma = 0$, then in view of (7.3.13), we have

$$[\widehat{\delta}\mathscr{C}, \chi_\epsilon] = \epsilon^{-1}\chi'_\epsilon \begin{pmatrix} 0 & \beta^{-2}\alpha + \mathcal{O}(\alpha^3) & \beta^{-1}\alpha^{-1} + \mathcal{O}(\alpha) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta^{-2} + \mathcal{O}(\alpha^2) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.3.34}$$

and since therefore only the (TT) and (NT) components of $[\widehat{\delta}\mathscr{C}, \chi_\epsilon]\widetilde{u}$ are non-zero, we merely compute

$$\begin{aligned} (\widehat{\delta}\mathscr{C}\widetilde{u})_{TT} &= -\delta_Y\widetilde{u}_{TT} - \alpha^{-1}\partial_\alpha^*\alpha^2\widetilde{u}_{TN} - \beta\alpha^{-1}\partial_\alpha^*\widetilde{u}_{NT} \\ &\in -\delta_Y\widetilde{u}_{TT} + 2\beta^{-2}\widetilde{u}_{TN} - \beta\alpha^{-1}\partial_\alpha^*\widetilde{u}_{NT} + \alpha\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y), \\ (\widehat{\delta}\mathscr{C}\widetilde{u})_{NT} &= \alpha^{-1}\delta_Y\widetilde{u}_{NT} + \partial_\alpha^*\widetilde{u}_{NN} \in \alpha^{-1}\delta_Y\widetilde{u}_{NT} + \mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y). \end{aligned}$$

Notice here that $\alpha^{-1}\partial_\alpha = 2\partial_\mu$ indeed preserves elements of $\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y)$. Now in (7.3.33), the pairing corresponding to the $(1,2)$-component of (7.3.34) is of the form (7.3.29) (recall that the volume density is $\alpha|dh| = \alpha\beta\, d\alpha|dk|$) and hence vanishes in the limit $\epsilon \to 0$, and we conclude that

$$\begin{aligned}\lim_{\epsilon\to 0}\langle\widehat{\delta}u,[\widehat{\delta},\chi_\epsilon]u\rangle &= -\langle\delta_Y\widetilde{u}_{TT}|_Y,\widetilde{u}_{NT}|_Y\rangle + 2\beta^{-2}\langle\widetilde{u}_{TN}|_Y,\widetilde{u}_{NT}|_Y\rangle \\ &\quad - \beta\langle(\alpha^{-1}\partial_\alpha^*\widetilde{u}_{NT})|_Y,\widetilde{u}_{NT}|_Y\rangle - \beta^{-1}\langle\delta_Y\widetilde{u}_{NT}|_Y,\widetilde{u}_{NN}|_Y\rangle,\end{aligned} \tag{7.3.35}$$

where we use the $L^2(Y,|dk|;K)$ inner product on the right hand side; we absorbed the factor of β from the volume density $\alpha\beta\,d\alpha|dk|$ into the functions in the pairings.

In a similar vein, we can use (7.3.12) to compute

$$[\widehat{d}\mathscr{C},\chi_\epsilon] = \epsilon^{-1}\chi'_\epsilon\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&-\alpha^{-1}&0\end{pmatrix} \tag{7.3.36}$$

and

$$\begin{aligned}(\widehat{d}\mathscr{C}\widetilde{u})_{TN} &= \partial_\alpha\widetilde{u}_{TT} - \alpha d_Y\widetilde{u}_{TN} - \alpha^{-1}d_Y\beta\widetilde{u}_{NT} \\ &\in -\beta\alpha^{-1}d_Y\widetilde{u}_{NT} + \mathcal{C}^\infty(\overline{X}_{\text{even}};\Lambda Y), \\ (\widehat{d}\mathscr{C}\widetilde{u})_{NN} &= -\alpha^{-1}\partial_\alpha\widetilde{u}_{NT} + d_Y\widetilde{u}_{NN}.\end{aligned}$$

Correspondingly,

$$\begin{aligned}\lim_{\epsilon\to 0}\langle\widehat{d}u,[\widehat{d},\chi_\epsilon]u\rangle &= -\langle d_Y\widetilde{u}_{NT}|_Y,\widetilde{u}_{TT}|_Y\rangle + \beta^{-1}\langle(\alpha^{-1}\partial_\alpha\widetilde{u}_{NT})|_Y,\widetilde{u}_{NT}|_Y\rangle \\ &\quad - \beta^{-1}\langle d_Y\widetilde{u}_{NN}|_Y,\widetilde{u}_{NT}|_Y\rangle,\end{aligned} \tag{7.3.37}$$

where we again use the $L^2(Y,|dk|;K)$ inner product on the right hand side; notice with regard to the powers of β that on the (TN) and (NN) components, the fiber inner product is $\beta^{-2}K$.

As a consequence of these computations, we conclude that the pairings in (7.3.33) stay bounded as $\epsilon\to 0$, hence $\widehat{d}u,\widehat{\delta}u\in L^2(\alpha|dh|;H\oplus H)$ by Fatou's Lemma. Looking at the most singular terms of $\widehat{d}\mathscr{C}\widetilde{u}$ and $\widehat{\delta}\mathscr{C}\widetilde{u}$ (again using (7.3.12) and (7.3.13)), this necessitates

$$d_Y\widetilde{u}_{NT}|_Y = 0, \quad \delta_Y\widetilde{u}_{NT}|_Y = 0. \tag{7.3.38}$$

Therefore, taking (7.3.35) and (7.3.37) into account, (7.3.33) simplifies to

$$\begin{aligned}0 = \|\widehat{\delta}u\|^2 + &\|\widehat{d}u\|^2 + \beta^{-1}\langle(\alpha^{-1}\partial_\alpha\widetilde{u}_{NT})|_Y, \widetilde{u}_{NT}|_Y\rangle \\ &- \beta\langle(\alpha^{-1}\partial_\alpha^*\widetilde{u}_{NT})|_Y, \widetilde{u}_{NT}|_Y\rangle + 2\beta^{-2}\langle\widetilde{u}_{TN}|_Y, \widetilde{u}_{NT}|_Y\rangle.\end{aligned} \tag{7.3.39}$$

Moreover, the fourth component of the equation $(\widehat{d}+\widehat{\delta})\mathscr{C}\widetilde{u} = 0$ yields

$$-(\alpha^{-1}\partial_\alpha\widetilde{u}_{NT})|_Y + d_Y\widetilde{u}_{NN}|_Y - \delta_Y\widetilde{u}_{NN}|_Y = 0,$$

which we can pair with $\widetilde{u}_{NT}|_Y$ relative to $L^2(Y, |dk|; K)$, and then an integration by parts together with (7.3.38) shows that the first boundary pairing in (7.3.39) vanishes. Likewise, the first component of $(\widehat{d}+\widehat{\delta})\mathscr{C}\widetilde{u} = 0$ gives

$$d_Y\widetilde{u}_{TT}|_Y - \delta_Y\widetilde{u}_{TT}|_Y + 2\beta^{-2}\widetilde{u}_{TN}|_Y - \beta(\alpha^{-1}\partial_\alpha^*\widetilde{u}_{NT})|_Y = 0,$$

which we can again pair with $\widetilde{u}_{NT}|_Y$, and in view of (7.3.38), we conclude that the second line of (7.3.39) vanishes as well. Thus, finally, (7.3.39) implies that $\widehat{d}u = 0$ and $\widehat{\delta}u = 0$.

Conversely, every $u \in \mathcal{C}^\infty_{(0)}$ satisfying $\widehat{d}u = 0$ and $\widehat{\delta}u = 0$ trivially lies in the kernel of $\widehat{d}+\widehat{\delta}$. ☐

The above proof in particular shows:

Corollary 7.3.9. *Let $u = \mathscr{C}\widetilde{u} \in \mathcal{C}^\infty_{(0)}$ be such that $\widehat{d}\,\widehat{\delta}u = 0$ (resp. $\widehat{\delta}\,\widehat{d}u = 0$), and assume that $\widetilde{u}_{NT}|_Y = 0$, or equivalently $u \in L^2(\alpha|dh|)$. Then $\widehat{\delta}u = 0$ (resp. $\widehat{d}u = 0$). In particular,* $\ker_{\mathcal{C}^\infty_{(0)}\cap L^2}\widehat{\Box} = \mathcal{H}\cap L^2$.

Proof. Suppose $\widehat{d}\,\widehat{\delta}u = 0$. With a cutoff function χ_ϵ as above, we obtain

$$0 = -\lim_{\epsilon\to 0}\langle\chi_\epsilon\widehat{d}\,\widehat{\delta}u, u\rangle = \lim_{\epsilon\to 0}\|\chi_\epsilon^{1/2}\widehat{\delta}u\|^2 + \lim_{\epsilon\to 0}\langle\widehat{\delta}u, [\widehat{\delta}, \chi_\epsilon]u\rangle.$$

In view of (7.3.35) and $\widetilde{u}_{NT}|_Y = 0$, the second term on the right hand side vanishes, and we deduce $\widehat{\delta}u = 0$. The proof that $\widehat{\delta}\,\widehat{d}u = 0$ implies $\widehat{d}u = 0$ is similar and uses (7.3.37). ☐

Corollary 7.3.10. *We have* $\ker\widehat{\Box} = \ker\widehat{d}\,\widehat{\delta}\cap\ker\widehat{\delta}\,\widehat{d}$.

Proof. If $u \in \ker\widehat{\Box}$, then $(\widehat{d}+\widehat{\delta})u \in \mathcal{H}$, thus $\widehat{\delta}(\widehat{d}+\widehat{\delta})u = \widehat{\delta}\,\widehat{d}u = 0$ and $\widehat{d}\,\widehat{\delta}u = 0$. ☐

We record another setting in which the boundary terms in the proof of Proposition 7.3.8 vanish:

Lemma 7.3.11. *Suppose $v \in \mathcal{C}^\infty_{(0)}$ is a solution of $\widehat{\delta}\,\widehat{\widehat{d}\,\widehat{\delta}}v = 0$. Then $\widehat{\widehat{d}\,\widehat{\delta}}v = 0$. Likewise, if $v \in \mathcal{C}^\infty_{(0)}$ is a solution of $\widehat{\widehat{d}\,\widehat{\delta}}\,\widehat{d}v = 0$, then $\widehat{\delta}\,\widehat{d}v = 0$.*

Proof. Write $w = \widehat{\delta}v \in \mathcal{C}^\infty_{(0)}$. Then $\widehat{\delta}\,\widehat{d}w = 0$ implies, by the proof of Proposition 7.3.8, that $\widehat{d}w \in L^2(\alpha|dh|; H \oplus H)$. Writing $w = \mathscr{C}\widetilde{w}$, this in particular implies $d_Y \widetilde{w}_{NT}|_Y = 0$; but writing $v = \mathscr{C}\widetilde{v}$, we have

$$\widetilde{w}_{NT} = (\mathscr{C}^{-1}\widehat{\delta}\mathscr{C}\widetilde{v})_{NT} = \delta_Y \widetilde{v}_{NT} + \alpha\partial^*_\alpha \widetilde{v}_{NN},$$

as follows from (7.3.15). Restricting to Y, we thus have $\widetilde{w}_{NT}|_Y = \delta_Y\widetilde{v}_{NT}|_Y$, and hence $0 = d_Y\delta_Y\widetilde{v}_{NT}|_Y$. We pair this in $L^2(Y, |dk|; K)$ with $\widetilde{v}_{NT}$ and integrate by parts, obtaining $\delta_Y\widetilde{v}_{NT}|_Y = 0$. But this implies that $\widetilde{w}_{NT}|_Y = 0$. By Corollary 7.3.9, this gives $\widehat{d}w = \widehat{\widehat{d}\,\widehat{\delta}}v = 0$.

For the second part, we proceed analogously: Letting $w = \widehat{d}v \in \mathcal{C}^\infty_{(0)}$, we have $\widehat{\widehat{d}\,\widehat{\delta}}w = 0$, thus $\widehat{\delta}w \in L^2$. This gives $\delta_Y\widetilde{w}_{NT}|_Y = 0$; but by (7.3.14), $\widetilde{w}_{NT}|_Y = -d_Y\widetilde{v}_{NT}|_Y$, therefore $\delta_Y\widetilde{w}_{NT}|_Y = 0$ implies $d_Y\widetilde{v}_{NT}|_Y = 0$, so $\widetilde{w}_{NT}|_Y = 0$, which in turn gives $\widehat{\delta}w = 0$ by Corollary 7.3.9, hence $\widehat{\delta}\,\widehat{d}v = 0$. □

7.3.4 Cohomological interpretation of zero resonant states

In this section, we will always work with $\sigma = 0$ and hence simply write $\widehat{d} = \widehat{d}(0)$, $\widehat{\delta} = \widehat{\delta}(0)$, $\widetilde{d} = \widetilde{d}(0)$, $\widetilde{\delta} = \widetilde{\delta}(0)$, $\widehat{\Box} = \widehat{\Box}(0)$ and $\widetilde{\Box} = \widetilde{\Box}(0)$.

The space $\mathcal{H}$ defined in Proposition 7.3.8 is graded by the form degree, i.e.

$$\mathcal{H} = \bigoplus_{k=0}^{n} \mathcal{H}^k, \tag{7.3.40}$$

where $\mathcal{H}^k$ is the space of all $u \in \mathcal{H}$ of pure form degree k. In the decomposition (7.3.1), this means that u_T is a differential k-form on X, and u_N is a differential $(k-1)$-form. Likewise, $\mathcal{K} := \ker\widehat{\Box}$ is graded by form degree, and we write

$$\ker_{\mathcal{C}^\infty_{(0)}} \widehat{\Box} = \bigoplus_{k=0}^{n} \mathcal{K}^k. \tag{7.3.41}$$

We aim to relate the spaces $\mathcal{H}^k$ and $\mathcal{K}^k$ to certain cohomology groups associated with $\overline{X}$. As in the Riemannian setting, the central tool is a Hodge type decomposition adapted to $\widehat{d}$ and $\widehat{\delta}$:

Lemma 7.3.12. *The following Hodge type decomposition holds on X:*

$$\mathcal{C}^\infty_{(0)} = \ker_{\mathcal{C}^\infty_{(0)}} \widehat{\Box} \oplus \operatorname{ran}_{\mathcal{C}^\infty_{(0)}} \widehat{\Box}. \tag{7.3.42}$$

Proof. We first claim that such a decomposition holds on $\widetilde{X}$, i.e. we claim that

$$\mathcal{C}^\infty(\widetilde{X}) = \ker \widetilde{\Box} \oplus \operatorname{ran} \widetilde{\Box}. \tag{7.3.43}$$

First of all, since $\widetilde{\Box}$ is Fredholm with index 0, its range is closed, and the codimension of the range equals the dimension of the kernel. Hence, in order to show (7.3.43), we merely need to check that the intersection of $\ker \widetilde{\Box}$ and $\operatorname{ran} \widetilde{\Box}$ is trivial. Thus, let $\widetilde{u} \in \ker \widetilde{\Box} \cap \operatorname{ran} \widetilde{\Box}$, and write $\widetilde{u} = \widetilde{\Box}\widetilde{v}$. Let $v = \widetilde{v}|_X$. Then $\widetilde{u} \in \ker \widetilde{\Box}$ means, restricting to X and using Corollary 7.3.10, that $\widehat{d}\,\widehat{\delta}\,\widehat{d}\,\widehat{\delta} v = 0$ and $\widehat{\delta}\,\widehat{d}\,\widehat{\delta}\,\widehat{d} v = 0$. Repeated application of Lemma 7.3.11 thus implies $\widehat{\delta}\,\widehat{d} v = 0$ and $\widehat{d}\,\widehat{\delta} v = 0$, hence $\widetilde{\delta}\,\widetilde{d}\widetilde{v}$ and $\widetilde{d}\,\widetilde{\delta}\widetilde{v}$ are supported in $\widetilde{X} \setminus X$. (This argument shows the uniqueness of the decomposition (7.3.42).) Therefore $\widetilde{u}$ is a solution of $\widetilde{\Box}\widetilde{u} = 0$ which is supported in $\widetilde{X} \setminus X$. By unique continuation at infinity on the asymptotically de Sitter side $\widetilde{X} \setminus X$ of $\widetilde{X}$, this implies $\widetilde{u} \equiv 0$, as claimed.

Now if $u \in \mathcal{C}^\infty_{(0)}$ is given, extend it arbitrarily to $\widetilde{u} \in \mathcal{C}^\infty(\widetilde{X})$, apply (7.3.43) and restrict both summands back to X. This establishes (7.3.42). $\Box$

Remark 7.3.13. The decomposition (7.3.42) does *not* hold if we replace $\widehat{\Box}$ in (7.3.42) by $\widehat{d}+\widehat{\delta}$. Indeed, if it did hold, this would say that $\widehat{\Box} u = 0$ implies $(\widehat{d}+\widehat{\delta})u = 0$, since $(\widehat{d}+\widehat{\delta})u$ lies both in $\ker(\widehat{d}+\widehat{\delta})$ and $\operatorname{ran}(\widehat{d}+\widehat{\delta})$ in this case. Since certainly $(\widehat{d}+\widehat{\delta})u = 0$ conversely implies $\widehat{\Box} u = 0$, this would mean that $\ker \widehat{\Box} = \ker(\widehat{d}+\widehat{\delta})$. Now by Lemmas 7.2.1 and 7.2.2, this in turn would give $\ker \widetilde{\Box} = \ker(\widetilde{d}+\widetilde{\delta})$. Now since $\widetilde{\Box}$ and $\widetilde{d}+\widetilde{\delta}$ are Fredholm with index 0, we could further deduce $\ker \widetilde{\Box}^* = \ker(\widetilde{d}+\widetilde{\delta})^*$, where the adjoints act on the space $\dot{\mathcal{C}}^{-\infty}(\widetilde{X})$ of supported distributions at the (artificial) Cauchy hypersurface $\partial\widetilde{X}$, see [64, Appendix B]. Since we have $\ker(\widetilde{d}+\widetilde{\delta})^* \subset \ker \widetilde{\Box}^*$ unconditionally, we can show the absurdity of this last equality by exhibiting an element u in $\ker \widetilde{\Box}^*$ which does not lie in $\ker(\widetilde{d}+\widetilde{\delta})^*$. This however is easy: Just let $u = 1_X$ be the characteristic function of X. Then from (7.3.14) and (7.3.15), we see that $(\widetilde{d}+\widetilde{\delta})u = \widetilde{d}u$ is a non-zero delta distribution supported at Y which is annihilated by $\widetilde{\delta}$.

This argument shows that we always have $\ker \widehat{\Box} \supsetneq \ker(\widehat{d}+\widehat{\delta})$. It is possible though that $\mathcal{H}^k = \mathcal{K}^k$ for *some* form degrees k (but this must fail for some value of k). For instance, this holds for $k=0$ by Corollary 7.3.9. We will give a more general statement below, see in particular Remark 7.3.18.

We now define a complex whose cohomology we will relate to the spaces $\mathcal{H}^k$ and $\mathcal{K}^k$: The space $\mathcal{C}^\infty_{(0)} \cap L^2(\alpha|dh|)$ of smooth forms $u = \mathscr{C}\widetilde{u}$ with $\widetilde{u}_{NT}|_Y = 0$ has a grading corresponding to form degrees, thus

$$\mathcal{D} := \mathcal{C}^\infty_{(0)} \cap L^2(\alpha|dh|) = \bigoplus_{k=0}^n \mathcal{D}^k.$$

Since in the above notation $u \in L^2(\alpha|dh|)$ (and thus $\widetilde{u}_{NT}|_Y = 0$) is equivalent to $\widetilde{u}_{NT} \in \alpha^2\mathcal{C}^\infty(\overline{X}_{\mathrm{even}}; \Lambda Y)$ near Y, one can easily check using (7.3.14) that $\widehat{d}$ acts on $\mathcal{C}^\infty_{(0)} \cap L^2(\alpha|dh|)$. We can then define the complex

$$0 \to \mathcal{D}^0 \xrightarrow{\widehat{d}} \mathcal{D}^1 \to \ldots \xrightarrow{\widehat{d}} \mathcal{D}^n \to 0.$$

We denote its cohomology by

$$\mathcal{H}^k_{L^2,\mathrm{dR}} = \ker(\widehat{d}\colon \mathcal{D}^k \to \mathcal{D}^{k+1})/\operatorname{ran}(\widehat{d}\colon \mathcal{D}^{k-1} \to \mathcal{D}^k). \tag{7.3.44}$$

There is a natural map from $\mathcal{H}^k_{L^2,\mathrm{dR}}$ into $\mathcal{H}^k$:

Lemma 7.3.14. *Every cohomology class $[u] \in \mathcal{H}^k_{L^2,\mathrm{dR}}$ has a unique representative $u' \in \mathcal{H}^k$, and the map $i\colon [u] \mapsto u'$ is injective.*

Proof. Let $[u] \in \mathcal{H}^k_{L^2,\mathrm{dR}}$, hence $\widehat{d}u = 0$ and, writing $u = \mathscr{C}\widetilde{u}$, $\widetilde{u}_{NT}|_Y = 0$. We first show the existence of a representative, i.e. an element $u - \widehat{d}v$ with $v \in \mathcal{D}$, which is annihilated by $\widehat{\delta}$. (Since it is clearly annihilated by $\widehat{d}$, this means $u - \widehat{d}v \in \mathcal{H}^k$.) That is, we need to solve the equation $\widehat{\delta}\,\widehat{d}v = \widehat{\delta}u$ with $v \in \mathcal{D}$. To achieve this, we use Lemma 7.3.12 to write

$$u = u_1 + (\widehat{d}\,\widehat{\delta} + \widehat{\delta}\,\widehat{d})u_2, \quad u_1 \in \ker \widehat{\Box}.$$

By our assumption on u and Corollary 7.3.10, u and u_1 are annihilated by $\widehat{\delta}\,\widehat{d}$, giving $\widehat{\delta}\,\widehat{d}\,\widehat{\delta}\,\widehat{d}u_2 = 0$. By Lemma 7.3.11, this implies $\widehat{\delta}\,\widehat{d}u_2 = 0$, hence

$$u = u_1 + \widehat{d}\,\widehat{\delta}u_2. \tag{7.3.45}$$

Applying $\widehat{d}\widehat{\delta}$, we obtain

$$\widehat{d}\widehat{\delta}\widehat{d}\widehat{\delta}u_2 = \widehat{d}\widehat{\delta}u \in L^2. \tag{7.3.46}$$

Now writing $u_2 = \mathscr{C}\widetilde{u}_2$, and noting that for any $w = \mathscr{C}\widetilde{w} \in \mathcal{C}^\infty_{(0)}$, $(\mathscr{C}^{-1}\widehat{d}\mathscr{C}\widetilde{w})_{NT}|_Y = -d_Y\widetilde{w}_{NT}|_Y$ as well as $(\mathscr{C}^{-1}\widehat{\delta}\mathscr{C}\widetilde{w})_{NT}|_Y = \delta_Y\widetilde{w}_{NT}|_Y$ by (7.3.14) and (7.3.15), the (NT) component of $\mathscr{C}^{-1}$ times equation (7.3.46) reads $d_Y\delta_Y d_Y\delta_Y\widetilde{u}_{2,NT}|_Y = 0$, which yields $\delta_Y\widetilde{u}_{2,NT}|_Y = 0$. As a consequence of this, $v := \widehat{\delta}u_2 \in L^2$ and therefore $\widehat{d}\widehat{\delta}u_2 \in L^2$. Hence (7.3.45) gives $u_1 \in L^2$; by Corollary 7.3.9 then, $u_1 \in \mathcal{H}$, in particular u_1 is annihilated by $\widehat{\delta}$. Therefore, applying $\widehat{\delta}$ to (7.3.45) yields $\widehat{\delta}(u - \widehat{d}v) = 0$, as desired.

Next, we show that the representative is unique: Thus, suppose $u - \widehat{d}v_1, u - \widehat{d}v_2 \in \mathcal{H}^k$ with $u, v_1, v_2 \in \mathcal{D}$, then with $v = v_1 - v_2 \in \mathcal{D}$, we have $\widehat{d}v \in \mathcal{H}^k$, thus $\widehat{\delta}\widehat{d}v = 0$, and by Corollary 7.3.9, we obtain $\widehat{d}v = 0$. Therefore, $u - \widehat{d}v_1 = u - \widehat{d}v_2$, establishing uniqueness, which in particular shows that the map i is well-defined.

Finally, we show the injectivity of i: Suppose $u \in \mathcal{D}$ satisfies $\widehat{d}u = 0$. There exists an element $v \in \mathcal{D}$ such that $u - \widehat{d}v \in \mathcal{H}^k$. Now if $i[u] = 0$, this precisely means that $u - \widehat{d}v = 0$; but then $[u] = [\widehat{d}v] = 0$ in $\mathcal{H}^k_{L^2,\mathrm{dR}}$. □

From the definition of the space $\mathcal{D}$, it is clear that $u \in \mathcal{H}^k$ lies in the image of i if and only if $u \in L^2$, i.e. if and only if $r(u) = 0$, where r is the map

$$r\colon \mathcal{C}^\infty_{(0)} \to \mathcal{C}^\infty(Y;\Lambda Y), \quad u = \mathscr{C}\widetilde{u} \mapsto \widetilde{u}_{NT}|_Y. \tag{7.3.47}$$

Thus, r extracts the singular part of u and thereby measures the failure of a given form $u \in \mathcal{C}^\infty_{(0)}$ to lie in $\mathcal{D}$. Observe that if $u = \mathscr{C}\widetilde{u} \in \mathcal{H}^k$, then $d_Y\widetilde{u}_{NT}|_Y = 0$ and $\delta_Y\widetilde{u}_{NT}|_Y = 0$, i.e. $r(u)$ is a harmonic form on Y. Since the space $\ker(\Delta_{Y,k-1})$ of harmonic forms on the closed manifold Y is isomorphic to the cohomology group $H^{k-1}(Y)$ by standard Hodge theory, we thus obtain:

Proposition 7.3.15. *The sequence*

$$0 \to \mathcal{H}^k_{L^2,\mathrm{dR}} \xrightarrow{i} \mathcal{H}^k \xrightarrow{r} H^{k-1}(Y) \tag{7.3.48}$$

is exact. Here, i is the map defined in Lemma 7.3.14, and r is the restriction map (7.3.47) (composed with the identification $\ker(\Delta_{Y,k-1}) \cong H^{k-1}(Y)$). Moreover, the map $i\colon \mathcal{H}^k_{L^2,\mathrm{dR}} \to \mathcal{H}^k \cap \mathcal{D}$ is an isomorphism with inverse $\mathcal{H}^k \cap \mathcal{D} \ni u \mapsto [u] \in \mathcal{H}^k_{L^2,\mathrm{dR}}$.

Proof. We only need to check the last claim. If $u \in \mathcal{H}^k \cap \mathcal{D}$, then $[u]$ does define a cohomology class in $\mathcal{H}^k_{L^2,\mathrm{dR}}$, and $i([u])$ is the unique representative of $[u]$ which lies in $\mathcal{H}^k$. Since u itself is such a representative, we must have $i([u]) = u$. For the converse, we note that for any $[u] \in \mathcal{H}^k_{L^2,\mathrm{dR}}$ we have $i([u]) = u - \widehat{d}v$ for some $v \in \mathcal{D}$, hence $[i([u])] = [u - \widehat{d}v] = [u]$. □

We can make a stronger statement: If we merely have $u \in \ker \widehat{\Box}$, then the proof of Proposition 7.3.8 shows that $\widehat{d}u, \widehat{\delta}u \in L^2$, hence $r(u)$ is harmonic.

Proposition 7.3.16. *We have a short exact sequence*

$$0 \to \mathcal{H}^k_{L^2,\mathrm{dR}} \xrightarrow{i} \mathcal{K}^k \xrightarrow{r} H^{k-1}(Y) \to 0, \tag{7.3.49}$$

where the first map is i defined in Lemma 7.3.14 (composed with the inclusion $\mathcal{H}^k \hookrightarrow \mathcal{K}^k$), and the second map is the restriction r, defined in (7.3.47) (composed with the identification $\ker(\Delta_{Y,k-1}) \cong H^{k-1}(Y)$).

Proof. The second map is well-defined by the comment preceding the statement of the proposition. Since the range of $\mathcal{H}^k_{L^2,\mathrm{dR}}$ in $\mathcal{K}^k$ consists of L^2 forms, we have $r \circ i = 0$. Moreover, if $u \in \ker r$, then u is an L^2 element of $\ker \widehat{\Box}$, thus $u \in \mathcal{H}^k$ by Corollary 7.3.9. By the remark following the proof of Lemma 7.3.14, therefore $u \in \operatorname{ran} i$.

It remains to show the surjectivity of r: Thus, let $w \in \ker(\Delta_{Y,k-1})$, and let $u' = \mathscr{C}\widetilde{u}' \in \mathcal{C}^\infty_{(0)}$ be any extension of w, i.e. $\widetilde{u}'_{NT}|_Y = w$. Then $(\widehat{d} + \widehat{\delta})u' \in \mathcal{D}$, since its (NT) component vanishes, and thus $\widehat{\Box}u' \in \mathcal{D}$. Writing $u' = u_1 + \widehat{\Box}u_2$ with $u_1 \in \ker \widehat{\Box}$, we conclude that $\widehat{\Box}u' = \widehat{\Box}^2 u_2$; taking the (NT) component of this equation gives $0 = \Delta_Y^2 \widetilde{u}_{2,NT}|_Y$ (where we write $u_2 = \mathscr{C}\widetilde{u}_2$ as usual), hence $d_Y \widetilde{u}_{2,NT}|_Y = 0$ and $\delta_Y \widetilde{u}_{2,NT}|_Y = 0$. But then $\widehat{\Box}u_2 \in L^2$. Therefore, $w = r(u') = r(u_1 + \widehat{\Box}u_2) = r(u_1)$. Since the degree k part of u_1 lies in $\mathcal{K}^k$ by the definition of u_1, we are done. □

Remark 7.3.17. Remark 7.3.13, which states that $\mathcal{H}^k \subsetneq \mathcal{K}^k$ for some values of k, implies in particular that the last map of (7.3.48) is not always onto.

Remark 7.3.18. Since $\dim Y = n - 2$, we have $H^{k-1}(Y) = 0$ for $k = 0$ and $k = n$. Hence, for these extreme values of k, Propositions 7.3.15 and 7.3.16 show $\mathcal{H}^k = \mathcal{K}^k \cong \mathcal{H}^k_{L^2,\mathrm{dR}}$, and this holds more generally for all k for which $H^{k-1}(Y) = 0$.

The spaces $\mathcal{H}^k_{L^2,\mathrm{dR}}$ are related to standard cohomology groups associated with the manifold with boundary $\overline{X}$: First, notice that elements of the space $\mathcal{D} = \mathcal{C}^\infty_{(0)} \cap L^2$ are not subject to any matching condition on singular terms, simply because the singular term ($\widetilde{u}_{NT}|_Y$ in the notation used above) vanishes. This means that we can split $\mathcal{D}$ into tangential and normal forms, $\mathcal{D} = \mathcal{D}_T \oplus \mathcal{D}_N$, thereby identifying elements $(u_T, u_N) \in \mathcal{D}_T \oplus \mathcal{D}_N$ with $u_T + \alpha\, dt \wedge u_N \in \mathcal{D}$, where $\mathcal{D}_T$ consists of all $u_T \in \mathcal{C}^\infty(\overline{X}; \Lambda\overline{X})$ which are of the form

$$u_T = \begin{pmatrix} u_{TT} \\ \alpha u_{TN} \end{pmatrix}, \quad u_{TT}, u_{TN} \in \mathcal{C}^\infty(\overline{X}_{\text{even}}; \Lambda Y),$$

near Y. Thus, elements $u_T \in \mathcal{D}_T$ are forms of the type $u_T = u_{TT} + d\alpha \wedge \alpha u_{TN} = u_{TT} + \frac{1}{2} d\mu \wedge u_{TN}$ with u_{TT}, u_{TN} smooth ΛY-valued forms on $\overline{X}_{\text{even}}$; hence, we simply have $\mathcal{D}_T = \mathcal{C}^\infty(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}})$. Likewise, $\mathcal{D}_N$ consists of all $u_N \in \mathcal{C}^\infty(\overline{X}; \Lambda\overline{X})$ which are of the form

$$u_N = \begin{pmatrix} \alpha u_{NT} \\ u_{NN} \end{pmatrix}, \quad u_{NT}, u_{NN} \in \mathcal{C}^\infty(\overline{X}_{\text{even}}; \Lambda Y),$$

near Y. Thus, elements $u_N \in \mathcal{D}_T$ are forms of the type $\alpha u_N = \mu u_{NT} + \frac{1}{2} d\mu \wedge u_{NN}$; therefore, $\alpha \mathcal{D}_N = \mathcal{C}^\infty_R(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}}) := \{u \in \mathcal{C}^\infty(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}}) \colon j^* u = 0\}$, where $j \colon \partial\overline{X}_{\text{even}} \hookrightarrow \overline{X}_{\text{even}}$ is the inclusion.

Since the differential $\widehat{d}$ on $\mathcal{D}$ acts as $d_X \oplus (-\alpha^{-1} d_X \alpha)$ on $\mathcal{D}_T \oplus \mathcal{D}_N$, the cohomology of the complex $(\mathcal{D}, \widehat{d})$ in degree k is the direct sum of the cohomology of $(\mathcal{D}_T, d_X)$ in degree k and of $(\alpha\mathcal{D}_N, d_X)$ in degree $(k-1)$. Since we identified $\mathcal{D}_T$ as simply the space of smooth forms on $\overline{X}_{\text{even}}$, the cohomology of $(\mathcal{D}_T, d_X)$ in degree k equals the absolute cohomology $H^k(\overline{X}_{\text{even}}) \cong H^k(\overline{X})$. (Here, we use that $\overline{X}_{\text{even}}$ is diffeomorphic to $\overline{X}$, with diffeomorphism given by gluing the map $\alpha^2 \mapsto \alpha$ near Y to the identity map away from Y.) Moreover, since $\mathcal{D}_N$ is the space of smooth forms on $\overline{X}_{\text{even}}$ which vanish at the boundary in the precise sense described above, the cohomology of $(\alpha\mathcal{D}_N, d_X)$ in degree k equals the relative cohomology $H^k(\overline{X}_{\text{even}}; \partial\overline{X}_{\text{even}}) \cong H^k(\overline{X}; \partial\overline{X})$ (see e.g. [108, §5.9]). In summary:

Proposition 7.3.19. *With $\mathcal{H}^k_{L^2,\text{dR}}$ defined in (7.3.44), there is a canonical isomorphism*

$$\mathcal{H}^k_{L^2,\text{dR}} \cong H^k(\overline{X}) \oplus H^{k-1}(\overline{X}, \partial\overline{X}). \tag{7.3.50}$$

Let us summarize the results obtained in the previous sections:

Theorem 7.3.20. *The only resonance of $d + \delta$ in $\operatorname{Im}\sigma \geq 0$ is $\sigma = 0$, and 0 is a simple* *resonance. Zero resonant states of the extended operator ($d+\delta$ on $\widetilde{M}$) are uniquely determined by their restriction to X, and the space $\mathcal{H}$ of these resonant states on X is equal to $\ker_{\mathcal{C}^\infty_{(0)}} \widehat{d}(0) \cap \ker_{\mathcal{C}^\infty_{(0)}} \widehat{\delta}(0)$. Also, resonant states on $\widetilde{X}$ are elements of $\ker \widetilde{d}(0) \cap \ker \widetilde{\delta}(0)$. Using the grading $\mathcal{H} = \bigoplus_{k=0}^n \mathcal{H}^k$ of $\mathcal{H}$ by form degrees, there is a canonical exact sequence*

$$0 \to H^k(\overline{X}) \oplus H^{k-1}(\overline{X}, \partial\overline{X}) \to \mathcal{H}^k \to H^{k-1}(\partial\overline{X}), \tag{7.3.51}$$

where the first map is the composition of the isomorphism (7.3.50) with the map i defined in Lemma 7.3.14, and the second map is the composition of the map r defined in (7.3.47) with the isomorphism $\ker(\Delta_{\partial\overline{X},k-1}) \cong H^{k-1}(\partial\overline{X})$.

Furthermore, the only resonance of $\Box_g$ in $\operatorname{Im}\sigma \geq 0$ is $\sigma = 0$. Elements of $\ker \widetilde{\Box}(0)$, which are the zero resonant states of the extended operator ($\Box_g$ on $\widetilde{M}$) if the zero resonance is simple, are uniquely determined by their restriction to X. The space $\mathcal{K} = \bigoplus_{k=0}^n \mathcal{K}^k \subset \mathcal{C}^\infty_{(0)}$ of these resonant states on X, graded by form degree, satisfying $\mathcal{K}^k \supset \mathcal{H}^k$, fits into the short exact sequence

$$0 \to H^k(\overline{X}) \oplus H^{k-1}(\overline{X}, \partial\overline{X}) \to \mathcal{K}^k \to H^{k-1}(\partial\overline{X}) \to 0, \tag{7.3.52}$$

with maps as above. We moreover have

$$\mathcal{K}^k \cap L^2 = \mathcal{H}^k \cap L^2 \cong H^k(\overline{X}) \oplus H^{k-1}(\overline{X}, \partial\overline{X})$$

where $L^2 = L^2(X, \alpha|dh|; H \oplus H)$. More precisely then, the summand $H^k(\overline{X})$ in (7.3.51) and (7.3.52) corresponds to the tangential components (in the decomposition (7.3.1)) of elements of $\mathcal{H}^k \cap L^2$, and the summand $H^{k-1}(\overline{X}, \partial\overline{X})$ to the normal components.

Lastly, the Hodge star operator on M induces isomorphisms $\mathcal{H}^k \xrightarrow{\cong} \mathcal{H}^{n-k}$ and $\mathcal{K}^k \xrightarrow{\cong} \mathcal{K}^{n-k}$, $k = 0, \ldots, n$.

Proof. We prove the statement about resonant states for $d+\delta$ on the extended space $\widetilde{M}$: Thus, if $\widetilde{u} \in \ker(\widetilde{d}(0) + \widetilde{\delta}(0))$, then the restriction of $\widetilde{u}$ to X lies in $\ker \widehat{d}(0) \cap \ker \widehat{\delta}(0)$, therefore $\widetilde{d}(0)\widetilde{u} = -\widetilde{\delta}(0)\widetilde{u}$ is supported in $\widetilde{X} \setminus X$; but then $\widetilde{\Box}(0)(\widetilde{d}(0)\widetilde{u}) = \widetilde{d}(0)\widetilde{\delta}(0)\widetilde{d}(0)\widetilde{u} = 0$ and the asymptotically de Sitter nature of $\widetilde{X} \setminus X$ implies $\widetilde{d}(0)\widetilde{u} \equiv 0$, hence also $\widetilde{\delta}(0)\widetilde{u} \equiv 0$, as claimed.

The only remaining part of the statement that has not yet been proved is the last: Viewing $u \in \mathcal{H}^k$ as a t-independent k-form on $M = \mathbb{R}_t \times X$ (with the metric (7.1.1)), we have $(d+\delta)u = 0$, and for any t-independent k-form u on M, we have that $(d+\delta)u = 0$ implies $u \in \mathcal{H}^k$, where we view the t-independent form as a form on X valued in the form bundle of M, as explained in §7.2. Then $u \in \mathcal{H}^k$ is equivalent to $du = 0$, $\delta u = 0$, which in turn is equivalent to $\delta(\star u) = 0$, $d(\star u) = 0$, and thus $\star u \in \mathcal{H}^{n-k}$. The proof for the spaces $\mathcal{K}^k$ is the same and uses $\star\Box = \Box\star$. □

This in particular proves Theorem 7.1.3.

7.4 Results for static de Sitter and Schwarzschild-de Sitter spacetimes

We now supplement the results obtained in the previous section by high energy estimates for the inverse normal operator family from Chapter 6 and deduce expansions and decay for solutions to Maxwell's equations as well as for more general linear waves on de Sitter and Schwarzschild-de Sitter backgrounds. The rather detailed description of asymptotics in the Schwarzschild-de Sitter setting will be essential in our discussion of Kerr-de Sitter space in §7.5.

7.4.1 de Sitter space

We recall from §2.2 that de Sitter space is the hyperboloid $\{z_1^2 + \cdots + z_n^2 - z_{n+1}^2 = 1\}$ in $(n+1)$-dimensional Minkowski space, equipped with the induced Lorentzian metric. Introducing a boundary defining function $\tau = z_{n+1}^{-1}$ of future infinity, and adding the $\tau = 0$ to the spacetime, we obtain the bordified space $N = [0,1)_\tau \times Z$ with $Z = \mathbb{S}^{n-1}$, modifying τ slightly, the metric has the form

$$g^0 = \tau^{-2}\bar{g}, \quad \bar{g} = d\tau^2 - h^0(\tau, x, dx),$$

with h^0 even in τ, i.e. h^0 is a metric on Z which depends smoothly on τ^2; this is of course in particular an example of an even asymptotically de Sitter-like space, see §2.2.2. Thus, g^0 is a 0-metric in the sense of Mazzeo and Melrose [81]. Fixing a point p at future infinity, the static model of de Sitter space, which we denote by M here, consistent with the notation in this chapter (see however Remark 7.1.2), is the interior of the backward light cone from p. We introduce static coordinates on M, denoted $(t,x) \in \mathbb{R} \times X$, where $X = B_1 \subset \mathbb{R}^{n-1}$

is the open unit ball in $\mathbb{R}^{n-1}$ and $x \in \mathbb{R}^{n-1}$ are the standard coordinates on $\mathbb{R}^{n-1}$, with respect to which the induced metric on M is given by

$$g = \alpha^2\,dt^2 - h, \quad \alpha = (1-|x|^2)^{1/2},$$
$$h = dx^2 + \frac{1}{1-|x|^2}(x\cdot dx)^2 = \alpha^{-2}\,dr^2 + r^2\,d\omega^2,$$

using polar coordinates (r,ω) on $\mathbb{R}^{n-1}_x$ near $r=1$, and denoting the round metric on the unit sphere $\mathbb{S}^{n-2}$ by $d\omega^2$. We compactify X to the closed unit ball $\overline{X}_{\mathrm{even}} = \overline{B_1} \subset \mathbb{R}^{n-1}$, and denote by $\overline{X}$ the space which is $\overline{X}_{\mathrm{even}}$ topologically, but with α added to the smooth structure. In order to see that the metric g fits into the framework of Theorem 7.3.20, note that $dr = -\alpha r^{-1}\,d\alpha$, so

$$h = r^{-2}\,d\alpha^2 + r^2\,d\omega^2,$$

and $r = (1-\alpha^2)^{1/2}$, thus h is an even metric on the space $\overline{X}$ and has the form (7.1.2) with $\beta = 1$. Using Theorem 7.3.20, we can now easily compute the spaces of resonances:

Theorem 7.4.1. *On an n-dimensional static de Sitter spacetime, $n \geq 4$, the spaces of resonances of $\Box$ and $d+\delta$ are*

$$\mathcal{K}^0 = \mathcal{H}^0 = \langle 1\rangle, \quad \mathcal{K}^n = \mathcal{H}^n = \langle r^{n-2}\,dt \wedge dr \wedge \omega\rangle,$$

where ω denotes the volume form on the round sphere $\mathbb{S}^{n-2}$. Furthermore,

$$\mathcal{K}^1 = \langle -\alpha^{-2} r\,dr + \alpha^{-1}\,dt\rangle, \mathcal{H}^1 = 0, \quad \mathcal{K}^{n-1} = \langle \star(-\alpha^{-2} r\,dr + \alpha^{-1}\,dt)\rangle, \mathcal{H}^{n-1} = 0,$$
$$\mathcal{K}^k = \mathcal{H}^k = 0, \quad k = 2,\ldots,n-2.$$

In particular, on 4-dimensional static de Sitter space, if u is a solution of $(d+\delta)u = 0$ with smooth initial data, then the degree 0 component of u decays exponentially to a constant, the degree 1, 2 and 3 components decay exponentially to 0, and the degree 4 component decays exponentially to a constant multiple of the volume form. Analogous statements hold on any n-dimensional static de Sitter space, $n \geq 5$.

Proof. We compute the cohomological data that appear in (7.3.51) and (7.3.52) using $\overline{X} \cong$

$\overline{B_1}$ and $\partial\overline{X} \cong \mathbb{S}^{n-2}$:

$$\dim H^{k-1}(\partial\overline{X}) = \begin{cases} 0, & k = 0, 2, \dots, n-2, n, \\ 1, & k = 1, n-1 \end{cases}$$

$$\dim H^{k}(\overline{X}) = \begin{cases} 1, & k = 0 \\ 0, & 1 \le k \le n, \end{cases}$$

$$\dim H^{k-1}(\overline{X}, \partial\overline{X}) = \begin{cases} 0, & 0 \le k \le n-1 \\ 1, & k = n. \end{cases}$$

Thus, we immediately deduce

$$\dim \mathcal{K}^0 = \dim \mathcal{K}^1 = \dim \mathcal{K}^{n-1} = \dim \mathcal{K}^n = 1, \quad \dim \mathcal{K}^k = 0, \quad 2 \le k \le n-2,$$
$$\dim \mathcal{H}^0 = \dim \mathcal{H}^n = 1, \quad \dim \mathcal{H}^k = 0, \quad 2 \le k \le n-2.$$

Now, since $d + \delta$ annihilates constants, we find $1 \in \mathcal{K}^0 = \mathcal{H}^0$ and $\star 1 \in \mathcal{K}^n = \mathcal{H}^n$, which in view of the 1-dimensionality of these spaces already concludes their computation.

In order to compute $\mathcal{K}^1$, notice that we have $\mathcal{K}^1 \cong H^0(\partial\overline{X})$ from (7.3.52), thus an element u spanning $\mathcal{K}^1$ has non-trivial singular components at $\alpha = 0$. One is led to the guess $u = \alpha^{-1}\,d\alpha + \alpha^{-1}\,dt = -\alpha^{-2} r\,dr + \alpha^{-1}\,dt$, which is indeed annihilated by $\Box$; we will give full details for this computation in the next section when discussing Schwarzschild-de Sitter spacetimes, which in the case of vanishing black hole mass are static de Sitter spacetimes, with a point removed, see in particular the calculations following (7.4.9); but since u as defined above is smooth at $r = 0$, we obtain $\Box u = 0$ at $r = 0$ as well by continuity. Since $\mathcal{K}^1$ is 1-dimensional, we therefore deduce $\mathcal{K}^1 = \langle u \rangle$. One can then check that $(d+\delta)u \neq 0$, and this implies $\mathcal{H}^1 = 0$. The corresponding statements for $\mathcal{K}^{n-1}$ and $\mathcal{H}^{n-1}$ are immediate consequences of this and the fact that the Hodge star operator induces isomorphisms $\mathcal{H}^1 \cong \mathcal{H}^{n-1}$ and $\mathcal{K}^1 \cong \mathcal{K}^{n-1}$.

The high energy estimates for $d + \delta$ required to deduce asymptotic expansions for solutions of $(d+\delta)u = 0$ follow from those of its square $\Box$, which is principally scalar and fits directly into the framework described in [114, §2-4]. Thus, a contour deformation argument as in the proof of Theorem 5.2.3 finishes the proof. □

By studying the space of *dual resonant states* of $d + \delta$, one can in fact easily show that the 0-resonance of $\Box$ is simple and thus deduce exponential decay of smooth solutions to $\Box u = 0$ to an element of $\mathcal{K}^k$ in all form degrees $k = 0, \ldots, n$. We give details in the next section on Schwarzschild-de Sitter space.

In the present de Sitter setting, one can deduce asymptotics very easily in a different manner using the global de Sitter space picture, by analyzing indicial operators in the 0-calculus: Concretely, we write differential k-forms (by which we mean smooth sections of the k-th exterior power of the 0-cotangent bundle of N) as

$$u = \tau^{-k} u_T + \frac{d\tau}{\tau} \wedge \tau^{1-k} u_N, \tag{7.4.1}$$

where u_T and u_N are smooth forms on Z of form degrees k and $(k-1)$, respectively. One readily computes the differential d_k acting on k-forms to be

$$d_k = \begin{pmatrix} \tau d_Z & 0 \\ -k + \tau\partial_\tau & -\tau d_Z \end{pmatrix}.$$

Furthermore, by the choice of basis in (7.4.1), the inner product on k-forms induced by g^0 is given by

$$G_k^0 = \begin{pmatrix} (-1)^k H_k^0 & 0 \\ 0 & (-1)^{k-1} H_{k-1}^0 \end{pmatrix}.$$

Using that the volume density is $|dg^0| = \tau^{-n}\, d\tau |dh^0|$, we compute the codifferential δ_k acting on k-forms to be

$$\delta_k = \begin{pmatrix} -\tau\delta_Z & -(k-1) + \tau^{n-1}\tau\partial_\tau^*\tau^{1-n} \\ 0 & \tau\delta_Z \end{pmatrix} = \begin{pmatrix} -\tau\delta_Z & n - k - \tau\partial_\tau + \mathcal{O}_{\mathcal{C}^\infty(N)}(\tau) \\ 0 & \tau\delta_Z \end{pmatrix},$$

where ∂_τ^* is the $L^2(N, |d\bar{g}|)$-adjoint (suppressing the bundles in the notation) of ∂_τ, and we use the even-ness of g^0 in the second step to deduce $\partial_\tau^* = -\partial_\tau + \mathcal{O}_{\mathcal{C}^\infty(N)}(\tau)$. Therefore, the indicial roots of $d + \delta$ on the degree k-part of the form bundle are k and $n - k$.

Next, for $0 \leq k \leq n$, we compute the Hodge d'Alembertian, dealing with the cases $k = 0$ and $k = n$ simultaneously with $1 \leq k \leq n-1$ by implicitly assuming that for $k = 0$, only the $(1,1)$-part of this operator is present, acting on 0-forms, and for $k = n$, only the $(2,2)$-part is present, acting on n-forms:

$$\Box_k = d_{k-1}\delta_k + \delta_{k+1} d_k$$

$$= \begin{pmatrix} -\tau d_Z \tau \delta_Z - \tau \delta_Z \tau d_Z - P_k & \tau d_Z \\ -\tau \delta_Z & -\tau d_Z \tau \delta_Z - \tau \delta_Z \tau d_Z - P_{k-1} \end{pmatrix} + \mathcal{O}_{\mathrm{Diff}_0^1}(\tau)$$

where $P_k = (\tau\partial_\tau)^2 - (n-1)\tau\partial_\tau + k(n-k-1)$. Thus, the indicial polynomial of $\Box_k$ is

$$I(\Box_k)(s) = \begin{pmatrix} s^2 - (n-1)s + k(n-k-1) & 0 \\ 0 & s^2 - (n-1)s + (k-1)(n-k) \end{pmatrix}.$$

On tangential forms, the indicial roots of $\Box_k$ are therefore $k, n-1-k$, and on normal forms, they are $k-1, n-k$. We thus have:

form degree	0	1	$2 \leq k \leq n-2$	$n-1$	n
tgt. ind. roots	$0, n-1$	$1, n-2$	$k, n-1-k$	$0, n-1$	–
norm. ind. roots	–	$0, n-1$	$k-1, n-k$	$1, n-2$	$0, n-1$

Hence in particular, all roots are ≥ 0, and 0 is never a double root. Thus, the arguments of [111] (which are in the scalar setting, but work in the current setting as well with only minor modifications) show that solutions u to the wave equation on differential k-forms on N with smooth initial data at $\tau = \tau_0 > 0$ decay exponentially (in $-\log\tau$) if 0 is not an indicial root, and decay to a stationary state if 0 is an indicial root. (Of course, since we know all indicial roots, we could be much more precise in describing the asymptotics, but we only focus on the 0-resonance here.) Explicitly, scalar waves decay to a smooth function on Z, 1-form waves decay to an element of $\frac{d\tau}{\tau}\mathcal{C}^\infty(Z)$, k-form waves decay exponentially to 0 for $2 \leq k \leq n-2$, $(n-1)$-form waves decay to an element of $\mathcal{C}^\infty(Z; \Lambda^{n-1}Z)$, and n-form waves finally decay to an element of $\frac{d\tau}{\tau} \wedge \mathcal{C}^\infty(Z; \Lambda^{n-1}Z)$.

Since the static model of de Sitter space arises by blowing up a point p at future infinity of compactified de Sitter space and considering the backward light cone from p, we can find the resonant states for the static model by simply finding the space of restrictions to p of the asymptotic states described above; but since the fibers of $\Lambda^0(Z)$ and $\Lambda^{n-1}(Z)$ are 1-dimensional, hence we have reproved Theorem 7.4.1.

We point out that if one wants to analyze differential form-valued waves or solutions to Maxwell's equations on Schwarzschild-de Sitter space, there is no global picture (in the sense of a 0-differential problem) as in the de Sitter case. Thus, the direct approach outlined in the proof of Theorem 7.4.1 is the only possible one in this case, and it is very instructive as it shows even more clearly how the cohomological interpretation of the space of zero resonant states can be used very effectively.

7.4.2 Schwarzschild-de Sitter space

The computation of resonant states for Schwarzschild-de Sitter spacetimes of any dimension is no more difficult than the computation in 4 dimensions, thus we directly treat the general case of $n \geq 4$ spacetime dimensions. Recall from §2.3 that the metric of n-dimensional Schwarzschild-de Sitter space $M = \mathbb{R}_t \times X$, $X = (r_-, r_+)_r \times \mathbb{S}^{n-2}_\omega$, is given by

$$g = \mu\, dt^2 - (\mu^{-1}\, dr^2 + r^2\, d\omega^2),$$

where $d\omega^2$ is the round metric on the sphere $\mathbb{S}^{n-2}$, and $\mu = 1 - \frac{2M_\bullet}{r^{n-3}} - \lambda r^2$, $\lambda = \frac{2\Lambda}{(n-2)(n-1)}$, where the black hole mass $M_\bullet$ and the cosmological constant Λ are positive. We assume the non-degeneracy condition (2.3.2), which guarantees that μ has two unique positive roots $0 < r_- < r_+$.

As in §2.3, we define $\alpha = \mu^{1/2}$, thus $d\alpha = \frac{1}{2}\mu'\alpha^{-1}\, dr$, and

$$\beta_\pm := \mp\frac{2}{\mu'(r_\pm)} > 0,$$

then the metric g can be written as

$$g = \alpha^2\, dt^2 - h, \quad h = \widetilde{\beta}_\pm^2\, d\alpha^2 + r^2\, d\omega^2,$$

where $\widetilde{\beta}_\pm = \mp 2/\mu'(r)$. Thus, if we let $\overline{X}_{\mathrm{even}} = [r_-, r_+]_r \times \mathbb{S}^{n-2}_\omega$ with the standard smooth structure, then $\widetilde{\beta}_\pm = \beta_\pm$ modulo $\alpha^2\mathcal{C}^\infty(\overline{X}_{\mathrm{even}})$, where we note that r is a smooth function of μ, thus an *even* function of α, near $r = r_\pm$ in view of $\mu'(r_\pm) \neq 0$. The manifold $\overline{X}$ is $\overline{X}_{\mathrm{even}}$ topologically, but with smooth functions of $\alpha = \mu^{1/2}$ added to the smooth structure. We denote $Y = \partial\overline{X} = \mathbb{S}^{n-2} \sqcup \mathbb{S}^{n-2}$.

By the analysis in §7.2, all zero resonant states u, written in the form (7.3.7) near Y, lie in the space $\mathcal{C}^\infty_{(0)}$, defined in (7.3.9). In the current setting, it is more natural to write differential forms as

$$u = u_{TT} + \alpha^{-1}\, dr \wedge u_{TN} + \alpha\, dt \wedge u_{NT} + \alpha\, dt \wedge \alpha^{-1}\, dr \wedge u_{NN}, \tag{7.4.2}$$

since $\alpha^{-1}\, dr$ has squared norm -1 (with respect to the metric g). We compute how the matching condition on the singular terms of u, encoded in the $\beta_\pm\alpha^{-1}$ entry of the matrix $\mathscr{C}$, changes when we thus change the basis of the form bundle: Namely, we have $\beta_\pm\alpha^{-1}\, d\alpha =$

$(\mp 1 + \alpha^2 \mathcal{C}^\infty(\overline{X}_{\text{even}}))\alpha^{-1}\alpha^{-1}\,dr$; thus, for u written as in (7.4.2), we have

$$u \in \mathcal{C}^\infty_{(0)} \iff \begin{pmatrix} u_{TT} \\ u_{TN} \\ u_{NT} \\ u_{NN} \end{pmatrix} \in \mathscr{C}_\pm \begin{pmatrix} \mathcal{C}^\infty(\overline{X}_{\text{even}}; \Lambda\mathbb{S}^{n-2}) \\ \mathcal{C}^\infty(\overline{X}_{\text{even}}; \Lambda\mathbb{S}^{n-2}) \\ \mathcal{C}^\infty(\overline{X}_{\text{even}}; \Lambda\mathbb{S}^{n-2}) \\ \mathcal{C}^\infty(\overline{X}_{\text{even}}; \Lambda\mathbb{S}^{n-2}) \end{pmatrix}$$

near $r = r_\pm$, where

$$\mathscr{C}_\pm = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & \mp\alpha^{-1} & 0 \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.4.3}$$

We now proceed to compute the explicit form of the operators d_p, δ_p and $\Box_p$, where the subscript p indicates the form degree on which the operators act. First, we recall (7.3.2) and (7.3.4) in the form

$$d_p = \begin{pmatrix} d_{X,p} & 0 \\ \alpha^{-1}\partial_t & -\alpha^{-1} d_{X,p-1}\alpha \end{pmatrix}, \quad \delta_p = \begin{pmatrix} -\alpha^{-1}\delta_{X,p}\alpha & -\alpha^{-1}\partial_t \\ 0 & \delta_{X,p-1} \end{pmatrix},$$

and these operators act on forms $u = u_T + \alpha\, dt \wedge u_N$, with u_T and u_N differential forms on X. Writing forms on X as $v = v_T + \alpha^{-1}\, dr \wedge v_N$, we have

$$d_{X,p} = \begin{pmatrix} d_{\mathbb{S}^{n-2},p} & 0 \\ \alpha\partial_r & -d_{\mathbb{S}^{n-2},p-1} \end{pmatrix}. \tag{7.4.4}$$

In order to compute the codifferential, we observe that the volume density on X induced by h is given by $\alpha^{-1} r^{n-2}\, dr|d\omega|$, while the induced inner product on the fibers on the bundle of p-forms is

$$H_p = \begin{pmatrix} r^{-2p}\Omega_p & 0 \\ 0 & r^{-2(p-1)}\Omega_{p-1} \end{pmatrix},$$

where Ω_p is the fiber inner product on the p-form bundle on $\mathbb{S}^{n-2}$. Therefore,

$$\begin{gathered} \delta_{X,p} = \begin{pmatrix} r^{-2}\delta_{\mathbb{S}^{n-2},p} & \partial^*_{r,p-1} \\ 0 & -r^{-2}\delta_{\mathbb{S}^{n-2},p-1} \end{pmatrix}, \\ \partial^*_{r,p-1} = -\alpha r^{-(n-2)} r^{2(p-1)} \partial_r r^{-2(p-1)} r^{n-2}. \end{gathered} \tag{7.4.5}$$

We obtain:

Lemma 7.4.2. *In the bundle decomposition* (7.4.2), *we have*

$$d_p = \begin{pmatrix} d_{\mathbb{S}^{n-2},p} & 0 & 0 & 0 \\ \alpha\partial_r & -d_{\mathbb{S}^{n-2},p-1} & 0 & 0 \\ \alpha^{-1}\partial_t & 0 & -d_{\mathbb{S}^{n-2},p-1} & 0 \\ 0 & \alpha^{-1}\partial_t & -\partial_r\alpha & d_{\mathbb{S}^{n-2},p-2} \end{pmatrix} \tag{7.4.6}$$

and

$$\delta_p = \begin{pmatrix} -r^{-2}\delta_{\mathbb{S}^{n-2},p} & -\alpha^{-1}\partial^*_{r,p-1}\alpha & -\alpha^{-1}\partial_t & 0 \\ 0 & r^{-2}\delta_{\mathbb{S}^{n-2},p-1} & 0 & -\alpha^{-1}\partial_t \\ 0 & 0 & r^{-2}\delta_{\mathbb{S}^{n-2},p-1} & \partial^*_{r,p-2} \\ 0 & 0 & 0 & -r^{-2}\delta_{\mathbb{S}^{n-2},p-2} \end{pmatrix}. \tag{7.4.7}$$

Moreover,

$$\begin{aligned} -r^2\Box_p = &\begin{pmatrix} \Delta_p & -2\alpha r d_{p-1} & 0 & 0 \\ -2\alpha r^{-1}\delta_p & \Delta_{p-1} & -r^2\mu^{-1}\mu'\partial_t & 0 \\ 0 & -r^2\mu^{-1}\mu'\partial_t & \Delta_{p-1} & -2\alpha r d_{p-2} \\ 0 & 0 & -2\alpha r^{-1}\delta_{p-1} & \Delta_{p-2} \end{pmatrix} \\ &+ \begin{pmatrix} r^2\alpha^{-1}\partial^*_{r,p}\alpha^2\partial_r & 0 & 0 & 0 \\ 0 & r^2\alpha\partial_r\alpha^{-1}\partial^*_{r,p-1}\alpha & 0 & 0 \\ 0 & 0 & r^2\partial^*_{r,p-1}\partial_r\alpha & 0 \\ 0 & 0 & 0 & r^2\partial_r\alpha\partial^*_{r,p-2} \end{pmatrix} \\ &+ \begin{pmatrix} r^2\mu^{-1}\partial_{tt} & 0 & 0 & 0 \\ 0 & r^2\mu^{-1}\partial_{tt} & 0 & 0 \\ 0 & 0 & r^2\mu^{-1}\partial_{tt} & 0 \\ 0 & 0 & 0 & r^2\mu^{-1}\partial_{tt} \end{pmatrix}. \end{aligned} \tag{7.4.8}$$

We can now compute the spaces $\mathcal{K}$ and $\mathcal{H}$ of zero resonances for $\Box$ and $d+\delta$ and deduce asymptotics for solutions of $(d+\delta)u = 0$:

Theorem 7.4.3. *On an n-dimensional Schwarzschild-de Sitter spacetime, $n \geq 4$, there exist two linearly independent 1-forms $u_\pm = f_{1,\pm}(r)\mu^{-1}\,dr + f_{2,\pm}(r)\,dt \in \mathcal{K}^1 = \ker\widehat{\Box}_1 \subset \mathcal{C}^\infty_{(0)}$, and* *we then have:*

$$\mathcal{K}^0 = \mathcal{H}^0 = \langle 1 \rangle, \quad \mathcal{K}^n = \mathcal{H}^n = \langle r^{n-2}\, dt \wedge dr \wedge \omega \rangle,$$

where ω *denotes the volume form on the round sphere* $\mathbb{S}^{n-2}$*. Furthermore,*

$$\mathcal{K}^1 = \langle u_+, u_- \rangle, \mathcal{H}^1 = 0, \quad \mathcal{K}^{n-1} = \langle \star u_+, \star u_- \rangle, \mathcal{H}^{n-1} = 0,$$
$$\mathcal{K}^k = \mathcal{H}^k = 0, \quad k = 3, \ldots, n-3.$$

For $n = 4$*,*

$$\mathcal{K}^2 = \mathcal{H}^2 = \langle \omega, r^{-2}\, dt \wedge dr \rangle,$$

while for $n > 4$*,*

$$\mathcal{K}^2 = \mathcal{H}^2 = \langle r^{-(n-2)}\, dt \wedge dr \rangle, \quad \mathcal{K}^{n-2} = \mathcal{H}^{n-2} = \langle \omega \rangle.$$

In particular, on 4*-dimensional Schwarzschild-de Sitter space, if* u *is a solution of* $(d + \delta)u = 0$ *with smooth initial data, then the degree* 0 *component of* u *decays exponentially to a constant, the degree* 1 *and degree* 3 *components decay exponentially to* 0*, the degree* 2 *component decays exponentially to a linear combination of* ω *and* $r^{-2}\, dt \wedge dr$*, and the degree* 4 *component decays exponentially to a constant multiple of the volume form. Analogous statements hold on any* n*-dimensional Schwarzschild-de Sitter space,* $n \geq 5$*.*

The forms $u_\pm$ in fact have a simple explicit form, see (7.4.9) and the parenthetical remark following (7.4.10).

Proof of Theorem 7.4.3. First, we observe that

$$H^k(\overline{X}) \cong H^k(\mathbb{S}^{n-2}), \quad H^{k-1}(\overline{X}, \partial\overline{X}) \cong H^{n-k}(\overline{X}) \cong H^{n-k}(\mathbb{S}^{n-2})$$

by Poincaré duality, and

$$H^{k-1}(\partial\overline{X}) \cong H^{k-1}(\mathbb{S}^{n-2}) \oplus H^{k-1}(\mathbb{S}^{n-2}).$$

Thus, the short exact sequence (7.3.52) immediately gives the dimensions of the spaces $\mathcal{K}^k$, and (7.3.51) gives the dimensions of $\mathcal{H}^k$ for all values of k except $k = 1$ and $k = n-1$.

We now compute $\mathcal{H}$ and $\mathcal{K}$ in the case $n = 4$. For $k = 0$, the short exact sequence (7.3.52) reads $0 \to H^0(\overline{X}) \oplus 0 \to \mathcal{K}^0 \to 0 \to 0$, and since $H^0(\overline{X}) = \langle [1] \rangle$, this suggests 1 as a resonant state for $\Box$ on 0-forms (i.e. functions), and indeed $\Box 1 = 0$, hence $\mathcal{K}^0 = \langle 1 \rangle$. Theorem 7.3.20 also shows that $\mathcal{H}^0 = \mathcal{K}^0$. Then we immediately obtain $\mathcal{H}^4 = \mathcal{K}^4 = \langle \star 1 \rangle = \langle r^2\, dt \wedge dr \wedge \omega \rangle$.

Next, we treat the form degree $k = 2$. Then (7.3.52) reads $0 \to H^2(\overline{X}) \oplus H^1(\overline{X}, \partial\overline{X}) \to \mathcal{K}^2 \to 0 \to 0$. Now $H^2(\overline{X}) = \langle [\omega] \rangle$, and a generator of $H^1(\overline{X}, \partial\overline{X})$ is given by the Poincaré dual of ω (which generates $H^2(\overline{X})$). This suggests the ansatz $u = f(r)\omega$ for an element of $\mathcal{K}^2 = \mathcal{H}^2$ (the latter equality following from (7.3.51)), and then $\star u$ will be the second element of a basis of $\mathcal{K}^2$. Now, in the decomposition (7.4.2), we compute using Lemma 7.4.2 that $\widehat{\delta}_2(0)u = 0$ for $u = f(r)\omega$, and

$$\widehat{d}_2(0)u = \widehat{d}_2(0)\begin{pmatrix} f(r)\omega \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha f'(r)\omega \\ 0 \\ 0 \end{pmatrix},$$

which vanishes precisely if $f(r)$ is constant.

The analysis of resonant states in form degree $k = 1$ is just a bit more involved. Since (7.3.52) now reads $0 \to 0 \oplus 0 \to \mathcal{K}^1 \to H^0(\mathbb{S}^2 \sqcup \mathbb{S}^2) \to 0$, every non-trivial element u of $\mathcal{K}^1$ fails to be in $L^2(\alpha|dh|)$, and in fact the singular behavior is expected to be $u = \mathscr{C}_\pm \widetilde{u}$ with $\widetilde{u}_{NT}|_{r=r_\pm} = c_\pm \in \mathbb{C}$, since $H^0(\mathbb{S}^2 \sqcup \mathbb{S}^2)$ is generated by locally constant functions, which are therefore constant on $r = r_-$ as well as on $r = r_+$. We thus make the ansatz

$$u = \alpha^{-1} f_1(r)\, \alpha^{-1}\, dr + \alpha\, dt \wedge \alpha^{-1} f_2(r). \tag{7.4.9}$$

We then compute

$$-\widehat{\Box}_1(0)u = \begin{pmatrix} 0 \\ \alpha\partial_r \alpha^{-1} \partial_{r,0}^* f_1 \\ \partial_{r,0}^* \partial_r f_2 \\ 0 \end{pmatrix},$$

and by definition of $\partial_{r,p}^*$ in (7.4.5), this vanishes if and only if f_1 and f_2 satisfy the ODEs

$$\partial_r r^{-2} \partial_r r^2 f_1 = 0,$$
$$r^{-2} \partial_r r^2 \partial_r f_2 = 0.$$

The general form of the solution is

$$\begin{aligned} f_1(r) &= f_{11} r + f_{12} r^{-2}, \\ f_2(r) &= f_{21} + f_{22} r^{-1}, \end{aligned} \tag{7.4.10}$$

$f_{jk} \in \mathbb{C}$, $j,k = 1,2$. (On n-dimensional Schwarzschild-de Sitter space, the exponents 2 and -2 in these ODEs get replaced by $n-2$ and $2-n$, and the general forms of the solutions are $f_1(r) = f_{11}r + f_{12}r^{2-n}$ and $f_2(r) = f_{21} + f_{22}r^{3-n}$. The subsequent analysis of the matching conditions goes through with obvious modifications.) Now recall that resonant states are elements of $\mathcal{C}^\infty_{(0)}$ and thus satisfy a matching condition in the singular components, which is captured by the matrix (7.4.3). Concretely, we require $f_2(r_-) = f_1(r_-)$ and $f_2(r_+) = -f_1(r_+)$; in terms of f_{jk}, $j,k = 1,2$, these conditions translate into

$$\begin{pmatrix} r_- & r_-^{-2} & -1 & -r_-^{-1} \\ r_+ & r_+^{-2} & 1 & r_+^{-1} \end{pmatrix} \begin{pmatrix} f_{11} \\ f_{12} \\ f_{21} \\ f_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the 2×4 matrix on the left has rank 2, we get a 2-dimensional space of solutions. In fact, it is easy to see that we can freely specify the values $f_1(r_-)$ and $f_1(r_+)$, and f_1 and f_2 are then uniquely determined. To be specific, we can for instance define $u_+ \in \mathcal{K}^1$ to be the 1-form with $f_1(r_-) = 0$, $f_1(r_+) = 1$, and $u_- \in \mathcal{K}^1$ to be the 1-form with $f_1(r_-) = 1$, $f_1(r_+) = 0$, and we then have $\mathcal{K}^1 = \langle u_+, u_- \rangle$, as claimed.

Next, since $\mathcal{H}^1 \subset \mathcal{K}^1$, computing $\mathcal{H}^1$ simply amounts to finding all linear combinations of u_- and u_+ which are annihilated by both $\widehat{d_1}(0)$ and $\widehat{\delta_1}(0)$. But

$$\widehat{d_1}(0) \begin{pmatrix} 0 \\ \alpha^{-1} f_1(r) \\ \alpha^{-1} f_2(r) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\partial_r f_2 \end{pmatrix} = 0$$

requires f_2 to be constant, and

$$\widehat{\delta}_1(0)\begin{pmatrix}0\\ \alpha^{-1}f_1(r)\\ \alpha^{-1}f_2(r)\\ 0\end{pmatrix}=\begin{pmatrix}-\alpha^{-1}\partial_{r,0}^*f_1\\ 0\\ 0\\ 0\end{pmatrix}=0$$

implies $r^{-2}\partial_r r^2 f_1=0$, hence $f_1(r)=f_1(r_-)(r/r_-)^{-2}$. The matching condition requires $f_1(r_+)=f_1(r_-)(r_+/r_-)^{-2}=-f_2(r_+)=-f_2(r_-)=-f_1(r_-)$ and is therefore only satisfied if $f_1(r_-)=0$, which implies $f_1\equiv 0$ and $f_2\equiv 0$. This shows that $\mathcal{H}^1=0$ and finishes the computation of the spaces of resonances for $n=4$. The computation for spacetime dimensions $n\geq 5$ is completely analogous.

Finally, the statement about asymptotics of solutions to $(d+\delta)u=0$ follows from the above computations combined with high energy estimates for $d+\delta$, which follow from those for $\Box$, and Lemma 7.3.7. To see the relevance of the latter, recall that if $(\widehat{d}(\sigma)+\widehat{\delta}(\sigma))^{-1}$ had a second order pole at 0, then solutions to $(d+\delta)u=0$ would generically blow up linearly; the simplicity of the pole ensures that solutions stay bounded with the asymptotic stationary state given by an element of $\mathcal{H}$. The high energy estimates for $\Box$ acting on differential forms however were proved in Chapter 6 and follow from combining Theorem 6.4.8 with Dyatlov's result [42]. Recall that the problem is that one needs the subprincipal symbol of $\Box$ (or a conjugated version thereof), relative to a *positive definite* fiber inner product, at the trapping to be smaller than $\nu_{\min}/2$, where $\nu_{\min}$ is the minimal expansion rate in the normal direction at the trapped set, computed in (2.3.11) for the operator $-r^2\Box$. We briefly show how in certain situations, in particular in dimensions $n\geq 5$, one can use ordinary (rather than pseudodifferential) inner products to resolve this problem: Thus, we want to bound the imaginary part of $\mathcal{P}=-r^2\Box_g$ in terms of $\nu_{\min}$, in order to obtain high energy estimates below the real line. That is, we want to show that

$$Q:=|\tau|^{-1}\sigma_1\left(\frac{1}{2i}(\mathcal{P}-\mathcal{P}^*)\right)<\frac{\nu_{\min}}{2}$$

at the trapped set, cf. also the discussion in §9.2.6, where we take the adjoint with respect to some Riemannian inner product B, to be chosen, on the bundle $\Lambda^p\mathbb{S}^{n-2}\oplus\Lambda^{p-1}\mathbb{S}^{n-2}\oplus\Lambda^{p-1}\mathbb{S}^{n-2}\oplus\Lambda^{p-2}\mathbb{S}^{n-2}$; notice that Q is a self-adjoint section of the endomorphism bundle of this bundle. An obvious guess is to use $B = H \oplus H$ in the tangential-normal decomposition (7.3.6), thus

$$B = r^{-2p}\Omega_p \oplus r^{-2(p-1)}\Omega_{p-1} \oplus r^{-2(p-1)}\Omega_{p-1} \oplus r^{-2(p-2)}\Omega_{p-2}.$$

In this case, the expression (7.4.8) shows that the only parts of $\mathcal{P}$ that are not symmetric with respect to B at the spacetime trapped set

$$\Gamma = \{(t, r_p, \omega; \tau, 0, \eta) \colon \frac{r^4}{\Delta_r}\tau^2 = |\eta|^2\},$$

see (2.3.12) (we are using the notation of that section), are the $(2,3)$ and $(3,2)$ components; thus, taking adjoints with respect to B, we compute

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \pm r^2\mu^{-1}\mu' & 0 \\ 0 & \pm r^2\mu^{-1}\mu' & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

at Γ, with the sign depending the sign of τ. Now $(\mu/r^2)' = 0$ at $r = r_p$ implies $\mu^{-1}\mu' r_p^2 = 2r_p$ there; the eigenvalues of Q are therefore $\pm 2r_p$, and they are bounded by $\nu_{\min}/2$, see (2.3.11), if and only if

$$r_p^2\lambda > \frac{(5-n)(n-3)}{4(n-1)},$$

which in spacetime dimensions $n \geq 5$ is always satisfied. In dimension $n = 4$ however, the condition becomes $r_p^2\lambda > 1/12$, or

$$9M_\bullet\Lambda > \frac{1}{4},$$

while the non-degeneracy condition (2.3.2) requires $9M_\bullet\Lambda < 1$. Therefore, only for very massive black holes or very large cosmological constants does the above choice of positive definite inner product B yield a sufficiently small imaginary part of $\mathcal{P}$. In fact, for parameters $M_\bullet$ and Λ with $9M_\bullet\Lambda \leq 1/4$, the endomorphism Q is not bounded by $\nu_{\min}/2$ for *any* choice of B, and pseudodifferential inner products indeed become necessary to remove this restriction on $M_\bullet$ and Λ in $n = 4$. □

We can in fact prove boundedness and asymptotics for solutions of the wave equation on differential forms in all form degrees as well. To begin, write

$$(\widetilde{d}(\sigma)+\widetilde{\delta}(\sigma))^{-1}=\sigma^{-1}A_{-1}+\mathcal{O}(1),\quad A_{-1}=\sum_{j=1}^{4}\langle\cdot,\psi_j\rangle\phi_j, \tag{7.4.11}$$

near $\sigma=0$, where $\{\phi_j\}_{j=1,\ldots,4}$ is a basis of the space of resonant states and $\{\psi_j\}_{j=1,\ldots,4}$ is a basis of the space $\mathcal{H}_*=\ker(\widetilde{d}(0)+\widetilde{\delta}(0))^*$ of dual states. (After choosing the ϕ_j', the ψ_j' are uniquely determined, and vice versa, see Remark A.1.5.) Therefore, we need to understand the dual states of $d+\delta$ in order to understand the order and structure of the pole of $\widetilde{\Box}(\sigma)^{-1}=\big((\widetilde{d}(\sigma)+\widetilde{\delta}(\sigma))^{-1}\big)^2$ as $\sigma=0$. Notice here that the adjoint $(\widetilde{d}(\sigma)+\widetilde{\delta}(\sigma))^*$ acts on distributions on $\widetilde{X}$ which are *supported* at the Cauchy hypersurface $\partial\widetilde{X}$ (see [64, Appendix B] for this and related notions). In particular, an element $\widetilde{u}\in\ker(\widetilde{d}(\sigma)+\widetilde{\delta}(\sigma))^*$ satisfies $\widetilde{u}\in\ker\widetilde{\Box}(\sigma)$ and is a supported distribution at $\partial\widetilde{X}$, thus by local uniqueness, $\widetilde{u}$ vanishes in the hyperbolic region $\widetilde{X}\setminus X$, hence $\operatorname{supp}\widetilde{u}\subset\overline{X}$.

Lemma 7.4.4. *The spaces $\mathcal{H}_*$ and $\mathcal{K}_*$ of dual states for $d+\delta$ and $\Box$, respectively, on n-dimensional Schwarzschild-de Sitter space, $n\geq 4$, are graded by form degree, $\mathcal{H}_*=\bigoplus_{k=0}^n\mathcal{H}_*^k$, $\mathcal{K}_*=\bigoplus_{k=0}^n\mathcal{K}_*^k$, and have the following explicit descriptions:*

$$\mathcal{K}_*^0=\langle 1_X\rangle,\mathcal{H}_*^0=0,\quad \mathcal{K}_*^n=\langle 1_X r^{n-2}\,dt\wedge dr\wedge\omega\rangle,\mathcal{H}_*^n=0,$$
$$\mathcal{H}_*^1=\langle\delta_{r=r_-}\,dr,\delta_{r=r_+}\,dr\rangle,\quad \mathcal{H}_*^{n-1}=\langle\delta_{r=r_-}\,dr\wedge\omega,\delta_{r=r_+}\,dr\wedge\omega\rangle,$$
$$\mathcal{H}_*^k=0,\quad k=2,\ldots,n-2,$$

where ω denotes the volume form on the round sphere $\mathbb{S}^{n-2}$. Furthermore, $\mathcal{K}_*^1=\mathcal{H}_*^1$, $\mathcal{K}_*^{n-1}=\mathcal{H}_*^{n-1}$ and

$$\mathcal{K}_*^k=0,\quad k=3,\ldots,n-3.$$

For $n=4$,

$$\mathcal{K}_*^2=\langle 1_X\omega,1_X r^{-2}\,dt\wedge dr\rangle,$$

while for $n>4$,

$$\mathcal{K}_*^2=\langle 1_X r^{2-n}\,dt\wedge dr\rangle,\quad \mathcal{K}_*^{n-2}=\langle 1_X\omega\rangle.$$

We have $\langle\phi,\psi\rangle=0$ for all $\phi\in\mathcal{H}$, $\psi\in\mathcal{H}_*$.

Proof. For computing the dual resonant states, we need to compute the form of $\widetilde{\Box}(0)$ near the two components of $\partial\overline{X}=\mathbb{S}^{n-2}\sqcup\mathbb{S}^{n-2}$. Since dual states are supported in $\overline{X}_{\mathrm{even}}$, it suffices to compute $\mathscr{C}_\pm^{-1}\widehat{\Box}(0)\mathscr{C}_\pm$, since any smooth extension of this operator to $\widetilde{X}$ agrees with $\widetilde{\Box}(0)$ in X and to infinite order at $\partial\overline{X}_{\mathrm{even}}$, thus the difference annihilates dual states. Using Lemma 7.4.2, we compute

$$-\mathscr{C}_\pm^{-1}\widehat{\Box}_p(0)\mathscr{C}_\pm = r^{-2}\begin{pmatrix}\Delta_p & 0 & 0 & 0\\ 0 & \Delta_{p-1} & 0 & 0\\ 0 & 0 & \Delta_{p-1} & 0\\ 0 & 0 & 0 & \Delta_{p-2}\end{pmatrix}$$
$$+\begin{pmatrix}\alpha^{-1}\partial_{r,p}^*\alpha^2\partial_r & -2\alpha^2 r^{-1}d_{p-1} & \pm 2r^{-1}d_{p-1} & 0\\ -2r^{-3}\delta_p & \partial_r\alpha^{-1}\partial_{r,p-1}^*\alpha^2 & \pm(2(p-1)-(n-2))r^{-2} & \mp 2r^{-1}d_{p-2}\\ 0 & 0 & \alpha\partial_{r,p-1}^*\partial_r & -2\alpha^2 r^{-1}d_{p-2}\\ 0 & 0 & -2r^{-3}\delta_{p-1} & \partial_r\alpha\partial_{r,p-2}^*\end{pmatrix},$$

where the Laplace operators, differentials and codifferentials are the operators on $\mathbb{S}^{n-2}$. This does extend to an operator acting on smooth functions on $(r_\pm-\delta, r_\pm+\delta)\times\mathbb{S}^{n-2}$, $\delta>0$ small, near $r_\pm$.

Now for $p=0$, clearly $\alpha^{-1}\partial_{r,0}^*\alpha^2\partial_r 1_X = \mp\alpha^{-1}\partial_{r,0}^*(\mu\delta_{r=r_\pm}) = 0$, hence $\mathcal{K}_*^0 = \langle 1_X\rangle$. (Observe that since $\widetilde{\Box}_0(0)$ is Fredholm of index 0 and has a 1-dimensional kernel according to Theorem 7.4.3, the space of dual 0-form resonances is 1-dimensional as well.) Likewise, for $p=n$, we have

$$\partial_r\alpha\partial_{r,n-2}^*(1_X r^{n-2}\,dt\wedge dr\wedge\omega) = -\partial_r\mu r^{n-2}\partial_r(1_X\,dt\wedge dr\wedge\omega) = 0,$$

confirming $\mathcal{K}_*^n = \langle 1_X r^{n-2}\,dt\wedge dr\wedge\omega\rangle$. By completely analogous arguments, we find $1_X r^{2-n}\,dt\wedge dr\in\mathcal{K}_*^2$ and $1_X\omega\in\mathcal{K}_*^{n-2}$.

In order to proceed, notice that $\widetilde{d}(0)+\widetilde{\delta}(0)$ maps $\mathcal{K}_*$ into $\mathcal{H}_*$. Hence, we can find dual states for $d+\delta$ by applying $\widetilde{d}(0)+\widetilde{\delta}(0)$ to the dual states of $\Box$ that we have already identified. For this computation, we note

$$\mathscr{C}_\pm^{-1}\widehat{d}_p(0)\mathscr{C}_\pm = \begin{pmatrix}d_{\mathbb{S}^{n-2},p} & 0 & 0 & 0\\ \partial_r & -d_{\mathbb{S}^{n-2},p-1} & 0 & 0\\ 0 & 0 & -d_{\mathbb{S}^{n-2},p-1} & 0\\ 0 & 0 & -\partial_r & d_{\mathbb{S}^{n-2},p-2}\end{pmatrix},$$

$$\mathscr{C}_\pm^{-1}\widehat{\delta}_p(0)\mathscr{C}_\pm = \begin{pmatrix} -r^2\delta_{\mathbb{S}^{n-2},p} & -\alpha^{-1}\partial^*_{r,p-1}\alpha^2 & \pm\alpha^{-1}\partial^*_{r,p-1} & 0 \\ 0 & r^{-2}\delta_{\mathbb{S}^{n-2},p-1} & 0 & \pm\alpha^{-1}\partial^*_{r,p-2} \\ 0 & 0 & r^{-2}\delta_{\mathbb{S}^{n-2},p-1} & \alpha\partial^*_{r,p-2} \\ 0 & 0 & 0 & -r^{-2}\delta_{\mathbb{S}^{n-2},p-2} \end{pmatrix}.$$

Thus, $(\widetilde{d}_0(0)+\widetilde{\delta}_0(0))1_X$ and $(\widetilde{d}_2(0)+\widetilde{\delta}_2(0))(1_X r^{2-n}\,dt\wedge dr)$ are both linear combinations of $\delta_{r=r_\pm}\,dr$, hence $\delta_{r=r_\pm}\,dr \in \mathcal{H}^1_* \subset \mathcal{K}^1_*$, and similarly $(\widetilde{d}_n(0)+\widetilde{\delta}_n(0))(1_X \star 1)$ and $(\widetilde{d}_{n-2}(0)+\widetilde{\delta}_{n-2}(0))(1_X\omega)$ are both linear combinations of $\delta_{r=r_\pm}\,dr\wedge\omega$, hence $\delta_{r=r_\pm}\,dr\wedge\omega \in \mathcal{H}^{n-1}_* \subset \mathcal{K}^{n-1}_*$.

We have therefore identified 4 and 8 linearly independent dual states for $d+\delta$ and $\Box$, which is equal to the dimensions of $\mathcal{H}$ and $\mathcal{K}$, respectively, and since $\widetilde{d}(0)+\widetilde{\delta}(0)$ and $\widetilde{\Box}(0)$ have index 0, all dual states are linear combinations of these, i.e. we have thus identified a basis of the spaces of dual states. The orthogonality of resonant and dual states for $d+\delta$ follows immediately from the explicit forms of both derived in Theorem 7.4.3 and in this lemma: All dual states have form degree 1 or $n-1$, while all resonant states have form degree 0, 2, $n-2$ or n. $\Box$

The orthogonality statement in Lemma 7.4.4 combined with (7.4.11) immediately gives $A^2_{-1}=0$, hence the coefficient of σ^{-2} in the Laurent expansion of $\widehat{\Box}(\sigma)^{-1}$ at $\sigma=0$ vanishes. For precisely those form degrees $0\leq p\leq n$ for which $\mathcal{K}^p$ is non-trivial, $\widehat{\Box}(\sigma)^{-1}$ does have a simple pole at $\sigma=0$, and

$$\widehat{\Box}_p(\sigma)^{-1} = \sigma^{-1}\sum_{j=1}^{\dim\mathcal{K}^p}\langle\cdot,\psi'_j\rangle\phi'_j + \mathcal{O}(1),$$

where ϕ'_j and ψ'_j run over a basis of $\ker\widehat{\Box}_p(0)\cong\mathcal{K}^p$ and $\mathcal{K}^p_*=\ker\widehat{\Box}_p(0)^*$, respectively.

Theorem 7.4.5. *On 4-dimensional Schwarzschild-de Sitter space, if $0\leq p\leq 4$ and u is a differential form of degree p which solves $\Box u=0$ with smooth initial data, then u decays exponentially to*

- *a constant for $p=0$,*
- *a linear combination of u_+ and u_-, defined in the statement of Theorem 7.4.3, for $p=1$,*
- *a linear combination of ω and $r^{-2}\,dt\wedge dr$ for $p=2$,*

- *a linear combination of* $\star u_+$ *and* $\star u_-$ *for* $p = 3$ *and*
- *a constant multiple of* $r^2\, dt \wedge dr \wedge \omega$ *for* $p = 4$.

Analogous statements hold on any n*-dimensional Schwarzschild-de Sitter space,* $n \geq 5$.

7.5 Results for Kerr-de Sitter spacetimes

We now prove that some of the results obtained in the previous section for the 4-dimensional Schwarzschild-de Sitter spacetime are stable under perturbations, which allows us to draw conclusions about asymptotics for solutions of $(d + \delta)u = 0$ or $\Box u = 0$ on Kerr-de Sitter space with very little effort. Thus, fixing the black hole mass $M_\bullet$ and the cosmological constant $\Lambda > 0$, denote by g_a the Kerr-de Sitter metric with angular momentum a; thus, g_0 is the Schwarzschild-de Sitter metric. Assuming the non-degeneracy condition (2.3.2), which ensures that the cosmological horizon lies outside the black hole event horizon, the same will be true for small $|a|$, which is the setting in which work here. Only very basic facts about the Kerr-de Sitter metric will need to be used; we refer to §2.4 for details and further information. We will write δ_{g_a} for the codifferential with respect to the metric g_a. We furthermore denote by $M = \mathbb{R}_t \times X$ the domain of exterior communications, and by $\widetilde{M} = \mathbb{R}_{t_*} \times \widetilde{X}$ the 'extended' spacetime.

To begin, recall that the scalar wave equation (and by essentially the same arguments the wave equation on differential forms, since the principal symbol of the Hodge d'Alembertian is scalar, see also [112, §4] for a discussion in a related context) on the Kerr-de Sitter spacetime fits into the microlocal framework developed in [114]. In particular, asymptotics for waves follow directly from properties of the Mellin transformed normal operator family, and moreover the analysis of the latter is stable under perturbations: This means that the set of resonances in any compact subset of the complex plane depends continuously on the metric, see Appendix A, while the existence of the spectral gap is stable under perturbations in view of the r-normal hyperbolicity (for every r) of the trapped set of Schwarzschild-de Sitter (in fact, Kerr-de Sitter) space. In the present context, this concretely means that for any $\epsilon > 0$, there exists $a_\epsilon > 0$ such that for all angular momenta a with $|a| < a_\epsilon$, the meromorphic family of operators $\mathcal{R}_a(\sigma) := (\widetilde{d}(\sigma) + \widetilde{\delta_{g_a}}(\sigma))^{-1}$ has no poles in $|\sigma| \geq \epsilon$, $\operatorname{Im}\sigma \geq 0$, and such that moreover all poles in $|\sigma| < \epsilon$ are perturbations of the pole of $\mathcal{R}_0(\sigma)$ at 0, in the sense that the total rank of the poles of $\mathcal{R}_a(\sigma)$ in $|\sigma| < \epsilon$ is unchanged, namely, equal to 4 by Theorem 7.4.3. Now, Lemma 7.4.4 suggests considering dual resonant states instead, which have a simpler form; the same stability result as for $\mathcal{R}_a(\sigma)$ holds for $\mathcal{R}_a^*(\sigma) := ((\widetilde{d}(\sigma)+\widetilde{\delta_{g_a}}(\sigma))^*)^{-1}$. However, just as in the case of Schwarzschild-de Sitter space, we can immediately write down 4 linearly independent dual 0-resonant states for $d+\delta_{g_a}$: Namely, apply $\widetilde{d}(0)+\widetilde{\delta_{g_a}}(0)$ to 1_X (this is a dual resonant state for $\Box_{g_a}$), which produces a sum of δ-distributions supported at the horizons $r = r_\pm$, and splitting this up into the part supported at r_- and the part supported at r_+, we obtain 2 linearly independent dual resonant states for $d+\delta$ in form degree 1. The same procedure can be applied to $\star_{g_a} 1_X$, yielding 2 linearly independent dual resonant states for $d+\delta$ in form degree 3 (which are simply the Hodge duals of the dual states in form degree 1). Hence,

$$\mathcal{H}_a := \ker(\widetilde{d}(0)+\widetilde{\delta_{g_a}}(0)), \tag{7.5.1}$$

which has the same dimension as

$$\mathcal{H}_{a,*} := \ker(\widetilde{d}(0)+\widetilde{\delta_{g_a}}(0))^*, \tag{7.5.2}$$

is at least 4-dimensional for small $|a|$, but it is also at most 4-dimensional by the above perturbation stability argument! Hence, for small $|a|$, we deduce that 0 is the only pole of $\mathcal{R}_a(\sigma)$, i.e. the only resonance of $d+\delta_{g_a}$, in $\operatorname{Im}\sigma \geq 0$ (and also the only pole of $\mathcal{R}_a^*(\sigma)$ in this half space), and is simple due to the equality of the rank of the resonance and the dimension of the nullspace of $\widetilde{d}(0)+\widetilde{\delta_{g_a}}(0)$, see Lemma A.1.3.

We can use this in turn to prove the stability of the zero resonance for $\Box_{g_a}$ in all form degrees. Let $\pi_k\colon \mathcal{C}^\infty(\widetilde{M};\Lambda\widetilde{M}) \to \mathcal{C}^\infty(\widetilde{M};\Lambda\widetilde{M})$ denote the projection onto differential forms with pure form degree $k\in\{0,\ldots,4\}$, which induces a map on $\mathcal{C}^\infty(\widetilde{X};\Lambda\widetilde{X}\oplus\Lambda\widetilde{X})$. Let

$$\mathcal{K}_a := \ker\widetilde{\Box_{g_a}}(0) = \bigoplus_{k=0}^{4}\mathcal{K}_a^k \tag{7.5.3}$$

be the grading of the zero resonant space of $\Box_{g_a}$ by form degree, likewise

$$\mathcal{K}_{a,*} := \ker\widetilde{\Box_{g_a}}(0)^* = \bigoplus_{k=0}^{4}\mathcal{K}_{a,*}^k \tag{7.5.4}$$

for the space of dual resonant states. Observe that $\pi_k\mathcal{H}_a \subseteq \mathcal{K}_a^k$, since $u\in\mathcal{H}_a$ implies

$0 = \pi_k \Box_{g_a} u = \Box_{g_a} \pi_k u$. Now, since $\Box_{g_a} 1 = 0$, we have $\mathcal{K}_a^0 = \langle 1 \rangle$ for small $|a|$ by stability, likewise $\mathcal{K}_a^4 = \langle \star_{g_a} 1 \rangle$. Furthermore, $\mathcal{K}_a^2$ is at most 2-dimensional for small $|a|$ (since $\mathcal{K}_0^2$ is 2-dimensional), but also $\mathcal{K}_a^2 \supseteq \pi_2 \mathcal{H}_a$; now $\pi_2 \mathcal{H}_0$ is 2-dimensional by Theorem 7.4.3 and $\mathcal{H}_a$ depends smoothly on a, see also Remark A.1.7. Thus $\mathcal{K}_a^2 = \pi_2 \mathcal{H}_a$ is 2-dimensional for small $|a|$; therefore $\mathcal{K}_a^2 = \pi_2 \mathcal{H}_a$ is 2-dimensional. Finally, we have $\mathcal{H}_{a,*}^1 \subseteq \mathcal{K}_{a,*}^1$, hence by the analysis of $d + \delta_{g_a}$ above, $\mathcal{K}_{a,*}^1$, hence $\mathcal{K}_a^1$, is at least 2-dimensional, but since $\mathcal{K}_0^1$ is 2-dimensional, we must in fact have $\dim \mathcal{K}_a^1 = 2$ for small $|a|$; likewise $\dim \mathcal{K}_a^3 = 2$. Hence, we have $\dim \mathcal{K}_a^k = \dim \mathcal{K}_0^k$ for $k = 0, \ldots, 4$, which in particular means that the zero resonance of $\Box_{g_a}$ is the only resonance in $\operatorname{Im} \sigma \geq 0$, and the resonance is simple.

We now summarize the above discussion, including a small improvement. The following theorem is completely parallel to Theorem 7.4.3, Lemma 7.4.4 and Theorem 7.4.5 for Schwarzschild-de Sitter spacetimes, extending these to Kerr-de Sitter spacetimes with small angular momentum:

Theorem 7.5.1. *For small $|a|$, the only resonance of $d + \delta_{g_a}$ in $\operatorname{Im} \sigma \geq 0$ is a simple resonance at $\sigma = 0$, likewise for $\Box_{g_a}$. The spaces $\mathcal{H}_a$ and $\mathcal{H}_{a,*}$ of resonant and dual resonant states for $d + \delta_{g_a}$ are graded by form degree as $\mathcal{H}_a = \bigoplus_{k=0}^4 \mathcal{H}_a^k$, $\mathcal{H}_{a,*} = \bigoplus_{k=0}^4 \mathcal{H}_{a,*}^k$. In particular, denoting the degree of differential forms on which operators act by subscripts, we have $\mathcal{H}_a^k = \ker \widetilde{d}_k(0) \cap \ker (\widetilde{\delta_{g_a}})_k(0)$, with*

$$\mathcal{H}_a^0 = \langle 1 \rangle, \quad \mathcal{H}_a^1 = 0, \quad \mathcal{H}_a^2 = \langle u_{a,1}, u_{a,2} \rangle, \quad \mathcal{H}_a^3 = 0, \quad \mathcal{H}_a^4 = \langle \star_{g_a} 1 \rangle$$

for some 2-forms $u_{a,1}, u_{a,2}$, which can be chosen to depend smoothly on a, with $u_{0,1} = r^{-2}\, dt \wedge dr$, $u_{0,2} = \omega$ in the notation of Theorem 7.4.3, and

$$\begin{aligned} &\mathcal{H}_{a,*}^0 = 0, \quad \mathcal{H}_{a,*}^1 = \langle \delta_{r=r_-}\, dr, \delta_{r=r_+}\, dr \rangle, \\ &\mathcal{H}_{a,*}^2 = 0, \quad \mathcal{H}_{a,*}^3 = \star_{g_a} \mathcal{H}_{a,*}^1, \quad \mathcal{H}_{a,*}^4 = 0. \end{aligned}$$

For the spaces $\mathcal{K}_a$ and $\mathcal{K}_{a,*}$ of resonant and dual resonant states for $\Box_{g_a}$, we have

$$\mathcal{K}_a^0 = \mathcal{H}_a^0, \quad \mathcal{K}_a^1 = \langle u_{a,+}, u_{a,-} \rangle, \quad \mathcal{K}_a^2 = \mathcal{H}_a^2, \quad \mathcal{K}_a^3 = \star_{g_a} \mathcal{K}_a^1, \quad \mathcal{K}_a^4 = \mathcal{H}_a^4$$

for some 1-forms $u_{a,\pm}$, which can be chosen to depend smoothly on a, with $u_{0,\pm} = u_\pm$ in *the notation of Theorem 7.4.5, and*

$$\mathcal{K}^0_{a,*} = \langle 1_X \rangle, \quad \mathcal{K}^1_{a,*} = \mathcal{H}^1_{a,*},$$
$$\mathcal{K}^2_{a,*} = \langle 1_X u_{a,1}, 1_X u_{a,2} \rangle, \quad \mathcal{K}^3_{a,*} = \mathcal{H}^3_{a,*}, \quad \mathcal{K}^4_{a,*} = \langle \star_{g_a} 1_X \rangle.$$

In particular, the form degree k part of a solution u to $(d+\delta_{g_a})u = 0$, resp. $\Box_{g_a} u = 0$, with smooth initial data decays exponentially to an element of $\mathcal{H}^k_a$, resp. $\mathcal{K}^k_a$, for $k = 0, \ldots, 4$.

We derive an explicit expression for $u_{a,1}$ and $u_{a,2}$ in Remark 7.5.4 below.

Remark 7.5.2. Since for all $k = 0, \ldots, 4$, either $\mathcal{H}^k_a = 0$ or $\mathcal{H}^k_{a,*} = 0$, hence $\mathcal{H}_a$ and $\mathcal{H}_*$ are orthogonal, we obtain another proof, as in the Schwarzschild-de Sitter case, of the fact that $\Box_{g_a}$ acting on differential forms only has a simple resonance at 0.

Proof of Theorem 7.5.1. We only need to prove that the space $\mathcal{H}_a$ is graded by form degree: Let $\pi_{\text{even}} = \pi_0 + \pi_2 + \pi_4$ denote the projection onto even form degree parts, then since $d + \delta_{g_a}$ maps even degree forms to odd degree forms and vice versa, π_{even} maps $\mathcal{H}_a$ into itself. Now suppose $u \in \pi_{\text{even}}\mathcal{H}_a$, and write $u = u_0 + u_2 + u_4$ with $u_k = \pi_k u$, $k = 0, 2, 4$. Then $0 = \pi_1(d + \delta_{g_a})u = du_0 + \delta_{g_a} u_2$, using the identification of resonant states with t_*-independent forms as in the proof of Theorem 7.3.20. Applying δ_{g_a} to this equation gives $0 = \Box_{g_a} u_0$, which implies $u_0 \in \mathcal{K}^0_a$, i.e. u_0 is a constant, as discussed before the statement of the theorem. Likewise, $u_4 \in \mathcal{K}^4_a$, so u_4 is the Hodge dual of a constant. Therefore, $d+\delta_{g_a}$ annihilates both u_0 and u_4, hence $u_2 \in \mathcal{H}_a$. This argument shows that in fact $\pi_2\mathcal{H}_a \subset \mathcal{H}_a$. Since $\pi_2\mathcal{H}_a$ is 2-dimensional, as noted above, we have

$$\langle 1 \rangle \oplus \pi_2\mathcal{H}_a \oplus \langle \star_{g_a} 1 \rangle \subseteq \mathcal{H}_a,$$

with both sides having the same dimension (namely, 4), and thus equality holds, providing the grading of $\mathcal{H}_a$ by form degree. □

This in particular proves Theorem 7.1.1.

Remark 7.5.3. Observe that *all* ingredients in the Fredholm analysis of the normal operator family, which here in particular involves estimates at normally hyperbolic trapping, as well as *all* of the above arguments which lead to a characterization of the spaces of resonances are stable in the sense that they apply to *any* stationary perturbation of a given

Schwarzschild-de Sitter spacetime (4-dimensional for the above, but similar arguments apply in all spacetime dimensions ≥ 4), not only to slowly rotating Kerr-de Sitter black holes. In fact, using the analysis of operators with non-smooth coefficients developed in Chapters 8 and 9, we can deduce decay and expansions in the exact same form as in the above theorem for waves on spacetimes which are merely 'asymptotically stationary' and close to Schwarzschild-de Sitter, i.e. for which the metric tensor differs from a stationary metric close to Schwarzschild-de Sitter by an exponentially decaying symmetric 2-tensor (with suitable regularity).

This shows at once that *quasilinear* wave equations on differential forms of the form $\Box_{g(u,\nabla u)}u = q(u,\nabla u)$ with small initial data can be solved globally, provided $g(0,0)$ is close to the Schwarzschild-de Sitter metric, $g(u,\nabla u) = g(0,0)$ for 0-resonant states u of $\Box_{g(0,0)}$, and the non-linearity q annihilates 0-resonant states; to give an (artificial) example, on 2-forms, one could take $q(u,\nabla u) = |du|^2 u$, and $g(u,\nabla u) = g(0,0) + g'(du,\delta u)$, where g' is a smooth bundle map from the form bundle into symmetric 2-tensors which vanishes at least simply at $(0,0)$, so g' is merely an exponentially small perturbation. Notice that in this example, we force the asymptotic model to be fixed, since in general the space of zero resonant states may vary with u, which causes additional complications that are difficult to deal with in this generality.

Remark 7.5.4. In the case of the Kerr-de Sitter metric, we can in fact explicitly write down $u_{a,1} \in \mathcal{H}_a^2$ (and then take $u_{a,2} = \star_{g_a} u_{a,1}$ to obtain a basis of $\mathcal{H}_a^2$). Indeed, on the Kerr spacetime, Andersson and Blue [3] give the values of the spin coefficients of the Maxwell field for the Coulomb solution in [3, §3.1], and reconstructing the Maxwell field itself (in the basis given by wedge products of differentials of the Boyer-Lindquist coordinates t, r, θ, ϕ) is then an easy computation using the explicit form of the null tetrad given in [3, Introduction, §2.4].[27] A tedious but straightforward calculation shows that the resulting 2-form

$$\begin{aligned} u_{a,1} := F_{a,TR}(r,\theta)\,(dt - a\sin^2\theta\,d\phi)\wedge dr \\ + F_{a,\Theta\Phi}(r,\theta)\sin\theta\,d\theta\wedge(a\,dt - (r^2+a^2)\,d\phi) \end{aligned}$$

[27] In the definition of ϕ_0 in [3, §2.4], the second summand $F[\widehat{\mathbf{\Theta}}, \widehat{\mathbf{\Phi}}_{\mathrm{PNV}}]$ should be replaced by $F[\bar{\mathbf{m}}, \mathbf{m}]$ to yield the correct result, see also [18, Equation (2)†].

with

$$F_{a,TR}(r,\theta) = \frac{r^2 - a^2\cos^2\theta}{(r^2 + a^2\cos^2\theta)^2}, \quad F_{a,\Theta\Phi}(r,\theta) = \frac{2ar\cos\theta}{(r^2 + a^2\cos^2\theta)^2}$$

is a solution of Maxwell's equations on Kerr-de Sitter space as well, i.e. when the cosmological constant is positive.

Chapter 8

Pseudodifferential operators with rough coefficients

8.1 Introduction

We now prepare the analysis of *quasilinear* wave equations on non-trapping spacetimes (with normally hyperbolic trapping) according to Definition 2.5.1; the equations we will consider have the form

$$\Box_{g(u,du)}u = f + q(u, du),$$

that is, the metric tensor g is now allowed to depend on the solution u itself. Thus, even at the level of the principal symbol, such equations exhibit non-smooth behavior: If u only has limited regularity, then so does the metric $g(u, du)$, and correspondingly the characteristic set, the null-bicharacteristic flow etc. Since we showed in Chapter 5 that weighted b-Sobolev are natural spaces in the global study of linear waves on such spacetimes, we shall consider pseudodifferential operators with coefficients lying in such spaces here. Similar operators with rough coefficients were considered before on Euclidean space in the work of Beals and Reed [9], and we will follow many of their ideas. Note that nonlinear analysis in Fréchet spaces like $\mathcal{C}^\infty$ requires an even finer control of regularity properties for operators with non-smooth coefficients, since this necessitates an analysis of operators with *non-smooth* coefficients which is uniform in a certain sense as one varies the regularity requirement.

We will give the details on how one solves quasilinear equations using the technology of rough ps.d.o.s in Chapter 9, but roughly speaking, the plan is to repeat the analysis of semilinear equations in Chapter 5, replacing standard microlocal regularity results in the smooth setting by their rough counterparts, and use suitable iteration schemes. On spaces with normally hyperbolic trapping, any iteration scheme for general equations with derivatives in the non-linearity will lose derivatives, as we saw in §5.3.2, and our use of the Nash-Moser iteration scheme (see [60, 99] and the references therein) in §9.2 forces us to be rather precise in our regularity estimates; indeed, what we need are so-called *tame estimates*; see below.

Thus, the main ingredient of the framework in which will analyze b-operators with non-smooth coefficients on manifolds with boundary is a partial calculus for what we call *b-Sobolev b-pseudodifferential operators*; for brevity, we will refer to these as 'non-smooth operators' to distinguish them from 'smooth operators,' by which we mean standard b-pseudodifferential operators, discussed in §3.3. b-Sobolev b-ps.d.o.s are (generalizations of) b-ps.d.o.s with coefficients in b-Sobolev spaces.[28] This calculus allows us to prove the microlocal regularity results for b-Sobolev b-ps.d.o.s which we discussed in the smooth setting in §§3.2.1 (elliptic regularity), 3.2.2 (real principal type propagation), 3.2.3 (complex absorbing potentials), 3.3.1 (b-radial points) and 3.3.2 (normally hyperbolic trapping in the b-sense); see §§8.5 and 8.8.2 for their non-smooth counterparts. We only develop a local calculus for non-smooth operators on $\overline{\mathbb{R}^n_+}$ (and $\mathbb{R}^n$) rather than providing an invariant calculus on a manifold. For our applications on (static) de Sitter and Kerr-de Sitter spacetimes, which are diffeomorphic to open subsets of $\overline{\mathbb{R}^n_+}$ when compactified at future infinity, this is sufficient; further, elliptic regularity and the real principal type propagation of singularities are purely local results and do not require an invariant calculus, as we will explain in §8.9. Radial point and trapping estimates on the other hand have a non-local character in that the radial or trapped set may in general not be contained in a single coordinate patch. Hence, the non-smooth microlocal study of more general geometries requires additional arguments, and we show in §8.9 how to deduce radial point and trapping estimates on manifolds from the local results using simple partition of unity arguments.

We remark that paradifferential methods would give sharper results with respect to the regularity of the spaces on which we prove our microlocal regularity results, and correspondingly we do not make any substantial effort here to push the regularity down: Our entirely L^2-based method is both conceptually and technically relatively straightforward, powerful

[28] Beals and Reed [9] consider operators on Euclidean space with coefficients in microlocal Sobolev spaces; this generality is not needed for our purposes, even though including it would only require more care in bookkeeping.

enough for our purposes, and lends itself very easily to generalizations in other contexts.

The study of ps.d.o.s with non-smooth coefficients is not new: Beals and Reed [9] developed a partial calculus with coefficients in L^2-based Sobolev spaces on Euclidean space, which is the basis for our extension to manifolds with boundary. Marschall [77] gave an extension of the calculus to L^p-based Sobolev spaces (and even more general spaces) and in addition proved the invariance of certain classes of non-smooth operators under changes of coordinates. Witt [121] extended the L^2-based calculus to contain elliptic parametrices. Pseudodifferential calculi for coefficients in C^k spaces have been studied by Kumano-go and Nagase [72]. In a slightly different direction, paradifferential operators, pioneered by Bony [14] and Meyer [93], are a widely used tool in nonlinear PDE, often giving more precise results than rough ps.d.o.s, at the expense of significant technical complications; see e.g. Hörmander [63] and Taylor [108, 107] and the references therein.

We will now give the idea how to generalize b-ps.d.o.s to the non-smooth setting. First, recall that the action of a b-ps.d.o. $A \in \Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$ with smooth full symbol $a = a(x, y; \lambda, \eta) \in S^m_{\mathrm{b}}((\overline{\mathbb{R}^n_+})_{x,y} \times \mathbb{R}^n_{\lambda,\eta})$, see Definitions 3.3.1 and 3.3.2 for the notation used here, can be written as

$$Au(x, y) = \int_{\mathbb{R}\times\mathbb{R}^{n-1}\times(0,\infty)\times\mathbb{R}^{n-1}} e^{i(y-y')\eta} s^{i\lambda} a(x, y, \lambda, \eta) u(x/s, y')\, d\lambda\, d\eta\, \frac{ds}{s}\, dy', \qquad (8.1.1)$$

where we did not make the logarithmic change of coordinates that we used in §3.3. Recall then the asymptotic expansion for the composition of two b-ps.d.o.s $A, B \in \Psi_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$,

$$\sigma(A \circ B)(z, \zeta) \sim \sum_{\beta \geq 0} \frac{1}{\beta!} (\partial^\beta_\zeta a\, {}^{\mathrm{b}}D^\beta_z b)(z, \zeta), \qquad (8.1.2)$$

where a and b are the full symbols of A and B, and ${}^{\mathrm{b}}D_z = (xD_x, D_y)$, where $D = -i\partial$ as usual. The analogous non-smooth operators that we discuss in this chapter, b-Sobolev b-ps.d.o.s, are locally defined by (8.1.1), but we now allow the symbol a to be less regular. As an example, for many remainder terms in our computations, it will suffice to merely have

$$\left\| \frac{a(z, \zeta)}{\langle \zeta \rangle^m} \right\|_{H^s_{\mathrm{b}}((\overline{\mathbb{R}^n_+})_z)} \leq C, \text{ uniformly in } \zeta \in \mathbb{R}^n, \qquad (8.1.3)$$

which already implies that $A = a(z, {}^{\mathrm{b}}D_z)$ defines a continuous map

$$A\colon H_{\mathrm{b}}^{s'} \to H_{\mathrm{b}}^{s'-m}, \quad s \geq s' - m, s > n/2 + \max(0, m - s'); \tag{8.1.4}$$

see Proposition 8.2.9. Assuming more regularity of the symbols in ζ, we can study compositions of such non-smooth operators; the main tool here is the asymptotic expansion (8.1.2), which must be cut off after finitely many terms in view of the limited regularity of the symbols, and the remainder term will be estimated carefully. In §8.2, we will develop the (partial) calculus of b-Sobolev b-ps.d.o.s as far as needed for our applications in Chapter 9.

Next, recall that elliptic regularity, on unweighted spaces for the sake of brevity, states that if $u \in H_{\mathrm{b}}^{-\infty}$ satisfies $Pu \in H_{\mathrm{b}}^{\sigma-m}$ for $P \in \Psi_{\mathrm{b}}^m$ which is elliptic at a point ζ in the cosphere bundle, then u is in H_{b}^σ microlocally at ζ. The proof is an easy application of the symbolic calculus – one essentially takes the reciprocal of the symbol of P near ζ to obtain an approximate inverse of P there – and readily generalizes to the non-smooth setting as shown in §8.4; the main technical task is to understand reciprocals of non-smooth symbols, which we will deal with in §8.3.

Further, given an operator $P \in \Psi_{\mathrm{b}}^m$ with real homogeneous principal symbol p, we need to study the singularities for solutions $u \in H_{\mathrm{b}}^{-\infty}$ of $Pu = f \in H_{\mathrm{b}}^{\sigma-m+1}$ within the characteristic set $\Sigma = p^{-1}(0)$ of u, where we assume $dp \neq 0$ at Σ so that Σ is a smooth conic codimension 1 submanifold of ${}^{\mathrm{b}}T^*M \setminus o$. The propagation of singularities, in the setting of closed manifolds discussed in §3.2.2, then states that $\mathrm{WF}_{\mathrm{b}}^\sigma(u)$ is invariant under the flow of the Hamilton vector field H_p of p. In other words, $\mathrm{WF}_{\mathrm{b}}^\sigma(u)$ is the union of maximally extended null-bicharacteristics of P. We will generalize this statement to the case of non-smooth P in §8.5.3 in a way that is similar to the outline of the proof of Theorem 3.2.1. Since P now only acts on a certain range of b-Sobolev spaces, the allowed degrees σ of regularity that we can propagate have bounds both from above and from below in terms of the regularity s of the coefficients of P; also, since non-smooth operators like the ones given by symbols as in (8.1.3) have very restricted mapping properties on low or negative order spaces, see (8.1.4), we need to assume higher regularity H_{b}^s of the coefficients of P when we want to propagate low regularity H_{b}^σ of solutions u. The main bookkeeping overhead of the proof of the propagation of singularities thus comes from the need to make sense of all compositions, dual pairings, adjoints and actions of non-smooth operators that appear in the course of the positive commutator argument. On a more technical side, we strive to limit the number of non-smooth operators as much as possible and thus have to absorb certain non-smooth terms appearing in the argument into an additional error term F of symbolic order 2σ; by judiciously choosing the operators in the positivity and the a priori control regions (called B and E in the proof of Theorem 3.2.1), we can however ensure that the symbol of F in fact has a sign, thus the additional term $\langle Fu, u\rangle$ appearing in the positive commutator argument can be bounded by the sharp Gårding inequality, which we will prove for non-smooth operators in §8.5.1.

The propagation of singularities near radial points and near normally hyperbolic trapping in the b-sense, as present in our applications on asymptotically de Sitter and Kerr-de Sitter space, is proved for non-smooth operators in §§8.5.4 and 8.5.5. The proof again proceeds via positive commutators, thus similar comments about the interplay of regularities as in the real principal type setting apply.

In order to have analogues of the local and global energy estimates proved in Chapter 4 available in the non-smooth setting, we include a full proof of the standard local energy energy estimate in the non-smooth setting; see §8.6.

We now give an example of the kinds of estimates that the non-smooth microlocal arguments give: Elliptic regularity for the equation $Pu = f$, where $P = \mathrm{Op}(p)$ is of order m and has coefficients in H^s, schematically yields the quantitative statement

$$\|u\|_{H^\sigma} \leq C(1 + \|p\|_{H^s})\|Pu\|_{H^{\sigma-m}}, \quad 0 \leq \sigma \leq s, \tag{8.1.5}$$

dropping all localizers etc. As we will explain in §9.2, the Nash-Moser iteration scheme requires more precise estimates: Namely, we need the right hand side of (8.1.5) to be linear in high regularity norms, to wit, for $\sigma \approx s$,

$$\|u\|_{H^\sigma} \leq C(\|p\|_{H^{s_0}})\big(\|Pu\|_{H^{\sigma-m}} + \|p\|_{H^s}\|Pu\|_{H^{s_0}}\big), \tag{8.1.6}$$

where s_0 is *fixed* and independent of σ, s. In our applications, the coefficients of P depend on u; thus, when one uses smoothing operators (which have large operator norms) to improve the regularity of u – this being necessary if every iteration step loses derivatives –, the constant in the tame estimate (8.1.6) depends linearly on the operator norm of the smoothing operator (mapping a lower regularity space into a higher regularity space), rather than quadratically as in (8.1.5); this is what allows for the fast convergence of Newton iteration (which underlies the Nash-Moser scheme) to dominate the blow-up of the smoothing operator norms. See [99] for details.

The purpose of §§8.7 and 8.8 is to show estimates of the form (8.1.6). Morally speaking, the only serious non-smoothness of b-Sobolev b-ps.d.o.s is that of their coefficients (rather than of the dependence on the fiber variables), which get multiplied by derivatives of the function the ps.d.o. acts on. Thus, once one has a way to prove a tame estimate in L^2-based spaces setting using Fourier methods, i.e. an estimate of the form

$$\|uv\|_{H^s} \leq C(\|u\|_{H^s}\|v\|_{H^{s_0}} + \|u\|_{H^{s_0}}\|v\|_{H^s}), \quad s \geq s_0 > n/2, \quad u, v \in H^s(\mathbb{R}^n), \tag{8.1.7}$$

one can easily obtain tame estimates for operator compositions in the non-smooth calculus, which then give tame microlocal regularity estimates. The idea of the proof of the estimate (8.1.7), see Corollary 8.7.2, is to split the product uv up in the Fourier-domain into a piece where u is localized at high frequency and v in low frequency, and another piece where this relationship is reversed.

8.2 A calculus for operators with b-Sobolev coefficients

We work locally on $\overline{\mathbb{R}^n_+}$. To analyze the action of operators with non-smooth coefficients on b-Sobolev functions, we need a convenient formula. Given $A \in \Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+})$ with full symbol $a(x, y; \lambda, \eta) \in S^m(\overline{\mathbb{R}^n_+})$, compactly supported in x, y, we have for $u \in \dot{\mathcal{C}}_{\mathrm{c}}^\infty(\overline{\mathbb{R}^n_+})$

$$\begin{aligned} Au(x,y) &= \iiiint e^{i\lambda\log(x/x')} e^{i\eta(y-y')} a(x,y;\lambda,\eta) u(x',y') \,\frac{dx'}{x'}\, dy'\, d\lambda\, d\eta \\ &= \iint x^{i\lambda} e^{i\eta y} a(x,y;\lambda,\eta) \widehat{u}(\lambda,\eta)\, d\lambda\, d\eta. \end{aligned}$$

Writing $\widehat{a}$ for the Mellin transform in x and the Fourier transform in y, we obtain

$$\begin{aligned} \widehat{Au}(\sigma,\gamma) &= \iiiint x^{-i(\sigma-\lambda)} e^{-i(\gamma-\eta)} a(x,y;\lambda,\eta) \widehat{u}(\lambda,\eta)\, d\lambda\, d\eta \,\frac{dx}{x}\, dy \\ &= \iint \widehat{a}(\sigma-\lambda,\gamma-\eta;\lambda,\eta) \widehat{u}(\lambda,\eta)\, d\lambda\, d\eta. \end{aligned} \tag{8.2.1}$$

Even though this makes sense as a distributional pairing, it is technically inconvenient to use directly: The problem is that if a does not vanish at $x = 0$, then $\widehat{a}(\sigma, \gamma; \lambda, \eta)$ has a pole at $\sigma = 0$ (cf. [82, Proposition 5.27]). This is easily dealt with by decomposing

$$a = a^{(0)}(y;\lambda,\eta) + a^{(1)}(x,y;\lambda,\eta), \tag{8.2.2}$$

where $a^{(0)}(y;\lambda,\eta) = a(0,y;\lambda,\eta)$ and $a^{(1)}(x,y;\lambda,\eta) = x\widetilde{a}^{(1)}(x,y;\lambda,\eta)$ with $\widetilde{a}^{(1)} \in S^m$. (Of course, $a^{(0)}$ in general no longer has compact support; however, this will be completely irrelevant for the analysis, due to the fact that $a^{(0)}$ has 'nice' behavior in y, *independently* in x.) Then $\widehat{a^{(1)}}(\sigma,\gamma;\lambda,\eta)$ is smooth and rapidly decaying in (σ,γ), and we write

$$(A^{(1)}u)\widehat{\ }(\zeta) = \int \widehat{a^{(1)}}(\zeta-\xi;\xi)\widehat{u}(\xi)\,d\xi. \tag{8.2.3}$$

For $A^{(0)} = a^{(0)}(y, {}^{\mathrm{b}}D)$, we obtain

$$(A^{(0)}u)\widehat{\ }(\sigma,\gamma) = \int \mathscr{F}a^{(0)}(\gamma-\eta;\sigma,\eta)\widehat{u}(\sigma,\eta)\,d\eta, \tag{8.2.4}$$

and $\mathscr{F}a^{(0)}(\gamma;\sigma,\eta)$ is rapidly decaying in γ.

Remark 8.2.1. Either we read off equation (8.2.4) directly from equation (8.2.1), where we observe that the symbol $a^{(0)}$ is independent of x, thus the integrals over x and λ are Mellin transform and inverse Mellin transform, respectively, and therefore cancel; or we observe that, with $a^{(0)}(x,y;\lambda,\eta) := a^{(0)}(y;\lambda,\eta)$, we have $\widehat{a^{(0)}}(\sigma-\lambda,\gamma-\eta;\lambda,\eta) = 2\pi\delta_{\sigma=\lambda}\mathscr{F}a^{(0)}(0,\gamma-\eta;\lambda,\eta)$. The second argument also shows that many manipulations on integrals that compute $A^{(1)}u$ (or compositions of b-operators) also apply to the computation of $A^{(0)}u$ if one reads integrals as appropriate distributional pairings.

Notice that (8.2.3) is, with the change in meaning of $\widehat{a^{(1)}}$ and $\widehat{u}$ and keeping in mind that $a^{(1)} = x\widetilde{a}^{(1)}$ is a rather special symbol, the same formula as for pseudodifferential operators on a manifold without boundary used by Beals and Reed [9]. Since also the characterization of H_{b}^s functions in terms of their mixed Mellin and Fourier transform (Lemma 3.3.5) is completely analogous to the characterization of H^s functions in terms of their Fourier transform, the arguments presented in [9] carry over to this restricted b-setting. In order to introduce necessary notation and construct a (partial) calculus in the full b-setting, containing weights, we will go through most arguments of [9], extending and adapting them to the b-setting; and of course we will have to treat the term A_0 separately.

The class of operators we are interested in are b-differential operators whose coefficients lie in (weighted) b-Sobolev spaces of high order. Let us remark that we do not develop an invariant calculus that can be transferred to a manifold; in particular, all definitions are on $\overline{\mathbb{R}^n_+}$. For examples of invariant symbol classes and invariant calculi, see [77, §5] and [121]. However, as mentioned in the introduction and explained in detail in §8.9, our local calculus suffices even for non-local regularity estimates. We thus define the following classes of non-smooth symbols:

Definition 8.2.2. For $m, s \in \mathbb{R}$, define the spaces of symbols

$$H^s_{\mathrm{b}} S^m_{(\mathrm{b})} = \Big\{ \sum_{\text{finite}} a_j(z) p_j(z,\zeta) \colon a_j \in H^s_{\mathrm{b}}, p_j \in S^m_{(\mathrm{b})} \Big\},$$

and denote by $H^s_{\mathrm{b}}\Psi^m_{(\mathrm{b})}$ the corresponding spaces of operators, i.e.

$$H^s_{\mathrm{b}}\Psi^m_{(\mathrm{b})} = \{ a(z, {}^{\mathrm{b}}D) \colon a(z,\zeta) \in H^s_{\mathrm{b}} S^m_{(\mathrm{b})} \}.$$

Moreover, let $\Psi^m = \{ a(z, {}^{\mathrm{b}}D) \colon a(z,\zeta) \in S^m \}$.

Remark 8.2.3. In this chapter, we will only deal with operators that are quantizations of symbols on the b-cotangent bundle, and thus with Ψ^m we will *always* mean the space defined above.

Remark 8.2.4. In a large part of the development of the calculus for non-smooth b-ps.d.o.s in this section, we will keep track of additional information on the symbols of most ps.d.o.s, encoded in the space of symbols S^*_{b}, in order to ensure that they act on weighted b-Sobolev spaces. Although this requires a small conceptual overhead, it simplifies some computations later on.

The spaces $H^*_{\mathrm{b}}\Psi^*_{(\mathrm{b})}$ are not closed under compositions, in fact they are not even left Ψ^*_{b}-modules. To get around this, which will be necessary in order to develop a sufficiently powerful calculus, we will consider less regular spaces, which however are still small enough to allow for good analytic (i.e. mapping and composition) properties.

Definition 8.2.5. For $s, m \in \mathbb{R}, k \in \mathbb{N}_0$, define the space

$$\begin{aligned} S^{m;0} H^s_{\mathrm{b}} &= \Big\{ p(z,\zeta) \colon p \in \langle\zeta\rangle^m L^\infty_\zeta\left((H^s_{\mathrm{b}})_z\right) \Big\} \\ &= \Big\{ p(z,\zeta) \colon \frac{\langle\eta\rangle^s \widehat{p}(\eta;\zeta)}{\langle\zeta\rangle^m} \in L^\infty_\zeta L^2_\eta \Big\}. \end{aligned}$$

Let $S_{\mathrm{b}}^{m;0}H_{\mathrm{b}}^{s}$ be the space of all symbols $p(x,y;\lambda,\eta)\in S^{m;0}H_{\mathrm{b}}^{s}$ which are entire in λ with values in $\langle\eta\rangle^m L^\infty_\eta((H_{\mathrm{b}}^s)_z)$ such that for all N the following estimate holds:

$$\|p(z;\lambda+i\mu,\eta)\|_{H_{\mathrm{b}}^s}\leq C_N\langle\lambda,\eta\rangle^m,\quad |\mu|\leq N. \tag{8.2.5}$$

Finally, define the spaces

$$S_{(\mathrm{b})}^{m;k}H_{\mathrm{b}}^{s}=\Big\{p(z,\zeta)\colon \partial_\zeta^\beta p\in S_{(\mathrm{b})}^{m-|\beta|;0}H_{\mathrm{b}}^{s}, |\beta|\leq k\Big\}.$$

The spaces of operators which are left quantizations of these symbols are denoted by $\Psi^{m;0}H_{\mathrm{b}}^{s}$, $\Psi_{\mathrm{b}}^{m;0}H_{\mathrm{b}}^{s}$ and $\Psi_{(\mathrm{b})}^{m;k}H_{\mathrm{b}}^{s}$, respectively.

Weighted versions of these spaces, involving $H_{\mathrm{b}}^{s,\alpha}$ for $\alpha\in\mathbb{R}$, are defined analogously.

Compare (8.2.5) with (3.3.4). We shall occasionally write $\mathrm{Op}(a):=a(z,{}^{\mathrm{b}}D)$ for (left) quantizations of symbols.

We can also define similar symbol and operator classes for operators acting on bundles: Let E,F,G be the trivial (complex or real) vector bundles over $\overline{\mathbb{R}^n_+}$ of ranks d_E,d_F,d_G, respectively, equipped with a smooth metric (Hermitian for complex bundles) on the fibers which is the standard metric on the fibers over the complement of a compact subset of $\overline{\mathbb{R}^n_+}$, then we can define

$$H_{\mathrm{b}}^{s}S^m(\overline{\mathbb{R}^n_+};G):=\{(a_i)_{1\leq i\leq d_G}\colon a_i\in H_{\mathrm{b}}^sS^m\}.$$

We then define the space $H_{\mathrm{b}}^s\Psi^m(\overline{\mathbb{R}^n_+};E,F)$ to consist of left quantizations of symbols in $H_{\mathrm{b}}^sS^m(\overline{\mathbb{R}^n_+};\mathrm{Hom}(E,F))$; likewise for all other symbol and operator classes. We shall also write $H_{\mathrm{b}}^s\Psi^m(\overline{\mathbb{R}^n_+};E):=H_{\mathrm{b}}^s\Psi^m(\overline{\mathbb{R}^n_+};E,E)$.

Remark 8.2.6. If we considered, as an example, the wave operator corresponding to a non-smooth metric acting on differential forms, the natural metric on the fibers of the form bundle would be non-smooth. Even though this could be dealt with directly in this setting, we simplify our arguments by choosing an 'artificial' smooth metric to avoid regularity considerations when taking adjoints, etc.

The first step is to prove mapping properties of operators in the classes just defined; compositions will be discussed in §8.2.2.

8.2.1 Mapping properties

The mapping properties of operators in $\Psi^{m;0}H_{\mathrm{b}}^{s}$ are easily proved using the following simple integral operator estimate.

Lemma 8.2.7. *(Cf. [9, Lemma 1.4].) Let $g(\eta,\xi) \in L_\xi^\infty L_\eta^2$ and $G(\eta,\xi) \in L_\eta^\infty L_\xi^2$. Then the operator*

$$Tu(\eta) = \int G(\eta,\xi)g(\eta-\xi,\xi)u(\xi)\,d\xi$$

is bounded on L^2 with operator norm $\leq \|G\|_{L_\eta^\infty L_\xi^2}\|g\|_{L_\xi^\infty L_\eta^2}$.

Proof. Cauchy-Schwarz gives

$$\begin{aligned}
\|Tu\|_{L^2}^2 &\leq \int \left(\int |G(\eta,\xi)|^2\,d\xi\right)\left(\int |g(\eta-\xi,\xi)u(\xi)|^2\,d\xi\right)\,d\eta \\
&\leq \|G\|_{L_\eta^\infty L_\xi^2}^2 \int \left(\int |g(\eta-\xi,\xi)|^2\,d\eta\right)|u(\xi)|^2\,d\xi \\
&\leq \|G\|_{L_\eta^\infty L_\xi^2}^2 \|g\|_{L_\xi^\infty L_\eta^2}^2 \|u\|_{L^2}^2.
\end{aligned}$$

□

The most common form of G in this chapter is given by and estimated in the following lemma. We use the notation

$$a_+ := \max(a,0), \quad a \in \mathbb{R}. \tag{8.2.6}$$

Lemma 8.2.8. *Suppose $s,r \in \mathbb{R}$ are such that $s \geq r, s > n/2 + (-r)_+$, then*

$$G(\eta,\xi) = \frac{\langle\eta\rangle^r}{\langle\eta-\xi\rangle^s\langle\xi\rangle^r} \in L_\eta^\infty(\mathbb{R}^n; L_\xi^2(\mathbb{R}^n)).$$

Proof. First, suppose $r \geq 0$. Then

$$G(\eta,\xi)^2 \leq \frac{1}{\langle\eta-\xi\rangle^{2(s-r)}\langle\xi\rangle^{2r}} + \frac{1}{\langle\eta-\xi\rangle^{2s}}.$$

Since $s > n/2$, the ξ-integral of the second fraction is finite and η-independent. For the ξ-integral of the first fraction, we split the domain of integration into two parts and obtain

$$\begin{aligned}
&\int_{|\xi|\leq|\eta-\xi|} \frac{1}{\langle\eta-\xi\rangle^{2(s-r)}\langle\xi\rangle^{2r}}\,d\xi + \int_{|\eta-\xi|\leq|\xi|} \frac{1}{\langle\eta-\xi\rangle^{2(s-r)}\langle\xi\rangle^{2r}}\,d\xi \\
&\qquad \leq \int \frac{1}{\langle\xi\rangle^{2s}}\,d\xi + \int \frac{1}{\langle\eta-\xi\rangle^{2s}}\,d\xi \in L_\eta^\infty.
\end{aligned}$$

Next, if $r < 0$, then

$$G(\eta,\xi)^2 = \frac{\langle\xi\rangle^{-2r}}{\langle\eta-\xi\rangle^{2s}\langle\eta\rangle^{-2r}} \leq \frac{1}{\langle\eta-\xi\rangle^{2(s-(-r))}} + \frac{1}{\langle\eta-\xi\rangle^{2s}},$$

where in the first fraction, we discarded the term $\langle\eta\rangle^{-2r} \geq 1$. Since $s-(-r) > n/2$, the integrals of both fractions are finite, and the proof is complete. □

Proposition 8.2.9. *Let $m \in \mathbb{R}$. Suppose $s \geq s'-m$ and $s > n/2 + (m-s')_+$. Then every $A = a(z, {}^{\mathrm{b}}D) \in \Psi^{m;0} H^s_{\mathrm{b}}(\overline{\mathbb{R}^n_+}; E, F)$ is a bounded operator $H^{s'}_{\mathrm{b}}(\overline{\mathbb{R}^n_+}; E) \to H^{s'-m}_{\mathrm{b}}(\overline{\mathbb{R}^n_+}; F)$. If $A \in \Psi^{m;0}_{\mathrm{b}} H^s_{\mathrm{b}}(\overline{\mathbb{R}^n_+}; E, F)$, then A is also a bounded operator $H^{s',\alpha}_{\mathrm{b}}(\overline{\mathbb{R}^n_+}; E) \to H^{s'-m,\alpha}_{\mathrm{b}}(\overline{\mathbb{R}^n_+}; F)$ for all $\alpha \in \mathbb{R}$.*

Note that this proposition also deals with 'low' regularity in the sense that negative b-Sobolev orders are permitted in the target space. We shall have occasion to use this in arguments involving dual pairings in §8.5.

Proof of Proposition 8.2.9. Let us first prove the statement without bundles, i.e. for complex-valued symbols and functions. Let $u \in H^{s'}_{\mathrm{b}}$ be given. Then

$$\langle\zeta\rangle^{s'-m}\widehat{Au}(\zeta) = \int \frac{\langle\zeta\rangle^{s'-m}\langle\xi\rangle^m}{\langle\zeta-\xi\rangle^s\langle\xi\rangle^{s'}} a_0(\zeta-\xi;\xi)u_0(\xi)\,d\xi$$

for $a_0(\zeta;\xi) \in L^\infty_\xi L^2_\zeta$, $u_0 \in L^2$. Lemma 8.2.8 ensures that the fraction in the integrand is an element of $L^\infty_\zeta L^2_\xi$, and then Lemma 8.2.7 implies $\langle\zeta\rangle^{s'-m}\widehat{Au}(\zeta) \in L^2_\zeta$.

In order to prove the second statement, we write for $u \in \dot{\mathcal{C}}^\infty_{\mathrm{c}}$

$$\begin{aligned} a(x,y,xD_x,D_y)u(x,y) &= \iint_{\operatorname{Im}\lambda=0} e^{i\lambda\log x}e^{i\eta y}a(x,y;\lambda,\eta)\widehat{u}(\lambda,\eta)\,d\lambda\,d\eta \\ &= \iint_{\operatorname{Im}\lambda=0} \widetilde{a}(\lambda)(\eta;x,y)\widehat{u}(\lambda,\eta)\,d\lambda\,d\eta, \end{aligned}$$

where

$$\widetilde{a}(\lambda)(\eta;x,y) = x^{i\lambda}e^{i\eta y}a(x,y;\lambda,\eta);$$

we want to shift the contour of integration to $\operatorname{Im}\lambda = -\alpha$. Assuming that $\operatorname{supp}_{x,y} a$ is compact, we have that for any N,

$$\|\widetilde{a}(\lambda)(\eta,\cdot,\cdot)\|_{H^{s,-N}_{\mathrm{b}}} \leq C_N\langle\lambda,\eta\rangle^{m+s}, \quad |\operatorname{Im}\lambda| < N,$$

and $\widetilde{a}(\lambda)$ is holomorphic in λ with values in $H_{\mathrm{b}}^{s,-N}$ for fixed η. Since $\widehat{u}(\lambda,\eta)$ is rapidly decaying, we infer for all sufficiently large $M > 0$

$$\begin{aligned}\int \|\widetilde{a}(\lambda)(\eta,\cdot,\cdot)\|_{H_{\mathrm{b}}^{s,-N}} |\widehat{u}(\lambda,\eta)|\,d\eta &\leq C_N \int \langle\lambda,\eta\rangle^{m+s-M}\,d\eta \\ &= C_{NM}\langle\lambda\rangle^{m+s-M+n-1},\end{aligned}$$

thus

$$\widetilde{a}'(\lambda)(x,y) := \int \widetilde{a}(\lambda)(\eta;x,y)\widehat{u}(\lambda,\eta)\,d\eta \in \langle\lambda\rangle^{-M} L^\infty_\lambda(H_{\mathrm{b}}^{s,-N})$$

for all $M > 0$, and $\widetilde{a}' \colon \mathbb{C} \to H_{\mathrm{b}}^{s,-N}$ is holomorphic. Therefore, if we choose $N > |\alpha|$, we can shift the contour of integration to the horizontal line $\mathbb{R} - i\alpha$:

$$\begin{aligned}a(x,y,xD_x,D_y)u(x,y) &= \int_{\operatorname{Im}\lambda=-\alpha} \widetilde{a}'(\lambda)(x,y)\,d\lambda \\ &= x^\alpha \iint_{\operatorname{Im}\lambda=0} e^{i\lambda\log x} e^{i\eta y} a(x,y;\lambda-i\alpha,\eta)(x^{-\alpha}u)\widehat{\ }(\lambda,\eta)\,d\lambda\,d\eta.\end{aligned}$$

By definition, $a|_{\operatorname{Im}\lambda=-\alpha}$ satisfies symbolic bounds just like $a|_{\operatorname{Im}\lambda=0}$, thus we are done by the first half of the proof.

Adding bundles is straightforward: Write $A \in \Psi^{m;0}H_{\mathrm{b}}^s(\overline{\mathbb{R}^n_+};E,F)$ as $A = (A_{ij})$, $A_{ij} \in \Psi^{m;0}H_{\mathrm{b}}^s(\overline{\mathbb{R}^n_+})$ and $u \in H_{\mathrm{b}}^{s'}(\overline{\mathbb{R}^n_+};E)$ as $u = (u_j)$, $u_j \in H_{\mathrm{b}}^{s'}(\overline{\mathbb{R}^n_+})$. Then $Au = (\sum_{j=1}^{d_E} A_{ij}u_j)$, thus $Au \in H_{\mathrm{b}}^{s'-m}(\overline{\mathbb{R}^n_+};F)$ follows by component-wise application of what we just proved. □

Corollary 8.2.10. *Let $s > n/2$. Then $H_{\mathrm{b}}^s(\overline{\mathbb{R}^n_+};\mathrm{End}(E))$ is an algebra. Moreover, the space $H_{\mathrm{b}}^{s'}(\overline{\mathbb{R}^n_+};\mathrm{Hom}(E,F))$ is a left $H_{\mathrm{b}}^s(\overline{\mathbb{R}^n_+};\mathrm{End}(E))$- and a right $H_{\mathrm{b}}^s(\overline{\mathbb{R}^n_+};\mathrm{End}(F))$-module for $|s'| \leq s$.*

Proof. As in the proof of Proposition 8.2.9, we can reduce the proof to the case of complex-valued functions. For $s' \geq 0$, the claim follows from $H_{\mathrm{b}}^{s'} \subset \Psi_{\mathrm{b}}^{0;0}H_{\mathrm{b}}^{s'}$ and the previous Proposition. For $s' \leq 0$, use duality. □

A simple tame version of this is given in Corollary 8.7.2.

8.2.2 Operator compositions

The basic idea is to mimic the formula for the asymptotic expansion of the full symbol of an operator which is the composition of $P = p(z,{}^{\mathrm{b}}D) \in \Psi^m_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$ and $Q = q(z,{}^{\mathrm{b}}D) \in \Psi^{m'}_{\mathrm{lb}}(\overline{\mathbb{R}^n_+})$, namely

$$\sigma(P\circ Q)(z,\zeta)\sim\sum_{\beta\geq 0}\frac{1}{\beta!}(\partial_\zeta^\beta p\,{}^{\mathrm{b}}D_z^\beta q)(z,\zeta);$$

recall the notation (3.3.6). If p or q only have limited regularity in ζ or z, we only keep finitely many terms of this expansion and estimate the resulting remainder term carefully. More precisely, we compute for $u\in\dot{\mathcal{C}}_{\mathrm{c}}^\infty$, keeping Remark 8.2.1 in mind:

$$\begin{aligned}(PQu)\widehat{\ }(\eta)&=\iint\widehat{p}(\eta-\xi;\xi)\widehat{q}(\xi-\zeta;\zeta)\widehat{u}(\zeta)\,d\zeta\,d\xi\\&=\int\left(\int\widehat{p}(\eta-\zeta-\xi;\zeta+\xi)\widehat{q}(\xi;\zeta)\,d\xi\right)\widehat{u}(\zeta)\,d\zeta,\end{aligned}\tag{8.2.7}$$

and

$$\begin{aligned}[(\partial_\zeta^\beta p\,{}^{\mathrm{b}}D_z^\beta q)(z,{}^{\mathrm{b}}D)u]\widehat{\ }(\eta)&=\int(\partial_\zeta^\beta p\,{}^{\mathrm{b}}D_z^\beta q)\widehat{\ }(\eta-\zeta;\zeta)\widehat{u}(\zeta)\,d\zeta\\&=\int\left(\int\partial_\zeta^\beta\widehat{p}(\eta-\zeta-\xi;\zeta)\xi^\beta\widehat{q}(\xi;\zeta)\,d\xi\right)\widehat{u}(\zeta)\,d\zeta.\end{aligned}$$

We now apply Taylor's theorem to the second argument of $\widehat{p}$ at $\xi=0$ in the inner integral in (8.2.7), keeping track of terms up to order $k-1\in\mathbb{N}_0$ (the case $k=0$ is handled easily), and obtain a remainder

$$\widehat{r}(\eta-\zeta;\zeta)=\sum_{|\beta|=k}\frac{k}{\beta!}\int\left(\int_0^1(1-t)^{k-1}\partial_\zeta^\beta\widehat{p}(\eta-\zeta-\xi;\zeta+t\xi)\,dt\right)\xi^\beta\widehat{q}(\xi;\zeta)\,d\xi,$$

corresponding to the operator

$$r(z,{}^{\mathrm{b}}D)=P\circ Q-\sum_{|\beta|<k}\frac{1}{\beta!}(\partial_\zeta^\beta p\,{}^{\mathrm{b}}D_z^\beta q)(z,{}^{\mathrm{b}}D).\tag{8.2.8}$$

We rewrite the remainder as

$$\widehat{r}(\eta;\zeta)=\sum_{|\beta|=k}\frac{k}{\beta!}\int\left(\int_0^1(1-t)^{k-1}\partial_\zeta^\beta\widehat{p}(\eta-\xi;\zeta+t\xi)\,dt\right)({}^{\mathrm{b}}D_z^\beta q)\widehat{\ }(\xi;\zeta)\,d\xi.\tag{8.2.9}$$

We will start by analyzing the terms in an expansion like (8.2.8) when the symbols involved are not smooth. When we deal with smooth b-operators by using the decomposition (8.2.2) of their symbols, we will need multiple sets of dual variables of x and y. For clarity, we will stick to the following names for them:

$$\text{(Mellin-)dual variables of } x\colon \sigma, \lambda, \rho,$$
$$\text{(Fourier-)dual variables of } y\colon \gamma, \eta, \theta.$$

Lemma 8.2.11. *Let $s, s', m, m' \in \mathbb{R}$ be such that $s > n/2$, $|s'| \leq s$. Then*

$$S^{m;0}H_{\mathrm{b}}^{s} \cdot S^{m';0}H_{\mathrm{b}}^{s'} \subset S^{m+m';0}H_{\mathrm{b}}^{s'},$$
$$S^{m} \cdot S^{m';0}H_{\mathrm{b}}^{s'} \subset S^{m+m';0}H_{\mathrm{b}}^{s'}.$$

The same statements are true if all symbol classes are replaced by the corresponding b-symbol classes.

Proof. In light of the definitions of the symbol classes, we can assume $m = m' = 0$. The first statement then is an immediate consequence of Corollary 8.2.10. In order to prove the second statement, we simply observe that, given $p \in S^0$, $p(\cdot;\zeta)$ is a uniformly bounded family of multipliers on $H_{\mathrm{b}}^{s'}$. A direct proof of the sort that we will use in the sequel goes as follows: Decompose the symbol p as in (8.2.2). The part $p^{(1)} \in S^{0;0}H_{\mathrm{b}}^{\infty}$ can then be dealt with using the first statement. Thus, we may assume $p = p^{(0)}$, i.e. $p = p(y;\lambda,\eta)$ is x-independent. Let $q \in S^{0;0}H_{\mathrm{b}}^{s'}$ be given. Choose N large and put

$$p_0(\gamma;\lambda,\eta) = \langle\gamma\rangle^N |\mathscr{F}p(\gamma;\lambda,\eta)|, \quad q_0(\sigma,\gamma;\lambda,\eta) = \langle\sigma,\gamma\rangle^{s'} |\widehat{q}(\sigma,\gamma;\lambda,\eta)|,$$
$$r_0(\sigma,\gamma;\lambda,\eta) = \langle\sigma,\gamma\rangle^{s'} |\widehat{pq}(\sigma,\gamma;\lambda,\eta)|.$$

Then

$$\begin{aligned}\iint r_0(\sigma,\gamma;\lambda,\eta)^2\,d\sigma\,d\gamma \\ &\leq \iint \left(\int \frac{\langle\sigma,\gamma\rangle^{s'}}{\langle\gamma-\theta\rangle^N \langle\sigma,\theta\rangle^{s'}} p_0(\gamma-\theta;\lambda,\eta) q_0(\sigma,\theta;\lambda,\eta)\,d\theta\right)^2 d\sigma\,d\gamma \\ &\lesssim \|p_0(\gamma;\lambda,\eta)\|^2_{L^\infty_{\lambda,\eta}L^2_\gamma} \|q_0(\sigma,\theta;\lambda,\eta)\|^2_{L^\infty_{\lambda,\eta}L^2_{\sigma,\theta}}\end{aligned}$$

by Cauchy-Schwarz. $\square$

Recall Remark 8.2.3 for the notation used in the following theorem on the composition properties of non-smooth operators:

Theorem 8.2.12. *Let $m, m', s, s' \in \mathbb{R}, k, k' \in \mathbb{N}_0$. For two operators $P = p(z, {}^{\mathrm{b}}D)$ and $Q = q(z, {}^{\mathrm{b}}D)$ of orders m and m', respectively, let*

$$R = P \circ Q - \sum_{|\beta|<k} \frac{1}{\beta!} (\partial_\zeta^\beta p \, {}^{\mathrm{b}}D_z^\beta q)(z, {}^{\mathrm{b}}D).$$

Denote the sum of the terms in the expansion for which $|\beta| = j$ by E_j.

(1) *Composition of non-smooth operators, $k \geq m + k'$, $k \geq k'$.*

   (a) *Suppose $s > n/2$ and $s \leq s' - k$ [$s \leq s' - 2k + m + k'$]. If $P \in \Psi^{m;k} H_{\mathrm{b}}^s$, $Q \in \Psi^{m';0} H_{\mathrm{b}}^{s'}$, then*

   $$E_j \in \Psi^{m+m'-j;0} H_{\mathrm{b}}^s, \quad R \in \Psi^{m'-k';0} H_{\mathrm{b}}^s \; [\Psi^{m+m'-k;0} H_{\mathrm{b}}^s].$$

   (b) *If $P \in \Psi^{m;k} H_{\mathrm{b}}^\infty$, $Q \in \Psi^{m';0} H_{\mathrm{b}}^{s'}$, then*

   $$E_j \in \Psi^{m+m'-j;0} H_{\mathrm{b}}^{s'-j}, \quad R \in \Psi^{m'-k';0} H_{\mathrm{b}}^{s'-k} \cap \Psi^{m+m'-k;0} H_{\mathrm{b}}^{s'-2k+m+k'}.$$

(2) *Composition of smooth with non-smooth operators.*

   (a) *Suppose $k \geq m + k'$, $k \geq k'$. If $P \in \Psi^m$, $Q \in \Psi^{m';0} H_{\mathrm{b}}^{s'}$, then*

   $$E_j \in \Psi^{m+m'-j;0} H_{\mathrm{b}}^{s'-j}, \quad R \in \Psi^{m'-k';0} H_{\mathrm{b}}^{s'-k} \cap \Psi^{m+m'-k;0} H_{\mathrm{b}}^{s'-2k+m+k'}.$$

   (b) *Suppose $k \leq k'$ and $k' \geq m$. If $P \in \Psi^{m;k'} H_{\mathrm{b}}^s$, $Q \in \Psi^{m'}$, then*

   $$E_j \in \Psi^{m+m'-j;0} H_{\mathrm{b}}^s, \quad R \in \Psi^{m+m'-k;0} H_{\mathrm{b}}^s.$$

(3) *Composition of smooth with non-smooth operators, $k \leq m + k'$, $k \geq k'$. If $P \in \Psi^m$, $Q \in \Psi^{m';0} H_{\mathrm{b}}^{s'}$, then*

   $$E_j \in \Psi^{m+m'-j;0} H_{\mathrm{b}}^{s'-j}, \quad R = R_1 \Lambda_{m+k'-k} + \Lambda_{m+k'-k} R_2,$$

   where $R_1, R_2 \in \Psi^{m'-k';0} H_{\mathrm{b}}^{s'-k}$, and $\Lambda_s = \lambda_s({}^{\mathrm{b}}D)$ in the notation of Corollary 3.3.7.

Moreover, (1)-(2) hold as well if all operator spaces are replaced by the corresponding b-spaces. Also, all results hold, mutatis mutandis, *if P maps sections of F to sections of G,* *and Q maps sections of E to sections of F.*

Proof. The statements about the E_j follow from Lemma 8.2.11. It remains to analyze the remainder operators. We will only treat the case $k > 0$; the case $k = 0$ is handled in a similar way. We prove parts (1), (2a) and (3) of the theorem for $k' = 0$ first.

(1a). Consider the case $s \leq s' - k$. We use formula (8.2.9) and define

$$p_0(\eta, \xi; \zeta) = \sum_{|\beta|=k} \frac{k}{\beta!} \langle \eta \rangle^s \int_0^1 |\partial_\zeta^\beta \widehat{p}(\eta; \zeta + t\xi)| \, dt,$$
$$q_0(\xi; \zeta) = \frac{\langle \xi \rangle^{s'-k} |({}^{\mathrm{b}}D_z^k q)\widehat{\ }(\xi; \zeta)|}{\langle \zeta \rangle^{m'}},$$

where ${}^{\mathrm{b}}D_z^k$ denotes the vector $({}^{\mathrm{b}}D_z^\beta)_{|\beta|=k}$. Since $p_0 \in L^\infty_{\zeta,\xi} L^2_\eta$ in view of $k \geq m$, i.e. $\partial_\zeta^\beta p$ is a symbol of order $m - k \leq 0$, and $q_0 \in L^\infty_\zeta L^2_\xi$, we obtain

$$\frac{\langle \eta \rangle^s |\widehat{r}(\eta; \zeta)|}{\langle \zeta \rangle^{m'}} \leq \int \frac{\langle \eta \rangle^s}{\langle \eta - \xi \rangle^s \langle \xi \rangle^{s'-k}} p_0(\eta - \xi, \xi; \zeta) q_0(\xi; \zeta) \, d\xi \in L^\infty_\zeta L^2_\eta$$

by Lemma 8.2.7, as claimed. Next, if $s \leq s' - 2k + m$, we instead define

$$p_0(\eta, \xi; \zeta) = \sum_{|\beta|=k} \frac{k}{\beta!} \langle \eta \rangle^s \int_0^1 \langle \zeta + t\xi \rangle^{k-m} |\partial_\zeta^\beta \widehat{p}(\eta; \zeta + t\xi)| \, dt \in L^\infty_{\zeta,\xi} L^2_\eta, \tag{8.2.10}$$

thus

$$\frac{\langle \eta \rangle^s |\widehat{r}(\eta; \zeta)|}{\langle \zeta \rangle^{m+m'-k}} \leq \int \frac{\langle \eta \rangle^s}{\langle \eta - \xi \rangle^s \langle \xi \rangle^{s'-k}} \cdot \frac{\langle \zeta \rangle^{k-m}}{\inf_{0 \leq t \leq 1} \langle \zeta + t\xi \rangle^{k-m}} \times p_0(\eta - \xi, \xi; \zeta) q_0(\xi; \zeta) \, d\xi$$

with $q_0 \in L^\infty_\zeta L^2_\xi$ as above. Now

$$\frac{\langle \zeta \rangle^{k-m}}{\inf_{0 \leq t \leq 1} \langle \zeta + t\xi \rangle^{k-m}} \lesssim \langle \xi \rangle^{k-m}, \tag{8.2.11}$$

since for $|\xi| \leq |\zeta|/2$, the left hand side is uniformly bounded, and for $|\zeta| \leq 2|\xi|$, we estimate the infimum from below by 1 and the numerator from above by $\langle \xi \rangle^{k-m}$. Therefore, we get $r_0 \in L^\infty_\zeta L^2_\eta$ in this case as well.

(1b) is proved similarly: Define $q_0(\xi;\zeta)$ as above, and choose N large and put

$$p_0(\eta,\xi;\zeta)=\sum_{|\beta|=k}\frac{k}{\beta!}\langle\eta\rangle^N\int_0^1|\partial_\zeta^\beta\widehat{p}(\eta;\zeta+t\xi)|\,dt.$$

Then

$$\frac{\langle\eta\rangle^{s'-k}|\widehat{r}(\eta;\zeta)|}{\langle\zeta\rangle^{m'}}\leq\int\frac{\langle\eta\rangle^{s'-k}}{\langle\eta-\xi\rangle^N\langle\xi\rangle^{s'-k}}p_0(\eta-\xi,\xi;\zeta)q_0(\xi;\zeta)\,d\xi,$$

and the fraction in the integrand is an element of $L^\infty_\eta L^2_\xi$ by Lemma 8.2.8, thus an application of Lemma 8.2.7 yields $R\in\Psi^{m';0}H_{\mathrm{b}}^{s'-k}$. In a similar manner, now using (8.2.11), we obtain $R\in\Psi^{m+m'-k;0}H_{\mathrm{b}}^{s'-2k+m}$.

(2). Decomposing the smooth operator as in (8.2.2), the x-dependent part has coefficients in H_{b}^∞, thus we can apply part (1). Therefore, we may assume that the smooth operator is x-independent in both cases.

(2a). The remainder is

$$\widehat{r}(\sigma,\gamma;\lambda,\eta)=\sum_{|\beta|=k}\frac{k}{\beta!}\int\left(\int_0^1(1-t)^{k-1}\partial_{\lambda,\eta}^\beta\mathscr{F}p(\gamma-\theta;\lambda+t\sigma,\eta+t\theta)\,dt\right)\\ \times({}^{\mathrm{b}}D_z^\beta q)\widehat{\ }(\sigma,\theta;\lambda,\eta)\,d\theta;$$

therefore, choosing N large and defining

$$p_0(\gamma,\sigma,\theta;\lambda,\eta)=\sum_{|\beta|=k}\frac{k}{\beta!}\langle\gamma\rangle^N\int_0^1|\partial_{\lambda,\eta}^\beta\mathscr{F}p(\gamma;\lambda+t\sigma,\eta+t\theta)|\,dt\in L^\infty_{\sigma,\theta,\lambda,\eta}L^2_\gamma,$$

$$q_0(\sigma,\theta;\lambda,\eta)=\frac{\langle\sigma,\theta\rangle^{s'-k}|({}^{\mathrm{b}}D_z^\beta q)\widehat{\ }(\sigma,\theta;\lambda,\eta)|}{\langle\lambda,\eta\rangle^{m'}}\in L^\infty_{\lambda,\eta}L^2_{\sigma,\theta},$$

we get

$$\frac{\langle\sigma,\gamma\rangle^{s'-k}|\widehat{r}(\sigma,\gamma;\lambda,\eta)|}{\langle\lambda,\eta\rangle^{m'}}\\ \leq\int\frac{\langle\sigma,\gamma\rangle^{s'-k}}{\langle\gamma-\theta\rangle^N\langle\sigma,\theta\rangle^{s'-k}}p_0(\gamma-\theta,\sigma,\theta;\lambda,\eta)q_0(\sigma,\theta;\lambda,\eta)\,d\theta,$$

which is an element of $L^\infty_{\lambda,\eta}L^2_{\sigma,\gamma}$ by Lemmas 8.2.8 and 8.2.7. This proves $R\in\Psi^{m';0}H_{\mathrm{b}}^{s'-k}$, and in a similar way we obtain $R\in\Psi^{m+m'-k;0}H_{\mathrm{b}}^{s'-2k+m}$.

(2b). Here, the remainder is

$$\widehat{r}(\sigma,\gamma;\lambda,\eta) = \sum_{|\beta|=k} \frac{k}{\beta!} \int \left(\int_0^1 (1-t)^{k-1} \partial_{\lambda,\eta}^{\beta} \widehat{p}(\sigma,\gamma-\theta;\lambda,\eta+t\theta)\,dt \right) \times \mathscr{F}({}^{\mathrm{b}}D_z^{\beta} q)(\theta;\lambda,\eta)\,d\theta,$$

and arguments similar to those used in (a) give the desired conclusion if $k = k'$. If $k < k'$, we just truncate the expansion after E_{k-1} and note that the resulting remainder term, which is the sum of the remainder term after expanding to order k' and the expansion terms $E_k, \ldots, E_{k'-1}$, indeed lies in $\Psi^{m+m'-k;0} H_{\mathrm{b}}^{s}$.

(3). We again use formula (8.2.9) for the remainder term and put

$$\widehat{r}_1(\eta;\zeta) = \frac{\widehat{r}(\eta;\zeta)\chi(|\zeta| \geq |\eta+\zeta|)}{\lambda_{m-k}(\zeta)}, \quad \widehat{r}_2(\eta;\zeta) = \frac{\widehat{r}(\eta;\zeta)\chi(|\zeta| < |\eta+\zeta|)}{\lambda_{m-k}(\eta+\zeta)},$$

the point being that, by equation (8.2.3), for any $u \in \dot{\mathcal{C}}_{\mathrm{c}}^{\infty}$,

$$\begin{aligned}(r(z,{}^{\mathrm{b}}D)u)\widehat{\ }(\eta) &= \int \widehat{r}(\eta-\zeta,\zeta)\widehat{u}(\zeta)\,d\zeta \\ &= \int \widehat{r}_1(\eta-\zeta,\zeta)(\Lambda_{m-k}u)\widehat{\ }(\zeta)\,d\zeta + \lambda_{m-k}(\eta)\int \widehat{r}_2(\eta-\zeta,\zeta)\widehat{u}(\zeta)\,d\zeta \\ &= (r_1(z,{}^{\mathrm{b}}D)\Lambda_{m-k}u)\widehat{\ }(\eta) + (\Lambda_{m-k}r_2(z,{}^{\mathrm{b}}D)u)\widehat{\ }(\eta).\end{aligned}$$

It remains to prove that $r_1(z,{}^{\mathrm{b}}D), r_2(z,{}^{\mathrm{b}}D) \in \Psi^{m';0} H_{\mathrm{b}}^{s'-k}$. First, we treat the case $P \in x\Psi^m$. Then for any $N \in \mathbb{N}$, we obtain, using

$$\sup_{0 \leq t \leq 1} \langle \zeta + t\xi \rangle^{m-k} \lesssim \langle \zeta \rangle^{m-k} + \langle \xi \rangle^{m-k},$$

that

$$\begin{aligned}\frac{\langle \eta \rangle^{s'-k} |\widehat{r}_1(\eta,\zeta)|}{\langle \zeta \rangle^{m'}} &\lesssim \int \frac{\langle \eta \rangle^{s'-k}(1+\langle \xi \rangle^{m-k}/\langle \zeta \rangle^{m-k})}{\langle \eta-\xi \rangle^{N} \langle \xi \rangle^{s'-k}} \chi(|\zeta| \geq |\eta+\zeta|) \\ &\qquad \times p_0(\eta-\xi,\xi;\zeta) q_0(\xi;\zeta)\,d\xi \\ &\equiv \int G(\eta,\xi;\zeta) p_0(\eta-\xi,\xi;\zeta) q_0(\xi;\zeta)\,d\xi,\end{aligned}$$

where $p_0(\eta,\xi;\zeta) \in L^\infty_{\xi,\zeta} L^2_\eta$ is defined as in (8.2.10) (with s replaced by N) and $q_0 \in L^\infty_\zeta L^2_\xi$ as before. We have to show $G(\eta,\xi;\zeta) \in L^\infty_{\eta,\zeta} L^2_\xi$ in order to be able to apply Lemma 8.2.7. For $|\xi| \geq 2|\eta|$, we immediately get, for N large enough,

$$G(\eta,\xi;\zeta) \lesssim \frac{1}{\langle\xi\rangle^{N'}} \left(1 + \frac{\langle\xi\rangle^{m-k}}{\langle\zeta\rangle^{m-k}}\right) \in L^\infty_{\eta,\zeta} L^2_\xi(|\xi| \geq 2|\eta|),$$

where $N' = N - (k-s')_+$. On the other hand, if $|\xi| < 2|\eta|$, we estimate

$$G(\eta,\xi;\zeta) \lesssim \frac{\langle\eta\rangle^{s'-k}}{\langle\eta-\xi\rangle^N \langle\xi\rangle^{s'-k}} \left(1 + \frac{\langle\eta\rangle^{m-k}}{\langle\zeta\rangle^{m-k}}\right) \chi(|\zeta| \geq |\eta+\zeta|)$$

and use that $|\zeta| \geq |\eta+\zeta|$ implies $|\eta| \leq |\eta+\zeta| + |-\zeta| \leq 2|\zeta|$, hence the product of the last two factors is uniformly bounded, giving $G(\eta,\xi;\zeta) \in L^\infty_{\eta,\zeta} L^2_\xi(|\xi| < 2|\eta|)$ by Lemma 8.2.8.

In the case $P = p(0,y;xD_x,D_y)$, we get the estimate

$$\frac{\langle\sigma,\gamma\rangle^{s'-k} |\widehat{r}_1(\sigma,\gamma;\lambda,\eta)|}{\langle\lambda,\eta\rangle^{m'}} \leq \int G(\sigma,\gamma,\theta;\lambda,\eta) p_0(\gamma-\theta,\sigma,\theta;\lambda,\eta) q_0(\sigma,\theta;\lambda,\eta)\, d\theta,$$

where $p_0(\gamma,\sigma,\theta;\lambda,\eta) \in L^\infty_{\sigma,\theta,\lambda,\eta} L^2_\gamma$, $q_0(\sigma,\theta;\lambda,\eta) \in L^\infty_{\lambda,\eta} L^2_{\sigma,\theta}$, and

$$\begin{aligned} G(\sigma,\gamma,\theta;\lambda,\eta) = \frac{\langle\sigma,\gamma\rangle^{s'-k}}{\langle\gamma-\theta\rangle^N \langle\sigma,\theta\rangle^{s'-k}} &\left(1 + \frac{\langle\sigma,\theta\rangle^{m-k}}{\langle\lambda,\eta\rangle^{m-k}}\right) \\ &\times \chi\Big(|(\lambda,\eta)| \geq |(\sigma,\gamma) + (\lambda,\eta)|\Big). \end{aligned}$$

As above, separating the cases $|(\sigma,\theta)| \geq 2|(\sigma,\gamma)|$ and $|(\sigma,\theta)| < 2|(\sigma,\gamma)|$, one obtains $G \in L^\infty_{\sigma,\gamma,\lambda,\eta} L^2_\theta$, and we can again apply Lemma 8.2.7.

The second remainder term r_2 is handled in the same way.

Next, we prove that (1)-(2) also hold for the corresponding b-operator spaces. Using exactly the same estimates as above, one obtains the respective symbolic bounds for the remainders on each line $\operatorname{Im}\lambda = \alpha_0$. What remains to be shown is the holomorphicity of the remainder operator in λ. This is a consequence of the fact that the derivatives $\partial_\lambda \partial_\zeta^\beta p$, $|\beta| = k$, and $\partial_\lambda q$, satisfy the same (in the case of symbols of smooth b-ps.d.o.s, even better by one order) symbol estimates as $\partial_\zeta^\beta p$ and q, respectively. Indeed, for (1a), i.e. for non-smooth b-symbols, this follows from the Cauchy integral formula, which for $\partial_\lambda q$ gives

$$\partial_\lambda q(z;\lambda,\eta) = \frac{1}{2\pi i}\oint_{\gamma(\lambda)} \frac{q(z;\sigma,\eta)}{(\sigma-\lambda)^2}\,d\sigma$$

where $\gamma(\lambda)$ is the circle around λ with radius 1. Namely, since $|\sigma-\lambda| = 1$ for $\sigma \in \gamma(\lambda)$, we get the desired estimate for $\partial_\lambda q$ from the corresponding estimate for q itself. We handle $\partial_\lambda\partial_\zeta^\beta p$ similarly. (1b) and (2) for b-operators follow in the same way.

Finally, let us prove (1), (2a) and (3) for $k' > 0$ following the argument of Beals and Reed in [9, Corollary 1.6], starting with (1a): Choose a partition of unity on $\mathbb{R}^n$ consisting of smooth non-negative functions $\chi_0,\ldots,\chi_n$ with $\operatorname{supp}\chi_0 \subset \{|\zeta| \leq 2\}$, and $|\zeta_l| \geq 1$ on $\operatorname{supp}\chi_l$. Then

$$P \circ Q\chi_0({}^{\mathrm{b}}D) \in \Psi^{m;k} H_{\mathrm{b}}^s \circ \Psi^{-\infty;0} H_{\mathrm{b}}^{s'}$$

can be treated using (1a) with $k' = 0$, taking an expansion up to order $k \geq m + k' \geq m$; all terms in the expansion as well as the remainder term are elements of $\Psi^{-\infty;0}H_{\mathrm{b}}^s$, hence $P \circ Q\chi_0({}^{\mathrm{b}}D) \in \Psi^{-\infty;0}H_{\mathrm{b}}^s$ can be put into the remainder term of the claimed expansion.

Let us now consider $P \circ Q\chi_l({}^{\mathrm{b}}D)$. For brevity, let us replace Q by $Q\chi_l({}^{\mathrm{b}}D)$ and thus assume $|\zeta_l| \geq 1$ on $\operatorname{supp} q(z,\zeta)$. Then by the Leibniz rule,

$$P \circ Q\,{}^{\mathrm{b}}D_{z_l}^{k'} = P\,{}^{\mathrm{b}}D_{z_l}^{k'} \circ Q - \sum_{j=1}^{k'} c_{jk'} P\,{}^{\mathrm{b}}D_{z_l}^{k'-j} \circ ({}^{\mathrm{b}}D_{z_l}^j q)(z, {}^{\mathrm{b}}D)$$

for some constants $c_{jk'} \in \mathbb{R}$. Composing on the right with ${}^{\mathrm{b}}D_{z_l}^{-k'}$, or rather a regularized version thereof, ${}^{\mathrm{b}}D_{z_l}^{-k'}\widetilde{\chi}_l({}^{\mathrm{b}}D)$, where $\widetilde{\chi}_l \equiv 1$ on $\operatorname{supp}\chi_l$ and $|\zeta_l| \geq 1/2$ on $\operatorname{supp}\widetilde{\chi}_l$, thus shows that $P \circ Q$ is an element of the space

$$\sum_{j=0}^{k'} \Psi^{m+k'-j;0} H_{\mathrm{b}}^s \circ \Psi^{m'-k'} H_{\mathrm{b}}^{s'-j}.$$

In view of the part of (1a) already proved, the j-th summand has an expansion to order $k - j \geq m + k' - j$ with error term in $\Psi_{\mathrm{b}}^{m'-k';0} H_{\mathrm{b}}^s$ $[\Psi_{\mathrm{b}}^{m+m'-k;0} H_{\mathrm{b}}^s]$, where we use $k - j \geq k - k' \geq 0$ and $s \leq (s'-j) - (k-j)$ $[s \leq (s'-j) - 2(k-j) + (m+k'-j)]$. Using the same idea, one can prove (1b), (2a) and (3). □

Notice that we do not claim in (3) that R_1 and R_2 lie in b-operator spaces if q does. The issue is that $1/\lambda_m(\zeta)$ in general has singularities for non-real ζ. In applications later in this chapter, we will only need the proposition as stated, with the additional assumption that p is a b-symbol, since instead of letting the operators in the expansion and the remainder operator act on weighted spaces, we will conjugate P and Q by the weight before applying the theorem.

8.3 Reciprocals of and compositions with functions in b-Sobolev spaces

In this section, we recall some basic results about $1/u$ and, more generally, $F(u)$, for u in appropriate b-Sobolev spaces on an n-dimensional compact manifold with boundary M, and smooth/analytic functions F.

Remark 8.3.1. We will give direct proofs here which in particular do not give Moser-type bounds; see [108, §§13.3, 13.10] for examples of the latter. However, at least special cases of the results below (e.g. when $\mathcal{C}^\infty(M)$ is replaced by $\mathbb{C}$ or $\mathbb{R}$) can easily be proved in a way as to obtain such bounds: The point is that the analysis can be localized and thus reduced to the case $M = \overline{\mathbb{R}^n_+}$; a logarithmic change of coordinates then gives an isometric isomorphism of $H^s_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ and $H^s(\mathbb{R}^n)$, and on the latter space, Moser-type reciprocal/composition results are standard, see [108]. However, we will give 'tame' improvements in §8.7.3.

8.3.1 Reciprocals

Let M be a compact n-dimensional manifold with boundary.

Lemma 8.3.2. *Let $s > n/2 + 1$. Suppose $u, w \in H^s_{\mathrm{b}}(M)$ and $a \in \mathcal{C}^\infty(M)$ are such that $|a + u| \geq c_0 > 0$ on $\operatorname{supp} w$. Then $w/(a + u) \in H^s_{\mathrm{b}}(M)$, and one has an estimate*

$$\left\| \frac{w}{a+u} \right\|_{H^s_{\mathrm{b}}} \leq C_K \|w\|_{H^s_{\mathrm{b}}} \left(1 + \|u\|_{H^s_{\mathrm{b}}}\right)^{\lceil s \rceil} \left(1 + \left\| \frac{1}{a+u} \right\|_{L^\infty(K)}\right)^{\lceil s \rceil + 1} \tag{8.3.1}$$

for any neighborhood K of $\operatorname{supp} w$.

Proof. We can assume that $\operatorname{supp} w$ and $\operatorname{supp} u$ lie in a coordinate patch of M. Note that clearly $w/(a + u) \in L^2_{\mathrm{b}}$. We will give an iterative argument that improves on the regularity of $w/(a + u)$ by (at most) 1 at each step, until we can eventually prove H^s_{b}-regularity.

To set this up, let us assume $w/(a+u)\in H_{\mathrm{b}}^{s'-1}$ for some $1\leq s'\leq s$. Recall the operator $\Lambda_{s'}=\lambda_{s'}({}^{\mathrm{b}}D)$ from Corollary 3.3.7, and choose $\psi_0,\psi\in\mathcal{C}^\infty(M)$ such that $\psi_0\equiv 1$ on $\operatorname{supp} w$, $\psi\equiv 1$ on $\operatorname{supp}\psi_0$, and such that moreover $|a+u|\geq c_0'>0$ on $\operatorname{supp}\psi$, which can be arranged since $u\in H_{\mathrm{b}}^s\subset C^0$. Then for $K=\operatorname{supp}\psi$,

$$
\begin{aligned}
\Big\|\Lambda_{s'}\frac{w}{a+u}\Big\|_{L^2_{\mathrm{b}}} &\leq \Big\|(1-\psi)\Lambda_{s'}\frac{\psi_0 w}{a+u}\Big\|_{L^2_{\mathrm{b}}}+\Big\|\psi\Lambda_{s'}\frac{\psi_0 w}{a+u}\Big\|_{L^2_{\mathrm{b}}} \\
&\lesssim \Big\|\frac{w}{a+u}\Big\|_{L^2_{\mathrm{b}}}+\Big\|\frac{1}{a+u}\Big\|_{L^\infty(K)}\Big\|\psi(a+u)\Lambda_{s'}\frac{w}{a+u}\Big\|_{L^2_{\mathrm{b}}} \\
&\lesssim \Big\|\frac{w}{a+u}\Big\|_{L^2_{\mathrm{b}}}+\Big\|\frac{1}{a+u}\Big\|_{L^\infty(K)}\left(\|\psi\Lambda_{s'}w\|_{L^2_{\mathrm{b}}}+\Big\|\psi[\Lambda_{s'},a+u]\frac{w}{a+u}\Big\|_{L^2_{\mathrm{b}}}\right) \\
&\lesssim \Big\|\frac{w}{a+u}\Big\|_{L^2_{\mathrm{b}}}+\Big\|\frac{1}{a+u}\Big\|_{L^\infty(K)} \\
&\qquad\times\left(\|w\|_{H_{\mathrm{b}}^{s'}}+\Big\|\frac{w}{a+u}\Big\|_{H_{\mathrm{b}}^{s'-1}}+\Big\|\psi[\Lambda_{s'},u]\frac{w}{a+u}\Big\|_{L^2_{\mathrm{b}}}\right),
\end{aligned}
\tag{8.3.2}
$$

where we used that the support assumptions on ψ_0 and ψ imply $(1-\psi)\Lambda_{s'}\psi_0\in\Psi^{-\infty}$, and $\psi[\Lambda_{s'},a]\in\Psi^{s'-1}$. Hence, in order to prove that $w/(a+u)\in H_{\mathrm{b}}^{s'}$, it suffices to show that $[\Lambda_{s'},u]\colon H_{\mathrm{b}}^{s'-1}\to L^2_{\mathrm{b}}$. Let $v\in H_{\mathrm{b}}^{s'-1}$. Since

$$
\begin{aligned}
(\Lambda_{s'}uv)\hat{}\,(\zeta)&=\int\lambda_{s'}(\zeta)\widehat{u}(\zeta-\eta)\widehat{v}(\eta)\,d\eta \\
(u\Lambda_{s'}v)\hat{}\,(\zeta)&=\int\widehat{u}(\zeta-\eta)\lambda_{s'}(\eta)\widehat{v}(\eta)\,d\eta,
\end{aligned}
$$

we have, by taking a first order Taylor expansion of $\lambda_{s'}(\zeta)=\lambda_{s'}(\eta+(\zeta-\eta))$ around $\zeta=\eta$,

$$
([\Lambda_{s'},u]v)\hat{}\,(\zeta)=\sum_{|\beta|=1}\int\left(\int_0^1\partial_\zeta^\beta\lambda_{s'}(\eta+t(\zeta-\eta))\,dt\right)({}^{\mathrm{b}}D_z^\beta u)\hat{}\,(\zeta-\eta)\widehat{v}(\eta)\,d\eta.
$$

We will to prove that this is an element of L^2_ζ using Lemma 8.2.7. Since for $|\beta|=1$,

$$
|\partial_\zeta^\beta\lambda_{s'}(\eta+t(\zeta-\eta))|\lesssim\langle\eta+t(\zeta-\eta)\rangle^{s'-1},
$$

$$
|({}^{\mathrm{b}}D_z^\beta u)\hat{}\,(\zeta-\eta)|=\frac{u_0(\zeta-\eta)}{\langle\zeta-\eta\rangle^{s-1}},\quad |\widehat{v}(\eta)|=\frac{v_0(\eta)}{\langle\eta\rangle^{s'-1}}
$$

for $u_0, v_0 \in L^2$, it is enough to observe that

$$\frac{\langle \eta + t(\zeta-\eta)\rangle^{s'-1}}{\langle \zeta-\eta\rangle^{s-1}\langle\eta\rangle^{s'-1}} \lesssim \frac{1}{\langle\zeta-\eta\rangle^{s-1}} + \frac{1}{\langle\zeta-\eta\rangle^{s-s'}\langle\eta\rangle^{s'-1}} \in L^\infty_\zeta L^2_\eta,$$

uniformly in $t \in [0,1]$, since $s-1 > n/2$.

To obtain the estimate (8.3.1), we proceed inductively, starting with the obvious estimate

$$\|w/(a+u)\|_{L^2_{\mathrm{b}}} \le \|w\|_{L^2_{\mathrm{b}}}\|1/(a+u)\|_{L^\infty(K)} \le \|w\|_{H^s_{\mathrm{b}}}\left(1 + \left\|\frac{1}{a+u}\right\|_{L^\infty(K)}\right).$$

Then, assuming that for integer $1 \le m \le s$, one has

$$\|w/(a+u)\|_{H^{m-1}_{\mathrm{b}}} \lesssim \|w\|_{H^s_{\mathrm{b}}}\left(1 + \left\|\frac{1}{a+u}\right\|_{L^\infty(K)}\right)^m (1 + \|u\|_{H^s_{\mathrm{b}}})^{m-1}$$

we conclude, using the estimate (8.3.2),

$$\begin{aligned}&\left\|\frac{w}{a+u}\right\|_{H^m_{\mathrm{b}}} \\ &\quad\lesssim \left\|\frac{w}{a+u}\right\|_{L^2_{\mathrm{b}}} + \left\|\frac{1}{a+u}\right\|_{L^\infty(K)}\left(\|w\|_{H^s_{\mathrm{b}}} + (1+\|u\|_{H^s_{\mathrm{b}}})\left\|\frac{w}{a+u}\right\|_{H^{m-1}_{\mathrm{b}}}\right) \\ &\quad\lesssim \|w\|_{H^s_{\mathrm{b}}}\left(1 + \left\|\frac{1}{a+u}\right\|_{L^\infty(K)}\right)^{m+1}(1+\|u\|_{H^s_{\mathrm{b}}})^m.\end{aligned}$$

Thus, one gets such an estimate for $m = \lfloor s\rfloor$; then the same type of estimate gives (8.3.1), since one has control over the H^{s-1}_{b}-norm of $w/(a+u)$ in view of $s-1 < \lfloor s\rfloor$ and the bound on $\|w/(a+u)\|_{H^{\lfloor s\rfloor}_{\mathrm{b}}}$. □

In particular:

Corollary 8.3.3. *Let $s > n/2+1$.*

(1) If $u \in H^s_{\mathrm{b}}(M)$ does not vanish on $\operatorname{supp}\phi$, where $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(M)$, then $\phi/u \in H^s_{\mathrm{b}}(M)$.

(2) Let $\alpha \ge 0$. If $u \in H^{s,\alpha}_{\mathrm{b}}(M)$ is bounded away from -1, then $1/(1+u) \in 1 + H^{s,\alpha}_{\mathrm{b}}(M)$.

Proof. The second statement follows from

$$1 - \frac{1}{1+u} = \frac{u}{1+u} \in H^{s,\alpha}_{\mathrm{b}}(M).$$ □

We also obtain the following result on the inversion of non-smooth elliptic symbols:

Proposition 8.3.4. *Let $s > n/2+1$, $m \in \mathbb{R}$, $k \in \mathbb{N}_0$.*

(1) *Suppose $p(z,\zeta) \in S^{m;k}H_{\mathrm{b}}^s(\overline{\mathbb{R}^n_+};\mathrm{Hom}(E,F))$ and $a(z,\zeta) \in S^0$ are such that the ellipticity bound $\|p(z,\zeta)^{-1}\|_{\mathrm{Hom}(F,E)} \le c_0\langle\zeta\rangle^{-m}$, $c_0 < \infty$, holds on $\mathrm{supp}\, a$. Then*

$$ap^{-1} \in S^{-m;k}H_{\mathrm{b}}^s(\overline{\mathbb{R}^n_+};\mathrm{Hom}(F,E)).$$

(2) *Suppose that the symbols $p'(z,\zeta) \in S^{m;k}H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n_+};\mathrm{Hom}(E,F))$ with $\alpha \ge 0$, $p''(z,\zeta) \in S^m(\overline{\mathbb{R}^n_+};\mathrm{Hom}(E,F))$ and $a(z,\zeta) \in S^0$ are such that*

$$\|(p'')^{-1}\|_{\mathrm{Hom}(F,E)}, \|(p'+p'')^{-1}\|_{\mathrm{Hom}(F,E)} \le c_0\langle\zeta\rangle^{-m} \text{ on } \mathrm{supp}\, a.$$

Then

$$a(p'+p'')^{-1} \in a(p'')^{-1} + S^{-m;k}H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n_+};\mathrm{Hom}(F,E)).$$

Proof. By multiplying the symbols p and p' by $\langle\zeta\rangle^{-m}$, we may assume that $m = 0$.

(1) Let us first treat the case of complex-valued symbols. By Corollary 8.3.3, we have $a(\cdot,\zeta)/p(\cdot,\zeta) \in H_{\mathrm{b}}^s$ uniformly in ζ; thus $a/p \in S^{0;0}H_{\mathrm{b}}^s$. Moreover, for $|\alpha| \le k$,

$$\partial_\zeta^\alpha\left(\frac{a}{p}\right) = \sum c_{\beta_1\cdots\gamma_\nu}\frac{\prod_{j=1}^\mu \partial_\zeta^{\beta_j} a \prod_{l=1}^\nu \partial_\zeta^{\gamma_l} p}{p^{\nu+1}},$$

where the sum is over all $\beta_1+\cdots+\beta_\mu+\gamma_1+\cdots+\gamma_\nu = \alpha$ with $|\gamma_j| \ge 1$, $1 \le j \le \nu$. Hence, using that H_{b}^s is an algebra and that the growth order of the numerator is $-|\alpha|$, we conclude, again by Corollary 8.3.3, that $\partial_\zeta^\alpha(a/p) \in S^{-|\alpha|;0}H_{\mathrm{b}}^s$; thus $a/p \in S^{0;k}H_{\mathrm{b}}^s$. If p is bundle-valued, we obtain $ap^{-1} \in S^{0;0}H_{\mathrm{b}}^s(\overline{\mathbb{R}^n_+};\mathrm{Hom}(F,E))$ using the explicit formula for the inverse of a matrix and Corollaries 8.2.10 and 8.3.3; then, by virtue of

$$\partial_\zeta(ap^{-1}) = (\partial_\zeta a - ap^{-1}(\partial_\zeta p))p^{-1},$$

similarly for higher derivatives, we get $ap^{-1} \in S^{0;k}H_{\mathrm{b}}^s(\overline{\mathbb{R}^n_+};\mathrm{Hom}(F,E))$.

(2) Since $a(p'+p'')^{-1} = \big(a(p'')^{-1}\big)\big(I+p'(p'')^{-1}\big)^{-1}$, we may without loss of generality assume $p' \in S^{0;k}H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n_+};\mathrm{End}(F))$, $p'' = I$ and $a \in S^0(\overline{\mathbb{R}^n_+};\mathrm{Hom}(F,E))$, and we need to show

$$(I+p')^{-1} - I \in S^{0;k}H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n_+};\mathrm{End}(F)).$$

But we can write
$$(I+p')^{-1}-I=-p'(I+p')^{-1},$$
which is an element of $S^{0;0}H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n_+};\mathrm{End}(F))$ by assumption. Then, by an argument similar to the one employed in the first part, we obtain the higher symbol estimates. □

8.3.2 Compositions

Using the results of the previous subsection and the Cauchy integral formula, we can prove several results on the regularity of $F(u)$ for F smooth or holomorphic and u in a weighted b-Sobolev space. The main use of such results for us will be that they allow us to understand the regularity of the coefficients of wave operators associated to non-smooth metrics.

In all results in this section, we shall assume that M is a compact n-dimensional manifold with boundary, $s>n/2+1$, and $\alpha\geq 0$.

Proposition 8.3.5. *Let $u\in H_{\mathrm{b}}^{s,\alpha}(M)$. If $F\colon\Omega\to\mathbb{C}$ is holomorphic in a simply connected neighborhood Ω of $u(M)$, then $F(u)-F(0)\in H_{\mathrm{b}}^{s,\alpha}(M)$. Moreover, there exists $\epsilon>0$ such that $F(v)-F(0)\in H_{\mathrm{b}}^{s,\alpha}(M)$ depends continuously on $v\in H_{\mathrm{b}}^{s,\alpha}(M)$, $\|u-v\|_{H_{\mathrm{b}}^{s,\alpha}}<\epsilon$.*

Proof. Observe that $u(M)$ is compact. Let $\gamma\subset\mathbb{C}$ denote a smooth contour which is disjoint from $u(M)$, has winding number 1 around every point in $u(M)$, and lies within the region of holomorphicity of F. Then, writing $F(z)-F(0)=zF_1(z)$ with F_1 holomorphic in Ω, we have
$$F(u)-F(0)=\frac{u}{2\pi i}\oint_\gamma F_1(\zeta)\frac{1}{\zeta-u}\,d\zeta,$$
Since $\gamma\ni\zeta\mapsto u/(\zeta-u)\in H_{\mathrm{b}}^{s,\alpha}(M)$ is continuous by Lemma 8.3.2, we obtain the desired conclusion $F(u)-F(0)\in H_{\mathrm{b}}^{s,\alpha}$.

The continuous dependence of $F(v)-F(0)$ on v near u is a consequence of Lemma 8.3.2 and Corollary 8.2.10. □

Proposition 8.3.6. *Let $u'\in\mathcal{C}^\infty(M)$, $u''\in H_{\mathrm{b}}^{s,\alpha}(M)$; put $u=u'+u''$. If $F\colon\Omega\to\mathbb{C}$ is holomorphic in a simply connected neighborhood Ω of $u(M)$, then $F(u)\in\mathcal{C}^\infty(M)+H_{\mathrm{b}}^{s,\alpha}(M)$; in fact, $F(v)$ depends continuously on v in a neighborhood of u in the topology of $\mathcal{C}^\infty(M)+H_{\mathrm{b}}^{s,\alpha}(M)$.*

Proof. Let $\gamma \subset \mathbb{C}$ denote a smooth contour which is disjoint from $u(M)$, has winding number 1 around every point in $u(M)$, and lies within the region of holomorphicity of F. Since $u'' = 0$ at ∂M and u'' is continuous by the Riemann-Lebesgue lemma, we can pick $\phi \in \mathcal{C}^\infty(M)$, $\phi \equiv 1$ near ∂M, such that γ is disjoint from $\overline{u'(\operatorname{supp}\phi)}$. Then

$$\begin{aligned}\phi F(u) &= \frac{1}{2\pi i}\oint_\gamma \phi \frac{F(\zeta)/(\zeta - u')}{1 - u''/(\zeta - u')}\,d\zeta \\ &= \frac{1}{2\pi i}\oint_\gamma \phi \frac{F(\zeta)}{\zeta - u'}\,d\zeta + \frac{1}{2\pi i}\oint_\gamma \phi \frac{(F(\zeta)/(\zeta - u'))u''}{(\zeta - u') - u''}\,d\zeta;\end{aligned}$$

the first term equals $\phi F(u')$, and the second term is an element of $H_{\mathrm{b}}^{s,\alpha}$ by Corollary 8.3.3. Next, let $\widetilde{\phi} \in \mathcal{C}^\infty(M)$ be identically equal to 1 on $\operatorname{supp}(1-\phi)$, and $\widetilde{\phi} \equiv 0$ near ∂M. Then $\widetilde{\phi}u \in H_{\mathrm{b}}^s$; in fact, it lies in any weighted such space. Thus,

$$(1-\phi)F(u) = \frac{1}{2\pi i}\oint_\gamma \frac{(1-\phi)F(\zeta)}{\zeta - \widetilde{\phi}u}\,d\zeta \in H_{\mathrm{b}}^{s,\alpha},$$

and the proof is complete. □

If we only consider $F(u)$ for real-valued u, it is in fact sufficient to assume $F \in \mathcal{C}^\infty(\mathbb{R};\mathbb{C})$ using almost analytic extensions, see e.g. [35, Chapter 8]: For any such function F and an integer $N \in \mathbb{N}$, let us define

$$\widetilde{F}_N(x+iy) = \sum_{k=0}^N \frac{(iy)^k}{k!}(\partial_x^k F)(x)\chi(y), \quad x, y \in \mathbb{R}, \tag{8.3.3}$$

where $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R})$ is identically 1 near 0. Then, writing $z = x + iy$, we have for y close to 0:

$$\partial_{\bar z}\widetilde{F}_N(z) = \frac{1}{2}(\partial_x + i\partial_y)\widetilde{F}_N(z) = \frac{(iy)^N}{2N!}(\partial_x^{N+1}F)(x)\chi(y) = \mathcal{O}(|\operatorname{Im} z|^N). \tag{8.3.4}$$

Observe that all $u \in \mathcal{C}^\infty(M) + H_{\mathrm{b}}^{s,\alpha}(M)$ are bounded, hence in analyzing $F(u)$, we may assume without restriction that $F \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R};\mathbb{C})$.

Proposition 8.3.7. *Let $F \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R};\mathbb{C})$. Then for $u \in H_{\mathrm{b}}^{s,\alpha}(M;\mathbb{R})$, we have $F(u) - F(0) \in H_{\mathrm{b}}^{s,\alpha}(M)$; in fact, $F(u) - F(0)$ depends continuously on u.*

Proof. Write $F(x) - F(0) = xF_1(x)$. Then, with $(\widetilde{F}_1)_N$ defined as in (8.3.3), the Cauchy-Pompeiu formula gives the pointwise identity

$$F(u) - F(0) = -\frac{u}{\pi}\int_{\mathbb{C}} \frac{\partial_{\bar\zeta}(\widetilde{F}_1)_N(\zeta)}{\zeta - u}\,dx\,dy, \quad \zeta = x + iy.$$

Here, note that the integrand is compactly supported, and $1/(\zeta - u(z))$ is locally integrable for all z. In particular, we can rewrite

$$F(u) - F(0) = -\frac{1}{\pi}\lim_{\delta\searrow 0}\int_{|\operatorname{Im}\zeta|>\delta} \partial_{\bar\zeta}(\widetilde{F}_1)_N(\zeta)\frac{u}{\zeta - u}\,dx\,dy. \tag{8.3.5}$$

Now Lemma 8.3.2 gives

$$\Big\|\frac{u}{\zeta - u}\Big\|_{H_{\mathrm{b}}^{s,\alpha}} \lesssim C(\|u\|_{H_{\mathrm{b}}^{s,\alpha}})|\operatorname{Im}\zeta|^{-s-2},$$

since u is real-valued. Thus, if we choose $N \geq s + 2$, then

$$\mathbb{C}\setminus\mathbb{R} \ni \zeta \mapsto \partial_{\bar\zeta}(\widetilde{F}_1)_N(\zeta)\frac{u}{\zeta - u} \in H_{\mathrm{b}}^{s,\alpha}$$

is bounded by (8.3.4), hence integrable, and therefore the limit in (8.3.5) exists in $H_{\mathrm{b}}^{s,\alpha}$, proving the proposition. □

We also have an analogue of Proposition 8.3.6.

Proposition 8.3.8. *Let $F \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R};\mathbb{C})$, and $u' \in \mathcal{C}^\infty(M;\mathbb{R}), u'' \in H_{\mathrm{b}}^{s,\alpha}(M;\mathbb{R})$; put $u = u' + u''$. Then $F(u) \in \mathcal{C}^\infty(M) + H_{\mathrm{b}}^{s,\alpha}(M)$; in fact, $F(u)$ depends continuously on u.*

Proof. As in the proof of the previous proposition, we have the pointwise identity

$$\begin{aligned} &F(u' + u'') - F(u') \\ &\quad = -\frac{1}{\pi}\lim_{\delta\searrow 0}\int_{|\operatorname{Im}\zeta|>\delta} \partial_{\bar\zeta}(\widetilde{F}_1)_N(\zeta)\Big(\frac{1}{\zeta - u' - u''} - \frac{1}{\zeta - u'}\Big)\,dx\,dy \\ &\quad = -\frac{1}{\pi}\lim_{\delta\searrow 0}\int_{|\operatorname{Im}\zeta|>\delta} \frac{\partial_{\bar\zeta}(\widetilde{F}_1)_N(\zeta)}{\zeta - u'}\cdot\frac{u''}{(\zeta - u') - u''}\,dx\,dy \end{aligned}$$

Writing $f_N := \partial_{\bar\zeta}(\widetilde{F}_1)_N$, we estimate the $H_{\mathrm{b}}^{s,\alpha}$-norm of the integrand for $\zeta \in \mathbb{C}\setminus\mathbb{R}$ using

Lemma 8.3.2 by

$$\left\|\frac{f_N(\zeta)}{\zeta-u'}\right\|_{\mathcal{L}(H_{\mathrm{b}}^{s,\alpha})}\left\|\frac{u''}{(\zeta-u')-u''}\right\|_{H_{\mathrm{b}}^{s,\alpha}}\lesssim\left\|\frac{f_N(\zeta)}{\zeta-u'}\right\|_{\mathcal{L}(H_{\mathrm{b}}^{s,\alpha})}|\operatorname{Im}\zeta|^{-s-2};$$

here, we denote by $\|h\|_{\mathcal{L}(H_{\mathrm{b}}^{s,\alpha})}$, for a function h, the operator norm of multiplication by h on $H_{\mathrm{b}}^{s,\alpha}$. We claim that the operator norm

$$b_s := \left\|\frac{f_N(\zeta)}{\zeta-u'}\right\|_{\mathcal{L}(H_{\mathrm{b}}^{s,\alpha})}$$

is bounded by $|\operatorname{Im}\zeta|^{N-s-1}$; then choosing $N\geq 2s+3$ finishes the proof as before. To prove this bound, we use interpolation: First, since u' is real-valued, we have

$$b_0=\mathcal{O}(|\operatorname{Im}\zeta|^{-1}|f_N(\zeta)|)=\mathcal{O}(|\operatorname{Im}\zeta|^{N-1})$$

by (8.3.4). Next, for integer $k\geq 1$, the Leibniz rule gives

$$b_k\lesssim\sum_{j=0}^{k}|\operatorname{Im}\zeta|^{-1-j}|\partial_x^{k-j}f_N(\zeta)|\lesssim|\operatorname{Im}\zeta|^{N-k-1},$$

where we use that $|\partial_x^\ell f_N(\zeta)|=\mathcal{O}(|\operatorname{Im}\zeta|^N)$ for all ℓ, as follows directly from the definition of f_N. By interpolation, we thus obtain $b_s\lesssim|\operatorname{Im}\zeta|^{N-s-1}$, as claimed. □

8.4 Elliptic regularity

With the partial calculus developed in §8.2, it is straightforward to prove elliptic regularity for b-Sobolev b-pseudodifferential operators. Notice that operators with coefficients in H_{b}^s for $s>n/2$ must vanish at the boundary by the Riemann-Lebesgue lemma, thus they cannot be elliptic there. A natural class of operators which can be elliptic at the boundary is obtained by adding smooth b-ps.d.o.s to b-Sobolev b-ps.d.o.s, and we will deal with such operators in the second part of the following theorem.

Theorem 8.4.1. *Let $m,s,r\in\mathbb{R}$ and $\zeta_0\in{}^{\mathrm{b}}S^*\overline{\mathbb{R}^n_+}$. Suppose $\widetilde{P}=\widetilde{P}_m+\widetilde{R}$, where $\widetilde{P}_m\in H_{\mathrm{b}}^s\Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+};E,F)$ has principal symbol $\widetilde{p}$, and $\widetilde{R}\in\Psi_{\mathrm{b}}^{m-1;0}H_{\mathrm{b}}^{s-1}(\overline{\mathbb{R}^n_+};E,F)$.*

(1) Let $P=\widetilde{P}$, and suppose $p\equiv\widetilde{p}$ is elliptic at ζ_0, or

(2) *let* $P = P_0 + \widetilde{P}$, *where* $P_0 \in \Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+}; E, F)$ *has principal symbol* p_0, *and suppose* $p = \widetilde{p} + p_0$ *is elliptic at* ζ_0.

Let $\widetilde{s} \in \mathbb{R}$ *be such that* $\widetilde{s} \leq s - 1$ *and* $s > n/2 + 1 + (-\widetilde{s})_+$. *Then in both cases, if* $u \in H_{\mathrm{b}}^{\widetilde{s}+m-1,r}(\overline{\mathbb{R}^n_+}; E)$ *satisfies*

$$Pu = f \in H_{\mathrm{b}}^{\widetilde{s},r}(\overline{\mathbb{R}^n_+}; F),$$

it follows that $\zeta_0 \notin \mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m,r}(u)$.

Proof. We will only prove the theorem without bundles; adding bundles only requires simple notational changes. In both cases, we can assume that $r = 0$ by conjugating P by x^{-r}; moreover, $\widetilde{R}u \in H_{\mathrm{b}}^{\widetilde{s}}$ by Proposition 8.2.9 by the assumptions on s and $\widetilde{s}$, thus we can absorb $\widetilde{R}u$ into the right hand side and hence assume $\widetilde{R} = 0$. Choose $a_0 \in S^0$ elliptic at ζ_0 such that p is elliptic on $\operatorname{supp} a_0$ (and non-vanishing there, which only matters near the zero section).

(1) Let λ_m be as in Corollary 3.3.7. By Proposition 8.3.4,

$$q(z,\zeta) := a_0(z,\zeta)\lambda_m(\zeta)/p(z,\zeta) \in S^{0;\infty}H_{\mathrm{b}}^s.$$

Put $Q = q(z, {}^{\mathrm{b}}D)$. Then by Theorem 8.2.12 (1a), using $P = \widetilde{P}_m \in \Psi^{m;0}H_{\mathrm{b}}^s$,

$$Q \circ P = a_0(z, {}^{\mathrm{b}}D)\Lambda_m + R'$$

with $R' \in \Psi^{m-1;0}H_{\mathrm{b}}^{s-1}$, hence by[29] Proposition 8.2.9

$$a_0(z, {}^{\mathrm{b}}D)\Lambda_m u = Qf - R'u \in H_{\mathrm{b}}^{\widetilde{s}}.$$

Then standard microlocal ellipticity implies $\zeta_0 \notin \mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m}(u)$.

(2) If $\zeta_0 \notin {}^{\mathrm{b}}T^*_{\partial\overline{\mathbb{R}^n_+}}\overline{\mathbb{R}^n_+}$, then the proof of part (1) applies, since away from $\partial\overline{\mathbb{R}^n_+}$, one has $\Psi_{\mathrm{b}}^m \subset H_{\mathrm{b}}^s\Psi_{\mathrm{b}}^m$. Thus, assuming $\zeta_0 \in {}^{\mathrm{b}}T^*_{\partial\overline{\mathbb{R}^n_+}}\overline{\mathbb{R}^n_+}$, we note that the ellipticity of p at ζ_0 implies $p_0 \neq 0$ near ζ_0, since the function $\widetilde{p}$ vanishes at $\partial\overline{\mathbb{R}^n_+}$. Therefore,

[29] For $Qf \in H_{\mathrm{b}}^{\widetilde{s}}$, we need $s \geq \widetilde{s}$ and $s > n/2 + (-\widetilde{s})_+$. For $R'u \in H_{\mathrm{b}}^{\widetilde{s}}$, we need $s - 1 \geq \widetilde{s}$ and $s - 1 > n/2 + (-\widetilde{s})_+$.

Proposition 8.3.4 applies if one chooses $a_0 \in S^0$ as in the proof of part (1), yielding

$$q(z,\zeta) := a_0(z,\zeta)\lambda_m(\zeta)/p(z,\zeta) = \widetilde{q}_0(z,\zeta) + q_0(z,\zeta),$$

where $\widetilde{q}_0 \in S^{0;\infty}H_{\mathrm{b}}^s$, $q_0 \in S^0$. Put $\widetilde{Q}_0 = \widetilde{q}_0(z, {}^{\mathrm{b}}D), Q_0 = q_0(z, {}^{\mathrm{b}}D)$, then

$$(\widetilde{Q}_0 + Q_0) \circ (\widetilde{P}_m + P_0) = a_0(z, {}^{\mathrm{b}}D)\Lambda_m + R'$$

with

$$\begin{aligned} R' &\in \Psi^{m-1;0}H_{\mathrm{b}}^{s-1} + \Psi^{m-1;0}H_{\mathrm{b}}^{s} + \Psi^{m-1;0}H_{\mathrm{b}}^{s-1} + \Psi_{\mathrm{b}}^{m-1} \\ &\subset \Psi^{m-1;0}H_{\mathrm{b}}^{s-1} + \Psi_{\mathrm{b}}^{m-1}, \end{aligned}$$

where the terms are the remainders of the first order expansions of $\widetilde{Q}_0 \circ \widetilde{P}_m$, $\widetilde{Q}_0 \circ P_0$, $Q_0 \circ \widetilde{P}_m$ and $Q_0 \circ P_0$, in this order; to see this, we use Theorem 8.2.12 (1a), (2b), (2a) and composition properties of b-ps.d.o.s, respectively. Hence

$$a_0(z, {}^{\mathrm{b}}D)\Lambda_m u = \widetilde{Q}_0 f + Q_0 f - R'u \in H_{\mathrm{b}}^{\widetilde{s}},$$

which implies $\zeta_0 \notin \mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m}(u)$. □

Remark 8.4.2. Notice that it suffices to have only local $H_{\mathrm{b}}^{\widetilde{s},r}$-membership of f near the base point of ζ_0. Under additional assumptions, even microlocal assumptions are enough, see in particular [9, Theorem 3.1]; we will not need this generality though.

8.5 Propagation of singularities

We next study the propagation of singularities, equivalently the propagation of regularity, for certain classes of non-smooth operators. The results cover operators that are of real principal type (§8.5.3), have a specific radial point structure (§8.5.4) or normally hyperbolic trapping in the b-sense (§8.5.5). For a microlocally more complete picture, we also include a brief discussion of complex absorption. Beals and Reed [9] discuss the propagation of singularities on manifolds without boundary for non-smooth ps.d.o.s, and parts of §§8.5.1 and 8.5.3 follow their exposition closely.

8.5.1 Sharp Gårding inequalities

We will need various versions of the sharp Gårding inequality, which will be used to obtain one-sided bounds for certain terms in positive commutator arguments later. For the first result, we follow the proof of [9, Lemma 3.1]. We introduce the notation

$$a_+ = \max(a, 0), \quad a \in \mathbb{R}.$$

Proposition 8.5.1. *Let $s, m \in \mathbb{R}$ be such that $s \geq 2 - m$ and $s > n/2 + 2 + m_+$. Let $p(z,\zeta) \in S^{2m+1;2} H_{\mathrm{b}}^s(\overline{\mathbb{R}^n_+}; \mathrm{End}(E))$ be a symbol with non-negative real part, i.e.*

$$\mathrm{Re}\langle p(z,\zeta)e, e\rangle \geq 0 \quad z \in \overline{\mathbb{R}^n_+}, \zeta \in \mathbb{R}^n, e \in E,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on the fibers of E. Then there is $C > 0$ such that $P = p(z, {}^{\mathrm{b}}D)$ satisfies the estimate

$$\mathrm{Re}\langle Pu, u\rangle \geq -C\|u\|^2_{H_{\mathrm{b}}^m}, \quad u \in \dot{\mathcal{C}}_{\mathrm{c}}^\infty(\overline{\mathbb{R}^n_+}; E).$$

Proof. Let $q \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n)$ be a non-negative even function, supported in $|\zeta| \leq 1$, with $\int q^2(\zeta)\, d\zeta = 1$, and put

$$F(\zeta, \xi) = \frac{1}{\langle \zeta \rangle^{n/4}} q\left(\frac{\xi - \zeta}{\langle \zeta \rangle^{1/2}}\right).$$

Define the symmetrization of p to be

$$p_{\mathrm{sym}}(\eta, z, \zeta) = \int F(\eta, \xi) p(z, \xi) F(\zeta, \xi)\, d\xi.$$

Observe that the integrand has compact support in ξ for all η, z, ζ, therefore p_{sym} is well-defined. Moreover,

$$(p_{\mathrm{sym}}({}^{\mathrm{b}}D, z, {}^{\mathrm{b}}D)u)\widehat{\ }(\eta) = \int \widehat{p}_{\mathrm{sym}}(\eta, \eta - \zeta, \zeta)\widehat{u}(\zeta)\, d\zeta,$$

hence, writing $u = (u_j)$, $p = (p_{ij})$, $p_{\mathrm{sym}} = ((p_{\mathrm{sym}})_{ij})$, and summing over repeated indices,

$$\begin{aligned}\mathrm{Re}\langle p_{\mathrm{sym}}({}^{\mathrm{b}}D, z, {}^{\mathrm{b}}D)u, u\rangle &= \mathrm{Re} \iint \widehat{p}_{\mathrm{sym}}(\eta, \eta - \zeta, \zeta)_{ij} \widehat{u}(\zeta)_j \overline{\widehat{u}_i(\eta)}\, d\zeta\, d\eta \\ &= \mathrm{Re} \iint \left(\int e^{iz\zeta} F(\zeta, \xi)\widehat{u}(\zeta)\, d\zeta\right)_j \overline{\left(\int e^{iz\eta} F(\eta, \xi)\widehat{u}(\eta)\, d\eta\right)_i} p_{ij}(z, \xi)\, d\xi\, dz\end{aligned}$$

$$= \iint \mathrm{Re}\big\langle p(z,\xi)F({}^{\mathrm{b}}D;\xi)u(z), F({}^{\mathrm{b}}D;\xi)u(z)\big\rangle\, d\xi\, dz \geq 0.$$

Thus, putting $r(z, {}^{\mathrm{b}}D) = p_{\mathrm{sym}}({}^{\mathrm{b}}D, z, {}^{\mathrm{b}}D) - p(z, {}^{\mathrm{b}}D)$, it suffices to show that $r(z,\zeta) \in S^{2m;0}H_{\mathrm{b}}^{s-2}(\overline{\mathbb{R}^n_+}; \mathrm{End}(E))$, i.e.

$$\frac{\langle\eta\rangle^{s-2}\|\widehat{r}(\eta;\zeta)\|_{\mathrm{End}(E)}}{\langle\zeta\rangle^{2m}} \leq r_0(\eta;\zeta), \quad r_0(\eta;\zeta) \in L^\infty_\zeta L^2_\eta. \tag{8.5.1}$$

in order to conclude the proof, since Proposition 8.2.9 then implies the continuity of $r(z, {}^{\mathrm{b}}D) \colon H^m_{\mathrm{b}}(\overline{\mathbb{R}^n_+}; E) \to H^{-m}_{\mathrm{b}}(\overline{\mathbb{R}^n_+}; E)$. *From now on, we will suppress the bundle E in our notation and simply write $|\cdot|$ for $\|\cdot\|_{\mathrm{End}(E)}$.* Now, $r(z, {}^{\mathrm{b}}D)$ acts on $\dot{\mathcal{C}}^\infty_{\mathrm{c}}$ by

$$(r(z, {}^{\mathrm{b}}D)u)\widehat{\ }(\eta) = \int \widehat{r}(\eta-\zeta,\zeta)\widehat{u}(\zeta)\, d\zeta;$$

hence

$$\begin{aligned} \widehat{r}(\eta;\zeta) &= \widehat{p}_{\mathrm{sym}}(\eta+\zeta,\eta,\zeta) - \widehat{p}(\eta;\zeta) \\ &= \int F(\eta+\zeta,\xi)\widehat{p}(\eta;\xi)F(\zeta,\xi)\, d\xi - \widehat{p}(\eta;\zeta) \qquad (8.5.2)\\ &= \int F(\eta+\zeta,\xi)\big(\widehat{p}(\eta;\xi) - \widehat{p}(\eta;\zeta)\big)F(\zeta,\xi)\, d\xi \\ &\quad + \int \big(F(\eta+\zeta,\xi) - F(\zeta,\xi)\big)\widehat{p}(\eta;\zeta)F(\zeta,\xi)\, d\xi, \qquad (8.5.3) \end{aligned}$$

where we use $\int F(\zeta,\xi)^2\, d\xi = 1$. To estimate $\widehat{r}(\eta;\zeta)$, we use that

$$|\widehat{p}(\eta;\zeta)| = \frac{\langle\zeta\rangle^{2m+1}}{\langle\eta\rangle^s} p_0(\eta;\zeta), \quad p_0(\eta;\zeta) \in L^\infty_\zeta L^2_\eta.$$

We get a first estimate from (8.5.2):

$$|\widehat{r}(\eta;\zeta)| \lesssim \int_S \frac{1}{\langle\eta+\zeta\rangle^{n/4}\langle\zeta\rangle^{n/4}\langle\eta\rangle^s}\langle\xi\rangle^{2m+1}p_0(\eta;\xi)\, d\xi + \frac{\langle\zeta\rangle^{2m+1}}{\langle\eta\rangle^s}p_0(\eta;\zeta),$$

where S is the set

$$S = \{|\xi-\zeta| \leq \langle\zeta\rangle^{1/2}, |\xi-(\eta+\zeta)| \leq \langle\eta+\zeta\rangle^{1/2}\}.$$

In particular, we have $\langle\zeta\rangle \sim \langle\xi\rangle \sim \langle\eta+\zeta\rangle$ on S, which yields

$$|\widehat{r}(\eta;\zeta)| \lesssim \frac{\langle\zeta\rangle^{2m+1-n/2}}{\langle\eta\rangle^s} \int_{|\xi-\zeta|\le\langle\zeta\rangle^{1/2}} p_0(\eta;\xi)\,d\xi + \frac{\langle\zeta\rangle^{2m+1}}{\langle\eta\rangle^s} p_0(\eta;\zeta).$$

We contend that

$$p_0'(\eta;\zeta) := \langle\zeta\rangle^{-n/2} \int_{|\xi-\zeta|\le\langle\zeta\rangle^{1/2}} p_0(\eta;\xi)\,d\xi \in L^\infty_\zeta L^2_\eta.$$

Indeed, this follows from Cauchy-Schwarz:

$$\begin{aligned}\int \left|\int_{|\xi-\zeta|\le\langle\zeta\rangle^{1/2}} p_0(\eta;\xi)\,d\xi\right|^2 d\eta &\lesssim \int \langle\zeta\rangle^{n/2} \int_{|\xi-\zeta|\le\langle\zeta\rangle^{1/2}} |p_0(\eta;\xi)|^2\,d\xi\,d\eta \\ &\lesssim \langle\zeta\rangle^n \|p_0(\eta;\xi)\|^2_{L^\infty_\xi L^2_\eta}.\end{aligned}$$

We deduce

$$|\widehat{r}(\eta;\zeta)| \le \frac{\langle\zeta\rangle^{2m+1}}{\langle\eta\rangle^s} p_0''(\eta;\zeta), \quad p_0''(\eta;\zeta) \in L^\infty_\zeta L^2_\eta.$$

If $|\eta| \ge |\zeta|/2$, this implies

$$\frac{\langle\eta\rangle^{s-1}|\widehat{r}(\eta;\zeta)|}{\langle\zeta\rangle^{2m}} \le \frac{\langle\zeta\rangle}{\langle\eta\rangle} p_0''(\eta;\zeta) \lesssim p_0''(\eta;\zeta), \tag{8.5.4}$$

thus we obtain a forteriori the desired estimate (8.5.1) in the region $|\eta| \ge |\zeta|/2$.

From now on, let us thus assume $|\eta| \le |\zeta|/2$. We estimate the first integral in (8.5.3). By Taylor's theorem,

$$\begin{aligned}\widehat{p}(\eta;\xi) - \widehat{p}(\eta;\zeta) &= \partial_\zeta \widehat{p}(\eta;\zeta)\cdot(\xi-\zeta) \\ &\quad + \int_0^1 (1-t)\langle \xi-\zeta, \partial_\zeta^2 \widehat{p}(\eta;\zeta+t(\xi-\zeta))\cdot(\xi-\zeta)\rangle\,dt,\end{aligned}$$

and since $\langle\xi\rangle \sim \langle\zeta\rangle$ on $\operatorname{supp} F(\zeta,\xi)$, this gives

$$\widehat{p}(\eta;\xi) - \widehat{p}(\eta;\zeta) = \partial_\zeta \widehat{p}(\eta;\zeta)\cdot(\xi-\zeta) + |\xi-\zeta|^2 \mathcal{O}(\langle\zeta\rangle^{2m-1}) \text{ on } \operatorname{supp} F(\zeta,\xi),$$

where we say $f \in \mathcal{O}(g)$ if $|f| \le |g|h$ for some $h \in L^\infty_\zeta L^2_\eta$. The first integral in (8.5.3) can then be rewritten as

$$\partial_\zeta \widehat{p}(\eta;\zeta)\cdot \int (\xi-\zeta)\big(F(\eta+\zeta,\xi)-F(\zeta,\xi)\big)F(\zeta,\xi)\,d\xi \\ +\mathcal{O}(\langle\zeta\rangle^{2m-1})\int |\xi-\zeta|^2 F(\eta+\zeta,\xi)F(\zeta,\xi)\,d\xi,$$

where we use $\int(\xi-\zeta)F(\zeta,\xi)^2\,d\xi=0$, which is a consequence of q being even.

Taking the second integral in (8.5.3) into account, we obtain

$$|\widehat{r}(\eta;\zeta)|\lesssim (M_1+M_2+M_3)p_0'''(\eta;\zeta),\quad p_0'''(\eta;\zeta)\in L^\infty_\zeta L^2_\eta, \tag{8.5.5}$$

where

$$\begin{aligned} M_1(\eta,\zeta)&=\frac{\langle\zeta\rangle^{2m+1}}{\langle\eta\rangle^s}\int \frac{|\xi-\zeta|}{\langle\zeta\rangle}|F(\eta+\zeta,\xi)-F(\zeta,\xi)|F(\zeta,\xi)\,d\xi\\ M_2(\eta,\zeta)&=\frac{\langle\zeta\rangle^{2m+1}}{\langle\eta\rangle^s}\int \frac{|\xi-\zeta|^2}{\langle\zeta\rangle^2}F(\eta+\zeta,\xi)F(\zeta,\xi)\,d\xi\\ M_3(\eta,\zeta)&=\frac{\langle\zeta\rangle^{2m+1}}{\langle\eta\rangle^s}\left|\int\big(F(\eta+\zeta,\xi)-F(\zeta,\xi)\big)F(\zeta,\xi)\,d\xi\right|. \end{aligned}$$

M_2 is estimated easily: On the support of the integrand, one has $|\xi-\zeta|^2\le\langle\zeta\rangle$, thus

$$M_2(\eta,\zeta)\lesssim\frac{\langle\zeta\rangle^{2m}}{\langle\eta\rangle^s}\cdot\frac{\langle\zeta\rangle^{n/2}}{\langle\eta+\zeta\rangle^{n/4}\langle\zeta\rangle^{n/4}};$$

here, the term $\langle\zeta\rangle^{n/2}$ in the numerator is (up to a constant) an upper bound for the volume of the domain of integration. Since we are assuming $|\eta|\le|\zeta|/2$, we have $\langle\eta+\zeta\rangle\gtrsim\langle\zeta\rangle$, which gives $M_2(\eta,\zeta)\lesssim\langle\zeta\rangle^{2m}/\langle\eta\rangle^s$.

In order to estimate M_1 and M_3, we will use

$$\begin{aligned} \partial_\zeta F(\zeta,\xi)&=\frac{a_0(\zeta)}{\langle\zeta\rangle^{n/4+1}}q_1\left(\frac{\xi-\zeta}{\langle\zeta\rangle^{1/2}}\right)+\frac{a_1(\zeta)}{\langle\zeta\rangle^{n/4+1/2}}\partial_\zeta q\left(\frac{\xi-\zeta}{\langle\zeta\rangle^{1/2}}\right),\\ \partial_\zeta^2 F(\zeta,\xi)&=\frac{a_2(\zeta)}{\langle\zeta\rangle^{n/4+1}}q_2\left(\frac{\xi-\zeta}{\langle\zeta\rangle^{1/2}}\right), \end{aligned}$$

where the a_j are scalar-, vector- or matrix-valued symbols of order 0, and $q_j\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

Hence, writing $F(\eta+\zeta,\xi)-F(\zeta,\xi)=\eta\cdot\partial_\zeta F(\zeta+\bar t\eta,\xi)$ for some $0\le\bar t\le 1$, we get

$$M_1(\eta,\zeta)\lesssim\frac{\langle\zeta\rangle^{2m+1}}{\langle\eta\rangle^s}\cdot\frac{\langle\zeta\rangle^{n/2}|\eta|}{\langle\zeta\rangle^{1/2}\langle\zeta+\bar t\eta\rangle^{n/4+1/2}\langle\zeta\rangle^{n/4}}\lesssim\frac{\langle\zeta\rangle^{2m}}{\langle\eta\rangle^{s-1}},$$

where we again use $|\eta|<|\zeta|/2$ and $\langle\zeta+\bar t\eta\rangle\gtrsim\langle\zeta\rangle$.

Finally, to bound M_3, we write

$$F(\eta+\zeta,\xi)-F(\zeta,\xi)=\eta\cdot\partial_\zeta F(\zeta,\xi)+\int_0^1(1-t)\langle\eta,\partial_\zeta^2F(\zeta+t\eta,\xi)\cdot\eta\rangle\,dt$$

and deduce

$$\begin{aligned}M_3(\eta,\zeta)\lesssim\frac{\langle\zeta\rangle^{2m+1}}{\langle\eta\rangle^s}\Bigg(&\frac{\langle\zeta\rangle^{n/2}|\eta|}{\langle\zeta\rangle^{n/4+1}\langle\zeta\rangle^{n/4}}\\&+\frac{|\eta|}{\langle\zeta\rangle^{n/4+1/2}}\left|\int(\partial_\zeta q)\left(\frac{\xi-\zeta}{\langle\zeta\rangle^{1/2}}\right)q\left(\frac{\xi-\zeta}{\langle\zeta\rangle^{1/2}}\right)\,d\xi\right|\\&+\frac{\langle\zeta\rangle^{n/2}|\eta|^2}{\langle\zeta\rangle^{n/4+1}\langle\zeta\rangle^{n/4}}\Bigg)\\\lesssim\ &\frac{\langle\zeta\rangle^{2m}}{\langle\eta\rangle^{s-2}},\end{aligned}$$

where we use

$$\int(\partial_\zeta q)\left(\frac{\xi-\zeta}{\langle\zeta\rangle^{1/2}}\right)q\left(\frac{\xi-\zeta}{\langle\zeta\rangle^{1/2}}\right)\,d\xi=0,$$

which holds since q has compact support. Plugging the estimates for M_j, $j=1,2,3$, into (8.5.5) proves that (8.5.1) holds. The proof is complete. □

The idea of the proof can also be used to prove the sharp Gårding inequality for smooth b-ps.d.o.s:

Proposition 8.5.2. *Let $m\in\mathbb{R}$, and let $p(z,\zeta)\in S^{2m+1}(\overline{\mathbb{R}^n_+};\mathrm{End}(E))$ be a symbol with non-negative real part. Then there is $C>0$ such that $P=p(z,{}^{\mathrm{b}}D)$ satisfies the estimate*

$$\mathrm{Re}\langle Pu,u\rangle\ge -C\|u\|^2_{H^m_{\mathrm{b}}},\quad u\in\dot{\mathcal{C}}^\infty_{\mathrm{c}}(\overline{\mathbb{R}^n_+};E).$$

Proof. Write $p(x,y;\zeta)=p^{(0)}(y;\zeta)+p^{(1)}(x,y;\zeta)$, where $p^{(0)}(y;\zeta)=p(0,y;\zeta)$ and $p^{(1)}=x\widetilde{p}\in H^\infty_{\mathrm{b}}S^{2m+1}$. The symmetrized operator $p({}^{\mathrm{b}}D,z,{}^{\mathrm{b}}D)$, defined as in the proof of Proposition 8.5.1 is again non-negative, and the symbol of the remainder operator $r(z,{}^{\mathrm{b}}D)=$

$p_{\mathrm{sym}}({}^{\mathrm{b}}D, z, {}^{\mathrm{b}}D) - p(z, {}^{\mathrm{b}}D)$ is the sum of two terms $p^{(0)}_{\mathrm{sym}} - p^{(0)}$ and $p^{(1)}_{\mathrm{sym}} - p^{(1)}$. The proof of Proposition 8.5.1 shows that $p^{(1)}_{\mathrm{sym}} - p^{(1)} \in S^{2m;0}H^{\infty}_{\mathrm{b}}$. It thus suffices to assume that $p = p^{(0)}$ is independent of x, which implies that p_{sym} is independent of x as well, and to prove $r(y, {}^{\mathrm{b}}D) = (p_{\mathrm{sym}} - p)(y, {}^{\mathrm{b}}D)\colon H^m_{\mathrm{b}} \to H^{-m}_{\mathrm{b}}$.

Similarly to the proof of Proposition 8.5.1, we put

$$
\begin{aligned}
F(\lambda, \eta; \sigma, \gamma) &= \frac{1}{\langle \lambda, \eta\rangle^{n/4}} q\left(\frac{(\sigma - \lambda, \gamma - \eta)}{\langle \lambda, \eta\rangle^{1/2}}\right) \\
p_{\mathrm{sym}}(\rho, \theta; y; \lambda, \eta) &= \iint F(\rho, \theta; \sigma, \gamma) p(y; \sigma, \gamma) F(\lambda, \eta; \sigma, \gamma)\, d\sigma\, d\gamma
\end{aligned}
$$

and obtain

$$
\begin{aligned}
(p_{\mathrm{sym}}({}^{\mathrm{b}}D; y; {}^{\mathrm{b}}D)u)\widehat{\ }(\rho, \theta) &= \int \mathscr{F} p_{\mathrm{sym}}(\rho, \theta; \theta - \eta; \rho, \eta) \widehat{u}(\rho, \eta)\, d\eta \\
\mathscr{F} r(\theta; \lambda, \eta) &= \mathscr{F} p_{\mathrm{sym}}(\lambda, \theta + \eta; \theta; \lambda, \eta) - \mathscr{F} p(\theta; \lambda, \eta),
\end{aligned}
$$

thus

$$
\begin{aligned}
&\mathscr{F} r(\theta; \lambda, \eta) \\
&\quad = \iint F(\lambda, \theta + \eta; \sigma, \gamma)\big(\mathscr{F} p(\theta; \sigma, \gamma) - \mathscr{F} p(\theta; \lambda, \eta)\big) F(\lambda, \eta; \sigma, \gamma)\, d\sigma\, d\gamma \\
&\quad\quad + \iint \big(F(\lambda, \theta + \eta; \sigma, \gamma) - F(\lambda, \eta; \sigma, \gamma)\big) \mathscr{F} p(\theta; \lambda, \eta) F(\lambda, \eta; \sigma, \gamma)\, d\sigma\, d\gamma.
\end{aligned}
$$

Then, following the argument in the previous proof, we obtain

$$
|\mathscr{F} r(\theta; \lambda, \eta)| \le \frac{\langle \lambda, \eta\rangle^{2m}}{\langle \theta\rangle^{N}} r_0(\theta; \lambda, \eta), \quad r_0(\theta; \lambda, \eta) \in L^{\infty}_{\lambda,\eta} L^2_{\theta}, \tag{8.5.6}
$$

where we use

$$
|\mathscr{F} p(\theta; \lambda, \eta)| = \frac{\langle \lambda, \eta\rangle^{2m+1}}{\langle \theta\rangle^{N+2}} p_0(\theta; \lambda, \eta), \quad p_0(\theta; \lambda, \eta) \in L^{\infty}_{\lambda,\eta} L^2_{\theta},
$$

which holds for every integer N (with p_0 depending on the choice of N). An estimate similar to the one used in the proof of Proposition 8.2.9 shows that (8.5.6) implies $r(y, {}^{\mathrm{b}}D)\colon H^s_{\mathrm{b}} \to H^{s-2m}_{\mathrm{b}}$ for all $s \in \mathbb{R}$. □

Finally, we merge Propositions 8.5.1 and 8.5.2.

Corollary 8.5.3. *Let $s, m \in \mathbb{R}$ be such that $s \geq 2 - m$, $s > n/2 + 2 + m_+$. Let $p_0(z,\zeta) \in S^{2m+1}(\overline{\mathbb{R}^n_+}; \mathrm{End}(E))$ and $\widetilde{p}(z,\zeta) \in S^{2m+1;2} H^s_{\mathrm{b}}(\overline{\mathbb{R}^n_+}; \mathrm{End}(E))$ be symbols such that $p = p_0 + \widetilde{p}$ has non-negative real part. Then there is $C > 0$ such that $P = p(z, {}^{\mathrm{b}}D)$ satisfies the estimate*

$$\mathrm{Re}\langle Pu, u\rangle \geq -C\|u\|^2_{H^m_{\mathrm{b}}}, \quad u \in \dot{\mathcal{C}}^\infty_{\mathrm{c}}(\overline{\mathbb{R}^n_+}; E).$$

Proof. The symmetrized operator $p_{\mathrm{sym}}({}^{\mathrm{b}}D, z, {}^{\mathrm{b}}D)$ is again non-negative, and the symbol of the remainder operator $r(z, {}^{\mathrm{b}}D) = p_{\mathrm{sym}}({}^{\mathrm{b}}D, z, {}^{\mathrm{b}}D) - p(z, {}^{\mathrm{b}}D)$ is the sum of two terms $(p_0)_{\mathrm{sym}} - p_0$ and $\widetilde{p}_{\mathrm{sym}} - \widetilde{p}$. The proofs of Propositions 8.5.1 and 8.5.2 show that $((p_0)_{\mathrm{sym}} - p_0)(z, {}^{\mathrm{b}}D)$ and $(\widetilde{p}_{\mathrm{sym}} - \widetilde{p})(z, {}^{\mathrm{b}}D)$ map H^m_{b} to H^{-m}_{b}, hence $r(z, {}^{\mathrm{b}}D)$ maps H^m_{b} to H^{-m}_{b}, and the proof is complete. □

8.5.2 Mollifiers

In order to deal with certain kinds of non-smooth terms in §§8.5.3 and 8.5.4, we will need smoothing operators in order to smooth out and approximate non-smooth functions in a precise way. We only state the results for unweighted spaces, but the corresponding statements for weighted spaces hold true by the same proofs.

Lemma 8.5.4. *Let $s \in \mathbb{R}$, $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}_+)$. Then $\chi(x/\epsilon) \to 0$ strongly as a multiplication operator on $H^s_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ as $\epsilon \to 0$, and in norm as a multiplication operator from $H^{s,\alpha}_{\mathrm{b}}(\overline{\mathbb{R}^n_+}) \to H^s_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ for $\alpha > 0$.*

Proof. We start with the first half of the lemma: For $s = 0$, the statement follows from the dominated convergence theorem. For s a positive integer, we use that

$$(x\partial_x)^s \left(\chi\left(\frac{x}{\epsilon}\right)\right) = \sum_{j=1}^{s} c_{sj} \left(\frac{x}{\epsilon}\right)^j \chi^{(j)}\left(\frac{x}{\epsilon}\right), \quad c_{sj} \in \mathbb{R},$$

is bounded and converges to 0 pointwise in $x > 0$ as $\epsilon \to 0$, thus by virtue of the Leibniz rule and the dominated convergence theorem, we obtain $\chi(x/\epsilon)u(x,y) \to 0$ in $H^s_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ for $u \in H^s_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$. For $s \in -\mathbb{N}$, the statement follows by duality.

Finally, to treat the case of general s, we first show that $\chi(\cdot/\epsilon)$ is a uniformly bounded family (in $\epsilon > 0$) of multiplication operators on $H^s_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ for all $s \in \mathbb{R}$: For $s \in \mathbb{N}_0$, this follows from the above estimates, for $s \in \mathbb{Z}$ again by duality, and then for general $s \in \mathbb{R}$ by interpolation. Now, put $M = \sup_{0<\epsilon\leq 1} \|\chi(\cdot/\epsilon)\|_{H^s_{\mathrm{b}} \to H^s_{\mathrm{b}}} < \infty$. Let $w \in H^s_{\mathrm{b}}$ and $\delta > 0$

be given, and choose $w' \in H_{\mathrm{b}}^\infty$ such that $\|w'-w\|_{H_{\mathrm{b}}^s} < \delta/2M$. By what we have already proved, we can choose $\epsilon_0 > 0$ so small that

$$\|\chi(\cdot/\epsilon)w'\|_{H_{\mathrm{b}}^s} \leq \|\chi(\cdot/\epsilon)w'\|_{H_{\mathrm{b}}^{\lceil s\rceil}} < \delta/2, \quad \epsilon < \epsilon_0;$$

then

$$\|\chi(\cdot/\epsilon)w\|_{H_{\mathrm{b}}^s} \leq \|\chi(\cdot/\epsilon)(w-w')\|_{H_{\mathrm{b}}^s} + \|\chi(\cdot/\epsilon)w'\|_{H_{\mathrm{b}}^s} < M\frac{\delta}{2M} + \frac{\delta}{2} = \delta.$$

Concerning the second half of the lemma, the case $s=0$ is clear since $x^\alpha\chi(x/\epsilon) \to 0$ in $L^\infty(\mathbb{R}_+)$ as $\epsilon \to 0$; as above, this implies the statement for s a positive integer, and the case of real s again follows by duality and interpolation. □

Lemma 8.5.5. *Let M be a compact manifold with boundary. Then there exists a family of operators $J_\epsilon\colon \mathcal{C}^{-\infty}(M) \to \mathcal{C}_{\mathrm{c}}^\infty(M^\circ)$, $\epsilon > 0$, such that $J_\epsilon \in \Psi_{\mathrm{b}}^{-\infty}(M)$, and for all $s, r \in \mathbb{R}$, J_ϵ is a uniformly bounded family of operators on $H_{\mathrm{b}}^{s,r}(M)$ that converges strongly to the identity map I as $\epsilon \to 0$.*

Proof. Choosing a product decomposition $\partial M \times [0,\epsilon_0)_x$ near the boundary of M and $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R})$, $\chi \equiv 1$ near 0, $\operatorname{supp}\chi \subset [0,1/2]$, we can define the multiplication operators $\chi(x/\epsilon)$ globally on $H_{\mathrm{b}}^{-\infty}(M)$. By the previous lemma, $I-\chi(\cdot/\epsilon)$ converges strongly to I on $H_{\mathrm{b}}^s(M)$; moreover, $\operatorname{supp}(u-\chi(\cdot/\epsilon)u) \subset \{x \geq \epsilon\}$. Thus, if we let $\widetilde{J}_\epsilon$ be a family of mollifiers, $\widetilde{J}_\epsilon \in \Psi_{\mathrm{b}}^{-\infty}(M)$, $\widetilde{J}_\epsilon \to I$ in $\Psi_{\mathrm{b}}^{\delta'}(M)$ for $\delta' > 0$, such that on the support of the Schwartz kernel of $\widetilde{J}_\epsilon$, we have $|x_1-x_2| < \epsilon/2$ near $\partial M \times \partial M$ where x_1, x_2 are the lifts of x to the left and right factor of $M \times M$, then we have that $\widetilde{J}_\epsilon(u-\chi(\cdot/\epsilon)u)$ is an element of $H_{\mathrm{b}}^\infty(M)$ with support in $\{x \geq \epsilon/2\}$, thus is smooth. Therefore, the family $J_\epsilon := \widetilde{J}_\epsilon \circ (I-\chi(\cdot/\epsilon))$ satisfies all requirements. □

8.5.3 Real principal type propagation, complex absorption

We will prove real principal type propagation estimates of b-regularity for operators with non-smooth coefficients by following the arguments outlined in §3.2.2 in the smooth coefficient case as closely as possible. We make a more robust and flexible choice however, following [33].

Theorem 8.5.6. *Let $m, r, s, \widetilde{s} \in \mathbb{R}$, $\alpha > 0$. Suppose $\widetilde{P} = \widetilde{P}_m + \widetilde{P}_{m-1} + \widetilde{R}$, where $\widetilde{P}_m \in H_{\mathrm{b}}^{s,\alpha}\Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+}; E)$ has a real, scalar, homogeneous principal symbol $\widetilde{p}_m$; moreover, let $\widetilde{P}_{m-1} \in$*

$H_{\mathrm{b}}^{s-1,\alpha}\Psi_{\mathrm{b}}^{m-1}(\overline{\mathbb{R}^n_+};E)$ *and* $\widetilde{R}\in\Psi_{\mathrm{b}}^{m-2}(\overline{\mathbb{R}^n_+};E)+\Psi_{\mathrm{b}}^{m-2;0}H_{\mathrm{b}}^{s-1,\alpha}(\overline{\mathbb{R}^n_+};E)$. *Suppose* s *and* $\widetilde{s}$ *are such that*

$$\widetilde{s}\le s-1,\quad s>n/2+7/2+(2-\widetilde{s})_+. \tag{8.5.7}$$

(1) Let $P\equiv\widetilde{P}$ *and* $p\equiv\widetilde{p}_m$, *or*

(2) let $P=P_0+\widetilde{P}$, *where* $P_0\in\Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+};E)$ *has a real, scalar, homogeneous principal symbol* p_0. *Denote* $p=p_0+\widetilde{p}_m$.

In both cases, if $u\in H_{\mathrm{b}}^{\widetilde{s}+m-3/2,r}(\overline{\mathbb{R}^n_+};E)$ *is such that* $Pu\in H_{\mathrm{b}}^{\widetilde{s},r}(\overline{\mathbb{R}^n_+};E)$, *then* $\mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m-1,r}(u)$ *is a union of maximally extended null-bicharacteristics of* p, *i.e. of integral curves of the Hamilton vector field* H_p *within the characteristic set* $p^{-1}(0)\subset{}^{\mathrm{b}}T^*\overline{\mathbb{R}^n_+}\setminus o$.

The proof, which will occupy the remainder of this section, in fact gives an estimate for the $H_{\mathrm{b}}^{\widetilde{s}+m-1,r}$-norm of u: Suppose $A,B,G\in\Psi_{\mathrm{b}}^0$ are such that all forward or backward null-bicharacteristics from $\mathrm{WF}'_{\mathrm{b}}(B)$ reach the elliptic set of A while remaining in the elliptic set of G, and $\psi\in\mathcal{C}_{\mathrm{c}}^\infty(\overline{\mathbb{R}^n_+})$ is identically 1 on $\pi(\mathrm{WF}'_{\mathrm{b}}(B))$, where $\pi\colon{}^{\mathrm{b}}T^*\overline{\mathbb{R}^n_+}\to\overline{\mathbb{R}^n_+}$ is the projection, then

$$\begin{aligned}&\|Bu\|_{H_{\mathrm{b}}^{\widetilde{s}+m-1,r}}\\&\quad\le C(\|GPu\|_{H_{\mathrm{b}}^{\widetilde{s},r}}+\|Au\|_{H_{\mathrm{b}}^{\widetilde{s}+m-1,r}}+\|\psi Pu\|_{H_{\mathrm{b}}^{\widetilde{s}-1,r}}+\|u\|_{H_{\mathrm{b}}^{\widetilde{s}+m-3/2,r}})\end{aligned} \tag{8.5.8}$$

in the sense that if all quantities on the right hand side are finite, then so is the left hand side, and the inequality holds. In particular, it suffices to have only microlocal $H_{\mathrm{b}}^{\widetilde{s},r}$-membership of Pu near the parts of null-bicharacteristics along which we want to propagate $H_{\mathrm{b}}^{\widetilde{s}+m-1,r}$-regularity of u. The term involving ψPu comes from the local requirements for elliptic regularity, see Remark 8.4.2. The constant C depends on natural (semi-)norms of the spaces in which the coefficients of P lie; for the more precise 'tame' estimate, see Proposition 8.8.3.

Later in this section, we will add complex absorption and obtain the following statement.

Theorem 8.5.7. *Under the assumptions of Theorem 8.5.6, let* $Q\in\Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+};E)$, $Q=Q^*$. *Suppose* $A,B,G\in\Psi_{\mathrm{b}}^0$ *are such that all forward, resp. backward, bicharacteristics from* $\mathrm{WF}'_{\mathrm{b}}(B)$ *reach the elliptic set of* A *while remaining in the elliptic set of* G, *and suppose moreover that* $q\le 0$, *resp.* $q\ge 0$, *on* $\mathrm{WF}'_{\mathrm{b}}(G)$, *further let* $\psi\in\mathcal{C}_{\mathrm{c}}^\infty(\overline{\mathbb{R}^n_+})$ *be identically* 1 *on*

$\pi(\mathrm{WF}'_{\mathrm{b}}(B))$, *then*

$$\|Bu\|_{H_{\mathrm{b}}^{\tilde{s}+m-1,r}} \leq C(\|G(P-iQ)u\|_{H_{\mathrm{b}}^{\tilde{s},r}} + \|Au\|_{H_{\mathrm{b}}^{\tilde{s}+m-1,r}} + \|\psi(P-iQ)u\|_{H_{\mathrm{b}}^{\tilde{s}-1,r}} + \|u\|_{H_{\mathrm{b}}^{\tilde{s}+m-3/2,r}}) \tag{8.5.9}$$

in the sense that if all quantities on the right hand side are finite, then so is the left hand side, and the inequality holds.

In other words, we can propagate estimates from the elliptic set of A forward along the Hamilton flow to $\mathrm{WF}'_{\mathrm{b}}(B)$ if $q \geq 0$, and backward if $q \leq 0$.

Conjugating by x^r (where x is the standard boundary defining function), it suffices to prove Theorems 8.5.6 and 8.5.7 for $r = 0$. Moreover, as in the smooth setting, we can apply Theorem 8.4.1 on the elliptic set of P in both cases and deduce microlocal $H_{\mathrm{b}}^{\tilde{s}+m}$-regularity of u there, which implies that $\mathrm{WF}_{\mathrm{b}}^{\tilde{s}+m-1}(u)$ is a subset of the characteristic set of P, and thus we only need to prove the propagation result within the characteristic set. We will begin by proving the first part of Theorem 8.5.6; the proof is then easily modified to yield the second part of Theorem 8.5.6. To keep the notation simple, we will only consider the case of complex-valued symbols (hence, operators acting on functions); in the general, bundle-valued case, all arguments go through with purely notational changes.

Propagation in the interior

For brevity, denote $M = \overline{\mathbb{R}^n_+}$. We start with the first half of Theorem 8.5.6, where we can in fact assume $\alpha = 0$ since we are working away from the boundary, as explained below. Thus, let $P = P_m + P_{m-1} + R$, where we assume $m \geq 1$ for now,

$$\begin{aligned} P_m &\in H_{\mathrm{b}}^s \Psi_{\mathrm{b}}^m \text{ with real homogeneous principal symbol,} \\ P_{m-1} &\in H_{\mathrm{b}}^{s-1} \Psi_{\mathrm{b}}^{m-1}, \\ R &\in \Psi_{\mathrm{b}}^{m-2;0} H_{\mathrm{b}}^{s-1}, \end{aligned}$$

and let us assume that we are given a solution

$$u \in H_{\mathrm{b}}^{\sigma-1/2}, \tag{8.5.10}$$

to the equation

$$Pu = f \in H_{\mathrm{b}}^{\sigma-m+1},$$

where $\sigma = \widetilde{s} + m - 1$ with $\widetilde{s}$ as in the statement of Theorem 8.5.6. In fact, since

$$R \colon H_{\mathrm{b}}^{\sigma-1/2} \subset H_{\mathrm{b}}^{\sigma-1} \to H_{\mathrm{b}}^{\sigma-m+1}$$

by Proposition 8.2.9,[30] we may absorb the term Ru into the right hand side; thus, we can assume $R = 0$, hence $P = P_m + P_{m-1}$. We denote the symbol of P_m by p.

Let γ be a null-bicharacteristic of p, and assume that H_p is never radial on γ. Note that this in particular means that $\gamma \cap {}^{\mathrm{b}}T^*_{\partial M}M = \emptyset$ since p vanishes identically at the boundary, and in fact this setup is the correct one for the discussion of real principal type propagation in the interior of M. *All functions we construct in this section are implicitly assumed to have support away from ∂M.* Even though we are working away from the boundary, we will still employ the b-notation throughout this section, since the proof of the real principal type propagation result (near and) within the boundary will only require minor changes compared to the proof of the interior result given here.

The objective is to propagate microlocal H_{b}^{σ}-regularity along γ to a point $\zeta_0 \in {}^{\mathrm{b}}T^*M \setminus o$, assuming a priori knowledge of microlocal H_{b}^{σ}-regularity of u near a point ζ_* on the backward bicharacteristic from ζ_0; the location and size of this region will be specified later, see Proposition 8.5.8. We will use a positive commutator argument.

The idea, following [33, §2], is to arrange for $\mathsf{H}_p = \rho^{1-m}H_p$, $\rho = \langle\zeta\rangle$,

$$\mathsf{H}_p\mathsf{a} = -\mathsf{b}^2 + \mathsf{e} - \mathsf{f}, \tag{8.5.11}$$

where $\mathsf{a}, \mathsf{b}, \mathsf{e}$ are smooth symbols and f is a non-smooth symbol, absorbing non-smooth terms of $\mathsf{H}_p\mathsf{a}$ in an appropriate way, which however has a definite sign; by virtue of the sharp Gårding inequality, we will be able to bound terms involving f using the a priori regularity assumptions on u. As in the smooth case, terms involving e will be controlled by the a priori assumptions of u near ζ_*. If b is elliptic at ζ_0, we are thus able to prove the desired H_{b}^{σ}-regularity at ζ_0. The actual commutant to be used, which has the correct symbolic order and is regularized, will be constructed later; see Proposition 8.5.8 for its relevant properties.

[30] We need $s - 1 \geq \sigma - m + 1$ and $s - 1 > n/2 + (m - \sigma - 1)_+$.

The general strategy for choosing the non-smooth symbol f is as follows: Non-smooth terms T, which arise in the computation and are positive, say $T \geq c > 0$, are smoothed out using a mollifier J, giving a smooth function JT, but only as much as to still preserve some positivity $JT - c/4 \geq c/4 > 0$, and in such a way that the error $T - JT + c/4$ is non-negative; then $\mathsf{b}^2 = JT - c/4$ is a smooth, positive term, and $\mathsf{f} = T - JT + c/4$ is non-smooth, but has a sign, and $T = \mathsf{b}^2 + \mathsf{f}$. The mollifiers we shall use were constructed in Lemma 8.5.5.

To start, choose $\widetilde{\eta} \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$ with $\widetilde{\eta}(\zeta_0) = 0$, $\mathsf{H}_p\widetilde{\eta}(\zeta_0) > 0$, i.e. $\widetilde{\eta}$ measures, at least locally, propagation along the Hamilton flow. Choose $\sigma_j \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$, $j = 1, \ldots, 2n-2$, with $\sigma_j(\zeta_0) = 0$ and $\mathsf{H}_p\sigma_j(\zeta_0) = 0$, and such that $d\widetilde{\eta}, d\sigma_j$ span $T^*_{\zeta_0}({}^{\mathrm{b}}S^*M)$. Put $\omega = \sum_{j=1}^{2n-2}\sigma_j^2$, so that $\omega^{1/2}$ approximately measures how far away one is from the bicharacteristic through ζ_0. Thus, $|\widetilde{\eta}| + \omega^{1/2}$ is, near ζ_0, equivalent to the distance from ζ_0 with respect to any distance function given by a Riemannian metric on ${}^{\mathrm{b}}S^*M$. Then for $\delta \in (0,1), \epsilon \in (0,1], \beta \in (0,1]$ and $F > 0$ (large) to be chosen later, let

$$\phi = \widetilde{\eta} + \frac{1}{\epsilon^2\delta}\omega,$$

and, taking $\chi_0(t) = e^{-1/t}$ for $t > 0$, $\chi_0(t) = 0$ for $t \leq 0$, and $\chi_1 \in \mathcal{C}^\infty(\mathbb{R})$, $\chi_1 \geq 0$, $\sqrt{\chi_1} \in \mathcal{C}^\infty(\mathbb{R})$, $\operatorname{supp}\chi_1 \subset (0,\infty), \operatorname{supp}\chi_1' \subset (0,1)$, and $\chi_1 \equiv 1$ in $[1,\infty)$, consider

$$\mathsf{a} = \chi_0\left(F^{-1}\Big(2\beta - \frac{\phi}{\delta}\Big)\right)\chi_1\left(\frac{\widetilde{\eta}+\delta}{\epsilon\delta} + 1\right).$$

First, we observe that $\mathsf{H}_p\phi(\zeta_0) = \mathsf{H}_p\widetilde{\eta}(\zeta_0) > 0$; but $\chi_1\left(\frac{\widetilde{\eta}+\delta}{\epsilon\delta} + 1\right) \equiv 1$ near ζ_0, so

$$\mathsf{H}_p\mathsf{a}(\zeta_0) = -F^{-1}\delta^{-1}\mathsf{H}_p\phi(\zeta_0)\chi_0'(2F^{-1}\beta) < 0$$

has the right sign at ζ_0.

Next, we analyze the support of a: First of all, If $\zeta \in \operatorname{supp}\mathsf{a}$, then

$$\phi(\zeta) \leq 2\beta\delta, \quad \widetilde{\eta}(\zeta) \geq -\delta - \epsilon\delta \geq -2\delta.$$

Since $\omega \geq 0$, we get $\widetilde{\eta} = \phi - \omega/\epsilon^2\delta \leq \phi \leq 2\beta\delta \leq 2\delta$, thus $\omega = \epsilon^2\delta(\phi - \widetilde{\eta}) \leq 4\epsilon^2\delta^2$, i.e.

$$-\delta - \epsilon\delta \leq \widetilde{\eta} \leq 2\beta\delta, \quad \omega^{1/2} \leq 2\epsilon\delta \quad \text{on } \operatorname{supp}\mathsf{a}. \tag{8.5.12}$$

In particular, we can make $\operatorname{supp} \mathsf{a}$ to be arbitrarily close to ζ_0 by choosing $\delta > 0$ small, hence there is $\delta_0 > 0$ small such that $\mathsf{H}_p\widetilde{\eta} \geq c_0 > 0$ whenever $|\widetilde{\eta}| \leq 2\delta_0$ and $\omega^{1/2} \leq 2\delta_0$. The support of a becomes localized near $\omega = 0$ by choosing $\epsilon > 0$ small. The parameter β then allows one to localize $\operatorname{supp} \mathsf{a}$ near the segment $\widetilde{\eta} \in [-\delta; 0]$. Moreover, we have

$$-\delta - \epsilon\delta \leq \widetilde{\eta} \leq -\delta, \quad \omega^{1/2} \leq 2\epsilon\delta \quad \text{on } \operatorname{supp} \mathsf{a} \cap \operatorname{supp} \chi_1', \tag{8.5.13}$$

which is the region where we will assume a priori microlocal control on u. Observe that by taking $\epsilon > 0$ small, we can make this region arbitrarily closely localized at $\widetilde{\eta} = -\delta$, $\omega = 0$.

Choose $\widetilde{\chi}_1 \in \mathcal{C}^\infty(\mathbb{R})$, $\widetilde{\chi}_1 \geq 0$, such that $\widetilde{\chi}_1 \equiv 1$ on $\operatorname{supp} \chi_1'$, and $\operatorname{supp} \widetilde{\chi}_1 \subset [0, 1]$. Since the coefficients of H_p are continuous because of $s > n/2 + 1$, we can choose a mollifier J as in Lemma 8.5.5, acting on a function f defined on ${}^{\mathrm{b}}T^*\overline{\mathbb{R}^n_+}$ by $(Jf)(z, \zeta) = J(f(\cdot, \zeta))(z)$, such that for

$$\begin{aligned} \mathsf{e} &= \chi_0\left(F^{-1}\left(2\beta - \frac{\phi}{\delta}\right)\right)(J\mathsf{H}_p)\left(\chi_1\left(\frac{\widetilde{\eta}+\delta}{\epsilon\delta}+1\right)\right) + \widetilde{\chi}_1\left(\frac{\widetilde{\eta}+\delta}{\epsilon\delta}+1\right), \\ \mathsf{f}' &= \chi_0\left(F^{-1}\left(2\beta - \frac{\phi}{\delta}\right)\right)\Bigg[\widetilde{\chi}_1\left(\frac{\widetilde{\eta}+\delta}{\epsilon\delta}+1\right) \\ &\qquad + (J\mathsf{H}_p - \mathsf{H}_p)\left(\chi_1\left(\frac{\widetilde{\eta}+\delta}{\epsilon\delta}+1\right)\right)\Bigg], \end{aligned} \tag{8.5.14}$$

hence $\mathsf{e} - \mathsf{f}' = \chi_0 \mathsf{H}_p \chi_1$, we have $\mathsf{f}' \geq 0$. Note that $\mathsf{e} \in \mathcal{C}^\infty$ has support as indicated in (8.5.13), and $\mathsf{f}' \in H^{s-1}_{\mathrm{b}}$ in the base variables.

In order to have (8.5.11), it remains to prove that the remaining term of $\mathsf{H}_p\mathsf{a}$, namely $\chi_1\mathsf{H}_p\chi_0$, is non-positive; for this, it is sufficient to require $\mathsf{H}_p\phi \geq c_0/2$ on $\operatorname{supp} \mathsf{a}$ if $\delta < \delta_0$. From the definition of ϕ, this would follow provided

$$|\mathsf{H}_p\omega| \leq c_0\epsilon^2\delta/2 \tag{8.5.15}$$

on $\operatorname{supp} \mathsf{a}$. Now, since for $s > n/2 + 2$, $\mathsf{H}_p\sigma_j$ is Lipschitz continuous and vanishes at ζ_0, we have

$$|\mathsf{H}_p\omega| \leq 2\sum_{j=1}^{2n-2} |\sigma_j||\mathsf{H}_p\sigma_j| \leq C\omega^{1/2}\left(|\widetilde{\eta}| + \omega^{1/2}\right), \tag{8.5.16}$$

hence (8.5.15) holds if $2C\epsilon\delta(2\delta + 2\epsilon\delta) \leq c_0\epsilon^2\delta/2$, which is satisfied provided $16C\delta/c_0 \leq \epsilon$. Let us choose $\epsilon = 16C\delta/c_0$, with δ small enough such that $\epsilon \leq 1$. For later use, let us note that then near $\widetilde{\eta} = -\delta$, the 'width' of the support of a is

$$\omega^{1/2} \leq \frac{c_0 \epsilon^2 \delta/2}{C(\omega^{1/2} + |\widetilde{\eta}|)} \lesssim \delta^2, \tag{8.5.17}$$

hence by (8.5.13), the region where we will assume a priori microlocal control on u (i.e. $\operatorname{supp} \mathsf{e}$) has size $\sim \delta^2$.

Now, let

$$\begin{aligned}
\mathsf{b} &= (F\delta)^{-1/2} \sqrt{(J\mathsf{H}_p)\phi - c_0/4} \sqrt{\chi_0'\left(F^{-1}\left(2\beta - \frac{\phi}{\delta}\right)\right)} \sqrt{\chi_1\left(\frac{\widetilde{\eta} + \delta}{\epsilon\delta} + 1\right)}, \\
\mathsf{f}'' &= (F\delta)^{-1} \left((\mathsf{H}_p - J\mathsf{H}_p)\phi + c_0/4\right) \chi_0'\left(F^{-1}\left(2\beta - \frac{\phi}{\delta}\right)\right) \chi_1\left(\frac{\widetilde{\eta} + \delta}{\epsilon\delta} + 1\right),
\end{aligned}$$

where J is the same mollifier as used in (8.5.14); we assume it is close enough to I so that $|(\mathsf{H}_p - J\mathsf{H}_p)\phi| < c_0/8$, which implies $(J\mathsf{H}_p)\phi - c_0/4 \geq c_0/8 > 0$ and $\mathsf{f}'' \geq 0$. Putting $\mathsf{f} = \mathsf{f}' + \mathsf{f}''$, which is H_{b}^{s-1} in the base variables, we thus have achieved (8.5.11).

Next, we have to make the commutant, a, a symbol of order $2\sigma - (m-1)$, so that the 'principal symbol' of $i[P, A]$, i.e. $H_p a$, is of order 2σ, hence b has order σ, which is what we need, since we want to prove H_{b}^{σ}-regularity of u at ζ_0. Thus, define

$$\breve{a} = \rho^{\sigma - (m-1)/2} \mathsf{a}^{1/2},$$

and let

$$\varphi_t = (1 + t\rho)^{-1} \tag{8.5.18}$$

be a regularizer, $\varphi_t \in S^{-1}$ for $t > 0$, which is uniformly bounded in S^0 for $t \in [0, 1]$ and satisfies $\varphi_t \to 1$ in S^ℓ for $\ell > 0$ as $t \to 0$. We define the regularized symbols to be $\breve{a}_t = \varphi_t \breve{a}$ and $a_t = \varphi_t^2 \rho^{2\sigma - (m-1)} \mathsf{a} = \breve{a}_t^2$.

We compute $\mathsf{H}_p \varphi_t = -t\varphi_t^2 \mathsf{H}_p \rho$. Amending (8.5.11) by another term which will be used to absorb certain terms later on, we aim to show that we can choose b_t, e_t and f_t such that, in analogy to (8.5.11), for $M > 0$ fixed, to be specified later,

$$\begin{aligned}
H_p a_t &= \varphi_t^2 \rho^{2\sigma} \left(\mathsf{H}_p \mathsf{a} + \left((2\sigma - m + 1) - 2t\varphi_t \rho\right)(\rho^{-1} \mathsf{H}_p \rho) \mathsf{a}\right) \\
&= -b_t^2 - M^2 \rho^{m-1} a_t + e_t - f_t,
\end{aligned}$$

that is to say,

$$\varphi_t^2\rho^{2\sigma}\left(\mathsf{H}_p\mathsf{a}+\left[\big((2\sigma-m+1)-2t\varphi_t\rho\big)(\rho^{-1}\mathsf{H}_p\rho)+M^2\right]\mathsf{a}\right)=-b_t^2+e_t-f_t. \tag{8.5.19}$$

Here, note that, using the definition of φ_t, $t\rho\varphi_t$ is a uniformly bounded family of symbols of order 0. To achieve (8.5.19), let us take

$$\begin{aligned}&e_t=\varphi_t^2\rho^{2\sigma}\mathsf{e}\\&f_t=f_t'+f_t'',\quad f_t'=\varphi_t^2\rho^{2\sigma}\mathsf{f}',\end{aligned} \tag{8.5.20}$$

where e,f' are given by (8.5.14); we will define f_t'' momentarily. Using $\chi_0(t)=t^2\chi_0'(t)$, we obtain

$$\begin{aligned}&\mathsf{H}_p\mathsf{a}+\left[\big((2\sigma-m+1)-2t\varphi_t\rho\big)(\rho^{-1}\mathsf{H}_p\rho)+M^2\right]\mathsf{a}\\&\quad=\mathsf{e}-\mathsf{f}'-(\digamma\delta)^{-1}\Bigg(\mathsf{H}_p\phi\\&\qquad-\left[\big((2\sigma-m+1)-2t\varphi_t\rho\big)(\rho^{-1}\mathsf{H}_p\rho)+M^2\right]\digamma^{-1}\delta\Big(2\beta-\frac{\phi}{\delta}\Big)^2\Bigg)\\&\qquad\qquad\times\chi_0'\left(\digamma^{-1}\left(2\beta-\frac{\phi}{\delta}\right)\right)\chi_1\left(\frac{\widetilde{\eta}+\delta}{\epsilon\delta}+1\right)\end{aligned}$$

Thus, if $\digamma$ is large enough, the term in the large parentheses is bounded from below by $3c_0/8$ on $\operatorname{supp}\mathsf{a}$, since $|2\beta-\phi/\delta|\le 4$ there. (The last statement follows from $-2\delta\le\widetilde{\eta}\le\phi\le 2\beta\delta\le 2\delta$ and $\beta\le 1$.) Therefore, we can put

$$\begin{aligned}b_t&=(\digamma\delta)^{-1/2}\varphi_t\rho^\sigma\Bigg((J\mathsf{H}_p)\phi\\&\qquad-\left[\big((2\sigma-m+1)-2t\varphi_t\rho\big)(\rho^{-1}(J\mathsf{H}_p)\rho)+M^2\right]\\&\qquad\qquad\times\digamma^{-1}\delta\Big(2\beta-\frac{\phi}{\delta}\Big)^2-\frac{c_0}{8}\Bigg)^{1/2}\\&\qquad\times\sqrt{\chi_0'\left(\digamma^{-1}\left(2\beta-\frac{\phi}{\delta}\right)\right)}\sqrt{\chi_1\left(\frac{\widetilde{\eta}+\delta}{\epsilon\delta}+1\right)},\\f_t''&=(\digamma\delta)^{-1}\varphi_t^2\rho^{2\sigma}\Bigg((\mathsf{H}_p-J\mathsf{H}_p)\phi\\&\qquad-\left[\big((2\sigma-m+1)-2t\varphi_t\rho\big)(\rho^{-1}(\mathsf{H}_p-J\mathsf{H}_p)\rho)\right]\end{aligned} \tag{8.5.21}$$

$$\times F^{-1}\delta\Big(2\beta - \frac{\phi}{\delta}\Big)^2 + \frac{c_0}{8}\Big)$$
$$\times \chi_0'\left(F^{-1}\left(2\beta - \frac{\phi}{\delta}\right)\right)\chi_1\left(\frac{\widetilde{\eta}+\delta}{\epsilon\delta}+1\right),$$

with $f_t'' \geq 0$ if the mollifier J is close enough to I, and thus obtain (8.5.19).

We now summarize this construction, slightly rephrased, retaining only the important properties of the constructed symbols. Let us fix any Riemannian metric on ${}^{\mathrm{b}}S^*M$ near ζ_0 and denote the metric ball around a point p with radius r in this metric by $B(p,r)$.

Proposition 8.5.8. *There exist $\delta_0 > 0$ and $C_0 > 0$ such that for $0 < \delta \leq \delta_0$, the following holds: For any $M > 0$, there exist a symbol $\check{a} \in S^{\sigma-(m-1)/2}$ and uniformly bounded families of symbols $\check{a}_t = \varphi_t\check{a} \in S^{\sigma-(m-1)/2}$ (with φ_t defined by (8.5.18)), $b_t \in S^{\sigma}$, $e_t \in S^{2\sigma}$ and $f_t \in S^{2\sigma;\infty}H_{\mathrm{b}}^{s-1}$, $f_t \geq 0$, supported in a coordinate neighborhood (independent of δ) of ζ_0 and supported away from ∂M, that satisfy the following properties:*

(1) $\check{a}_t H_p \check{a}_t = -b_t^2 - M^2\rho^{m-1}\check{a}_t^2 + e_t - f_t$.

(2) $b_t \to b_0$ in $S^{\sigma+\ell}$ for $\ell > 0$, and b_0 is elliptic at ζ_0.

(3) The support of e_t is contained in $B(\zeta_0 - \delta\mathsf{H}_p(\zeta_0), C_0\delta^2)$.

(4) For $t > 0$, the symbols have lower order: $\check{a}_t \in S^{\sigma-(m-1)/2-1}$, $b_t \in S^{\sigma-1}$, $e_t \in S^{2\sigma-2}$ and $f_t \in S^{2\sigma-2;\infty}H_{\mathrm{b}}^{s-1}$.

The commutant given by this proposition will now be used to deduce the propagation of regularity in a direction which agrees with the Hamilton flow to first order.

Let $\check{A} \in \Psi_{\mathrm{b}}^{\sigma-(m-1)/2}$ be a quantization of $\check{a}$ with $\mathrm{WF}_{\mathrm{b}}'(\check{A}) \subset \operatorname{supp}\check{a}$, let Φ_t be a quantization of φ_t, i.e. $\Phi_t \in \Psi_{\mathrm{b}}^0$ is a uniformly bounded family, $\Phi_t \in \Psi_{\mathrm{b}}^{-1}$ for $t > 0$, and let $\check{A}_t = \check{A}\Phi_t$. Moreover, let $B_t \in \Psi_{\mathrm{b}}^{\sigma}$ be a quantization of b_t, with uniform b-microsupport contained in a conic neighborhood of γ, such that $B_t \in \Psi_{\mathrm{b}}^{\sigma}$ is uniformly bounded, and $B_t \in \Psi_{\mathrm{b}}^{\sigma-1}$ for $t > 0$. Similarly, let $E_t \in \Psi_{\mathrm{b}}^{2\sigma}$ be a quantization of e_t with uniform b-microsupport disjoint from $\mathrm{WF}_{\mathrm{b}}^{\sigma}(u)$ in the sense that

$$\|E_t u\|_{H_{\mathrm{b}}^{\sigma}} \text{ is uniformly bounded for } t > 0. \tag{8.5.22}$$

This is the requirement that u is in H_{b}^{σ} on a part of the backwards bicharacteristic from ζ_0, more precisely in the ball specified in Proposition 8.5.8.

In a sense that we will make precise below, the principal symbol of the commutator $i\check{A}_t^*[P_m, \check{A}_t]$ is given by $\check{a}_t H_p \check{a}_t$, which is what we described in Proposition 8.5.8. We compute for $t > 0$, following the proof of [9, Theorem 3.2]:

$$\begin{aligned}\operatorname{Re}\langle i\check{A}_t^*[P_m, \check{A}_t]u, u\rangle &= \operatorname{Re}\big(\langle iP_m\check{A}_t u, \check{A}_t u\rangle - \langle i\check{A}_t P_m u, \check{A}_t u\rangle\big) \\ &= \frac{1}{2}\langle i(P_m - P_m^*)\check{A}_t u, \check{A}_t u\rangle - \operatorname{Re}\langle i\check{A}_t f, \check{A}_t u\rangle + \operatorname{Re}\langle i\check{A}_t P_{m-1} u, \check{A}_t u\rangle, \end{aligned} \tag{8.5.23}$$

where $\langle\cdot,\cdot\rangle$ denotes the sesquilinear pairing between spaces which are dual to each other relative to L^2_{b}. The adjoints here are taken with respect to the b-density $\frac{dx}{x}\,dy$, and in the case where P acts on a vector bundle, we use the smooth metric in the fibers of E for the adjoint. This computation needs to be justified, namely we must check that all pairings are well-defined by the a priori assumptions on u so that we can perform the integrations by parts.

First, we observe that

$$\check{A}_t^*\check{A}_t P_m u \in \check{A}_t^*\check{A}_t H_{\mathrm{b}}^s \cdot H_{\mathrm{b}}^{\sigma-m-1/2} \subset H_{\mathrm{b}}^{-\sigma+1/2},$$

because of $s \geq |\sigma - m - 1/2|$ and $\check{A}_t^*\check{A}_t \in \Psi_{\mathrm{b}}^{2\sigma-m-1}$. Since $(\sigma - 1/2) + (-\sigma + 1/2) = 0$ is non-negative, the pairing $\langle \check{A}_t^*\check{A}_t P_m u, u\rangle$ is well-defined. By the same token, the pairing $\langle \check{A}_t P_m u, \check{A}_t u\rangle$ is well-defined, hence we can integrate by parts, justifying half of the first equality in (8.5.23). For the second half of the first equality, we use $P_m \in H_{\mathrm{b}}^s\Psi_{\mathrm{b}}^m$ and[31] Corollary 8.2.10 to obtain

$$\begin{gathered} P_m\check{A}_t u \in P_m H_{\mathrm{b}}^{m/2} \subset H_{\mathrm{b}}^{-m/2}, \\ \check{A}_t^* P_m \check{A}_t u \in H_{\mathrm{b}}^{-\sigma+1/2}, \end{gathered}$$

which by the same reasoning as above proves the first equality in (8.5.23). For the second equality, we write P_m as a sum of terms of the form wQ_m with $w \in H_{\mathrm{b}}^s$, $Q_m \in \Psi_{\mathrm{b}}^m$, for which we have

$$\langle \check{A}_t u, wQ_m\check{A}_t u\rangle = \langle \bar{w}\check{A}_t u, Q_m\check{A}_t u\rangle = \langle Q_m^*\bar{w}\check{A}_t u, \check{A}_t u\rangle, \tag{8.5.24}$$

[31] This requires $s \geq m/2$; recall that we are assuming $m \geq 1$.

where the first equality follows from $\check{A}_t u \in H_{\mathrm{b}}^{m/2}$ and $Q_m \check{A}_t u \in H_{\mathrm{b}}^{-m/2}$,[32] and for the second equality, one observes that the two pairings on the right hand side in (8.5.24) are well-defined, and we can integrate by parts, i.e. move Q_m to the other side, taking its adjoint.

Now, since the principal symbol of P_m is real, we can apply Theorem 8.2.12 (3) with $k = 1, k' = 0$ to obtain $P_m - P_m^* \in \Psi_{\mathrm{b}}^{m-1} \circ \Psi^{0;0} H_{\mathrm{b}}^{s-1} + \Psi^{m-1;0} H_{\mathrm{b}}^{s-1}$. Therefore, Proposition 8.2.9 implies that $P_m - P_m^*$ defines a continuous map from $H_{\mathrm{b}}^{(m-1)/2}$ to $H_{\mathrm{b}}^{-(m-1)/2}$,[33] thus

$$|\langle (P_m - P_m^*)\check{A}_t u, \check{A}_t u\rangle| \le C_1 \|\check{A}_t u\|^2_{H_{\mathrm{b}}^{(m-1)/2}} \tag{8.5.25}$$

with a constant C_1 only depending on P_m.

Looking at the next term in (8.5.23), we estimate

$$|\langle \check{A}_t f, \check{A}_t u\rangle| \le \frac{1}{4}\|\check{A}_t f\|^2_{H_{\mathrm{b}}^{-(m-1)/2}} + \|\check{A}_t u\|^2_{H_{\mathrm{b}}^{(m-1)/2}} \le C_2 + \|\check{A}_t u\|^2_{H_{\mathrm{b}}^{(m-1)/2}},$$

where we use that

$$\check{A}_t f \in H_{\mathrm{b}}^{\sigma - m + 1 - \sigma + (m-1)/2} = H_{\mathrm{b}}^{-(m-1)/2}$$

uniformly.

For the last term on the right hand side of (8.5.23), the well-definedness is easily checked.[34] To bound it, we rewrite it as

$$\langle \check{A}_t P_{m-1} u, \check{A}_t u\rangle = \langle P_{m-1}\check{A}_t u, \check{A}_t u\rangle + \langle [\check{A}_t, P_{m-1}]u, \check{A}_t u\rangle.$$

The first term on the right hand side is bounded by $C_3\|\check{A}_t u\|^2_{H_{\mathrm{b}}^{(m-1)/2}}$ for some constant C_3 only depending on P_{m-1}; indeed, $P_{m-1} \colon H_{\mathrm{b}}^{(m-1)/2} \to H_{\mathrm{b}}^{-(m-1)/2}$ is continuous.[35] For the second term, note that $P_{m-1}\check{A}_t \in H_{\mathrm{b}}^{s-1}\Psi_{\mathrm{b}}^{\sigma+(m-1)/2}$ can be expanded to zeroth order, the first (and only) term being $p_{m-1}\check{a}_t$ and the remainder being $R_1' \in H_{\mathrm{b}}^{s-1}\Psi_{\mathrm{b}}^{\sigma+(m-1)/2-1}$; for notational convenience, we drop the explicit t-dependence here; inclusions are understood to be statements about a t-dependent family of operators being uniformly bounded in the respective space. Next, we can expand $\check{A}_t P_{m-1}$ to zeroth order by Theorem 8.2.12 (3)

[32] We need $s \ge m/2$ and can then use Corollary 8.2.10.

[33] Provided $s - 1 \ge (m-1)/2$ and $s - 1 > n/2 + (m-1)/2$.

[34] We need $s - 1 \ge |\sigma - m + 1/2|$ and can then use Corollary 8.2.10 to obtain $P_{m-1}u \in H_{\mathrm{b}}^{\sigma - m + 1/2}$.

[35] This requires $s - 1 \ge (m-1)/2$ and $s - 1 > n/2$.

with[36] $k' = 0$ – again obtaining $p_{m-1}\check{a}_t$ as the first term – which yields a remainder term $R_1'' + R_2$, where

$$\begin{aligned} R_1'' &\in \Psi^{\sigma+(m-1)/2-1;0} H_{\mathrm{b}}^{s-2} \\ R_2 &\in \Psi_{\mathrm{b}}^{\sigma-(m-1)/2-1} \circ \Psi^{m-1;0} H_{\mathrm{b}}^{s-2}. \end{aligned} \tag{8.5.26}$$

We can then use Proposition 8.2.9 to conclude that

$$R_1 := R_1'' - R_1' \in \Psi^{\sigma+(m-1)/2-1;0} H_{\mathrm{b}}^{s-2}$$

is a uniformly bounded family of maps[37]

$$R_1 \colon H_{\mathrm{b}}^{\sigma-1/2} \to H_{\mathrm{b}}^{-m/2+1}.$$

which shows that $\langle R_1 u, \check{A}_t u\rangle$ is uniformly bounded. Moreover, we can apply Proposition 8.2.9 and use the mapping properties of smooth b-ps.d.o.s to prove that $R_2 u \in H_{\mathrm{b}}^{-(m-1)/2}$ is uniformly bounded.[38] We thus conclude that

$$|\langle [\check{A}_t, P_{m-1}]u, \check{A}_t u\rangle| \leq C_4(M) + \|\check{A}_t u\|^2_{H_{\mathrm{b}}^{(m-1)/2}}, \tag{8.5.27}$$

where C_4, while it depends on M in the sense that it depends on a seminorm of the M-dependent operator $\check{A}$ constructed in Proposition 8.5.8, is independent of t.

Plugging all these estimates into (8.5.23), we thus obtain

$$\mathrm{Re}\langle i\check{A}_t^*[P_m, \check{A}_t]u, u\rangle \geq -(C_2 + C_4(M)) - (C_1 + 1 + C_3 + 1)\|\check{A}_t u\|^2_{H_{\mathrm{b}}^{(m-1)/2}},$$

where all constants are independent of $t > 0$, and C_1, C_2, C_3 are in addition independent of the real number M in Proposition 8.5.8. Choosing $M^2 > C_1 + C_3 + 2$, this implies that there is a constant $C < \infty$ such that for all $t > 0$, we have

$$\mathrm{Re}\Big\langle \Big(i\check{A}_t^*[P_m, \check{A}_t] + M^2(\Lambda\check{A}_t)^*(\Lambda\check{A}_t)\Big)u, u\Big\rangle \geq -C, \tag{8.5.28}$$

[36] Assuming $\sigma - (m-1)/2 \geq 1$.

[37] The requirements are $s - 2 \geq -m/2 + 1$, $s - 2 > n/2 + (m/2 - 1)_+$.

[38] Indeed, we have $u \in H_{\mathrm{b}}^{\sigma-1/2} \subset H_{\mathrm{b}}^{\sigma-1}$, and $\Psi^{m-1;0}H_{\mathrm{b}}^{s-2} \colon H_{\mathrm{b}}^{\sigma-1} \to H_{\mathrm{b}}^{\sigma-m}$ is continuous if $s - 2 \geq \sigma - m$, $s - 2 > n/2 + (m-\sigma)_+$.

where $\Lambda := \Lambda_{(m-1)/2}$. Therefore,

$$\mathrm{Re}\left\langle\left(i\check{A}_t^*[P_m,\check{A}_t]+B_t^*B_t+M^2(\Lambda\check{A}_t)^*(\Lambda\check{A}_t)-E_t\right)u,u\right\rangle\geq -C+\|B_tu\|^2_{L^2_{\mathrm{b}}}. \tag{8.5.29}$$

Here, we use that $\langle E_tu,u\rangle$ is uniformly bounded by (8.5.22).

The next step is to exploit the commutator relation in Proposition 8.5.8 in order to find a t-independent upper bound for the left hand side of (8.5.29). Theorem 8.2.12 (3), gives[39]

$$i[P_m,\check{A}_t]=(H_p\check{a}_t)(z,{}^{\mathrm{b}}D)+\widetilde{R}_1+\widetilde{R}_2$$

with uniformly bounded families of operators

$$\widetilde{R}_1\in\Psi^{\sigma+(m-1)/2-1;0}H^{s-2}_{\mathrm{b}}$$
$$\widetilde{R}_2\in\Psi_{\mathrm{b}}^{\sigma-(m-1)/2-2}\circ\Psi^{m;0}H^{s-2}_{\mathrm{b}}.$$

Notice that $H_p\check{a}_t\in H^{s-1}_{\mathrm{b}}S^{\sigma+(m-1)/2}$ uniformly. If we applied Theorem 8.2.12 (3) directly to the composition $\check{A}_t^*(H_p\check{a}_t)(z,{}^{\mathrm{b}}D)$, the regularity of the remainder operator, say R, obtained by applying Theorem 8.2.12 (3), would be too weak in the sense that we could not bound $\langle Ru,u\rangle$. To get around this difficulty, choose

$$J^+\in\Psi_{\mathrm{b}}^{\sigma-(m-1)/2-1},\quad J^-\in\Psi_{\mathrm{b}}^{-\sigma+(m-1)/2+1}$$

with real principal symbols j^+,j^- such that

$$J^+J^-=I+\widetilde{R},\quad \widetilde{R}\in\Psi_{\mathrm{b}}^{-\infty}. \tag{8.5.30}$$

Observe that $J^-\check{A}_t^*$ is uniformly bounded in Ψ^1_{b}. Then by Theorem 8.2.12 (3),

$$iJ^-\check{A}_t^*[P_m,\check{A}_t]=(j^-\check{a}_tH_p\check{a}_t)(z,{}^{\mathrm{b}}D)+R_1+R_2+R_3+R_4, \tag{8.5.31}$$

[39]Applicable with $k=2,k'=0$ if $\sigma-(m-1)/2\geq 2$.

where

$$\begin{aligned} R_1 = J^- \check{A}_t^* \widetilde{R}_1 &\in \Psi_{\mathrm{b}}^1 \circ \Psi^{\sigma+(m-1)/2-1;0} H_{\mathrm{b}}^{s-2} \\ R_2 = J^- \check{A}_t^* \widetilde{R}_2 &\in \Psi_{\mathrm{b}}^{\sigma-(m-1)/2-1} \circ \Psi^{m;0} H_{\mathrm{b}}^{s-2} \\ R_3 &\in \Psi_{\mathrm{b}}^{\sigma+(m-1)/2;0} H_{\mathrm{b}}^{s-2} \\ R_4 &\in \Psi_{\mathrm{b}}^0 \circ \Psi^{\sigma+(m-1)/2;0} H_{\mathrm{b}}^{s-2}. \end{aligned} \tag{8.5.32}$$

Applying Proposition 8.2.9,[40] we conclude that R_j $(1 \leq j \leq 4)$ is a uniformly bounded family of operators

$$H_{\mathrm{b}}^{\sigma-1/2} \to H_{\mathrm{b}}^{-m/2},$$

thus, since $(J^+)^* \in H_{\mathrm{b}}^{m/2}$, the pairings $\langle R_j u, (J^+)^* u\rangle$ are uniformly bounded.

Hence, Proposition 8.5.8 implies

$$\begin{aligned} &J^+\Big(iJ^- \check{A}_t^*[P_m, \check{A}_t] + J^- B_t^* B_t + J^- M^2 (\Lambda \check{A}_t)^* (\Lambda \check{A}_t) - J^- E_t\Big) \\ &= J^+\Big([j^-(\check{a}_t H_p \check{a}_t + b_t^2 + M^2 \rho^{m-1} \check{a}_t^2 - e_t)](z, {}^{\mathrm{b}}D) + R + G\Big) \\ &= J^+\big((-j^- f_t)(z, {}^{\mathrm{b}}D) + R + G\big), \end{aligned} \tag{8.5.33}$$

where $R = R_1 + R_2 + R_3 + R_4$ and $G \in \Psi_{\mathrm{b}}^{\sigma+(m-1)/2}$; G appears because the principal symbols of the smooth operators on both sides are equal. We already proved that $\langle J^+ R u, u\rangle$ is uniformly bounded; also, $\langle J^+ G u, u\rangle$ is uniformly bounded, since $J^+ G \in \Psi_{\mathrm{b}}^{2\sigma-1}$ and $u \in H_{\mathrm{b}}^{\sigma-1/2}$.

It remains to prove a uniform lower bound on[41]

$$\mathrm{Re}\langle J^+(j^- f_t)(z, {}^{\mathrm{b}}D)u, u\rangle = \mathrm{Re}\langle (j^- f_t)(z, {}^{\mathrm{b}}D)u, (J^+)^* u\rangle.$$

In order to be able to apply the sharp Gårding inequality, Proposition 8.5.1, we need to rewrite this. Since j^+ is bounded away from 0, we can write

$$(j^- f_t)(z, {}^{\mathrm{b}}D) = \left[\frac{j^- f_t}{j^+}\right](z, {}^{\mathrm{b}}D) \circ (J^+)^* + R, \quad R \in \Psi^{\sigma+(m-1)/2;0} H_{\mathrm{b}}^{s-1}$$

[40] The conditions $s - 2 \geq -m/2 + 1$ and $s - 2 > n/2 + m/2$ are sufficient to treat R_1, R_3 and R_4. For R_2, we need $s - 2 \geq \sigma - m - 1/2$ and $s - 2 > n/2 + (m + 1/2 - \sigma)_+$.

[41] To justify the integration by parts here, note that $j^- f_t \in S^{\sigma+(m-1)/2-1;\infty} H_{\mathrm{b}}^{s-1}$ for $t > 0$, thus $(j^- f_t)(z, {}^{\mathrm{b}}D)u \in H_{\mathrm{b}}^{-m/2+1}$ provided $s - 1 \geq -m/2 + 1$, $s - 1 > n/2 + (m/2 - 1)_+$, which follows from the conditions in Footnote 40.

by Theorem 8.2.12 (2b), since $j^-f_t/j^+ \in S^{m+1;\infty}H_{\mathrm{b}}^{s-1}$. Now $\langle Ru, (J^+)^*u\rangle$ is uniformly bounded, since $(J^+)^*u \in H_{\mathrm{b}}^{m/2}$ and $Ru \in H_{\mathrm{b}}^{-m/2}$ are uniformly bounded.[42] We can now apply the sharp Gårding inequality to deduce that

$$\mathrm{Re}\left\langle \left[\frac{j^-f_t}{j^+}\right](z,{}^{\mathrm{b}}D)(J^+)^*u,(J^+)^*u\right\rangle \geq -C\|(J^+)^*u\|^2_{H_{\mathrm{b}}^{m/2}} \geq -C, \tag{8.5.34}$$

where the constant C only depends on the uniform $S^{2\sigma;\infty}H_{\mathrm{b}}^{s-1}$-bounds on f_t and the $H_{\mathrm{b}}^{\sigma-1/2}$-norm of u.[43]

Putting (8.5.29), (8.5.33) and (8.5.34) together by inserting $I = J^+J^- - \widetilde{R}$ in front of the large parenthesis in (8.5.29) and observing that the error term

$$\mathrm{Re}\left\langle \widetilde{R}\Big(i\check{A}_t^*[P_m,\check{A}_t] + B_t^*B_t + M^2(\Lambda\check{A}_t)^*(\Lambda\check{A}_t) - E_t\Big)u, u\right\rangle$$

is uniformly bounded,[44] we deduce that $\|B_tu\|_{L^2_{\mathrm{b}}}$ is uniformly bounded for $t > 0$. Therefore, a subsequence $B_{t_k}u$, $t_k \to 0$, converges weakly to $v \in L^2_{\mathrm{b}}$ as $k \to \infty$. On the other hand, $B_{t_k}u \to Bu$ in $H_{\mathrm{b}}^{-\infty}$; hence $Bu = v \in L^2_{\mathrm{b}}$, which implies that $u \in H_{\mathrm{b}}^{\sigma}$ microlocally on the elliptic set of B.

To eliminate the assumption that $m \geq 1$, notice that the above propagation estimate for a general m-th order operator can be deduced from the m_0-th order result for any $m_0 \geq 1$, simply by considering

$$P\Lambda^+(\Lambda^-u) = f + PRu,$$

where $\Lambda^+ \in \Psi_{\mathrm{b}}^{-(m-m_0)}$ is elliptic with parametrix $\Lambda^- \in \Psi_{\mathrm{b}}^{m-m_0}$, and $\Lambda^+\Lambda^- = I + R$, $R \in \Psi_{\mathrm{b}}^{-\infty}$. If we pass from P to $P\Lambda^+$, which means passing from m to m_0, we correspondingly have to pass from σ to $\sigma_0 = \sigma - m + m_0$ in equation (8.5.10); in other words, the difference $\sigma - m = \sigma_0 - m_0$ remains the same. Thus, let us collect the conditions on s and $\widetilde{s} = \sigma - m + 1$ as given in the footnotes in the course of the argument: All conditions are satisfied provided

$$3/2 - s \leq \widetilde{s} \leq s - 1, \quad \widetilde{s} \geq (5 - m_0)/2, \tag{8.5.35}$$

[42] For Ru, we need $s - 1 > n/2 + m/2$, which follows from the conditions in Footnote 40.

[43] This requires $s - 1 \geq 2 - m/2$ and $s - 1 > n/2 + 2 + m/2$.

[44] Indeed, $\check{A}_t^*\check{A}_tP_mu \in H_{\mathrm{b}}^{-\sigma-3/2}$ is uniformly bounded because of $s \geq |\sigma - m - 1/2|$; and $\check{A}_tu \in H_{\mathrm{b}}^{m/2-1}$ is uniformly bounded, hence so is $P_m\check{A}_tu \in H_{\mathrm{b}}^{-m/2-1}$ in view of $s \geq m/2+1$, which follows from the condition in Footnote 35, and therefore $\check{A}_t^*P_m\check{A}_tu \in H_{\mathrm{b}}^{-\sigma-3/2}$ is uniformly bounded.

$$s > n/2 + 2 + (3/2 - \widetilde{s})_+, \quad s > n/2 + 3 + m_0/2 \tag{8.5.36}$$

for some $m_0 \geq 1$. The optimal choice for m_0 is thus $m_0 = \max(1, 5 - 2\widetilde{s}) = 1 + 2(2 - \widetilde{s})_+$; plugging this in, we obtain the conditions in the statement of Theorem 8.5.6:

$$s > n/2 + 7/2 + (2 - \widetilde{s})_+, \quad \widetilde{s} \leq s - 1.$$

Thus, we have proved a propagation result which propagates estimates in a direction which is 'correct to first order.' To obtain the final form of the propagation result, we use an argument by Melrose and Sjöstrand [89, 90], in the form given in [33, Lemma 8.1]. This finishes the proof of the first part of Theorem 8.5.6.

Remark 8.5.9. For second order real principal type operators of the form considered above, with the highest order derivative having H_{b}^s-coefficients, the maximal regularity one can prove for a solution u with right hand side $f \in H_{\mathrm{b}}^{s-1}$ is $H_{\mathrm{b}}^{\widetilde{s}+1}$ with $\widetilde{s}$ being at most $s-1$, i.e. one can prove $u \in H_{\mathrm{b}}^s$, which is exactly what we will need in our quest to solve quasilinear wave equations.

Propagation near the boundary

We now aim to prove the corresponding propagation result (near and) within the boundary ∂M: Thus, let $P = P_0 + \widetilde{P}$, where $\widetilde{P} = \widetilde{P}_m + \widetilde{P}_{m-1} + \widetilde{R}$, with $P_0 \in \Psi_{\mathrm{b}}^m$ and $\widetilde{P}_m \in H_{\mathrm{b}}^{s,\alpha}\Psi_{\mathrm{b}}^m$ having real homogeneous principal symbols, $\widetilde{P}_{m-1} \in H_{\mathrm{b}}^{s-1,\alpha}\Psi_{\mathrm{b}}^{m-1}$ and $\widetilde{R} \in \Psi_{\mathrm{b}}^{m-2;0}H_{\mathrm{b}}^{s-1,\alpha}$ as before, and let us assume that we are given a solution

$$u \in H_{\mathrm{b}}^{\sigma-1/2}$$

to the equation

$$Pu = f \in H_{\mathrm{b}}^{\sigma-m+1},$$

where $\sigma = \widetilde{s} + m - 1$. In fact, since

$$\widetilde{R} \colon H_{\mathrm{b}}^{\sigma-1/2} \subset H_{\mathrm{b}}^{\sigma-1} \to H_{\mathrm{b}}^{\sigma-m+1},$$

we may absorb the term $\widetilde{R}u$ into the right hand side; thus, we can assume $\widetilde{R} = 0$, hence $\widetilde{P} = \widetilde{P}_m + \widetilde{P}_{m-1}$.

Moreover, let γ be a null-bicharacteristic of $p = p_0 + \widetilde{p}_m$; we assume H_p is never radial on γ. Since $H_{\widetilde{p}_m} = 0$ at ${}^{\mathrm{b}}T^*_{\partial M}M$, this in particular implies that H_{p_0} is not radial on $\gamma \cap {}^{\mathrm{b}}T^*_{\partial M}M$, and the positivity of the principal symbol $\check{a}_t H_p \check{a}_t$ of the commutator there comes from the positivity of $\check{a}_t H_{p_0} \check{a}_t$.

The proof of the interior propagation, with small adaptations, carries over to the new setting. We indicate the changes: First, using the same notation, $\mathsf{H}_p \sigma_j$ now only is Hölder continuous with exponent α, thus (8.5.16) becomes

$$|\mathsf{H}_p \omega| \leq C\omega^{1/2}\left(|\widetilde{\eta}| + \omega^{1/2}\right)^{\alpha}.$$

Hence, for (8.5.15) to hold, we need

$$C\omega^{1/2}(|\widetilde{\eta}| + \omega^{1/2})^{\alpha} \leq c_0 \epsilon^2 \delta/2,$$

which holds if $2^{1+2\alpha} C \delta^{1+\alpha} \leq c_0 \epsilon \delta/2$, suggesting the choice $\epsilon = 4^{1+\alpha} C \delta^{\alpha}/c_0$; in particular $\epsilon \leq 1$ for δ small enough. Thus, the size of the a priori control region near $\widetilde{\eta} = -\delta$, cf. (8.5.17), becomes

$$\omega^{1/2} \leq \frac{c_0 \epsilon^2 \delta}{2C(|\widetilde{\eta}| + \omega^{1/2})^{\alpha}} = C_\alpha \delta^{1+\alpha},$$

which is small enough for the argument in [33, Lemma 8.1] to work. Further, defining the commutant a as before, we replace the a priori control terms e, f' in (8.5.14) by

$$\begin{aligned} \mathsf{e} &= \chi_0(\mathsf{H}_{p_0} + J\mathsf{H}_{\widetilde{p}_m})\chi_1 + \widetilde{\chi}_1, \\ \mathsf{f}' &= \chi_0(J\mathsf{H}_{\widetilde{p}_m} - \mathsf{H}_{\widetilde{p}_m})\chi_1 + \widetilde{\chi}_1, \end{aligned} \tag{8.5.37}$$

where we choose the mollifier J to be so close to I that $\mathsf{f}' \geq 0$; here, we use that the first summand in the definition of f' is an element of H^{s-1}_{b} in the base variables, hence for $s > n/2 + 1$ in particular continuous and vanishing at the boundary ∂M, and can therefore be dominated by $\widetilde{\chi}_1$. We then let e_t and f'_t be defined as in (8.5.20) with the above e and f'. We change the terms b_t and f''_t in (8.5.21) in a similar way: We take

$$\begin{aligned} b_t = (\digamma\delta)^{-1/2}\varphi_t \rho^{\sigma}\Big(&(\mathsf{H}_{p_0} + J\mathsf{H}_{\widetilde{p}_m})\phi \\ &- \big[\big((2\sigma - m + 1) - 2t\varphi_t\rho\big)(\rho^{-1}(J\mathsf{H}_{\widetilde{p}_m} + \mathsf{H}_{p_0})\rho) + M^2\big] \end{aligned}$$

$$\times F^{-1}\delta\Big(2\beta-\frac{\phi}{\delta}\Big)^2-\frac{c_0}{8}\Big)^{1/2}$$

$$\times\sqrt{\chi_0'\left(F^{-1}\left(2\beta-\frac{\phi}{\delta}\right)\right)}\sqrt{\chi_1\left(\frac{\widetilde{\eta}+\delta}{\epsilon\delta}+1\right)},$$

$$f_t''=(F\delta)^{-1}\varphi_t^2\rho^{2\sigma}\Bigg((\mathsf{H}_{\widetilde{p}_m}-J\mathsf{H}_{\widetilde{p}_m})\phi$$

$$-\big[\big((2\sigma-m+1)-2t\varphi_t\rho\big)(\rho^{-1}(\mathsf{H}_{\widetilde{p}_m}-J\mathsf{H}_{\widetilde{p}_m})\rho)\big]\,F^{-1}\delta\Big(2\beta-\frac{\phi}{\delta}\Big)^2+\frac{c_0}{8}\Bigg)$$

$$\times\chi_0'\left(F^{-1}\left(2\beta-\frac{\phi}{\delta}\right)\right)\chi_1\left(\frac{\widetilde{\eta}+\delta}{\epsilon\delta}+1\right).$$

As before, we can control the term $\langle E_t u,u\rangle$ in (8.5.29) by the a priori assumptions on u. The new feature here is that $f_t',f_t''\geq 0$ are not just symbols with coefficients having regularity H_{b}^{s-1}, but there are additional smooth terms involving $\widetilde{\chi}_1$ and $c_0/8$. Thus, we need to appeal to the version of the sharp Gårding inequality given in Corollary 8.5.3 to obtain a uniform lower bound on the term $\langle J^+(j^-f_t)(z,{}^{\mathrm{b}}D)u,u\rangle$ in (8.5.33).

Since the computation of compositions and commutators in the proof of the previous section for P_0 is standard as P_0 is a smooth b-ps.d.o., and since $\widetilde{P}_m$ and $\widetilde{P}_{m-1}$ lie in the same spaces as the operators called P_m and P_{m-1} there, all arguments now go through after straightforward changes that take care of the smooth b-ps.d.o. P_0.

This finishes the proof of Theorem 8.5.6.

Complex absorption

We next aim to prove Theorem 8.5.7, namely we add a complex absorbing potential $Q=q(z,{}^{\mathrm{b}}D)\in\Psi_{\mathrm{b}}^m$ with $Q=Q^*$ and prove the propagation of H_{b}^{σ}-regularity of solutions $u\in H_{\mathrm{b}}^{\sigma-1/2}$ to the equation

$$(P-iQ)u=f\in H_{\mathrm{b}}^{\sigma-m+1},$$

where γ is a null-bicharacteristic of P, in a direction which depends on the sign of q near γ. Namely, we can propagate H_{b}^{σ}-regularity *forward* along the flow of the Hamilton vector field H_{p_m} if $q\geq 0$ near γ, and *backward* along the flow if $q\leq 0$ near γ.

Let Γ be an open neighborhood of γ. It suffices to consider the case that $q\geq 0$ in Γ; recall that the proof of Theorem 8.5.6 showed the propagation *forward* along the flow, so the only step there that we have to change is the right hand side of equation (8.5.23), where we have an additional term in view of $P_m u = f - P_{m-1}u + iQu$, namely

$$-\operatorname{Re}\langle i\check{A}_t\, iQu, \check{A}_t u\rangle = \operatorname{Re}\langle \check{A}_t Qu, \check{A}_t u\rangle = \operatorname{Re}\langle Q\check{A}_t u, \check{A}_t u\rangle + \operatorname{Re}\langle \check{A}_t^*[\check{A}_t, Q]u, u\rangle.$$

The first term on the right is bounded from below by $-C_5\|\check{A}_t u\|_{H_{\mathrm{b}}^{(m-1)/2}}$ and will be absorbed as in (8.5.28), and the second term is bounded by the a priori microlocal $H_{\mathrm{b}}^{\sigma-1/2}$-regularity of u in Γ, since

$$\operatorname{Re}\langle \check{A}_t^*[\check{A}_t, Q]u, u\rangle = \frac{1}{2}\langle \widetilde{Q}_t u, u\rangle$$

with

$$\begin{aligned}\widetilde{Q}_t &= \check{A}_t^*[\check{A}_t, Q] + [Q, \check{A}_t^*]\check{A}_t \\ &= (\check{A}_t^* - \check{A}_t)[\check{A}_t, Q] + [\check{A}_t, [\check{A}_t, Q]] + [Q, \check{A}_t^* - \check{A}_t]\check{A}_t\end{aligned}$$

uniformly bounded in $\Psi_{\mathrm{b}}^{2\sigma-1}$ in view of the principal symbol of $\check{A}_t$ being real and the presence of double commutators.

This finishes the proof of Theorem 8.5.7.

8.5.4 Propagation near radial points

We will only consider the class of radial points which will be relevant in our applications, cf. §§5.2 and 5.3. We recall the setting from §3.3.1, explicitly including bundles this time: There, we considered an operator $P_0 \in \Psi_{\mathrm{b}}^m(M; E)$ with real, scalar, homogeneous principal symbol p on a compact manifold M with boundary $X = \partial M$ and boundary defining function x. Here, we take $M = \overline{\mathbb{R}^n_+}$, and write x for the boundary defining function, since we are only working in a local model here. The assumptions on p_0 are as follows:

(1) At $p_0 = 0$, $dp_0 \neq 0$, and at ${}^{\mathrm{b}}S^*_X M \cap p_0^{-1}(0)$, dp_0 and dx are linearly independent; hence $\Sigma = p_0^{-1}(0) \subset {}^{\mathrm{b}}S^*M$ is a smooth codimension 1 submanifold transversal to ${}^{\mathrm{b}}S^*_X M$.

(2) $L = L_+ \cup L_-$, where $L_\pm$ are smooth disjoint submanifolds of ${}^{\mathrm{b}}S^*_X M$, given by $L_\pm = \mathcal{L}_\pm \cap {}^{\mathrm{b}}S^*_X M$, where $\mathcal{L}_\pm$ are smooth disjoint submanifolds of Σ transversal to ${}^{\mathrm{b}}S^*_X M$, defined locally near ${}^{\mathrm{b}}S^*_X M$. Moreover, $\mathsf{H}_{p_0} = \rho^{1-m} H_{p_0}$ is tangent to $\mathcal{L}_\pm$, where, as before, $\rho = \langle\zeta\rangle$.

(3) There are functions $\beta_0, \widetilde{\beta} \in \mathcal{C}^\infty(L_\pm)$, $\beta_0, \widetilde{\beta} > 0$, such that

$$\rho \mathsf{H}_{p_0} \rho^{-1}|_{L_\pm} = \mp\beta_0, \qquad -x^{-1}\mathsf{H}_{p_0} x|_{L_\pm} = \mp\widetilde{\beta}\beta_0. \tag{8.5.38}$$

(4) For a homogeneous degree 0 quadratic defining function ρ_0 of $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_-$ within Σ,

$$\mp \mathsf{H}_{p_0}\rho_0 - \beta_1\rho_0 \geq 0 \text{ modulo terms that vanish cubically at } L_\pm, \tag{8.5.39}$$

where $\beta_1 \in \mathcal{C}^\infty(\Sigma)$, $\beta_1 > 0$ at $L_\pm$.

(5) The imaginary part of the subprincipal symbol is homogeneous, and equals

$$\sigma_{\mathrm{b},m-1}\left(\frac{1}{2i}(P_0 - P_0^*)\right) = \pm\widehat{\beta}\beta_0\rho^{m-1} \text{ at } L_\pm, \tag{8.5.40}$$

where $\widehat{\beta} \in \mathcal{C}^\infty(L_\pm; \pi^* \operatorname{End}(E))$, $\pi\colon L_\pm \to M$ being the projection to the base; note that $\widehat{\beta}$ is self-adjoint at every point.

Recall that these conditions imply that $L_\pm$ is a sink, resp. source, for the bicharacteristic flow within ${}^{\mathrm{b}}S^*_X M$, in the sense that nearby null-bicharacteristics tend to $L_\pm$ in the forward, resp. backward, direction; but at $L_\pm$ there is also an unstable, resp. stable, manifold, namely $\mathcal{L}_\pm$.

In the non-smooth setting, we will make the exact same assumptions on the 'smooth part' of the operator; the guiding principle is that non-smooth operators with coefficients in $H_{\mathrm{b}}^{s,\alpha}$, $\alpha > 0$, $s > n/2 + 1$, have symbols and associated Hamilton vector fields that vanish at the boundary in view of the Riemann-Lebesgue lemma, thus would not affect the above conditions anyway, with the exception of condition (4), which in the proof however is only used close to, but away from $L_\pm$, just as in the proof of Proposition 3.3.8, and the positivity of $\mp\mathsf{H}_{p_0}\rho_0$ there is preserved when one adds small non-smooth terms in $H_{\mathrm{b}}^{s,\alpha}$ to p_0. In order to be able to give a concise expression for the threshold regularity (determining whether one can propagate into or out of the boundary), let us define for a function $b \in \mathcal{C}^\infty(L_\pm, \pi^* \operatorname{End}(E))$ with values in self-adjoint endomorphisms of the fiber,

$$\inf_{L_\pm} b := \inf\{\lambda \in \mathbb{R} \colon b \geq \lambda I \text{ everywhere on } L_\pm\},$$

$$\sup_{L_\pm} b := \sup\{\lambda \in \mathbb{R} \colon b \leq \lambda I \text{ everywhere on } L_\pm\}.$$

We then have the following theorem:

Theorem 8.5.10. *Let $m, r, s, \widetilde{s} \in \mathbb{R}$, $\alpha > 0$. Let $P = P_0 + \widetilde{P}$, where $P_0 \in \Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+}; E)$ has a real, scalar, homogeneous principal symbol p_0, further $\widetilde{P} = \widetilde{P}_m + \widetilde{P}_{m-1} + \widetilde{R}$ with $\widetilde{P}_m \in H_{\mathrm{b}}^{s,\alpha}\Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+}; E)$ having a real, scalar, homogeneous principal symbol $\widetilde{p}_m$, moreover $\widetilde{P}_{m-1} \in H_{\mathrm{b}}^{s-1,\alpha}\Psi_{\mathrm{b}}^{m-1}(\overline{\mathbb{R}^n_+}; E)$ and $\widetilde{R} \in \Psi_{\mathrm{b}}^{m-2}(\overline{\mathbb{R}^n_+}; E) + \Psi_{\mathrm{b}}^{m-2;0}H_{\mathrm{b}}^{s-1,\alpha}(\overline{\mathbb{R}^n_+}; E)$. Suppose that the above conditions (1)-(5) hold for p_0. Finally, assume that s and $\widetilde{s}$ satisfy*

$$\widetilde{s} \leq s - 1, \quad s > n/2 + 7/2 + (2 - \widetilde{s})_+. \tag{8.5.41}$$

Suppose $u \in H_{\mathrm{b}}^{\widetilde{s}+m-3/2,r}(\overline{\mathbb{R}^n_+}; E)$ is such that $Pu \in H_{\mathrm{b}}^{\widetilde{s},r}(\overline{\mathbb{R}^n_+}; E)$.

(1) If $\widetilde{s} + (m-1)/2 - 1 + \inf_{L_\pm}(\widehat{\beta} - r\widetilde{\beta}) > 0$, let us assume that in a neighborhood of $L_\pm$, $\mathcal{L}_\pm \cap \{x > 0\}$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m-1,r}(u)$.

(2) If $\widetilde{s} + (m-1)/2 + \sup_{L_\pm}(\widehat{\beta} - r\widetilde{\beta}) < 0$, let us assume that a punctured neighborhood of $L_\pm$, with $L_\pm$ removed, in $\Sigma \cap {}^{\mathrm{b}}S^*_{\partial\overline{\mathbb{R}^n_+}}\overline{\mathbb{R}^n_+}$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m-1,r}(u)$.

Then in both cases, $L_\pm$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m-1,r}(u)$.

Adjoints are again taken with respect to the b-density $\frac{dx}{x}\,dy$ and the smooth metric on the vector bundle E. In fact, condition (8.5.40) is insensitive to changes both of the b-density and the metric on E by the radiality of H_{p_0} at $L_\pm$; see [114, Footnote 19] for details.

Remark 8.5.11. Since $\mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m-1,r}(u)$ is closed, we in fact have the conclusion that a neighborhood of $L_\pm$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m-1,r}(u)$. As in the real principal type setting (see equation (8.5.8) in particular), one can also rewrite the wavefront set statement as an estimate on the L^2_{b} norm of an operator of order $\widetilde{s} + m - 1$, elliptic at $L_\pm$, applied to u. In particular, we will see that it suffices to have only microlocal $H_{\mathrm{b}}^{\widetilde{s},r}$-membership of Pu near the part of the radial set that we propagate to/from, and local membership in $H_{\mathrm{b}}^{\widetilde{s}-1}$, which comes from a use of elliptic regularity (Theorem 8.4.1) in our argument.

Moreover, as before, see Remark 3.3.11, the theorem also holds for operators P which are perturbations of those for which it directly applies: Even though the dynamical assumptions (1)-(4) are not stable under perturbations, the estimates derived from these are. Here, perturbations are to be understood in the sense that P_0 may be perturbed within Ψ_{b}^m, and $\widetilde{P}_m, \widetilde{P}_{m-1}$ and $\widetilde{R}$ may be changed arbitrarily, with the estimate corresponding to the wavefront set statement of the theorem being locally uniform.

Proof of Theorem 8.5.10. We again drop the bundle E from the notation. The proof is an adaptation of the proof of Proposition 3.3.8 to our non-smooth setting. Since $\widetilde{R}u \in H_{\mathrm{b}}^{\widetilde{s}}$ by the a priori regularity on u, we can absorb $\widetilde{R}u$ into $f = Pu$ and thus assume $\widetilde{R} = 0$. Finally, let us assume $m \geq 1$ and $r = 0$ for now; these conditions will be eliminated at the end of the proof.

Define the regularizer $\varphi_t(\rho) = (1+t\rho)^{-1}$ for $t \geq 0$ as in the proof of Theorem 8.5.6, put $\mathsf{p}_0 = \rho^{-m}p_0$ and $\sigma = \widetilde{s} + m - 1$, and consider the commutant

$$a_t = \varphi_t(\rho)\psi(\rho_0)\psi_0(\mathsf{p}_0)\psi_1(x)\rho^{\sigma-(m-1)/2},$$

where $\psi, \psi_0, \psi_1 \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R})$ are equal to 1 near 0 and have derivatives which are ≤ 0 on $[0,\infty)$; we will be more specific about the supports of ψ, ψ_0, ψ_1 below. Let us also assume that $\sqrt{-\psi\psi'}$ and $\sqrt{-\psi_1\psi_1'}$ are smooth in a neighborhood of $[0,\infty)$. As usual, we put $\mathsf{H}_{\widetilde{p}_m} = \rho^{1-m}H_{\widetilde{p}_m}$. We then compute, using $\mathsf{H}_{\widetilde{p}_m}\varphi_t = -t\varphi_t^2\mathsf{H}_{\widetilde{p}_m}\rho$:

$$\begin{aligned} a_t H_{\widetilde{p}_m} a_t = \varphi_t^2\rho^{2\sigma}\psi\psi_0\psi_1\big((\sigma - (m-1)/2 - t\rho\varphi_t)(\rho^{-1}\mathsf{H}_{\widetilde{p}_m}\rho)\psi\psi_0\psi_1 \\ + (x^{-1}\mathsf{H}_{\widetilde{p}_m}x)x\psi\psi_0\psi_1' + (\mathsf{H}_{\widetilde{p}_m}\rho_0)\psi'\psi_0\psi_1 + \mathsf{H}_{\widetilde{p}_m}(\mathsf{p}_0)\psi\psi_0'\psi_1\Big), \end{aligned}$$

and to compute $a_t H_{p_0} a_t$, we can use (8.5.38) to simplify the resulting expression.

To motivate the next step, recall that the objective is to obtain an estimate similar to (8.5.28); however, since in our situation, the weight $\rho^{\sigma-(m-1)/2}$ can only give a limited amount of positivity at $L_\pm$, we need to absorb error terms, in particular the ones involving $P - P^*$, into the commutator $a_t H_{p_m} a_t$. Thus, consider

$$\begin{aligned} a_t H_{p_m} a_t \pm \rho^{m-1}a_t^2\beta_0\widehat{\beta} &= \pm\varphi_t^2\rho^{2\sigma}\psi\psi_0\psi_1 \\ &\times \Big(\big[\beta_0(\sigma - (m-1)/2 - t\rho\varphi_t + \widehat{\beta}) \\ &\qquad \pm (\sigma - (m-1)/2 - t\rho\varphi_t)(\rho^{-1}\mathsf{H}_{\widetilde{p}_m}\rho)\big]\psi\psi_0\psi_1 \\ &\quad + (\widetilde{\beta}\beta_0 \pm x^{-1}\mathsf{H}_{\widetilde{p}_m}x)x\psi\psi_0\psi_1' \pm (\mathsf{H}_{p_0}\rho_0 + \mathsf{H}_{\widetilde{p}_m}\rho_0)\psi'\psi_0\psi_1 \\ &\quad + (-m\beta_0\mathsf{p}_0 \pm \mathsf{H}_{\widetilde{p}_m}\mathsf{p}_0)\psi\psi_0'\psi_1\Big). \end{aligned}$$

Recall that $t\rho\varphi_t$ is a bounded family of symbols in S^0, and we in fact have $|t\rho\varphi_t| \leq 1$ for all t. We now proceed to prove the first case of the theorem. Let us make the following assumptions:

- On $\mathrm{supp}(\psi \circ \rho_0) \cap \mathrm{supp}(\psi_0 \circ \mathsf{p}_0) \cap \mathrm{supp}(\psi_1 \circ x)$:

$$\beta_0(\sigma - (m-1)/2 - 1 + \widehat{\beta}) \geq c_0 > 0 \tag{8.5.42}$$
$$|(\sigma - (m-1)/2 - t\rho\varphi_t)(\rho^{-1}\mathsf{H}_{\widetilde{p}_m}\rho)| \leq c_0/4 \text{ for all } t > 0.$$

  The first condition is satisfied at $L_\pm$ by assumption, and the second condition is satisfied close to $X = \{x = 0\}$, since $\rho^{-1}\mathsf{H}_{\widetilde{p}_m}\rho = o(1)$ as $x \to 0$ by Riemann-Lebesgue.

- On $\mathrm{supp}\, d(\psi_1 \circ x) \cap \mathrm{supp}(\psi \circ \rho_0) \cap \mathrm{supp}(\psi_0 \circ \mathsf{p}_0)$:

$$\widetilde{\beta}\beta_0 \geq c_1 > 0, \quad |x^{-1}\mathsf{H}_{\widetilde{p}_m}x| \leq c_1/2.$$

  The second condition is satisfied close to X, since $x^{-1}\mathsf{H}_{\widetilde{p}_m}x = o(1)$ as $x \to 0$.

- On $\mathrm{supp}\, d(\psi \circ \rho_0) \cap \mathrm{supp}(\psi_1 \circ x) \cap \mathrm{supp}(\psi_0 \circ \mathsf{p}_0)$:

$$\mp \mathsf{H}_{p_0}\rho_0 \geq \frac{\beta_1}{2}\rho_0 \geq c_2 > 0, \quad |\mathsf{H}_{\widetilde{p}_m}\rho_0| \leq c_2/2. \tag{8.5.43}$$

- On $\mathrm{supp}\, d(\psi_0 \circ \mathsf{p}_0) \cap \mathrm{supp}(\psi \circ \rho_0) \cap \mathrm{supp}(\psi_1 \circ x)$:

$$|\rho^{-m}p_m| \geq c_3 > 0. \tag{8.5.44}$$

  This can be arranged as follows: First, note that we can ensure

$$|\mathsf{p}_0| \geq 2c_3 \tag{8.5.45}$$

  there; then, since $|\rho^{-m}\widetilde{p}_m| = o(1)$ as $x \to 0$, shrinking the support of ψ_1 if necessary guarantees (8.5.44).

We can ensure that all these assumptions are satisfied by first choosing ψ_1, localizing near ${}^{\mathrm{b}}S^*_X M$, then ψ, localizing near $L_\pm$ within the characteristic set $(p_0)^{-1}(0)$ of P''_m, such that the inequalities in (8.5.42) and (8.5.43) are strict on $(p_0)^{-1}(0)$, then choosing ψ_0 (localizing near $(p_0)^{-1}(0)$) such that strict inequalities hold in (8.5.42), (8.5.43) and (8.5.45), and finally shrinking the support of ψ_1, if necessary, such that all inequalities hold.

We can then write

$$a_t H_{p_m} a_t \pm \rho^{m-1}a_t^2\beta_0\widehat{\beta} = \pm\left(\frac{c_0}{8}\rho^{m-1}a_t^2 + b_{1,t}^2 + b_{2,t}^2 - b_{3,t}^2 + f_t + g_t\right), \tag{8.5.46}$$

where, with a mollifier J as in Lemma 8.5.5,

$$\begin{aligned} b_{1,t} &= \varphi_t \rho^\sigma \psi\psi_0\psi_1 \Big[\beta_0(\sigma - (m-1)/2 - t\rho\varphi_t + \widehat{\beta}) \\ &\qquad\qquad \pm (\sigma - (m-1)/2 - t\rho\varphi_t)(\rho^{-1} J\mathsf{H}_{\widetilde{p}_m}\rho) - \frac{c_0}{2}\Big]^{1/2}, \\ b_{2,t} &= \varphi_t \rho^\sigma \psi_0\psi_1\sqrt{-\psi\psi'}\left[\mp\left(\mathsf{H}_{p_0}\rho_0 + J\mathsf{H}_{\widetilde{p}_m}\rho_0\right) - \frac{c_2}{4}\right]^{1/2}, \\ b_{3,t} &= \varphi_t \rho^\sigma \psi\psi_0\sqrt{-\psi_1\psi_1'}\left[\left(\widetilde{\beta}\beta_0 \pm x^{-1} J\mathsf{H}_{\widetilde{p}_m} x + \frac{c_1}{4}\right)x\right]^{1/2}, \\ g_t &= \varphi_t^2\rho^{2\sigma}\psi^2\psi_0\psi_0'\psi_1^2(-m\beta_0\mathsf{p}_0 \pm \mathsf{H}_{\widetilde{p}_m}\mathsf{p}_0), \end{aligned}$$

and $f_t = f_{1,t} + f_{2,t} + f_{3,t}$ with

$$\begin{aligned} f_{1,t} &= \varphi_t^2\rho^{2\sigma}\psi^2\psi_0^2\psi_1^2 \\ &\qquad \times \left[\pm(\sigma - (m-1)/2 - t\rho\varphi_t)\big(\rho^{-1}(\mathsf{H}_{\widetilde{p}_m} - J\mathsf{H}_{\widetilde{p}_m})\rho\big) + \frac{3c_0}{8}\right], \\ f_{2,t} &= \varphi_t^2\rho^{2\sigma}\psi\psi'\psi_0^2\psi_1^2\left(\pm\big(\mathsf{H}_{\widetilde{p}_m} - J\mathsf{H}_{\widetilde{p}_m}\big)\rho_0 - \frac{c_2}{4}\right), \\ f_{3,t} &= \varphi_t^2\rho^{2\sigma}\psi^2\psi_0^2\psi_1\psi_1'\left(\pm x^{-1}\big(\mathsf{H}_{\widetilde{p}_m} - J\mathsf{H}_{\widetilde{p}_m}\big)x - \frac{c_1}{4}\right)x. \end{aligned}$$

In particular, $b_{1,t}, b_{2,t} \in S^\sigma$, $b_{3,t} \in x^{1/2}S^\sigma$, $f_t \in S^{2\sigma;\infty}H_{\mathrm{b}}^{s-1} + S^{2\sigma}$, $g_t \in H_{\mathrm{b}}^{s-1}S^{2\sigma} + S^{2\sigma}$ uniformly, with the symbol orders one lower if $t > 0$ for $b_{j,t}$, $j = 1,2,3$, and two lower for f_t, g_t. The term $b_{1,t}^2$ will give rise to an operator which is elliptic at $L_\pm$. The term $b_{2,t}^2$ (which has the same, 'advantageous,' sign as $b_{1,t}$) can be discarded, and the term $-b_{3,t}^2$, with a 'disadvantageous' sign, will be bounded using the a priori regularity assumptions on u. An important point here is that the non-smooth symbol f_t is non-negative if we choose the mollifier J to be close enough to I; in fact, we then have $f_{j,t} \geq 0$ for $j = 1,2,3$. Lastly, we will be able to estimate the contribution of the term g_t using elliptic regularity, noting that its support is disjoint from the characteristic set $p_m^{-1}(0)$ of P_m.

Let $A_t \in \Psi_{\mathrm{b}}^{\sigma-(m-1)/2}$, $B_{1,t}, B_{2,t}, B_{3,t} \in \Psi_{\mathrm{b}}^\sigma$ denote quantizations with uniform b-wave front set contained in the support of the respective full symbols $a_t, b_{1,t}, b_{2,t}$ and $b_{3,t}$. Then we compute as in the proof of real principal type propagation (see equation (8.5.23) there), letting $P_m = P_0 + \widetilde{P}_m$:

$$\operatorname{Re}\langle iA_t^*[P_m, A_t]u, u\rangle = -\left\langle \frac{1}{2i}(P_m - P_m^*)A_t u, A_t u\right\rangle$$

$$- \operatorname{Re}\langle iA_t f, A_t u\rangle + \operatorname{Re}\langle iA_t \widetilde{P}_{m-1} u, A_t u\rangle$$

We split the first term on the right hand side into two pieces corresponding to the decomposition $P_m = P_0 + \widetilde{P}_m$. The piece involving P_0 will be dealt with later. For the other piece, note that $\widetilde{P}_m$ is a sum of terms of the form $\tau^\alpha w Q_m$, where $w \in H_{\mathrm{b}}^s$ is real-valued and $Q_m = q_m(z, {}^{\mathrm{b}}D) \in \Psi_{\mathrm{b}}^m$ has a real principal symbol. Now,

$$\begin{aligned} &\tau^\alpha w Q_m - (\tau^\alpha w Q_m)^* \\ &\quad = \tau^\alpha w (Q_m - Q_m^*) + \tau^\alpha (w Q_m^* - Q_m^* w) + \tau^\alpha (Q_m^* - \tau^{-\alpha} Q_m^* \tau^\alpha) w, \end{aligned}$$

thus, using Theorem 8.2.12 (3) with $k = 1, k' = 0$ (applicable because we are assuming $m \geq 1$) to compute $Q_m^* w$ and with $k = 0, k' = 0$ to compute the last term, we get

$$i(\widetilde{P}_m - \widetilde{P}_m^*) = R_1 + R_2 + R_3,$$

where

$$R_1 \in H_{\mathrm{b}}^{s-1,\alpha} \Psi_{\mathrm{b}}^{m-1}, R_2 \in \Psi_{\mathrm{b}}^{m-1} \circ \Psi^{0;0} H_{\mathrm{b}}^{s-1,\alpha}, R_3 \in \Psi^{m-1;0} H_{\mathrm{b}}^{s-1,\alpha}.$$

Let $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}_+)$, $\chi \equiv 1$ near 0. Writing R_1 as the sum of terms of the form $w'Q'$, where $w' \in H_{\mathrm{b}}^{s-1,\alpha}$ and $Q' \in \Psi_{\mathrm{b}}^{m-1}$, we have for $\epsilon' > 0$, which we can choose to be as small as we like provided we shrink the support of the Schwartz kernel of A_t:

$$\langle w'(z) Q' A_t u, A_t u\rangle = \langle \chi(x/\epsilon') w'(z) Q' A_t u, A_t u\rangle;$$

by Lemma 8.5.4, this can be bounded by $c_{\epsilon'} \|A_t u\|^2_{H_{\mathrm{b}}^{(m-1)/2}}$, where $c_{\epsilon'} \to 0$ as $\epsilon' \to 0$.[45] In a similar manner, we can treat the terms involving R_2 and R_3. Hence, under the assumption that the Schwartz kernel of A_t is localized sharply enough near $\partial M \times \partial M$, we have

$$|\langle (\widetilde{P}_m - \widetilde{P}_m^*) A_t u, A_t u\rangle| \leq C_\delta + \delta \|A_t u\|^2_{H_{\mathrm{b}}^{(m-1)/2}} \tag{8.5.47}$$

for an arbitrarily small, but fixed $\delta > 0$.

[45] This argument requires that elements of H_{b}^{s-1} are multipliers on $H_{\mathrm{b}}^{(m-1)/2}$, which is the case if $s - 1 \geq (m-1)/2$.

Next, for $\delta > 0$, we estimate

$$|\langle A_t f, A_t u\rangle| \leq C_\delta + \delta \|A_t u\|^2_{H_{\mathrm{b}}^{(m-1)/2}},$$

using that $\|A_t f\|_{H_{\mathrm{b}}^{-(m-1)/2}}$ is uniformly bounded.

Finally, we can bound the term $\langle A_t \widetilde{P}_{m-1} u, A_t u\rangle$ as in the proof of Theorem 8.5.6, thus obtaining

$$|\langle A_t \widetilde{P}_{m-1} u, A_t u\rangle| \leq C_\delta + \delta \|A_t u\|^2_{H_{\mathrm{b}}^{(m-1)/2}}.$$

Therefore, writing $Q := \frac{1}{2i}(P_0 - P_0^*) \in \Psi_{\mathrm{b}}^{m-1}$, we get

$$\pm \operatorname{Re}\langle (iA_t^*[P_m, A_t] + A_t^* Q A_t)u, u\rangle \leq C_\delta + \delta \|A_t u\|^2_{H_{\mathrm{b}}^{(m-1)/2}}$$

Now, using that $|\langle B_{t,3}^* B_{t,3} u, u\rangle| = \|B_{t,3} u\|^2_{L^2_{\mathrm{b}}}$ is uniformly bounded because of the assumed a priori control of u in a neighborhood of $L_\pm$ in $\mathcal{L}_\pm \cap \{x > 0\}$, we deduce, using the operator $\Lambda = \Lambda_{(m-1)/2}$:

$$\begin{aligned} \operatorname{Re}\Big\langle \Big(\pm i A_t^*[P_m, A_t] \pm A_t^* Q A_t - \frac{c_0}{8}(\Lambda A_t)^*(\Lambda A_t) \\ - B_{1,t}^* B_{1,t} - B_{2,t}^* B_{2,t} + B_{3,t}^* B_{3,t}\Big) u, u\Big\rangle \\ \leq C_\delta + \Big(\delta - \frac{c_0}{8}\Big) \|A_t u\|^2_{H_{\mathrm{b}}^{(m-1)/2}} - \|B_{1,t} u\|^2_{L^2_{\mathrm{b}}}, \end{aligned} \tag{8.5.48}$$

where we discarded the negative term $-\langle B_{2,t}^* B_{2,t} u, u\rangle$ on the right hand side. If we choose $\delta < c_0/8$, then we can also discard the term on the right hand side involving $A_t u$, hence

$$\begin{aligned} \|B_{1,t} u\|^2_{L^2_{\mathrm{b}}} \leq C - \operatorname{Re}\Big\langle \Big(\pm i A_t^*[P_m, A_t] \pm A_t^* Q A_t - \frac{c_0}{8}(\Lambda A_t)^*(\Lambda A_t) \\ - B_{1,t}^* B_{1,t} - B_{2,t}^* B_{2,t} + B_{3,t}^* B_{3,t}\Big) u, u\Big\rangle. \end{aligned} \tag{8.5.49}$$

We now exploit the commutator relation (8.5.46) in the same way as in the proof of Theorem 8.5.6: If we introduce operators

$$J^+ \in \Psi_{\mathrm{b}}^{\sigma-(m-1)/2-1}, \quad J^- \in \Psi_{\mathrm{b}}^{-\sigma+(m-1)/2+1}$$

with real principal symbols j^+, j^-, satisfying $J^+ J^- = I + \widetilde{R}$, $\widetilde{R} \in \Psi_{\mathrm{b}}^{-\infty}$, we obtain, keeping in mind (8.5.40),

$$\begin{aligned}
&\mathrm{Re}\Big\langle J^-\Big(\pm iA_t^*[P_m,A_t]\pm A_t^*QA_t-\frac{c_0}{8}(\Lambda A_t)^*(\Lambda A_t)\\
&\qquad\qquad -B_{1,t}^*B_{1,t}-B_{2,t}^*B_{2,t}+B_{3,t}^*B_{3,t}\Big)u,(J^+)^*u\Big\rangle\\
&\geq \mathrm{Re}\Big\langle\Big[j^-\Big(\pm a_tH_{p_m}a_t+\rho^{m-1}a_t^2\beta_0\widehat{\beta}-\frac{c_0}{8}\rho^{m-1}a_t^2\\
&\qquad\qquad -b_{1,t}^2-b_{2,t}^2+b_{3,t}^2\Big)\Big](z,{}^{\mathrm{b}}D)u,(J^+)^*u\Big\rangle-C\\
&=\mathrm{Re}\langle(j^-f_t)(z,{}^{\mathrm{b}}D)u,(J^+)^*u\rangle+\mathrm{Re}\langle(j^-g_t)(z,{}^{\mathrm{b}}D)u,(J^+)^*u\rangle-C,
\end{aligned}$$

where we absorbed various error terms in the constant C; see the discussion around equation (8.5.33) for details. The term involving f_t is uniformly bounded from below as explained in the proof of Theorem 8.5.6 after equation (8.5.33). It remains to bound the term involving g_t. Note that we can write $(j^-g_t)(z,\zeta)$ as a sum of terms of the form $w(z)\varphi_t(\zeta)^2s(z,\zeta)$, where $w\in H_{\mathrm{b}}^{s-1}$, or $w\in\mathcal{C}^\infty$, and $s\in S^{\sigma+(m-1)/2+1}$, and we can assume

$$({}^{\mathrm{b}}S^*M\cap\operatorname{supp}s)\cap p_m^{-1}(0)=\emptyset,$$

since this holds for g_t in place of s. Thus, on ${}^{\mathrm{b}}S^*M\cap\operatorname{supp}s$, we can use elliptic regularity, Theorem 8.4.1, to conclude that $\mathrm{WF}_{\mathrm{b}}^{\sigma+1}(u)\cap({}^{\mathrm{b}}S^*M\cap\operatorname{supp}s)=\emptyset$; but this implies that

$$(w\varphi_t^2s)(z,{}^{\mathrm{b}}D)u\in H_{\mathrm{b}}^{-(m-1)/2}$$

is uniformly bounded. Therefore, we finally obtain from (8.5.49) a uniform bound on $\|B_{1,t}u\|_{L_{\mathrm{b}}^2}$, which implies $B_{1,0}u\in L_{\mathrm{b}}^2$ and thus the claimed microlocal regularity of u at $L_\pm$, finishing the proof of the first part of the theorem in the case $m\geq 1$, $r=0$.

The proof of the second part is similar, only instead of requiring (8.5.42), we require

$$\beta_0(\sigma-(m-1)/2+\widehat{\beta})\leq -c_0<0$$

on $\operatorname{supp}(\psi\circ\rho_0)\cap\operatorname{supp}(\psi_0\circ\mathsf{p}_0)\cap\operatorname{supp}(\psi_1\circ x)$, and we correspondingly define

$$\begin{aligned}
b_{1,t}=\varphi_t\rho^\sigma\psi\psi_0\psi_1\Big[&-\beta_0(\sigma-(m-1)/2-t\rho\varphi_t+\widehat{\beta})\\
&\mp(\sigma-(m-1)/2-t\rho\varphi_t)(\rho^{-1}\mathsf{J}\mathsf{H}_{\widetilde{p}_m}\rho)-\frac{c_0}{2}\Big]^{1/2}.
\end{aligned}$$

We also redefine

$$
\begin{aligned}
b_{3,t} &= \varphi_t \rho^\sigma \psi \psi_0 \sqrt{-\psi_1 \psi_1'} \left[\left(\widetilde{\beta}\beta_0 \pm x^{-1} J \mathsf{H}_{\widetilde{p}_m} x - \frac{c_1}{4} \right) x \right]^{1/2}, \\
f_{1,t} &= \varphi_t^2 \rho^{2\sigma} \psi^2 \psi_0^2 \psi_1^2 \\
&\qquad \left[\mp(\sigma - (m-1)/2 - t\rho\varphi_t) \big(\rho^{-1} (\mathsf{H}_{\widetilde{p}_m} - J\mathsf{H}_{\widetilde{p}_m})\rho \big) + \frac{3c_0}{8} \right], \\
f_{3,t} &= \varphi_t^2 \rho^{2\sigma} \psi^2 \psi_0^2 \psi_1 \psi_1' \left(\mp x^{-1} \big(\mathsf{H}_{\widetilde{p}_m} - J\mathsf{H}_{\widetilde{p}_m} \big) x - \frac{c_1}{4} \right) x.
\end{aligned}
$$

Equation (8.5.46) then becomes

$$
a_t H_{p_m} a_t \pm \rho^{m-1} a_t^2 \beta_0 \widehat{\beta} = \mp \left(\frac{c_0}{8} \rho^{m-1} a_t^2 + b_{1,t}^2 - b_{2,t}^2 + b_{3,t}^2 + f_t + g_t \right),
$$

and the rest of the proof proceeds as before, the most important difference being that now the term $b_{3,t}^2$ has an advantageous sign (namely, the same as $b_{1,t}^2$), whereas $-b_{2,t}^2$ does not, which is the reason for the microlocal regularity assumption on u in a punctured neighborhood of $L_\pm$ within ${}^{\mathrm{b}}S^*_{\partial\overline{\mathbb{R}^n_+}} \overline{\mathbb{R}^n_+}$.

The last step in the proof is to remove the restrictions on m (the order of the operator) and r (the growth rate of u and f). We accomplish this by rewriting the equation $Pu = f$ (without restrictions on m and r) as

$$
(x^{-r} P \Lambda^+ x^r)(x^{-r} \Lambda^- u) = x^{-r} f + x^{-r} P R x^r (x^{-r} u),
$$

where $\Lambda^\pm \in \Psi_{\mathrm{b}}^{\mp(m-m_0)}$, $m_0 \geq 1$, have principal symbols $\rho^{\mp(m-m_0)}$ and satisfy $\Lambda^+\Lambda^- = I + R$, $R \in \Psi_{\mathrm{b}}^{-\infty}$. Then $x^{-r} P \Lambda^+ x^r$ has order m_0, and, recalling $\widetilde{s} = \sigma - m + 1$,

$$
x^{-r} f \in H_{\mathrm{b}}^{\widetilde{s}}, \quad x^{-r} \Lambda^- u \in H_{\mathrm{b}}^{\widetilde{s} + m_0 - 3/2}
$$

lie in unweighted b-Sobolev spaces. The principal symbol of $P_{0,r} := x^{-r} P_0 \Lambda^+ x^r$ is an elliptic multiple of the principal symbol of P_0, hence the Hamilton vector fields of $P_{0,r}$ and P_0 agree, up to a positive non-vanishing factor, on the characteristic set of P_0; in particular, even though β_0 in equation (8.5.38) may be different for $P_{0,r}$ than for P_0, $\widetilde{\beta}$ does not change, at least on $L_\pm$. However, the imaginary part of the subprincipal symbol, hence $\widehat{\beta}$, does change, resulting in a shift of the threshold values in the statement of the theorem: Concretely, we claim

$$\sigma_{\mathrm{b},m_0-1}\left(\frac{1}{2i}(P_{0,r}-P_{0,r}^*)\right)=\pm\rho^{m_0-1}\beta_0\left(\widehat{\beta}+\frac{m-m_0}{2}-r\widetilde{\beta}\right)\ \text{at}\ L_\pm. \tag{8.5.50}$$

Granted this, the threshold quantity is the sup, resp. inf, over $L_\pm$ of

$$\widetilde{s}+(m_0-1)/2+\widehat{\beta}+(m-m_0)/2-r\widetilde{\beta}=\widetilde{s}+(m-1)/2+\widehat{\beta}-r\widetilde{\beta}.$$

To prove (8.5.50), let us write $P_0=P'_m+P'_{m-1}$, where we can assume that Λ^+ and P'_m are (formally) self-adjoint by letting $P'_m=(P_0+P_0^*)/2$ and $P'_{m-1}=(P_0-P_0^*)/2$. We then compute

$$\begin{aligned}&\sigma_{\mathrm{b},m_0-1}\left(\frac{1}{2i}\big(x^{-r}P'_{m-1}\Lambda^+x^r-x^r\Lambda^+(P'_{m-1})^*x^{-r}\big)\right)\\&\qquad=\rho^{m_0-1}\rho^{1-m}\sigma_{\mathrm{b},m-1}\left(\frac{1}{2i}(P'_{m-1}-(P'_{m-1})^*)\right)=\pm\rho^{m_0-1}\beta_0\widehat{\beta},\end{aligned}$$

and

$$\begin{aligned}&\sigma_{\mathrm{b},m_0-1}\left(\frac{1}{2i}\big(x^{-r}P'_m\Lambda^+x^r-x^r\Lambda^+P'_mx^{-r}\big)\right)\\&=\sigma_{\mathrm{b},m_0-1}\left(\frac{1}{2i}[P'_m,\Lambda^+]\right)+\sigma_{\mathrm{b},m_0-1}\left(\frac{1}{2i}\big(x^{-r}[P'_m\Lambda^+,x^r]-x^r[\Lambda^+P'_m,x^{-r}]\big)\right)\\&=\pm\frac{m-m_0}{2}\beta_0\rho^{m_0-1}-rx^{-1}H_{p_0\rho^{m_0-m}}x\\&=\pm\left(\frac{m-m_0}{2}\beta_0-r\widetilde{\beta}\beta_0\right)\rho^{m_0-1}-rp_0x^{-1}H_{\rho^{m_0-m}}x.\end{aligned}$$

The last term on the right hand side involving p_0 vanishes at $L_\pm$, proving (8.5.50).

Lastly, the regularities needed for the proof to go through are that the conditions in (8.5.35) hold for some $m_0\geq 1$; thus, choosing $m_0=\max(1,5-2\widetilde{s})=1+2(2-\widetilde{s})_+$, we obtain the conditions (8.5.41). □

8.5.5 Normally hyperbolic trapping

We now extend the proof of non-trapping estimates on weighted b-Sobolev spaces at normally hyperbolically trapped sets given in Theorem 3.3.14, more specifically the estimates (3.3.23) and (3.3.24), to the non-smooth setting.

To set this up, let $P_0 \in \Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+})$ with

$$\frac{1}{2i}(P_0 - P_0^*) = E_1 \in \Psi_{\mathrm{b}}^{m-1}(\overline{\mathbb{R}^n_+}), \tag{8.5.51}$$

where the adjoint is taken with respect to a fixed smooth b-density; an example to keep in mind here and in what follows is $P_0 = \Box_g$ for a smooth Lorentzian b-metric g on $\overline{\mathbb{R}^n_+}$, considered a coordinate patch of Kerr-de Sitter space, in which case $E_1 = 0$, and the threshold weight in Theorem 8.5.12 below is $r = 0$. Let p_0 be the principal symbol of P_0. Let us use the coordinates $(z;\zeta) = (x, y; \lambda, \eta)$ on ${}^{\mathrm{b}}T^*\overline{\mathbb{R}^n_+}$ as usual and write $M = \overline{\mathbb{R}^n_+}$, $X = \partial\overline{\mathbb{R}^n_+}$. With $\Sigma \subset {}^{\mathrm{b}}S^*M$ denoting the characteristic set of P_0, we assume that P_0 has normally hyperbolic trapping in the b-sense at $\Gamma \subset \Sigma \cap {}^{\mathrm{b}}S^*_X M$, see Definition 2.3.1, with $\Gamma_+ \subset \Sigma \cap {}^{\mathrm{b}}S^*_X M$ denoting the unstable manifold and $\Gamma_- \subset \Sigma$ the stable manifold at Γ. Recall that $\Gamma \cap {}^{\mathrm{b}}T^*X = \emptyset$, so xD_x is elliptic near Γ; thus

$$\rho = \langle\lambda\rangle \text{ near } \Gamma$$

extends to the inverse of a boundary defining function of ${}^{\mathrm{b}}S^*M$ in ${}^{\mathrm{b}}\overline{T}^*M$. The rescaled Hamilton vector field

$$V = \rho^{-m+1} H_{p_0}$$

is tangent to $\Gamma_\pm$, and Γ_+ is backward trapped for the Hamilton flow, while Γ_- is forward trapped, and Γ is trapped. We recall the corresponding quantitative assumptions: Let $\phi_+ \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$ be a defining function of Γ_+ in ${}^{\mathrm{b}}S^*_X M$, and let $\phi_- \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$ be a defining function of Γ_-. Thus, Γ_+ is defined within ${}^{\mathrm{b}}S^*M$ by $x = 0, \phi_+ = 0$. Let

$$\widehat{p}_0 = \rho^{-m} p_0.$$

We then assume that ϕ_+ and ϕ_- satisfy

$$V\phi_+ = -c_+^2\phi_+ + \mu_+ x + \nu_+\widehat{p}_0, \quad V\phi_- = c_-^2\phi_- + \nu_-\widehat{p}_0, \tag{8.5.52}$$

with $c_\pm > 0$ smooth near Γ and $\mu_+, \nu_\pm$ smooth near Γ, consistent with the (in)stability of Γ_- (Γ_+); further, x satisfies

$$Vx = -c_\partial x, \quad c_\partial > 0, \tag{8.5.53}$$

which is consistent with the stability of Γ_-, and near Γ,

$$\rho^{-1}V\rho = c_f x \tag{8.5.54}$$

for some smooth c_f, which holds in view of our choice of ρ. These definitions are entirely analogous to (3.3.13), (3.3.15) and (3.3.19).

We now perturb P_0 by a non-smooth operator $\widetilde{P}$, that is, we consider the operator

$$P = P_0 + \widetilde{P}, \quad \widetilde{P} = \widetilde{P}_m + \widetilde{P}_{m-1} + \widetilde{R}, \tag{8.5.55}$$

where for some fixed $\alpha > 0$, we have $\widetilde{P}_{m-j} \in H_{\mathrm{b}}^{s-j,\alpha}\Psi_{\mathrm{b}}^{m-j}$, $j = 0,1$, and $\widetilde{R} \in \Psi_{\mathrm{b}}^{m-2;0}H_{\mathrm{b}}^{s-1,\alpha}$.

We then have the following tame non-trapping estimate at Γ:

Theorem 8.5.12. *Using the above notation and making the above assumptions, let $s, \widetilde{s} \in \mathbb{R}$ be such that*

$$\widetilde{s} \le s - 1, \quad s > n/2 + 7/2 + (2 - \widetilde{s})_+. \tag{8.5.56}$$

Suppose $u \in H_{\mathrm{b}}^{\widetilde{s}+m-3/2,r}(\overline{\mathbb{R}^n_+})$ is such that $Pu = f \in H_{\mathrm{b}}^{\widetilde{s},r}(\overline{\mathbb{R}^n_+})$.

Then for $r < -\sup_\Gamma \rho^{-m+1}\sigma_{\mathrm{b},m-1}(E_1)/c_\partial$ and for any neighborhood U of Γ, there exists a set $U_2 \subset U$ with $U_2 \cap \Gamma_+ = \emptyset$ such that if $\mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m-1,r}(u) \cap U_2 = \emptyset$, then in fact $\mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m-1,r}(u) \cap \Gamma = \emptyset$; thus, we can propagate microlocal regularity into Γ.

On the other hand, for $r > -\inf_\Gamma \rho^{-m+1}\sigma_{\mathrm{b},m-1}(E_1)/c_\partial$ and for a suitable set $U_2 \subset U$ with $U_2 \cap \Gamma_- = \emptyset$, we can again propagate regularity of u into Γ.

Proof. The main part of the argument, in particular the choice of the commutant, is a slight modification of the positive commutator argument used in §3.3.2; the handling of the non-smooth terms is a modification of the proof of the radial point estimate, Theorem 8.5.10. In particular, the positivity comes from differentiating the weight x^{-r} in the commutant. To avoid working in weighted b-Sobolev spaces for the non-smooth problem, we will conjugate P by x^{-r}, giving an advantageous (here meaning negative) contribution to the imaginary part of the subprincipal symbol of the conjugated operator near Γ.

Throughout this proof, we denote operators and their symbols by the corresponding capital and lower case letters, respectively.

Concretely, put $\sigma = \widetilde{s} + m - 1$, and define

$$u_r := x^{-r}u \in H_{\mathrm{b}}^{\sigma-1/2}, \quad f_r := x^{-r}f \in H_{\mathrm{b}}^{\sigma-m+1},$$

$$P_r := x^{-r} P x^r = P_{0,r} + \widetilde{P}_r, \quad P_{0,r} = x^{-r} P_0 x^r, \widetilde{P}_r = x^{-r} \widetilde{P} x^r,$$

where

$$\widetilde{P}_r = \widetilde{P}_{m,r} + \widetilde{P}_{m-1,r} + \widetilde{R}_r, \quad \widetilde{P}_{m-j,r} \in H_{\mathrm{b}}^{s-j,\alpha} \Psi_{\mathrm{b}}^{m-j}, \widetilde{R}_r \in \Psi_{\mathrm{b}}^{m-2;0} H_{\mathrm{b}}^{s-1,\alpha};$$

then $P_r u_r = f_r$, and we must show a non-trapping estimate for u_r on *unweighted* b-Sobolev spaces. A simple computation shows that

$$\frac{1}{2i}(P_{0,r} - P_{0,r}^*) - \left(\frac{1}{2i}(P_0 - P_0^*) - \mathrm{Op}(r x^{-1} H_{p_0} x) \right) \in \Psi_{\mathrm{b}}^{m-2};$$

but $x^{-1} H_{p_0} x = -\rho^{m-1} c_\partial$ with $c_\partial > 0$ near Γ by (8.5.53), hence, using (8.5.51),

$$\frac{1}{2i}(P_{0,r} - P_{0,r}^*) = E_1 + E_1' + B \tag{8.5.57}$$

with $B, E_1' \in \Psi_{\mathrm{b}}^{m-1}$, where B has principal symbol $b = r c_\partial \rho^{m-1}$ near Γ, and $\mathrm{WF}_{\mathrm{b}}'(E_1') \cap \Gamma = \emptyset$. Notice that by assumption on r, $B + E_1$ is elliptic on Γ.

We now turn to the positive commutator argument: Fix $0 < \beta < \min(1, \alpha)$ and define

$$\rho_+ = \phi_+^2 + x^\beta.$$

Let $\chi_0(t) = e^{-1/t}$ for $t > 0$ and $\chi_0(t) = 0$ for $t < 0$, further $\chi \in \mathcal{C}_{\mathrm{c}}^\infty([0, R))$ for $R > 0$ to be chosen below, $\chi \equiv 1$ near 0, $\chi' \leq 0$, and finally $\psi \in \mathcal{C}_{\mathrm{c}}^\infty((-R, R))$, $\psi \equiv 1$ near 0. Define for $\kappa > 0$, specified later,

$$a = \rho^{\sigma - (m-1)/2} \chi_0(\rho_+ - \phi_-^2 + \kappa) \chi(\rho_+) \psi(\widehat{p}_0).$$

On $\operatorname{supp} a$, we have $\rho_+ \leq R$, thus the argument of χ_0 is bounded above by $R + \kappa$. Moreover, $\phi_-^2 \leq R + \kappa$ and $x \leq R^{1/\beta}$, therefore a is supported in any given neighborhood of Γ if one chooses R and κ small. Notice that a is merely a *conormal* symbol which does not grow at the boundary, but we showed in §3.3.5 that we have a full symbolic calculus for such symbols as well. We also remark that the proofs of composition results of smooth and non-smooth b-ps.d.o.s presented in §8.2 go through without changes if one uses b-ps.d.o.s with non-growing conormal, instead of smooth, symbols.[46]

[46] A somewhat more direct way of dealing with this issue goes as follows: Assume, as one may, that

Define the regularizer $\varphi_\delta(\zeta) = (1+\delta\rho)^{-1}$ near Γ, and put $a_\delta = \varphi_\delta a$. Put $\widetilde{V} = \rho^{-m+1}H_{\widetilde{p}_{m,r}}$ and define $\widetilde{c}_\partial, \widetilde{c}_f \in H_{\mathrm{b}}^{s-1,\alpha}$ near Γ by $\widetilde{V}x = -\widetilde{c}_\partial x$, $\rho^{-1}\widetilde{V}\rho = \widetilde{c}_f x$. Then, with $p_{m,r} = p_{0,r} + \widetilde{p}_{m,r}$, we obtain, using (8.5.52)-(8.5.54):

$$
\begin{aligned}
a_\delta H_{p_{m,r}} a_\delta &= \varphi_\delta^2 \rho^{2\sigma}\chi_0^2\chi^2\psi^2(\sigma-(m-1)/2-\delta\rho\varphi_\delta)(c_f+\widetilde{c}_f)x \\
&\quad - \varphi_\delta^2\rho^{2\sigma}\chi_0\chi_0'\chi^2\psi^2(2c_+^2\phi_+^2+\beta c_\partial x^\beta - 2\mu_+\phi_+ x - 2\nu_+\phi_+\widehat{p}_0 \\
&\qquad + 2c_-^2\phi_-^2 + 2\nu_-\phi_-\widehat{p}_0 - \widetilde{V}\phi_+^2 + \beta\widetilde{c}_\partial x^\beta + \widetilde{V}\phi_-^2) \\
&\quad + \varphi_\delta^2\rho^{2\sigma}\chi_0^2\chi\chi'\psi^2(V\rho_+ + \widetilde{V}\rho_+) + \varphi_\delta^2\rho^{2\sigma}\chi_0^2\chi^2\psi\psi'(V\widehat{p}_0+\widetilde{V}\widehat{p}_0) \\
&= -c_+^2 a_{+,\delta}^2 - c_-^2 a_{-,\delta}^2 + a_{+,\delta}h_{+,\delta}p_{m,r} + a_{-,\delta}h_{-,\delta}p_{m,r} + e_\delta + g_\delta - f_\delta,
\end{aligned} \tag{8.5.58}
$$

where, writing $\widehat{p}_0 = \rho^{-m}p_{m,r} - \rho^{-m}\widetilde{p}_{m,r}$ in the second and third line,

$$
\begin{aligned}
a_{\pm,\delta} &= \varphi_\delta\rho^\sigma\sqrt{2\chi_0\chi_0'}\chi\psi\phi_\pm, \\
h_{\pm,\delta} &= \pm\varphi_\delta\rho^{\sigma-m}\sqrt{2\chi_0\chi_0'}\chi\psi\nu_\pm, \\
e_\delta &= \varphi_\delta^2\rho^{2\sigma}\chi_0^2\chi\chi'\psi^2(V\rho_+ + \widetilde{V}\rho_+), \\
g_\delta &= \varphi_\delta^2\rho^{2\sigma}\chi_0^2\chi^2\psi\psi'(V\widehat{p}_0 + \widetilde{V}\widehat{p}_0), \\
f_\delta &= \varphi_\delta^2\rho^{2\sigma}\chi_0\chi^2\psi^2\Big[\big(\beta(c_\partial+\widetilde{c}_\partial)x^\beta - 2\mu_+\phi_+x - \widetilde{V}\phi_+^2 + \widetilde{V}\phi_-^2 \\
&\qquad + 2(\nu_+\phi_+ - \nu_-\phi_-)\rho^{-m}\widetilde{p}_{m,r}\big)\chi_0' \\
&\qquad\quad - (\sigma-(m-1)/2-\delta\rho\varphi_\delta)(c_f+\widetilde{c}_f)x\chi_0\Big]
\end{aligned}
$$

Note that in the definition of f_δ, by the choice of β and using the fact that χ_0 is bounded by a constant multiple of χ_0' on its support, the constant being uniform for $R+\kappa<1$, the term $c_\partial x^\beta$ dominates all other terms on the support of $f_\delta \in S^{2\sigma;\infty}H_{\mathrm{b}}^{s-1}$ for R and κ small enough, hence $f_\delta \geq 0$, and its contribution will be controlled by virtue of the sharp Gårding inequality. The term arising from e_δ will be controlled using the a priori regularity

$\ell := \beta^{-1} \in \mathbb{N}$. Then, even though a is not a smooth symbol of $\overline{\mathbb{R}^n_+}$ with the standard smooth structure, it becomes smooth if one changes the smooth structure of $\overline{\mathbb{R}^n_+}$ by blowing up the boundary to the ℓ-th order, i.e. by taking $x' = x^\beta$ as a boundary defining function, thus obtaining a manifold M_ℓ, which is $\overline{\mathbb{R}^n_+}$ as a topological manifold, but with a different smooth structure; in particular, the function $x = (x')^\ell$ is smooth on M_ℓ in view of $\ell \in \mathbb{N}$. Moreover, the blow-down map $M_\ell \to \overline{\mathbb{R}^n_+}$ induces isomorphisms (see e.g. [82, §4.18])

$$H_{\mathrm{b}}^{s',\gamma}(\overline{\mathbb{R}^n_+}) \cong H_{\mathrm{b}}^{s',\ell\gamma}(M_\ell), \quad s',\gamma \in \mathbb{R}.$$

Therefore, one can continue to work on $\overline{\mathbb{R}^n_+}$, tacitly assuming that all functions and operators live on, and all computations are carried out on, M_ℓ.

assumption of u_r on Γ_-, and g_δ, which is supported away from the characteristic set, will be controlled using elliptic regularity.

Proceeding with the argument, we first make the simplification $\widetilde{R}_r = 0$ by replacing f by $f - \widetilde{R}_r u_r$, and we assume $m \geq 1$ and $\widetilde{s} \geq (5-m)/2$ for now. Then we have, as in the proof of Theorem 8.5.10,

$$\begin{aligned}\mathrm{Re}\langle iA_\delta^*[P_{0,r} + \widetilde{P}_{m,r}, A_\delta]u_r, u_r\rangle + \left\langle \frac{1}{2i}(P_{0,r} - P_{0,r}^*)A_\delta u_r, A_\delta u_r\right\rangle \\ = -\left\langle \frac{1}{2i}(\widetilde{P}_{m,r} - \widetilde{P}_{m,r}^*)A_\delta u_r, A_\delta u_r\right\rangle \\ - \mathrm{Re}\langle iA_\delta f, A_\delta u_r\rangle + \mathrm{Re}\langle iA_\delta \widetilde{P}_{m-1,r}u_r, A_\delta u_r\rangle.\end{aligned}$$

Estimating each term on the right hand side as in the proof of Theorem 8.5.10 and using (8.5.57), we obtain for any $\mu > 0$:

$$\mathrm{Re}\left\langle \big(A_\delta^*(i[P_{0,r} + \widetilde{P}_{m,r}, A_\delta] + E_1 + E_1' + B)A_\delta\big)u_r, u_r\right\rangle \geq -C_\mu - \mu\|A_\delta u_r\|^2_{(m-1)/2}. \quad (8.5.59)$$

Here and in what follows, we in particular absorb all terms involving $\|u_r\|_{\sigma-1/2}$ into the constant C_μ. On the left hand side, the E_1'-term can be dropped because of $\mathrm{WF}_\mathrm{b}'(E_1') \cap \mathrm{WF}_\mathrm{b}'(A) = \emptyset$ for sufficiently localized a. Moreover, the principal symbol of $E_1 + B$ near Γ is $e_1 + b = -q^2$ with q an elliptic symbol of order $(m-1)/2$, since, by assumption on r, we have $e_1 + rc_\partial \rho^{m-1} < 0$ near Γ. Therefore, we can write $E_1 + B = -Q^*Q + E_1'' + E_2$, where $E_1'' \in \Psi_\mathrm{b}^{m-1}$, $E_2 \in \Psi_\mathrm{b}^{m-2}$, $\mathrm{WF}_\mathrm{b}'(E_1'') \cap \Gamma = \emptyset$. Again, the resulting term in the pairing (8.5.59) involving E_1'' can be dropped; also, the term involving E_2 can be dropped at the cost of changing the constant C_μ, since $u_r \in H_\mathrm{b}^{\sigma-1/2}$.

Hence, introducing $J^\pm \in \Psi_\mathrm{b}^{\pm(\sigma-(m-1)/2-1)}$, with real principal symbols, satisfying $I - J^+J^- \in \Psi_\mathrm{b}^{-\infty}$, we get

$$\mathrm{Re}\,\big\langle \mathrm{Op}(j^- a_\delta H_{p_{m,r}} a_\delta)u_r, (J^+)^* u_r\big\rangle - \|QA_\delta u_r\|_0^2 \geq -C_\mu - \mu\|A_\delta u_r\|^2_{(m-1)/2}. \quad (8.5.60)$$

We now plug the commutator relation (8.5.58) into this estimate. We obtain several terms, which we bound as follows: First, since $j^- e_\delta \in (\mathcal{C}^\infty + H_\mathrm{b}^{s-1,\alpha})S^{\sigma+(m-1)/2+1}$ uniformly, $\mathrm{Op}(j^- e_\delta)$ is a uniformly bounded family of maps $H_\mathrm{b}^\sigma \to H_\mathrm{b}^{-(m+1)/2}$; thus, choosing $\widetilde{E} \in \Psi_\mathrm{b}^0$

with $\mathrm{WF}'_{\mathrm{b}}(\widetilde{E}) \subset U$ and with $\mathrm{WF}'_{\mathrm{b}}(I-\widetilde{E})$ disjoint from $\operatorname{supp} e_\delta$, we conclude

$$|\langle \mathrm{Op}(j^- e_\delta)u_r, (J^+)^* u_r\rangle| \le C + |\langle \mathrm{Op}(j^- e_\delta)u_r, (J^+)^* \widetilde{E} u_r\rangle| \le C + \|B_2 u_r\|_\sigma^2$$

for some $B_2 \in \Psi_{\mathrm{b}}^0$ with $\mathrm{WF}'_{\mathrm{b}}(B_2) \cap \Gamma_+ = \emptyset$.

Next, the term $\langle \mathrm{Op}(j^- g_\delta)u_r, (J^+)^* u_r\rangle$ is uniformly bounded, as detailed in the proof of Theorem 8.5.10. Moreover, by the sharp Gårding inequality, see the argument in the proof of Theorem 8.5.6,

$$\mathrm{Re}\langle \mathrm{Op}(-j^- f_\delta)u_r, (J^+)^* u_r\rangle \le C.$$

Further, we obtain two terms involving $h_{\pm,\delta}$; introducing $B_3 \in \Psi_{\mathrm{b}}^0$ elliptic on $\mathrm{WF}'_{\mathrm{b}}(A)$, these can be bounded for $\mu > 0$ by

$$\begin{aligned}
&|\langle \mathrm{Op}(j^- a_{\pm,\delta} h_{\pm,\delta} p_{m,r})u_r, (J^+)^* u_r\rangle| \\
&\qquad \le C + |\langle \mathrm{Op}(j^- a_{\pm,\delta} h_{\pm,\delta})(P_{0,r} + \widetilde{P}_{m,r})u_r, (J^+)^* u_r\rangle| \\
&\qquad \le C + |\langle H_{\pm,\delta} f_r, A_{\pm,\delta} u_r\rangle| + |\langle \mathrm{Op}(j^- a_{\pm,\delta} h_{\pm,\delta})\widetilde{P}_{m-1,r} u_r, (J^+)^* u_r\rangle| \\
&\qquad \le C + \mu \|A_{\pm,\delta} u_r\|_0^2 + C_\mu \|B_3 f_r\|_{\sigma-m}^2.
\end{aligned}$$

Here, for the first estimate, we employ Theorem 8.2.12 (3) to obtain

$$\begin{aligned}
&\mathrm{Op}(j^- a_{\pm,\delta} h_{\pm,\delta})\widetilde{P}_{m,r} - \mathrm{Op}(j^- a_{\pm,\delta} h_{\pm,\delta} \widetilde{p}_{m,r}) \\
&\qquad\qquad =: \Upsilon_\delta \in \Psi_{\mathrm{b}}^{\sigma+(m-1)/2;0} H_{\mathrm{b}}^{s-1} + \Psi_{\mathrm{b}}^{\sigma-(m-1)/2-1} \circ \Psi_{\mathrm{b}}^{m;0} H_{\mathrm{b}}^{s-1},
\end{aligned}$$

and Υ_δ is easily seen to be uniformly bounded from $H_{\mathrm{b}}^{\sigma-1/2}$ to $H_{\mathrm{b}}^{-m/2}$, whereas $(J^+)^* u_r \in H_{\mathrm{b}}^{m/2}$, thus $|\langle \Upsilon_\delta u_r, (J^+)^* u_r\rangle| \le C$. For the second estimate, we simply use $(P_{0,r} + \widetilde{P}_{m,r})u_r = f_r - \widetilde{P}_{m-1,r} u_r$, and for the third estimate, we apply the Peter–Paul inequality to the first pairing; to bound the second pairing, we use the boundedness of $\widetilde{P}_{m-1,r} \colon H_{\mathrm{b}}^{\sigma-1/2} \to H_{\mathrm{b}}^{\sigma-m+1/2}$.

Finally, including the terms $c_\pm^2 a_{\pm,\delta}^2$ into the estimate obtained from (8.5.60) by making use of the above estimates, we obtain

$$\begin{aligned}
&\|C_+ A_{+,\delta} u_r\|_0^2 + \|C_- A_{-,\delta} u_r\|_0^2 + \|Q A_\delta u_r\|_0^2 \\
&\qquad \le C_\mu + \mu \|A_{+,\delta} u_r\|_0^2 + \mu \|A_{-,\delta} u_r\|_0^2 + \mu \|A_\delta u_r\|_{(m-1)/2}^2 \\
&\qquad\qquad + \|B_2 u_r\|_\sigma^2 + \|B_1 f_r\|_{\sigma-m+1}^2 + C_\mu \|\chi f_r\|_{\sigma-m}^2,
\end{aligned}$$

where $B_1 \in \Psi_{\mathrm{b}}^0$ is elliptic on $\mathrm{WF}'_{\mathrm{b}}(A)$ with $\mathrm{WF}'_{\mathrm{b}}(B_1) \subset U$, and $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(M)$ is identically 1 near the projection of $\Gamma \subset {}^{\mathrm{b}}S^*M$ to the base M. Since c_+ and c_- have positive lower bounds near Γ, we can absorb the terms on the right involving $A_{\pm,\delta}$ into the left hand side by choosing μ sufficiently small, at the cost of changing the constant C_μ; likewise, $\rho^{-(m-1)/2}q$ has a positive lower bound near $\operatorname{supp} a$, hence the term on the right involving A_δ can be absorbed into the left hand side for small μ. Dropping the first two terms on the left hand side, we obtain the H_{b}^σ-regularity of u_r at Γ, hence $\mathrm{WF}_{\mathrm{b}}^{\sigma,r}(u) \cap \Gamma = \emptyset$.

Next, we remove the restriction $m \geq 1$: Let $m_0 \geq 1$. The idea, as before, is to rewrite $Pu = f$ as $P\Lambda^+(\Lambda^- u) = f + PRu$, where $\Lambda^\pm \in \Psi_{\mathrm{b}}^{\pm(m_0-m)}$, with real principal symbols, satisfy $\Lambda^+\Lambda^- = I + R$. We now have to be a bit careful though to not change the imaginary part of the subprincipal symbol of $P\Lambda^+$ at Γ. Concretely, we choose Λ^+ self-adjoint with principal symbol $\lambda^+ = \rho^{m_0-m}$ near Γ; then

$$P_0\Lambda^+ - (P_0\Lambda^+)^* = \Lambda^+(P_0 - P_0^*) + [P_0, \Lambda^+].$$

Clearly, $\Lambda^+(P_0 - P_0^*) \in x\Psi_{\mathrm{b}}^{m_0-1} + \Psi_{\mathrm{b}}^{m_0-2}$, and the principal symbol of the second term is

$$\sigma_{\mathrm{b},m_0-1}([P_0, \Lambda^+]) = -iH_{p_0}\lambda^+ = -ix(m_0 - m)\rho^{m_0-1}c_f$$

near Γ by (8.5.54), hence, using (8.5.51),

$$P_0\Lambda^+ - (P_0\Lambda^+)^* = \Lambda^+ E_1 + xE_1' + E_1'' + E_2$$

with $E_1', E_1'' \in \Psi_{\mathrm{b}}^{m_0-1}$, $E_2 \in \Psi_{\mathrm{b}}^{m_0-2}$ and $\mathrm{WF}'_{\mathrm{b}}(E_1'') \cap \Gamma = \emptyset$; therefore, the first part of the proof with P and u replaced by $P\Lambda^+$ and $\Lambda^- u$, respectively, applies. The proof of the theorem in the case $r < -\sup_\Gamma \rho^{-m+1}e_1/c_\partial$ is complete.

When the role of Γ_+ and Γ_- is reversed, there is an overall sign change, and we thus get a advantageous (now meaning positive) contribution to the subprincipal part of the conjugated operator P_r for $r > -\inf_\Gamma \rho^{-m+1}e_1/c_\partial$; the rest of the argument is unchanged. □

8.6 Energy estimates

Let (M, g) be a compact manifold with boundary equipped with a Lorentzian b-metric g satisfying

$$g \in \mathcal{C}^\infty(M; S^{2\mathrm{b}}T^*M) + H_{\mathrm{b}}^s(M; S^{2\mathrm{b}}T^*M),$$

where the b-Sobolev space here is defined using an arbitrary fixed smooth b-density on M. Let $U \subset M$ be open, and suppose $\mathfrak{t}\colon U \to (t_0, t_1)$ is a proper function such that $d\mathfrak{t}$ is timelike on U. We consider the operator

$$P = \Box_g + L, \quad L \in (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1})\mathrm{Diff}_{\mathrm{b}}^1 + (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-2}).$$

For $s > n/2$, one obtains using Lemma 8.3.2 and Corollary 8.2.10 that in any coordinate system the coefficients G^{ij} of the dual metric G are elements of $\mathcal{C}^\infty + H_{\mathrm{b}}^s$, and all Christoffel symbols are elements of $\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1}$. Therefore, by definition of $\Box_g$, one easily obtains that

$$\Box_g \in (\mathcal{C}^\infty + H_{\mathrm{b}}^s)\mathrm{Diff}_{\mathrm{b}}^2 + (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1})\mathrm{Diff}_{\mathrm{b}}^1,$$

thus

$$P \in (\mathcal{C}^\infty + H_{\mathrm{b}}^s)\mathrm{Diff}_{\mathrm{b}}^2 + (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1})\mathrm{Diff}_{\mathrm{b}}^1 + (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-2}). \tag{8.6.1}$$

We prove the following energy estimate, analogous to [114, Proposition 3.8]; we restrict ourselves to operators acting on functions for brevity, but the proof works for bundles as well.

Proposition 8.6.1. *Let $t_0 < T_0 < T_0' < T_1 < t_1$ and $r \in \mathbb{R}$, and suppose $s > n/2+2$. Then there exists a constant $C > 0$ such that for all $u \in H_{\mathrm{b}}^{2,r}(M)$, the following estimate holds:*

$$\|u\|_{H_{\mathrm{b}}^{1,r}(\mathfrak{t}^{-1}([T_0',T_1]))} \leq C(\|Pu\|_{H_{\mathrm{b}}^{0,r}(\mathfrak{t}^{-1}([T_0,T_1]))} + \|u\|_{H_{\mathrm{b}}^{1,r}(\mathfrak{t}^{-1}([T_0,T_0']))}).$$

This also holds with P replaced by P^*. If one replaces C by any $C' > C$, the estimate also holds for small perturbations of P in the space indicated in (8.6.1).

Proof. Let us work in a coordinate system $z_1 = x, z_2 = y_1, \ldots, z_n = y_{n-1}$, where x is a boundary defining function in case we are working near the boundary. By piecing together estimates from coordinate patches, one can deduce the full result. Write ${}^{\mathrm{b}}\partial_j = \partial_{z_j}$ for $2 \leq j \leq n$, and ${}^{\mathrm{b}}\partial_1 = x\partial_x$ if we are working near the boundary, ${}^{\mathrm{b}}\partial_1 = \partial_x$ otherwise.

Moreover, let us fix the *Riemannian* b-metric

$$\widetilde{g} = \frac{dx^2}{x^2} + dy^2$$

near the boundary, $\widetilde{g} = dx^2 + dy^2$ away from it. *We adopt the summation convention in this proof.*

We will imitate the proof of [114, Proposition 3.8], which proves a similar result in a smooth, semiclassical setting. Thus, consider the commutant $V = -iZ$, where $Z = x^{-2r}\chi(\mathfrak{t})W$ with $\chi \in \mathcal{C}^\infty(\mathbb{R})$, chosen later in the proof, and $W = G(-, {}^{\mathrm{b}}d\mathfrak{t})$, which is timelike in U. We will compute the 'commutator'

$$-i(V^*P - P^*V) = -i(V^*\Box_g - \Box_g^*V) - iV^*L + iL^*V, \tag{8.6.2}$$

where the adjoints are taken with respect to the (b-)metric $\widetilde{g}$. First, we need to make sense of all appearing operator compositions. Notice that $V \in x^{-2r}(\mathcal{C}^\infty + H_{\mathrm{b}}^s)\mathrm{Diff}_{\mathrm{b}}^1$, and writing $V = -iZ^j\,{}^{\mathrm{b}}\partial_j$, we get

$$V^* = -i\,{}^{\mathrm{b}}\partial_j Z^j = V - i({}^{\mathrm{b}}\partial_j Z^j) \in x^{-2r}(\mathcal{C}^\infty + H_{\mathrm{b}}^s)\mathrm{Diff}_{\mathrm{b}}^1 + x^{-2r}(\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1}),$$

similarly

$$\Box_g, \Box_g^*, P^* \in (\mathcal{C}^\infty + H_{\mathrm{b}}^s)\mathrm{Diff}_{\mathrm{b}}^2 + (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1})\mathrm{Diff}_{\mathrm{b}}^1 + (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-2});$$

now, since

$$(\mathcal{C}^\infty + H_{\mathrm{b}}^{s-j})\mathrm{Diff}_{\mathrm{b}}^j(\mathcal{C}^\infty + H_{\mathrm{b}}^{s-k})\mathrm{Diff}_{\mathrm{b}}^k \subset \sum_{l \leq j}(\mathcal{C}^\infty + H_{\mathrm{b}}^{s-j}H_{\mathrm{b}}^{s-k-l})\mathrm{Diff}_{\mathrm{b}}^{j+k-l},$$

it suffices to require $s > n/2 + 2$, since then $H_{\mathrm{b}}^{s-j}H_{\mathrm{b}}^{s-k-j} \subset H_{\mathrm{b}}^{s-k-j}$ for $0 \leq j, k \leq 2$, $0 \leq j + k \leq 3$.

Returning to the computation of (8.6.2), we conclude that $-i(V^*\Box_g - \Box_g^*V) \in (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-3,-2r})\mathrm{Diff}_{\mathrm{b}}^2$, and thus its principal symbol is defined. Since it is a formally self-adjoint (with respect to $\widetilde{g}$) operator with real coefficients that vanishes on constants, it equals

${}^{\mathrm{b}}d^*C\,{}^{\mathrm{b}}d$ provided the principal symbols are equal. To compute it, let us write

$$-i(V^*\Box_g - \Box_g^* V) = -({}^{\mathrm{b}}\partial_k Z^k)\Box_g + i[\Box_g, V] - i(\Box_g - \Box_g^*)V.$$

(See [113, §3-4] for a similar computation.) We define $S^i \in \mathcal{C}^\infty + H_{\mathrm{b}}^{s-1}$ by

$$\sigma_{\mathrm{b}}^2(-i(\Box_g - \Box_g^*)V) = 2S^iZ^j\zeta_i\zeta_j = (S^iZ^j + S^jZ^i)\zeta_i\zeta_j.$$

Moreover, with H_G denoting the Hamilton vector field of the dual metric of g,

$$H_G = G^{ij}\zeta_i{}^{\mathrm{b}}\partial_j + G^{ij}\zeta_j{}^{\mathrm{b}}\partial_i - ({}^{\mathrm{b}}\partial_k G^{ij})\zeta_i\zeta_j\partial_{\zeta_k},$$

we find $\sigma_{\mathrm{b}}^2(-i(V^*\Box_g - \Box_g^*V)) = B^{ij}\zeta_i\zeta_j$ with

$$\begin{aligned} B^{ij} = -{}^{\mathrm{b}}\partial_k(Z^kG^{ij}) + G^{ik}({}^{\mathrm{b}}\partial_k Z^j) + G^{jk}({}^{\mathrm{b}}\partial_k Z^i) + S^iZ^j + S^jZ^i& \\ \in x^{-2r}(\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1})&, \end{aligned}$$

thus

$$-i(V^*\Box_g - \Box_g^*V) = {}^{\mathrm{b}}d^*C\,{}^{\mathrm{b}}d, \quad C_i^j = B^{ij}.$$

Let us now plug $Z = x^{-2r}\chi W$ into the definition of B^{ij} and separate the terms with derivatives falling on χ, the idea being that the remaining terms, considered error terms, can then be dominated by choosing χ' large compared to χ. We get

$$\begin{aligned} B^{ij} = x^{-2r}({}^{\mathrm{b}}\partial_k\chi)&(G^{ik}W^j + G^{jk}W^i - G^{ij}W^k) \\ &+ \chi\bigl(G^{ik}({}^{\mathrm{b}}\partial_k x^{-2r}W^j) + G^{jk}({}^{\mathrm{b}}\partial_k x^{-2r}W^i) \\ &\qquad - {}^{\mathrm{b}}\partial_k(x^{-2r}W^kG^{ij}) + x^{-2r}(S^iW^j + S^jW^i)\bigr). \end{aligned}$$

Notice here that for a b-1-form $\omega \in \mathcal{C}^\infty(M; {}^{\mathrm{b}}T^*M)$, the quantity

$$\begin{aligned} E_{W,{}^{\mathrm{b}}d\chi}(\omega) &:= \frac{1}{2}({}^{\mathrm{b}}\partial_k\chi)(G^{ik}W^j + G^{jk}W^i - G^{ij}W^k)\omega_i\overline{\omega_j} \\ &= \frac{1}{2}\bigl[(\omega, {}^{\mathrm{b}}d\chi)_G\overline{\omega(W)} + \omega(W)({}^{\mathrm{b}}d\chi, \omega)_G - {}^{\mathrm{b}}d\chi(W)(\omega,\omega)_G\bigr] \\ &= \chi'(\mathsf{t})E_{W,{}^{\mathrm{b}}dt}(\omega) \end{aligned}$$

is related to the sesquilinear energy-momentum tensor

$$E_{W,{}^{\mathrm{b}}d\mathfrak{t}}(\omega) = \mathrm{Re}\big((\omega, {}^{\mathrm{b}}d\mathfrak{t})_G\overline{\omega(W)}\big) - \frac{1}{2}{}^{\mathrm{b}}d\mathfrak{t}(W)(\omega,\omega)_G,$$

where $(\cdot,\cdot)_G$ is the sesquilinear inner product on ${}^{\mathbb{C}\mathrm{b}}T^*M$. This quantity, rewritten in terms of b-vector fields as

$$E_{X,Y}(\omega) = \mathrm{Re}(\omega(X)\overline{\omega(Y)}) - \frac{1}{2}\langle X,Y\rangle(\omega,\omega)_G,$$

is well-known to be positive definite provided X and Y are both future (or both past) timelike, see e.g. [1]. In our setting, we thus have $E_{W,{}^{\mathrm{b}}d\mathfrak{t}} = E_{W,W} > 0$ by our definition of W. Correspondingly,

$$C = x^{-2r}\chi' A + x^{-2r}\chi R$$

with A positive definite and R symmetric.

We obtain[47]

$$\langle -i(V^*P - P^*V)u, u\rangle = \langle C\,{}^{\mathrm{b}}du, {}^{\mathrm{b}}du\rangle - \langle iLu, Vu\rangle + \langle iVu, Lu\rangle. \tag{8.6.3}$$

We now finish the proof by making χ' large compared to χ on $\mathfrak{t}^{-1}([T_0', T_1])$, as follows: Pick $T_1' \in (T_1, t_1)$ and let

$$\widetilde{\chi}(s) = \widetilde{\chi}_1\left(\frac{s-T_0}{T_0'-T_0}\right)\chi_0(-\digamma^{-1}(s-T_1')), \quad \chi(s) = \widetilde{\chi}(s)H(T_1-s),$$

where H is the Heaviside step function, $\chi_0(s) = e^{-1/s}H(s) \in \mathcal{C}^\infty(\mathbb{R})$ (which satisfies $\chi_0'(s) = s^{-2}\chi_0(s)$) and $\widetilde{\chi}_1 \in \mathcal{C}^\infty(\mathbb{R})$ equals 0 on $(-\infty,0]$ and 1 on $[1,\infty)$; see Figure 8.1.

Then in (T_0', T_1'),

$$\begin{aligned}\chi'(s) &= -\digamma^{-1}\chi_0'(-\digamma^{-1}(s-T_1'))H(T_1-s) - \chi_0(-\digamma^{-1}(T_1-T_1'))\delta_{T_1}\\ &= -\digamma(s-T_1')^{-2}\chi(s) - \chi_0(-\digamma^{-1}(T_1-T_1'))\delta_{T_1},\end{aligned}$$

[47] The integrations by parts here and further below are readily justified using $s > n/2+2$: In fact, since we are assuming $u \in H_{\mathrm{b}}^{2,r}$, we have $Vu \in H_{\mathrm{b}}^{1,-r}$ for $s > n/2, s \geq 1$, and then $P^*Vu \in H_{\mathrm{b}}^{-1,-r}$ provided multiplication with an H_{b}^{s-j} function is continuous $H_{\mathrm{b}}^1 \to H_{\mathrm{b}}^{1-j}$ for $j = 0,1,2$, which is true for $s > n/2+1$; similarly, one has $Pu \in H_{\mathrm{b}}^{0,r}$ provided $s > n/2$, and then $V^*Pu \in H_{\mathrm{b}}^{-1,-r}$ if multiplication by an H_{b}^{s-j} function is continuous $H_{\mathrm{b}}^{-j} \to H_{\mathrm{b}}^{-1}$ for $j = 0,1$, which holds for $s > n/2, s \geq 1$.

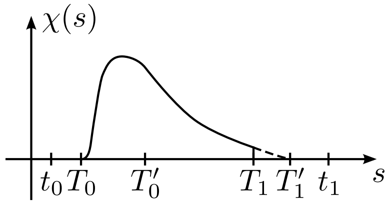

Figure 8.1: Graph of the commutant χ. The dashed line is the graph of the part of $\widetilde{\chi}$ that is cut off using the Heaviside function in the definition of χ.

in particular $\chi(s) = -\digamma^{-1}(s - T_1')^2\chi'(s)$ on (T_0', T_1); hence for any $\gamma > 0$, we can choose $\digamma > 0$ so large that $\chi \leq -\gamma\chi'$ on (T_0', T_1); therefore

$$-(\chi' A + \chi R) \geq -\frac{1}{2}\chi'\widetilde{\chi}_1 A \text{ on } (T_0', T_1).$$

Put $\chi_1(s) = \widetilde{\chi}_1(s)H(T_1 - s)$, then

$$\begin{aligned}
-\langle C{}^{\mathrm{b}}du, {}^{\mathrm{b}}du\rangle &\geq \frac{1}{2}\langle x^{-2r}(-\chi'\chi_1)A{}^{\mathrm{b}}du, {}^{\mathrm{b}}du\rangle \\
&\quad + \chi_0(-\digamma^{-1}(T_1 - T_1'))\langle x^{-2r}A\delta_{T_1}{}^{\mathrm{b}}du, {}^{\mathrm{b}}du\rangle - C'\|{}^{\mathrm{b}}du\|^2_{H_{\mathrm{b}}^{0,r}(\mathfrak{t}^{-1}([T_0, T_0']))},
\end{aligned}$$

and the term on the right hand side involving δ_{T_1} is positive, thus can be dropped. Hence, using equation (8.6.3) and the positivity of A,

$$\begin{aligned}
c_0\|\sqrt{-\chi'\chi_1}{}^{\mathrm{b}}du\|^2_{H_{\mathrm{b}}^{0,r}} &\leq \frac{1}{2}\langle x^{-2r}(-\chi'\chi_1)A{}^{\mathrm{b}}du, {}^{\mathrm{b}}du\rangle \\
&\leq C'\|{}^{\mathrm{b}}du\|^2_{H_{\mathrm{b}}^{0,r}(\mathfrak{t}^{-1}([T_0, T_0']))} + C'\|\chi^{1/2}Pu\|_{H_{\mathrm{b}}^{0,r}}\|\chi^{1/2\mathrm{b}}du\|_{H_{\mathrm{b}}^{0,r}} \\
&\qquad + C'\|\chi^{1/2\mathrm{b}}du\|^2_{H_{\mathrm{b}}^{0,r}} + C'\|\chi^{1/2\mathrm{b}}du\|_{H_{\mathrm{b}}^{0,r}}\|\chi^{1/2}u\|_{H_{\mathrm{b}}^{0,r}} \\
&\leq C''\|u\|^2_{H_{\mathrm{b}}^{1,r}(\mathfrak{t}^{-1}([T_0, T_0']))} + C'\|\chi^{1/2}Pu\|^2_{H_{\mathrm{b}}^{0,r}} + C'\gamma\|\sqrt{-\chi'\chi_1}{}^{\mathrm{b}}du\|^2_{H_{\mathrm{b}}^{0,r}} \\
&\qquad + C'\gamma\|\sqrt{-\chi'\chi_1}u\|^2_{H_{\mathrm{b}}^{0,r}},
\end{aligned} \tag{8.6.4}$$

where the norms are on $\mathfrak{t}^{-1}([T_0, T_1])$ unless otherwise specified. Choosing $\digamma$ large and thus γ small allows us to absorb the second to last term on the right into the left hand side. To finish the proof, we need to treat the last term, as follows: We compute, using $W\chi = \chi' G({}^{\mathrm{b}}d\mathfrak{t}, {}^{\mathrm{b}}d\mathfrak{t}) \equiv m\chi'$ with $m \in \mathcal{C}^\infty + H_{\mathrm{b}}^s$ positive,

$$\langle (W^* x^{-2r}\chi + x^{-2r}\chi W)u, u\rangle$$

$$
\begin{aligned}
&= -\langle (Wx^{-2r}\chi - x^{-2r}\chi W)u, u\rangle - \langle (\mathrm{div}_{\widetilde{g}} W)x^{-2r}\chi u, u\rangle \\
&\geq -\langle x^{-2r} m\chi' u, u\rangle_{H^0_{\mathrm{b}}(\mathfrak{t}^{-1}([T_0,T_1]))} - \langle x^{-2r} w\chi u, u\rangle - \langle (\mathrm{div}_{\widetilde{g}} W)x^{-2r}\chi u, u\rangle \\
&\geq \|\sqrt{-\chi'\chi_1} m^{1/2} u\|^2_{H^{0,r}_{\mathrm{b}}(\mathfrak{t}^{-1}([T_0',T_1]))} - \|\sqrt{|\chi'|} m^{1/2} u\|^2_{H^{0,r}_{\mathrm{b}}(\mathfrak{t}^{-1}([T_0,T_0']))} \\
&\qquad - C\|\sqrt{\chi} u\|^2_{H^{0,r}_{\mathrm{b}}(\mathfrak{t}^{-1}([T_0,T_1]))},
\end{aligned}
$$

where $w = x^{2r} W x^{-2r} \in \mathcal{C}^\infty + H^s_{\mathrm{b}}$. Similarly as above, we now choose $\digamma$ large to obtain

$$
\|\sqrt{-\chi'\chi_1} u\|^2_{H^{0,r}_{\mathrm{b}}(\mathfrak{t}^{-1}([T_0',T_1]))} \leq C\|\sqrt{\chi}\,{}^{\mathrm{b}}du\|^2_{H^{0,r}_{\mathrm{b}}} + C\|\sqrt{\chi + |\chi'|} u\|^2_{H^{0,r}_{\mathrm{b}}(\mathfrak{t}^{-1}([T_0,T_0']))}.
$$

Plugging this estimate into (8.6.4), we can absorb one of the resulting terms, namely $\gamma\|\sqrt{\chi}\,{}^{\mathrm{b}}du\|^2_{H^{0,r}_{\mathrm{b}}(\mathfrak{t}^{-1}([T_0',T_1]))}$, into the left hand side and thus finish the proof of the estimate, since $\sqrt{-\chi'\chi_1}$ has a positive lower bound on $\mathfrak{t}^{-1}([T_0',T_1])$.

That the estimate holds for perturbations of P follows simply from the observation that all constants in this proof depend on finitely many seminorms of the coefficients of P, hence the constants only change by small amounts if one makes a small perturbation of P. □

8.7 Tame estimates in the non-smooth calculus

In this section we prove tame estimates for the H_{b}-coefficient, or simply non-smooth, b-pseudodifferential operators defined in §8.2. Such estimates, as e.g. in [60, 99], are crucial for applications in a Nash-Moser iteration scheme, as we will see in Chapter 9.

8.7.1 Mapping properties

We start with the tame mapping estimate, Proposition 8.7.1, which essentially states that for non-smooth pseudodifferential operators A, a high regularity norm of Au can be estimated by a high regularity norm of A times a low regularity norm of u, plus a low regularity norm of A times a high regularity norm of u. This is stronger than the a priori continuity estimate one gets from the bilinear map $(A, u) \mapsto Au$, which would require a product of high norms of both. In case A is a multiplication operator, this is essentially a b-version of a (weak) Moser estimate, see Corollary 8.7.2.

We continue to work on the half space $\overline{\mathbb{R}^n_+}$ with coordinates $z = (x, y) \in [0, \infty) \times \mathbb{R}^{n-1}$; the coordinates in the fiber of the b-cotangent bundle are denoted $\zeta = (\lambda, \eta)$, i.e. we write b-covectors as $\lambda\,\frac{dx}{x} + \eta\,dy$. For brevity, we will use the following notation for Sobolev, symbol class and operator class norms, with the distinction between symbolic and b-Sobolev norms being clear from the context:

$$\begin{aligned} \|u\|_s &:= \|u\|_{H_{\mathrm{b}}^s}, & \|u\|_{s,r} &:= \|u\|_{H_{\mathrm{b}}^{s,r}}, \\ \|a\|_{m,s} &:= \|a\|_{S^{m;0}H_{\mathrm{b}}^s}, & \|a\|_{(m;k),s} &:= \|a\|_{S^{m;k}H_{\mathrm{b}}^s}, \\ \|A\|_{m,s} &:= \|A\|_{\Psi^{m;0}H_{\mathrm{b}}^s}, & \|A\|_{(m;k),s} &:= \|A\|_{\Psi^{m;k}H_{\mathrm{b}}^s}. \end{aligned}$$

Recall Definition 8.2.5 for the definitions of these symbol and operator classes. If A is a b-operator acting on an element of a weighted b-Sobolev space with weight r (which will be apparent from the context), then $\|A\|_{m,s}$ is to be understood as $\|x^{-r}Ax^r\|_{m,s}$, similarly for $\|A\|_{(m;k),s}$. Lastly, for $A \in H_{\mathrm{b}}^s\Psi_{\mathrm{b}}^m$, we write $\|A\|_{H_{\mathrm{b}}^s\Psi_{\mathrm{b}}^m}$, by an abuse of notation, for an unspecified $H_{\mathrm{b}}^s\Psi_{\mathrm{b}}^m$-seminorm of A.

Recall the notation $x_+ = \max(a, 0)$ for $a \in \mathbb{R}$.

Proposition 8.7.1. *(Extension of 8.2.9.) Let $s \in \mathbb{R}$, $A = \mathrm{Op}(a) \in \Psi^{m;0}H_{\mathrm{b}}^s$, and suppose $s' \in \mathbb{R}$ is such that $s \geq s' - m$, $s > n/2 + (m - s')_+$. Then A defines a bounded map $H_{\mathrm{b}}^{s'} \to H_{\mathrm{b}}^{s'-m}$, and for all fixed μ, ν with*

$$\mu > n/2 + (m - s')_+, \quad \nu > n/2 + (m - s')_+ + s' - s,$$

there is a constant $C > 0$ such that

$$\|Au\|_{s'-m} \leq C(\|A\|_{m,\mu}\|u\|_{s'} + \|A\|_{m,s}\|u\|_\nu). \tag{8.7.1}$$

Observe that by the assumptions on s and s', the intervals of allowed μ, ν are always non-empty (since they contain $\mu = s$ and $\nu = s'$). Estimates of the form (8.7.1) are precisely the aforementioned 'tame estimates.'

Proof of Proposition 8.7.1. We compute

$$\begin{aligned} \|Au\|_{s'-m}^2 &= \int \langle\zeta\rangle^{2(s'-m)}|\widehat{Au}(\zeta)|^2\,d\zeta \\ &\leq \int \langle\zeta\rangle^{2(s'-m)}\left(\int |\widehat{a}(\zeta-\xi,\xi)\widehat{u}(\xi)|\,d\xi\right)^2 d\zeta. \end{aligned}$$

We split the inner integral into two pieces, corresponding to the domains of integration $|\zeta - \xi| \leq |\xi|$ and $|\xi| \leq |\zeta - \xi|$, which can be thought of as splitting up the action of A on u

into a low-high and a high-low frequency interaction. We estimate

$$
\begin{aligned}
&\int \langle\zeta\rangle^{2(s'-m)} \left(\int_{|\zeta-\xi|\leq|\xi|} |\widehat{a}(\zeta-\xi,\xi)\widehat{u}(\xi)|\,d\xi \right)^2 d\zeta \\
&\quad \leq \int \left(\int_{|\zeta-\xi|\leq|\xi|} \frac{\langle\zeta\rangle^{2(s'-m)}\langle\xi\rangle^{2m}}{\langle\zeta-\xi\rangle^{2\mu}\langle\xi\rangle^{2s'}}\,d\xi \right) \\
&\qquad\qquad \times \left(\int \frac{\langle\zeta-\xi\rangle^{2\mu}|\widehat{a}(\zeta-\xi,\xi)|^2}{\langle\xi\rangle^{2m}} \langle\xi\rangle^{2s'}|\widehat{u}(\xi)|^2\,d\xi \right) d\zeta,
\end{aligned} \tag{8.7.2}
$$

and we claim that the integral which is the first factor on the right hand side is uniformly bounded in ζ: Indeed, if $s'-m\geq 0$, then we use $|\zeta|\leq 2|\xi|$ on the domain of integration, thus

$$
\int_{|\zeta-\xi|\leq|\xi|} \frac{\langle\zeta\rangle^{2(s'-m)}}{\langle\zeta-\xi\rangle^{2\mu}\langle\xi\rangle^{2(s'-m)}}\,d\xi \lesssim \int \frac{1}{\langle\zeta-\xi\rangle^{2\mu}}\,d\xi \in L^\infty_\zeta,
$$

since $\mu > n/2$; if, on the other hand, $s'-m\leq 0$, then $|\xi|\leq|\zeta-\xi|+|\zeta|$ gives

$$
\int_{|\zeta-\xi|\leq|\xi|} \frac{\langle\xi\rangle^{2(m-s')}}{\langle\zeta-\xi\rangle^{2\mu}\langle\zeta\rangle^{2(m-s')}}\,d\xi \lesssim \int \frac{1}{\langle\zeta-\xi\rangle^{2(\mu-(m-s'))}} + \frac{1}{\langle\zeta-\xi\rangle^{2\mu}}\,d\xi \in L^\infty_\zeta,
$$

since $\mu > n/2+(m-s')$; hence, from (8.7.2), the $H^{s'-m}_{\mathrm{b}}$ norm of the low-high frequency interaction in Au is bounded by $C_\mu\|a\|_{m,\mu}\|u\|_{s'}$.

We estimate the norm of high-low interaction in a similar way: We have

$$
\begin{aligned}
&\int \langle\zeta\rangle^{2(s'-m)} \left(\int_{|\xi|\leq|\zeta-\xi|} |\widehat{a}(\zeta-\xi,\xi)\widehat{u}(\xi)|\,d\xi \right)^2 d\zeta \\
&\quad \leq \int \left(\int_{|\xi|\leq|\zeta-\xi|} \frac{\langle\zeta\rangle^{2(s'-m)}\langle\xi\rangle^{2m}}{\langle\zeta-\xi\rangle^{2s}\langle\xi\rangle^{2\nu}}\,d\xi \right) \\
&\qquad\qquad \times \left(\int \frac{\langle\zeta-\xi\rangle^{2s}|\widehat{a}(\zeta-\xi,\xi)|^2}{\langle\xi\rangle^{2m}} \langle\xi\rangle^{2\nu}|\widehat{u}(\xi)|^2\,d\xi \right) d\zeta.
\end{aligned} \tag{8.7.3}
$$

If $s'-m\geq 0$, the first inner integral on the right hand side is bounded by

$$
\int_{|\xi|\leq|\zeta-\xi|} \frac{1}{\langle\zeta-\xi\rangle^{2(s-s'+m)}\langle\xi\rangle^{2(\nu-m)}}\,d\xi \leq \int \frac{1}{\langle\xi\rangle^{2(s-s'+\nu)}}\,d\xi,
$$

where we use $s\geq s'-m$, and this integral is finite in view of $\nu > n/2+s'-s$; if $s'-m\leq 0$, then

$$\int_{|\xi|\leq|\zeta-\xi|} \frac{1}{\langle\zeta\rangle^{2(m-s')}\langle\zeta-\xi\rangle^{2s}\langle\xi\rangle^{2(\nu-m)}}\,d\xi \leq \int \frac{1}{\langle\xi\rangle^{2(\nu-m+s)}}\,d\xi,$$

which is finite in view of $\nu > n/2+m-s$. In summary, we need $\nu > n/2+\max(m,s')-s = n/2+(m-s')_+ + s' - s$ and can then bound the $H_{\mathrm{b}}^{s'-m}$ norm of the high-low interaction by $C_\nu \|a\|_{m,s}\|u\|_\nu$. The proof is complete. □

Using $H_{\mathrm{b}}^s \subset S^{0;0}H_{\mathrm{b}}^s$, we obtain the following weak version (compared to [108, Proposition 13.3.7]) of the Moser estimate for the product of two b-Sobolev functions:

Corollary 8.7.2. *(Extension of Corollary 8.2.10.) Let $s > n/2$, $|s'| \leq s$. If $u \in H_{\mathrm{b}}^s, v \in H_{\mathrm{b}}^{s'}$, then $uv \in H_{\mathrm{b}}^{s'}$, and one has an estimate*

$$\|uv\|_{s'} \leq C(\|u\|_\mu \|v\|_{s'} + \|u\|_s \|v\|_\nu)$$

for fixed $\mu > n/2 + (-s')_+, \nu > n/2 + s'_+ - s$. In particular, for $u, v \in H_{\mathrm{b}}^s$,

$$\|uv\|_s \leq C(\|u\|_\mu \|v\|_s + \|u\|_s \|v\|_\mu)$$

for fixed $\mu > n/2$.

8.7.2 Operator compositions

We give a tame estimate for the norms of expansion and remainder terms arising in the composition of two non-smooth operators, see Theorem 8.2.12 (1a).

Proposition 8.7.3. *Suppose $s, m, m' \in \mathbb{R}$, $k, k' \in \mathbb{N}_0$ are such that $s > n/2$, $s \leq s' - k$ and $k \geq m + k'$. Suppose $P = p(z, {}^{\mathrm{b}}D) \in \Psi^{m;k}H_{\mathrm{b}}^s$, $Q = q(z, {}^{\mathrm{b}}D) \in \Psi^{m';0}H_{\mathrm{b}}^{s'}$. Put*

$$E_j := \sum_{|\beta|=j} \frac{1}{\beta!}(\partial_\zeta^\beta p\, {}^{\mathrm{b}}D_z^\beta q)(z, {}^{\mathrm{b}}D),$$
$$R := P \circ Q - \sum_{0\leq j<k} E_j.$$

Then $E_j \in \Psi^{m+m'-j;0}H_{\mathrm{b}}^s$ and $R \in \Psi^{m'-k';0}H_{\mathrm{b}}^s$, and for $\mu > n/2$ fixed,

$$\|E_j\|_{m+m'-j,s} \leq C(\|P\|_{(m;j),\mu}\|Q\|_{m',s+j} + \|P\|_{(m;j),s}\|Q\|_{m',\mu+j}),$$
$$\|R\|_{m'-k',s} \leq C(\|P\|_{(m;k),\mu}\|Q\|_{m',s+k} + \|P\|_{(m;k),s}\|Q\|_{m',\mu+k}).$$

Proof. The statements about the E_j follow from Corollary 8.7.2. For the purpose of proving the estimate for R, we define

$$p_0 = \partial_\zeta^k p \in S^{m-k;0} H_{\mathrm{b}}^s, \quad {}^{\mathrm{b}}D_z^k q \in S^{m';0} H_{\mathrm{b}}^{s'-k},$$

where we write $\partial_\zeta^k = (\partial_\zeta^\beta)_{|\beta|=k}$, similarly for ${}^{\mathrm{b}}D_z^k$. Notice that in particular $p_0 \in S^{0;0} H_{\mathrm{b}}^s$. Then $R = r(z, {}^{\mathrm{b}}D)$ with

$$|\widehat{r}(\eta;\zeta)| \lesssim \int \left(\int_0^1 p_0(\eta - \xi; \zeta + t\xi)\, dt \right) q_0(\xi;\zeta)\, d\xi$$

by Taylor's formula, hence

$$\begin{aligned}
&\int \frac{\langle \eta \rangle^{2s} |\widehat{r}(\eta;\zeta)|^2}{\langle \zeta \rangle^{2m'}}\, d\eta \\
&\quad \lesssim \int \left(\int_{|\eta-\xi|\le|\xi|} \frac{\langle \eta \rangle^{2s}}{\langle \eta - \xi \rangle^{2\mu} \langle \xi \rangle^{2s}}\, d\xi \right) \\
&\qquad\qquad \times \left(\int \left(\int_0^1 \langle \eta - \xi \rangle^{2\mu} |p_0(\eta - \xi, \zeta + t\xi)|^2\, dt \right) \frac{\langle \xi \rangle^{2s} |q_0(\xi;\zeta)|^2}{\langle \zeta \rangle^{2m'}}\, d\xi \right) d\eta \\
&\qquad + \int \left(\int_{|\xi|\le|\eta-\xi|} \frac{\langle \eta \rangle^{2s}}{\langle \eta - \xi \rangle^{2s} \langle \xi \rangle^{2\mu}}\, d\xi \right) \\
&\qquad\qquad \times \left(\int \left(\int_0^1 \langle \eta - \xi \rangle^{2s} |p_0(\eta - \xi, \zeta + t\xi)|^2\, dt \right) \frac{\langle \xi \rangle^{2\mu} |q_0(\xi;\zeta)|^2}{\langle \zeta \rangle^{2m'}}\, d\xi \right) d\eta,
\end{aligned}$$

which implies the claimed estimate for $k' = 0$. For $k' > 0$, we use the same trick of Beals and Reed [9] as in the proof of Theorem 8.2.12 to reduce the statement to the case $k' = 0$: Recall that the idea is to split up $q(z, \zeta)$ into a 'trivial' part q_0 with compact support in ζ and n parts q_i, where q_i has support in $|\zeta_i| \ge 1$, and then writing

$$P \circ Q_i = \sum_{j=0}^{k'} c_{jk'} P\, {}^{\mathrm{b}}D_{z_i}^{k'-j} \circ ({}^{\mathrm{b}}D_{z_i}^j q_i)(z, {}^{\mathrm{b}}D)\, {}^{\mathrm{b}}D_{z_i}^{-k'}$$

for some constants $c_{jk'} \in \mathbb{R}$ using the Leibniz rule; then what we have proved above for $k' = 0$ can be applied to the j-th summand on the right hand side, which we expand to order $k - j$, giving the result. □

8.7.3 Reciprocals of and compositions with H_b^s functions

We also need sharper bounds for reciprocals and compositions of b-Sobolev functions on a compact n-dimensional manifold with boundary. Localizing using a partition of unity, we can simply work on $\overline{\mathbb{R}^n_+}$.

Proposition 8.7.4. *(Extension of Lemma 8.3.2.) Let $s > n/2+1$, $u, w \in H_b^s$, $a \in \mathcal{C}^\infty$, and suppose that $|a+u| \geq c_0$ near $\operatorname{supp} w$. Then $w/(a+u) \in H_b^s$, and one has an estimate*

$$\left\|\frac{w}{a+u}\right\|_s \leq C(\|u\|_\mu, \|a\|_{C^N}) c_0^{-1} \max(c_0^{-\lceil s\rceil}, 1)\big(\|w\|_s + \|w\|_\mu(1+\|u\|_s)\big). \tag{8.7.4}$$

for any fixed $\mu > n/2+1$ and some s-dependent $N \in \mathbb{N}$.

Proof. Choose $\psi_0, \psi \in \mathcal{C}^\infty$ such that $\psi_0 \equiv 1$ on $\operatorname{supp} w$, $\psi \equiv 1$ on $\operatorname{supp}\psi_0$, and such that moreover $|a+u| \geq c_0 > 0$ on $\operatorname{supp}\psi$. Then we have $\|w/(a+u)\|_0 \leq c_0^{-1}\|w\|_0$. We now iteratively prove higher regularity of $w/(a+u)$ as in the proof of Lemma 8.3.2, but now we keep track of constants in order to prove an accompanying 'tame' estimate: Let us assume $w/(a+u) \in H_b^{s'-1}$ for some $1 \leq s' \leq s$. Let $\Lambda_{s'} = \lambda_{s'}({}^{\mathrm{b}}D) \in \Psi_b^{s'}$ be an operator with principal symbol $\langle\zeta\rangle^{s'}$. Then

$$\begin{aligned}
\left\|\Lambda_{s'}\frac{w}{a+u}\right\|_0 &\leq \left\|(1-\psi)\Lambda_{s'}\frac{\psi_0 w}{a+u}\right\|_0 + \left\|\psi\Lambda_{s'}\frac{\psi_0 w}{a+u}\right\|_0 \\
&\lesssim \left\|\frac{w}{a+u}\right\|_0 + c_0^{-1}\left\|\psi(a+u)\Lambda_{s'}\frac{w}{a+u}\right\|_0 \\
&\leq c_0^{-1}\|w\|_0 + c_0^{-1}\left(\|\psi\Lambda_{s'}w\|_0 + \left\|\psi[\Lambda_{s'}, a+u]\frac{w}{a+u}\right\|_0\right) \\
&\lesssim c_0^{-1}\left(\|w\|_{s'} + \left\|\frac{w}{a+u}\right\|_{s'-1} + \left\|\psi[\Lambda_{s'}, u]\frac{w}{a+u}\right\|_0\right),
\end{aligned} \tag{8.7.5}$$

where we used that the support assumptions on ψ_0 and ψ imply $(1-\psi)\Lambda_{s'}\psi_0 \in \Psi_b^{-\infty}$, and $\psi[\Lambda_{s'}, a] \in \Psi_b^{s'-1}$. Hence, in order to prove that $w/(a+u) \in H_b^{s'}$, it suffices to show that $[\Lambda_{s'}, u] \colon H_b^{s'-1} \to H_b^0$. Let $v \in H_b^{s'-1}$. Since

$$\begin{aligned}
(\Lambda_{s'}uv)\hat{}\,(\zeta) &= \int \lambda_{s'}(\zeta)\widehat{u}(\zeta-\xi)\widehat{v}(\xi)\,d\xi \\
(u\Lambda_{s'}v)\hat{}\,(\zeta) &= \int \widehat{u}(\zeta-\xi)\lambda_{s'}(\xi)\widehat{v}(\xi)\,d\xi,
\end{aligned}$$

we have, by taking a first order Taylor expansion of $\lambda_{s'}(\zeta) = \lambda_{s'}(\xi + (\zeta - \xi))$ around $\zeta = \xi$,

$$([\Lambda_{s'}, u]v)\widehat{\ }(\zeta) = \sum_{|\beta|=1} \int \left(\int_0^1 \partial_\zeta^\beta \lambda_{s'}(\xi + t(\zeta - \xi))\, dt \right) ({}^{\mathrm{b}}D_z^\beta u)\widehat{\ }(\zeta - \xi)\widehat{v}(\xi)\, d\xi,$$

thus, writing $u' = {}^{\mathrm{b}}D_z u \in H_{\mathrm{b}}^{s-1}$,

$$|([\Lambda_{s'}, u]v)\widehat{\ }(\zeta)| \lesssim \int \left(\int_0^1 \langle \xi + t(\zeta - \xi) \rangle^{s'-1}\, dt \right) \left|\widehat{u'}(\zeta - \xi)\right| |\widehat{v}(\xi)|\, d\xi.$$

To obtain a tame estimate for the L^2_ζ norm of this expression, we again use the method of decomposing the integral into low-high and high-low components: The low-high component is bounded by

$$\int \left(\int_{|\zeta-\xi|\leq|\xi|} \frac{\sup_{0\leq t\leq 1} \langle \xi + t(\zeta-\xi) \rangle^{2(s'-1)}}{\langle \zeta - \xi \rangle^{2(\mu-1)} \langle \xi \rangle^{2(s'-1)}}\, d\xi \right) \times \left(\int \langle \zeta - \xi \rangle^{2(\mu-1)} \left|\widehat{u'}(\zeta - \xi)\right|^2 \langle \xi \rangle^{2(s'-1)} |\widehat{v}(\xi)|^2\, d\xi \right) d\zeta;$$

the first inner integral, in view of $s' \geq 1$, so the sup is bounded by $\langle \xi \rangle^{2(s'-1)}$, which cancels the corresponding term in the denominator, is finite for $\mu > n/2 + 1$. For the high-low component, we likewise estimate

$$\int \left(\int_{|\xi|\leq|\zeta-\xi|} \frac{\sup_{0\leq t\leq 1} \langle \xi + t(\zeta-\xi) \rangle^{2(s'-1)}}{\langle \zeta - \xi \rangle^{2(s-1)} \langle \xi \rangle^{2\nu}}\, d\xi \right) \times \left(\int \langle \zeta - \xi \rangle^{2s} \left|\widehat{u'}(\zeta - \xi)\right|^2 \langle \xi \rangle^{2\nu} |\widehat{v}(\xi)|^2\, d\xi \right) d\zeta,$$

and the first inner integral on the right hand side is bounded by

$$\int_{|\xi|\leq|\zeta-\xi|} \frac{1}{\langle \zeta - \xi \rangle^{2(s-s')} \langle \xi \rangle^{2\nu}}\, d\xi \leq \int \frac{1}{\langle \xi \rangle^{2(s-s'+\nu)}}\, d\xi$$

because of $s \geq s'$, which is finite for $\nu > n/2 + s' - s$. We conclude that

$$\|[\Lambda_{s'}, u]v\|_0 \leq C_{\mu\nu}(\|u\|_\mu \|v\|_{s'-1} + \|u\|_{s'} \|v\|_\nu),$$

for $\mu > n/2+1, \nu > n/2+s'-s$. Plugging this into (8.7.5) yields

$$\left\|\frac{w}{a+u}\right\|_{s'} \lesssim c_0^{-1}\left(\|w\|_{s'} + (1+\|u\|_\mu)\left\|\frac{w}{a+u}\right\|_{s'-1} + \|u\|_{s'}\left\|\frac{w}{a+u}\right\|_\nu\right),$$

where the implicit constant in the inequality is independent of c_0, w and u. Using the abbreviations $q_\sigma := \|w/(a+u)\|_\sigma$, $u_\sigma = \|u\|_\sigma$, $w_\sigma = \|w\|_\sigma$ and fixing $\mu > n/2+1$, this means

$$q_{s'} \lesssim c_0^{-1}(w_{s'} + (1+u_\mu)q_{s'-1} + u_{s'}q_\nu), \quad \nu > n/2+s'-s,$$

with the implicit constant being independent of c_0, w, a, u, μ. We will use this for $s' \le \gamma := \lfloor n/2\rfloor + 1$ with $\nu = s'-1$, and for $s' > \gamma$, we will take $\nu = \gamma$, thus obtaining a tame estimate for q_s. In more detail, for $1 \le s' \le \gamma$, we have

$$q_{s'} \lesssim c_0^{-1}(w_{s'} + (1+u_{s'})q_{s'-1}),$$

which gives, with $C_0 = \max(1, c_0^{-1})$,

$$q_\gamma \lesssim c_0^{-1} w_\gamma \sum_{j=0}^{\gamma-1}(c_0^{-1}(1+u_\gamma))^j + (c_0^{-1}(1+u_\gamma))^\gamma q_0 \lesssim c_0^{-1}C_0^\gamma w_\gamma(1+u_\gamma)^\gamma$$

using the bound $q_0 \le c_0^{-1}w_0 \le c_0^{-1}w_\gamma$. For $\gamma < s' \le s$, we have

$$q_{s'} \lesssim c_0^{-1}(w_s + u_s q_\gamma + (1+u_\mu)q_{s'-1}),$$

thus for integer $k \ge 1$ with $\gamma + k \le s$,

$$\begin{aligned} q_{\gamma+k} &\le c_0^{-1}(w_s + u_s q_\gamma)\sum_{j=0}^{k-1}(c_0^{-1}(1+u_\mu))^j + (c_0^{-1}(1+u_\mu))^k q_\gamma \\ &\lesssim c_0^{-1}C_0^{k-1}(1+u_\mu)^k(w_s + (1+u_s)q_\gamma) \\ &\lesssim c_0^{-1}C_0^{\gamma+k}(1+u_\mu)^{\gamma+k}(w_s + (1+u_s)w_\gamma), \end{aligned}$$

where we used $\mu > \gamma$ in the last inequality, thus proving the estimate (8.7.4) in case s is an integer; in the general case, we just use $q_{\gamma'} \le q_\gamma$ for $\gamma' < \gamma$, in particular for $\gamma' = s - \lceil s-\gamma\rceil$, and use the above with $q_{\gamma+k}$ replaced by $q_{\gamma'+k}$. □

As in §8.3.2, one thus obtains regularity results for compositions, but now with sharper estimates. To illustrate how to obtain these, let us prove an extension of Proposition 8.3.5. Let M be a compact n-dimensional manifold with boundary, $s > n/2+1$, $\alpha \geq 0$.

Proposition 8.7.5. *Let $u \in H_{\mathrm{b}}^{s,\alpha}(M)$. If $F\colon \Omega \to \mathbb{C}$, $F(0)=0$, is holomorphic in a simply connected neighborhood Ω of the range of u, then $F(u) \in H_{\mathrm{b}}^{s,\alpha}(M)$, and*

$$\|F(u)\|_{s,\alpha} \leq C(\|u\|_{\mu,\alpha})(1+\|u\|_{s,\alpha}) \tag{8.7.6}$$

for fixed $\mu > n/2+1$. Moreover, there exists $\epsilon > 0$ such that $F(v) \in H_{\mathrm{b}}^{s,\alpha}(M)$ depends continuously on $v \in H_{\mathrm{b}}^{s,\alpha}(M)$, $\|u-v\|_{s,\alpha} < \epsilon$.

Proof. Observe that $u(M)$ is compact. Let $\gamma \subset \mathbb{C}$ denote a smooth contour which is disjoint from $u(M)$, has winding number 1 around every point in $u(M)$, and lies within the region of holomorphicity of F. Then, writing $F(z) = zF_1(z)$ with F_1 holomorphic in Ω, we have

$$F(u) = \frac{1}{2\pi i}\oint_\gamma F_1(\zeta)\frac{u}{\zeta-u}\,d\zeta,$$

Since $\gamma \ni \zeta \mapsto u/(\zeta-u) \in H_{\mathrm{b}}^{s,\alpha}(M)$ is continuous by Proposition 8.7.4, we obtain, using the estimate (8.7.4),

$$\|F(u)\|_{s,\alpha} \leq C(\|u\|_\mu)\big(\|u\|_{s,\alpha} + \|u\|_{\mu,\alpha}(1+\|u\|_s)\big),$$

which implies (8.7.6) in view of $\alpha \geq 0$. The continuous (in fact, Lipschitz) dependence of $F(v)$ on v is a consequence of Proposition 8.7.4 and Corollary 8.7.2. □

We also study compositions $F(u)$ for $F \in \mathcal{C}^\infty(\mathbb{R};\mathbb{C})$ and real-valued u.

Proposition 8.7.6. *(Extension of Proposition 8.3.7.) Let $F \in \mathcal{C}^\infty(\mathbb{R};\mathbb{C})$, $F(0)=0$. Then for $u \in H_{\mathrm{b}}^{s,\alpha}(M;\mathbb{R})$, we have $F(u) \in H_{\mathrm{b}}^{s,\alpha}(M)$, and one has an estimate*

$$\|F(u)\|_{s,\alpha} \leq C(\|u\|_{\mu,\alpha})(1+\|u\|_{s,\alpha}) \tag{8.7.7}$$

for fixed $\mu > n/2+1$. In fact, $F(u)$ depends continuously on u.

Proof. The proof is the same as the proof of Proposition 8.3.7 only we now use the sharper estimate (8.7.4) to obtain (8.7.7). □

Proposition 8.7.7. *(Extension of Proposition 8.3.8.) Let* $F \in \mathcal{C}^\infty(\mathbb{R};\mathbb{C})$*, and* $u' \in \mathcal{C}^\infty(M;\mathbb{R}), u'' \in H_{\mathrm{b}}^{s,\alpha}(M;\mathbb{R})$*; put* $u = u' + u''$*. Then* $F(u) \in \mathcal{C}^\infty(M) + H_{\mathrm{b}}^{s,\alpha}(M)$*, and one has an estimate*

$$\|F(u) - F(u')\|_{s,\alpha} \leq C(\|u'\|_{C^N}, \|u''\|_{\mu,\alpha})(1 + \|u''\|_{s,\alpha})$$

for fixed $\mu > n/2 + 1$ *and some* $N \in \mathbb{N}$*. In fact,* $F(u)$ *depends continuously on* u*.*

Proof. The proof is the same as the proof of Proposition 8.3.8, but now uses the sharper estimate (8.7.4). □

8.8 Tame microlocal regularity estimates

When stating microlocal regularity estimates (like elliptic regularity, real principal type propagation, etc.) for operators with coefficients in $H_{\mathrm{b}}^s(\overline{\mathbb{R}^n_+})$, we will give two quantitative statements, one for 'low' regularities $\sigma \lessapprox n/2$, in which we will not make use of any tame estimates established earlier, and one for 'high' regularities $n/2 \lessapprox \sigma \lessapprox s$, in which the tame estimates will be used.

To concisely write down tame estimates, we use the following notation: The right hand side of a tame estimate will be a real-valued function, denoted by L, of the form

$$\begin{aligned} L(p_1^\ell, \ldots, p_a^\ell; p_1^h, \ldots, p_b^h; u_1^\ell, \ldots, u_c^\ell; u_1^h, \ldots, u_d^h) \\ = \sum_{j=1}^d c_j(p_1^\ell, \ldots, p_a^\ell) u_j^h + \sum_{j=1}^b \sum_{k=1}^c c_{jk}(p_1^\ell, \ldots, p_a^\ell) p_j^h u_k^\ell \end{aligned} \tag{8.8.1}$$

here, the c_j and c_{jk} are continuous functions. In applications, $p_j^{\ell/h}$ will be a low/high regularity norm of the coefficients of a non-smooth operator, and $u_j^{\ell/h}$ will be a low/high regularity norm of a function that an operator is applied to. The important feature of such functions L is that they are linear in the $u_j^{\ell/h}$, and all p_j^h, u_j^h, corresponding to high regularity norms, only appear in the first power.

8.8.1 Elliptic regularity

Concretely, we have the following quantitative elliptic estimate:

Proposition 8.8.1. *(Tame version of Theorem 8.4.1.) Let $m, s, r \in \mathbb{R}$ and $\zeta_0 \in {}^{\mathrm{b}}S^*\overline{\mathbb{R}^n_+}$. Suppose $P = \widetilde{P}_m + \widetilde{R}$, where $\widetilde{P}_m \in H^s_{\mathrm{b}}\Psi^m_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ has a homogeneous principal symbol p, and $\widetilde{R} \in \Psi^{m-1;0}_{\mathrm{b}} H^{s-1}_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$. Suppose p is elliptic at ζ_0. Let $\widetilde{s} \in \mathbb{R}$ be such that $\widetilde{s} \leq s-1$ and $s > n/2 + 1 + (-\widetilde{s})_+$, and suppose that $u \in H^{\widetilde{s}+m-1,r}_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ satisfies*

$$Pu = f \in H^{\widetilde{s},r}_{\mathrm{b}}(\overline{\mathbb{R}^n_+}).$$

Then there exists $B \in \Psi^0_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ elliptic at ζ_0 such that $Bu \in H^{\widetilde{s}+m}_{\mathrm{b}}$, and for $\widetilde{s} \leq n/2 + t$, $t > 0$, the estimate

$$\begin{aligned} \|Bu\|_{\widetilde{s}+m,r} \leq C(&\|P'\|_{(m;1),n/2+1+(-\widetilde{s})_+ +t}, \|R\|_{m-1,n/2+(-\widetilde{s})_+ +t}) \\ &\times (\|u\|_{\widetilde{s}+m-1,r} + \|f\|_{\widetilde{s},r}) \end{aligned} \tag{8.8.2}$$

holds. For $\widetilde{s} > n/2$, $\epsilon > 0$, there is a tame estimate

$$\begin{aligned} \|Bu\|_{\widetilde{s}+m,r} \leq L(&\|P'\|_{(m;1),n/2+1+\epsilon}, \|R\|_{m-1,n/2+\epsilon}; \|P'\|_{(m;1),s}, \|R\|_{m-1,s-1}; \\ &\|u\|_{n/2+m-1+\epsilon,r}, \|f\|_{n/2-1+\epsilon,r}; \|u\|_{\widetilde{s}+m-1,r}, \|f\|_{\widetilde{s},r}). \end{aligned} \tag{8.8.3}$$

Remark 8.8.2. In our application of such an estimate to the study of nonlinear equations it will be irrelevant what exactly the low regularity norms in (8.8.3) are; in fact, it will be sufficient to know that there is *some* tame estimate of the general form (8.8.3), and this in turn is clear without any computation, namely it follows directly from the fact that we have tame estimates for all 'non-smooth' operations involved in the proof of this proposition. The same remark applies to all further tame microlocal regularity results below. The only point where the precise numerology does matter is when we want to find an explicit bound on the number of required derivatives in our quasilinear applications in Chapter 9, see in particular Theorems 9.2.2, 9.2.3 and 9.2.4.

Proof of Proposition 8.8.1. We can assume that $r = 0$ by conjugating P by x^{-r}. Choose $a_0 \in S^0$ elliptic at ζ_0 such that p_m is elliptic on $\operatorname{supp} a_0$. Let $\Lambda_m \in \Psi^m_{\mathrm{b}}$ be a b-ps.d.o. with full symbol $\lambda_m(\zeta)$ independent of z, whose principal symbol is $\langle\zeta\rangle^m$, and define

$$q(z,\zeta) := a_0(z,\zeta)\lambda_m(\zeta)/p_m(z,\zeta) \in S^{0;\infty}H^s_{\mathrm{b}}, \quad Q = q(z, {}^{\mathrm{b}}D),$$

then by Proposition 8.7.4 and Corollary 8.7.2, we have

$$\|Q\|_{(0;k),\sigma} \leq C(\|P'\|_{(m;k),n/2+1+\epsilon})(1+\|P'\|_{(m;k),\sigma}), \quad \sigma > n/2+1, \epsilon > 0. \tag{8.8.4}$$

Put $B = a_0(z, {}^{\mathrm{b}}D)\Lambda_m$, then

$$Q \circ P' = B + R'$$

with $R' \in \Psi^{m-1;0}H_{\mathrm{b}}^{s-1}$; by Proposition 8.7.3, we have for $n/2 < \sigma \leq s-1$

$$\|R'\|_{m-1,\sigma} \lesssim \|Q\|_{(0;1),\mu}\|P'\|_{(m;1),\sigma+1} + \|Q\|_{(0;1),\sigma}\|P'\|_{(m;1),\mu+1}, \quad \mu > n/2. \tag{8.8.5}$$

Now, since $Bu = QP'u - R'u = Qf - QRu - R'u$, we need to estimate the $H_{\mathrm{b}}^{\widetilde{s}}$-norms of Qf, QRu and $R'u$, which we will do using Proposition 8.7.1. In the low regularity regime, we have, for $t > 0$ and $\widetilde{s} \leq n/2+t$, using (8.8.4) and (8.8.5):

$$\begin{aligned}
\|Qf\|_{\widetilde{s}} &\lesssim \|Q\|_{0,n/2+(-\widetilde{s})_++t}\|f\|_{\widetilde{s}} \leq C(\|P'\|_{m,n/2+1+(-\widetilde{s})_++t})\|f\|_{\widetilde{s}}, \\
\|R'u\|_{\widetilde{s}} &\lesssim \|R'\|_{m-1,n/2+(-\widetilde{s})_++t}\|u\|_{\widetilde{s}+m-1} \\
&\qquad \leq C(\|P'\|_{(m;1),n/2+1+(-\widetilde{s})_++t})\|u\|_{\widetilde{s}+m-1}, \\
\|QRu\|_{\widetilde{s}} &\leq C(\|P'\|_{m,n/2+1+(-\widetilde{s})_++t})\|R\|_{m-1,n/2+(-\widetilde{s})_++t}\|u\|_{\widetilde{s}+m-1},
\end{aligned}$$

giving (8.8.2). In the high regularity regime, in fact for $0 \leq \widetilde{s} \leq s-1$, we have, for $\epsilon > 0$,

$$\begin{aligned}
\|Qf\|_{\widetilde{s}} &\lesssim \|Q\|_{0,n/2+\epsilon}\|f\|_{\widetilde{s}} + \|Q\|_{0,s}\|f\|_{n/2-1+\epsilon} \\
&\qquad \leq C(\|P'\|_{m,n/2+1+\epsilon})(\|f\|_{\widetilde{s}} + (1+\|P'\|_{m,s})\|f\|_{n/2-1+\epsilon}), \\
\|R'u\|_{\widetilde{s}} &\lesssim \|R'\|_{m-1,n/2+\epsilon}\|u\|_{\widetilde{s}+m-1} + \|R'\|_{m-1,s-1}\|u\|_{n/2+m-1+\epsilon} \\
&\qquad \leq C(\|P'\|_{(m;1),n/2+1+\epsilon})(\|u\|_{\widetilde{s}+m-1} + (1+\|P'\|_{(m;1),s})\|u\|_{n/2+m-1+\epsilon}), \\
\|QRu\|_{\widetilde{s}} &\leq L(\|P'\|_{m,n/2+1+\epsilon}, \|R\|_{m-1,n/2+\epsilon}; \|P'\|_{m,s}, \|R\|_{m-1,s-1}; \\
&\qquad\qquad \|u\|_{n/2+m-1+\epsilon}; \|u\|_{\widetilde{s}+m-1}),
\end{aligned}$$

giving (8.8.3). The proof is complete. □

There is a similar tame microlocal elliptic estimate for operators of the form $P = P_0 + \widetilde{P}$ with $\widetilde{P}$ being of the same form as P above, and $P_0 \in \Psi_{\mathrm{b}}^m$, as in the second part of Theorem 8.4.1, where the tame estimate now also involves the C^N-norm of the 'smooth part' P_0 of the operator for some (s-dependent) N. Since in our application P_0 will only depend on finitely many complex parameters, there is no need to prove an estimate which is also tame with respect to the C^N-norm of P_0; however, this could easily be done in principle.

8.8.2 Real principal type propagation, radial points, normally hyperbolic trapping

Tame estimates for propagation type statements, i.e. real principal type propagation, propagation near radial points as well as near normally hyperbolic trapping, can be deduced from a careful analysis of the proofs of the corresponding results in §8.5. The main observation is that the regularity requirements, given in the footnotes to the proofs of these results in §8.5, indicate what regularity is needed to estimate the corresponding terms: For example, an operator in $A \in \Psi^{m;0} H_{\mathrm{b}}^s$ with $m \geq 0$ maps $H_{\mathrm{b}}^{m/2}$ to $H_{\mathrm{b}}^{-m/2}$ under the condition $s > n/2 + m/2$, which is to say that one has a bound

$$\|Au\|_{-m/2} \lesssim \|A\|_{m-1,n/2+m/2+\epsilon} \|u\|_{m/2}, \quad \epsilon > 0.$$

This means that the *only* places where one needs to use tame operator bounds for operators with coefficients of regularity s are those where the condition for mapping properties etc. to hold reads $s \gtrapprox \sigma$ where σ is the regularity of the target space, i.e. where σ is comparable to the regularity s of the coefficients.

We again only prove the tame real principal type estimate in the interior, adopting the notation of the corresponding proof of the first part of Theorem 8.5.6. The estimate near the boundary is proved in the same way, see also the discussion at the end of §8.8.1.

Proposition 8.8.3. *(Tame version of Theorem 8.5.6.) Let $m, r, s, \widetilde{s} \in \mathbb{R}$. Suppose $P_m \in H_{\mathrm{b}}^s \Psi_{\mathrm{b}}^m(\overline{\mathbb{R}_+^n})$ has a real, scalar, homogeneous principal symbol p, and let $P = P_m + P_{m-1} + R$, where $P_{m-1} \in H_{\mathrm{b}}^{s-1} \Psi_{\mathrm{b}}^{m-1}(\overline{\mathbb{R}_+^n})$, $R \in \Psi_{\mathrm{b}}^{m-2;0} H_{\mathrm{b}}^{s-1}(\overline{\mathbb{R}_+^n})$. Assume that s and $\widetilde{s}$ are such that*

$$\widetilde{s} \leq s - 1, \quad s > n/2 + 7/2 + (2 - \widetilde{s})_+,$$

and let us assume that $u \in H_{\mathrm{b}}^{\widetilde{s}+m-3/2,r}(\overline{\mathbb{R}_+^n})$ satisfies $Pu = f \in H_{\mathrm{b}}^{\widetilde{s},r}(\overline{\mathbb{R}_+^n})$. Suppose $\zeta_0 \notin \mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m-1,r}(u)$, and let $\gamma\colon [0,T] \to {}^{\mathrm{b}}T^*\overline{\mathbb{R}_+^n} \setminus o$ be a segment of a null-bicharacteristic of p with $\gamma(0) = \zeta_0$, then $\gamma(t) \notin \mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m-1,r}(u)$ for all $t \in [0,T]$. Moreover, for all $A \in \Psi_{\mathrm{b}}^0$

elliptic at ζ_0 *there exist* $B \in \Psi_{\mathrm{b}}^0$ *elliptic at* $\gamma(T)$ *and* $G \in \Psi_{\mathrm{b}}^0$ *elliptic on* $\gamma([0,T])$ *such that for* $\widetilde{s} \leq n/2+1$, $\epsilon > 0$,

$$\begin{aligned} &\|Bu\|_{\widetilde{s}+m-1,r} \\ &\quad \leq C(\|P_m\|_{H_{\mathrm{b}}^{n/2+7/2+(2-\widetilde{s})_++\epsilon}\Psi_{\mathrm{b}}^m}, \|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+(3/2-\widetilde{s})_++\epsilon}\Psi_{\mathrm{b}}^{m-1}}, \|R\|_{n/2+1+(-\widetilde{s})_+}) \\ &\qquad \times (\|u\|_{\widetilde{s}+m-3/2,r} + \|Au\|_{\widetilde{s}+m-1,r} + \|Gf\|_{\widetilde{s},r}). \end{aligned} \tag{8.8.6}$$

Moreover, for $\widetilde{s} > n/2+1$, $\epsilon > 0$, *there is a tame estimate*

$$\begin{aligned} \|Bu\|_{\widetilde{s}+m-1,r} \leq L(&\|P_m\|_{H_{\mathrm{b}}^{n/2+7/2+\epsilon}\Psi_{\mathrm{b}}^m}, \|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+\epsilon}\Psi_{\mathrm{b}}^{m-1}}, \|R\|_{n/2+\epsilon}; \\ &\|P_m\|_{H_{\mathrm{b}}^s\Psi_{\mathrm{b}}^m}, \|P_{m-1}\|_{H_{\mathrm{b}}^{s-1}\Psi_{\mathrm{b}}^{m-1}}, \|R\|_{m-2,s-1}; \\ &\|u\|_{n/2-1/2+m+\epsilon}; \|u\|_{\widetilde{s}+m-3/2,r}, \|Au\|_{\widetilde{s}+m-1,r}, \|Gf\|_{\widetilde{s},r}). \end{aligned} \tag{8.8.7}$$

Proof. We follow the proof of Theorem 8.5.6 and state the estimates needed to establish (8.8.6) and (8.8.7) along the way. Using the notation of the proof of Theorem 8.5.6, but now calling the regularization parameter δ, in particular $\check{A}_\delta \in \Psi_{\mathrm{b}}^{\widetilde{s}+(m-1)/2}$ is the regularized commutant, which depends on a positive constant M chosen below, and putting $\widetilde{f} = f - Ru$, we have, assuming $m \geq 1$ and $\widetilde{s} \geq (5-m)/2$ for now,

$$\begin{aligned} &\mathrm{Re}\langle i\check{A}_\delta^*[P_m, \check{A}_\delta]u, u\rangle \\ &\quad = \frac{1}{2}\langle i(P_m - P_m^*)\check{A}_\delta u, \check{A}_\delta u\rangle - \mathrm{Re}\langle i\check{A}_\delta \widetilde{f}, \check{A}_\delta u\rangle + \mathrm{Re}\langle i\check{A}_\delta P_{m-1}u, \check{A}_\delta u\rangle \\ &\quad \equiv I + II + III. \end{aligned}$$

For $\epsilon > 0$, we can bound the first term by

$$|I| \lesssim \|P_m\|_{H_{\mathrm{b}}^{n/2+1+(m-1)/2+\epsilon}\Psi_{\mathrm{b}}^m} \|\check{A}_\delta u\|_{(m-1)/2}^2,$$

the second one by

$$|II| \lesssim \|\check{A}_\delta f\|_{-(m-1)/2}^2 + \|Ru\|_{\widetilde{s}}^2 + \|\check{A}_\delta u\|_{(m-1)/2}^2,$$

where in turn

$$\|Ru\|_{\widetilde{s}} \lesssim \begin{cases} \|R\|_{m-2;n/2+(-\widetilde{s})_++t}\|u\|_{\widetilde{s}+m-2}, & \widetilde{s} \leq n/2+t, \\ \|R\|_{m-2;n/2+\epsilon}\|u\|_{\widetilde{s}+m-2} + \|R\|_{m-2;s-1}\|u\|_{n/2+m-2+\epsilon}, & \widetilde{s} \geq 0 \end{cases}$$

for $t > 0$ by Proposition 8.7.1. We estimate the third term by

$$|III| \lesssim \|P_{m-1}\|_{H_{\mathrm{b}}^{\max(n/2+\epsilon,(m-1)/2)}\Psi_{\mathrm{b}}^{m-1}}\|\check{A}_\delta u\|^2_{(m-1)/2} + |\langle[\check{A}_\delta, P_{m-1}]u, \check{A}_\delta u\rangle|$$

and further, with $R_2 \in \Psi_{\mathrm{b}}^{\widetilde{s}+(m-1)/2-1} \circ \Psi^{m-1;0}H_{\mathrm{b}}^{s-2}$ denoting a part of the expansion of $[\check{A}_\delta, P_{m-1}]$ as defined in (8.5.26),

$$\begin{aligned} |\langle[\check{A}_\delta, P_{m-1}]u, \check{A}_\delta u\rangle| \leq C(M)\|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+(m/2-1)_++\epsilon}\Psi_{\mathrm{b}}^{m-1}}\|u\|^2_{\widetilde{s}+m-3/2} \\ + \|R_2 u\|^2_{-(m-1)/2} + \|\check{A}_\delta u\|^2_{(m-1)/2}, \end{aligned}$$

where

$$\begin{aligned} &\|R_2 u\|_{-(m-1)/2} \\ &\leq C(M) \begin{cases} \|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+(1-\widetilde{s})_++\epsilon}\Psi_{\mathrm{b}}^{m-1}}\|u\|_{\widetilde{s}+m-2}, & \widetilde{s} \leq n/2+1+\epsilon, \\ \|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+\epsilon}\Psi_{\mathrm{b}}^{m-1}}\|u\|_{\widetilde{s}+m-2} & \\ \qquad +\|P_{m-1}\|_{H_{\mathrm{b}}^{s-1}\Psi_{\mathrm{b}}^{m-1}}\|u\|_{n/2+m-1+\epsilon}, & \widetilde{s} \geq 1. \end{cases} \end{aligned}$$

Therefore, we obtain, see equation (8.5.29),

$$\begin{aligned} \mathrm{Re}\Big\langle\Big(i\check{A}_\delta^*[P_m, \check{A}_\delta] + B_\delta^* B_\delta + M^2(\Lambda\check{A}_\delta)^*(\Lambda\check{A}_\delta) - E_\delta\Big)u, u\Big\rangle \\ \geq -|\langle E_\delta u, u\rangle| - \|\check{A}_\delta f\|^2_{-(m-1)/2} - L^2 + \|B_\delta u\|^2_{L^2_{\mathrm{b}}}, \end{aligned} \tag{8.8.8}$$

where

$$M = M(\|P_m\|_{H_{\mathrm{b}}^{n/2+1+(m-1)/2+\epsilon}\Psi_{\mathrm{b}}^m}, \|P_{m-1}\|_{H_{\mathrm{b}}^{\max(n/2+\epsilon,(m-1)/2)}\Psi_{\mathrm{b}}^{m-1}}),$$

and L is 'tame'; more precisely, for $\widetilde{s} \leq n/2+t$, $t > 0$,

$$L \leq C(M, \|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+\max(m/2-1,1-\widetilde{s})_++\epsilon}\Psi_{\mathrm{b}}^{m-1}}, \|R\|_{m-2;n/2+(-\widetilde{s})_++t})\|u\|_{\widetilde{s}+m-3/2},$$

and for $\widetilde{s} \geq 1$,

$$L = L(M, \|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+(m/2-1)_++\epsilon}\Psi_{\mathrm{b}}^{m-1}}, \|R\|_{m-2,n/2+\epsilon}; \\ \|P_{m-1}\|_{H_{\mathrm{b}}^{s-1}\Psi_{\mathrm{b}}^{m-1}}, \|R\|_{m-2;s-1}; \|u\|_{n/2+m-1+\epsilon}; \|u\|_{\widetilde{s}+m-3/2}).$$

Next, in order to exploit the positive commutator of the principal symbols of P_m and $\check{A}_\delta$ in the estimate (8.8.8), we introduce operators $J^\pm \in \Psi_{\mathrm{b}}^{\pm(\widetilde{s}+(m-1)/2-1)}$ with principal symbols $j^\pm$ such that $J^+J^- - I \in \Psi_{\mathrm{b}}^{-\infty}$; then

$$iJ^-\check{A}_\delta^*[P_m, \check{A}_\delta] = \mathrm{Op}(j^-\check{a}_\delta H_p \check{a}_\delta) + R_1 + R_2 + R_3 + R_4,$$

see equation (8.5.32), where

$$|\langle R_j u, (J^+)^* u\rangle| \leq C(M)\|P_m\|_{H_{\mathrm{b}}^{n/2+2+m/2+\epsilon}\Psi_{\mathrm{b}}^m}\|u\|_{\widetilde{s}+m-3/2}^2, \quad j = 1, 3, 4,$$

and $R_2 \in \Psi_{\mathrm{b}}^{\widetilde{s}+(m-1)/2-1} \circ \Psi^{m;0}H_{\mathrm{b}}^{s-2}$, hence

$$\begin{aligned} &|\langle R_2 u, (J^+)^* u\rangle| \\ &\quad \leq C(M)\begin{cases} (1 + \|P_m\|_{H_{\mathrm{b}}^{n/2+2+(3/2-\widetilde{s})_++\epsilon}\Psi_{\mathrm{b}}^m}^2)\|u\|_{\widetilde{s}+m-3/2}^2 & \forall\ \widetilde{s}, \\ (1 + \|P_m\|_{H_{\mathrm{b}}^{n/2+2+\epsilon}\Psi_{\mathrm{b}}^m}^2)\|u\|_{\widetilde{s}+m-3/2}^2 & \\ \qquad\qquad + \|P_m\|_{H_{\mathrm{b}}^s\Psi_{\mathrm{b}}^m}^2\|u\|_{n/2-1/2+m+\epsilon}^2 & \widetilde{s} \geq 3/2. \end{cases} \end{aligned}$$

Thus, further following the proof of Theorem 8.5.6 to equation (8.5.33) and beyond, it remains to bound

$$\mathrm{Re}\langle \mathrm{Op}(j^- f_\delta / j^+)(J^+)^* u, (J^+)^* u\rangle + \mathrm{Re}\langle R' u, (J^+)^* u\rangle, \quad R' \in \Psi^{\widetilde{s}+3(m-1)/2;0}H_{\mathrm{b}}^{s-1},$$

from below, which is accomplished by

$$|\langle R' u, (J^+)^* u\rangle| \leq C(M)\|P_m\|_{H_{\mathrm{b}}^{n/2+1+m/2+\epsilon}\Psi_{\mathrm{b}}^m}\|u\|_{\widetilde{s}+m-3/2}^2,$$
$$\mathrm{Re}\langle \mathrm{Op}(j^- f_\delta / j^+)(J^+)^* u, (J^+)^* u\rangle \geq -C(M)\|P_m\|_{H_{\mathrm{b}}^{n/2+3+m/2+\epsilon}\Psi_{\mathrm{b}}^m}\|u\|_{\widetilde{s}+m-3/2}^2.$$

Lastly, for general $m \in \mathbb{R}$, we rewrite the equation $Pu = f$ as $P\Lambda^+(\Lambda^- u) = f + PRu$ with

$\Lambda^\pm \in \Psi_{\mathrm{b}}^{\mp(m-m_0)}$, $R \in \Psi_{\mathrm{b}}^{-\infty}$, where $m_0 \geq 1$; hence, replacing P by $P\Lambda^+$, u by $\Lambda^- u$ and m by m_0 in the above estimates is equivalent to just replacing m by m_0 in the b-Sobolev norms of the coefficients of P. Choosing $m_0 = 1 + 2(2-\widetilde{s})_+$ then implies the estimates (8.8.6) and (8.8.7) with $B = B_0$, G an elliptic multiple of $\check{A}_0$, and A elliptic on the microsupport of E_0. $\square$

In a similar manner, we can analyze the proof of the radial point estimate, Theorem 8.5.10, obtaining the following tame estimates:

Proposition 8.8.4. *(Tame version of Theorem 8.5.10.) Let $m, r, s, \widetilde{s} \in \mathbb{R}$, $\alpha > 0$. Let $P = P_0 + \widetilde{P}$, where $P_0 \in \Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+})$ has a real, scalar, homogeneous principal symbol p_0, further $\widetilde{P} = \widetilde{P}_m + \widetilde{P}_{m-1} + \widetilde{R}$ with $\widetilde{P}_m \in H_{\mathrm{b}}^{s,\alpha}\Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+})$ having a real, scalar, homogeneous principal symbol $\widetilde{p}_m$, moreover $\widetilde{P}_{m-1} \in H_{\mathrm{b}}^{s-1,\alpha}\Psi_{\mathrm{b}}^{m-1}(\overline{\mathbb{R}^n_+})$ and $\widetilde{R} \in \Psi_{\mathrm{b}}^{m-2}(\overline{\mathbb{R}^n_+}) + \Psi_{\mathrm{b}}^{m-2;0}H_{\mathrm{b}}^{s-1,\alpha}(\overline{\mathbb{R}^n_+})$. Suppose that the conditions (1)-(5) in §8.5.4 hold for p_0. Finally, assume that s and $\widetilde{s}$ satisfy*

$$\widetilde{s} \leq s - 1, \quad s > n/2 + 7/2 + (2-\widetilde{s})_+. \tag{8.8.9}$$

Suppose $u \in H_{\mathrm{b}}^{\widetilde{s}+m-3/2,r}(\overline{\mathbb{R}^n_+})$ is such that $Pu \in H_{\mathrm{b}}^{\widetilde{s},r}(\overline{\mathbb{R}^n_+})$.

(1) If $\widetilde{s} + (m-1)/2 - 1 + \inf_{L_\pm}(\widehat{\beta} - r\widetilde{\beta}) > 0$, let us assume that in a neighborhood of $L_\pm$, $\mathcal{L}_\pm \cap \{x > 0\}$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m-1,r}(u)$.

(2) If $\widetilde{s} + (m-1)/2 + \sup_{L_\pm}(\widehat{\beta} - r\widetilde{\beta}) < 0$, let us assume that a punctured neighborhood of $L_\pm$, with $L_\pm$ removed, in $\Sigma \cap {}^{\mathrm{b}}S^*_{\partial\overline{\mathbb{R}^n_+}}\overline{\mathbb{R}^n_+}$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m-1,r}(u)$.

Then in both cases, $L_\pm$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{\widetilde{s}+m-1,r}(u)$.

Quantitatively, for every neighborhood U of $L_\pm$, there exist $B_0, B_1 \in \Psi_{\mathrm{b}}^0$ elliptic at $L_\pm$, $A \in \Psi_{\mathrm{b}}^0$ with microsupport in the respective a priori control region in the two cases above, with $\mathrm{WF}'_{\mathrm{b}}(A), \mathrm{WF}'_{\mathrm{b}}(B_j) \subset U$, $j = 1, 2$, and $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(U)$, such for $\widetilde{s} \leq n/2 + 1$, $\epsilon > 0$, we have, with implicit dependence of the appearing constants on seminorms of the smooth operator P_0:

$$\begin{aligned} \|B_0 u\|_{\widetilde{s}+m-1,r} \leq C(&\|\widetilde{P}_m\|_{H_{\mathrm{b}}^{n/2+7/2+(2-\widetilde{s})_++\epsilon,\alpha}\Psi_{\mathrm{b}}^m}, \\ &\|\widetilde{P}_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+(3/2-\widetilde{s})_++\epsilon,\alpha}\Psi_{\mathrm{b}}^{m-1}}, \|\widetilde{R}\|_{m-2,n/2+1+(-\widetilde{s})_+}) \\ &\times (\|u\|_{\widetilde{s}+m-3/2,r} + \|Au\|_{\widetilde{s}+m-1,r} + \|B_1 f\|_{\widetilde{s},r} + \|\chi f\|_{\widetilde{s}-1,r}). \end{aligned} \tag{8.8.10}$$

Moreover, for $\widetilde{s} > n/2+1$, $\epsilon > 0$, *there is a tame estimate*

$$\begin{aligned}\|B_0u\|_{\widetilde{s}+m-1,r} \leq L(&\|\widetilde{P}_m\|_{H_{\mathrm{b}}^{n/2+7/2+\epsilon,\alpha}\Psi_{\mathrm{b}}^m}, \|\widetilde{P}_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+\epsilon,\alpha}\Psi_{\mathrm{b}}^{m-1}}, \|\widetilde{R}\|_{m-2,n/2+\epsilon};\\ &\|\widetilde{P}_m\|_{H_{\mathrm{b}}^{s,\alpha}\Psi_{\mathrm{b}}^m}, \|\widetilde{P}_{m-1}\|_{H_{\mathrm{b}}^{s-1,\alpha}\Psi_{\mathrm{b}}^{m-1}}, \|\widetilde{R}\|_{m-2,s-1}; \|u\|_{n/2-1/2+m+\epsilon}, \|f\|_{n/2-1+\epsilon};\\ &\|u\|_{\widetilde{s}+m-3/2,r}, \|Au\|_{\widetilde{s}+m-1,r}, \|B_1f\|_{\widetilde{s},r}, \|\chi f\|_{\widetilde{s}-1,r}).\end{aligned} \tag{8.8.11}$$

Proof. One detail changes as compared to the previous proof: While it still suffices to only assume microlocal regularity $B_2f \in H_{\mathrm{b}}^{\widetilde{s},r}$ at $L_\pm$, we now in addition need to assume local regularity $\chi f \in H_{\mathrm{b}}^{\widetilde{s}-1,r}$, which is due to the use of elliptic regularity in the proof given in §8.4. □

Likewise, we have the following tame non-trapping estimate at Γ:

Proposition 8.8.5. *(Tame version of Theorem 8.5.12.) Under the assumptions of Theorem 8.5.12, and using the notation used there, let* $s, \widetilde{s} \in \mathbb{R}$ *be such that*

$$\widetilde{s} \leq s-1, \quad s > n/2+7/2+(2-\widetilde{s})_+.$$

Suppose $u \in H_{\mathrm{b}}^{\widetilde{s}+m-3/2,r}(\overline{\mathbb{R}^n_+})$ *is such that* $Pu = f \in H_{\mathrm{b}}^{\widetilde{s},r}(\overline{\mathbb{R}^n_+})$.

Then for $r < -\sup_\Gamma \rho^{-m+1}\sigma_{\mathrm{b},m-1}(E_1)/c_\partial$ *and for any neighborhood* U *of* Γ, *there exist* $B_0 \in \Psi_{\mathrm{b}}^0(M)$ *elliptic at* Γ *and* $B_1, B_2 \in \Psi_{\mathrm{b}}^0(M)$ *with* $\mathrm{WF}_{\mathrm{b}}'(B_j) \subset U$, $j=0,1,2$, $\mathrm{WF}_{\mathrm{b}}'(B_2) \cap \Gamma_+ = \emptyset$, *and* $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(U)$, *such that the following estimate holds for* $\widetilde{s} \leq n/2+1$, $\epsilon > 0$:

$$\begin{aligned}\|B_0u\|_{\widetilde{s}+m-1,r} \leq C(&\|\widetilde{P}_m\|_{H_{\mathrm{b}}^{n/2+7/2+(2-\widetilde{s})_++\epsilon,\alpha}\Psi_{\mathrm{b}}^m},\\ &\|\widetilde{P}_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+(3/2-\widetilde{s})_++\epsilon,\alpha}\Psi_{\mathrm{b}}^{m-1}}, \|\widetilde{R}\|_{m-2,n/2+1+(-\widetilde{s})_+})\\ &\times(\|u\|_{\widetilde{s}+m-3/2,r} + \|B_2u\|_{\widetilde{s}+m-1,r} + \|B_1f\|_{\widetilde{s},r} + \|\chi f\|_{\widetilde{s}-1,r}).\end{aligned} \tag{8.8.12}$$

Moreover, for $\widetilde{s} > n/2+1$, $\epsilon > 0$, *there is a tame estimate*

$$\begin{aligned}\|B_0u\|_{\widetilde{s}+m-1,r} \leq L(&\|\widetilde{P}_m\|_{H_{\mathrm{b}}^{n/2+7/2+\epsilon,\alpha}\Psi_{\mathrm{b}}^m}, \|\widetilde{P}_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+\epsilon,\alpha}\Psi_{\mathrm{b}}^{m-1}}, \|\widetilde{R}\|_{m-2,n/2+\epsilon};\\ &\|\widetilde{P}_m\|_{H_{\mathrm{b}}^{s,\alpha}\Psi_{\mathrm{b}}^m}, \|\widetilde{P}_{m-1}\|_{H_{\mathrm{b}}^{s-1,\alpha}\Psi_{\mathrm{b}}^{m-1}}, \|\widetilde{R}\|_{m-2,s-1}; \|u\|_{n/2-1/2+m+\epsilon}, \|f\|_{n/2-1+\epsilon};\\ &\|u\|_{\widetilde{s}+m-3/2,r}, \|B_2u\|_{\widetilde{s}+m-1,r}, \|B_1f\|_{\widetilde{s},r}, \|\chi f\|_{\widetilde{s}-1,r}).\end{aligned} \tag{8.8.13}$$

On the other hand, for $r > -\inf_\Gamma \rho^{-m+1}\sigma_{\mathrm{b},m-1}(E_1)/c_\partial$ *and for appropriate* B_2 *with*

$\mathrm{WF}'_{\mathrm{b}}(B_2) \cap \Gamma_- = \emptyset$, *the estimates* (8.8.12) *and* (8.8.13) *hold as well. These estimates are understood in the sense that if all quantities on the right hand side are finite, then so is the left hand side, and the inequality holds.*

8.9 Regularity estimates on manifolds

Strengthening the assumptions on our non-smooth operators in the local (on $\overline{\mathbb{R}^n_+}$) non-smooth microlocal regularity results established in the previous sections slightly, making them invariant, we now show how to deduce these results on a compact manifold M with boundary by making use of a partition of unity and the local theory.

To illustrate the idea, we begin by discussing elliptic regularity, Theorem 8.4.1, and for brevity only the interior result (1) in the scalar case. Thus, we assume

$$P = \widetilde{P}_m + \widetilde{R}, \quad \widetilde{P}_m \in H^s_{\mathrm{b}}\Psi^m_{\mathrm{b}}(M),\ \widetilde{R} \in H^{s-1}_{\mathrm{b}}\Psi^{m-1}_{\mathrm{b}}(M),$$

which strengthens the assumption of $\widetilde{R}$; we stress that the class of operators $H^s_{\mathrm{b}}\Psi^m_{\mathrm{b}}(M)$ is invariantly defined on M. Then, with $\widetilde{s} \in \mathbb{R}$ such that $\widetilde{s} \leq s-1$, $s > n/2 + 1 + (-\widetilde{s})_+$, and $u \in H^{\widetilde{s}+m-1}_{\mathrm{b}}(M)$ such that

$$Pu = f \in H^{\widetilde{s}}_{\mathrm{b}}(M),$$

and assuming that P is elliptic at $\zeta_0 \in {}^{\mathrm{b}}S^*M$, we obtain $\zeta_0 \notin \mathrm{WF}^{\widetilde{s}+m}_{\mathrm{b}}(u)$ as follows: First, we reduce to the case $R = 0$ by absorbing $Ru \in H^{\widetilde{s}}_{\mathrm{b}}(M)$ into the right hand side f. Next, pick $\phi^{(0)} \in \mathcal{C}^\infty_{\mathrm{c}}(M)$ localizing within a coordinate chart, with $\phi^{(0)} \equiv 1$ near the base point of ζ_0, and $\phi^{(1)} \in \mathcal{C}^\infty_{\mathrm{c}}(M)$ with support in the same chart, and $\phi^{(1)} \equiv 1$ near $\operatorname{supp} \phi^{(0)}$. In order to show microlocal elliptic regularity for u solving $\phi^{(0)}Pu = \phi^{(0)}f$ at ζ_0, we insert $\phi^{(1)}$ by writing

$$\phi^{(0)}Pu = \phi^{(0)}P\phi^{(1)}u + \phi^{(0)}P(1-\phi^{(1)})u;$$

now $\phi^{(0)}P(1-\phi^{(1)}) \in H^s_{\mathrm{b}}\Psi^{-\infty}_{\mathrm{b}}(M)$ maps u into $H^s_{\mathrm{b}}(M) \subset H^{\widetilde{s}}_{\mathrm{b}}(M)$, hence can be absorbed into the right hand side. Picking $\phi^{(2)} \in \mathcal{C}^\infty_{\mathrm{c}}(M)$ with $\phi^{(2)} \equiv 1$ near $\operatorname{supp} \phi^{(1)}$, localizing in the same coordinate chart, we have thus reduced to proving elliptic regularity at ζ_0 for u solving

$$(\phi^{(0)}P\phi^{(1)})(\phi^{(2)}u) \in H^{\widetilde{s}}_{\mathrm{b}}(M), \tag{8.9.1}$$

which is now completely localized in a single coordinate chart, hence Theorem 8.4.1 gives the desired conclusion. Hence, we circumvent the development of an invariant non-smooth calculus by simply analyzing the mapping properties of error terms like $\phi^{(0)}P(1-\phi^{(1)})$, rather than finding a suitable symbol class which they belong to.

The arguments for propagation estimates are very similar. Since the radial point and trapping estimates in Theorems 8.5.10 and 8.5.12 are modifications of the real principal type propagation result, Theorem 8.5.6, we only indicate how to make the proof of the latter work on manifolds. Again, we suppress bundles and only consider the interior propagation result, thus in the notation of the proof of Theorem 8.5.6 in this case,

$$P = P_m + P_{m-1} + R, \quad P_m \in H_{\mathrm{b}}^s \Psi_{\mathrm{b}}^m(M),\ P_{m-1} \in H_{\mathrm{b}}^{s-1}\Psi_{\mathrm{b}}^{m-1}(M),\ R \in H_{\mathrm{b}}^{s-1}\Psi_{\mathrm{b}}^{m-2}(M);$$

note again the invariant assumption on R here. First, we note that the symbolic construction of the commutant leading up to Proposition 8.5.8 (as well as of the commutants used for radial point and trapping estimates) is invariant, so the task is merely to obtain estimates for the non-smooth operators arising in the course of the positive commutator argument. Following the proof of Theorem 8.5.6, we begin by establishing a bound on $P_m - P_m^* \in \mathcal{L}(H_{\mathrm{b}}^{(m-1)/2}(M), H_{\mathrm{b}}^{-(m-1)/2}(M))$, see (8.5.25), which we obtain by writing

$$P_m - P_m^* = \sum_j \phi_j P_m - P_m^* \phi_j,$$

where $\{\phi_j\}$ is a partition of unity on M subordinated to a cover by local coordinate charts; then, writing $\phi^{(0)} = \phi_j$ for any fixed j, and choosing $\phi^{(1)} \in \mathcal{C}_{\mathrm{c}}^\infty(M)$ supported in the same coordinate chart as $\phi^{(0)}$ and identically 1 near $\operatorname{supp}\phi^{(0)}$, we further write

$$\phi^{(0)}P_m - P_m^*\phi^{(0)} = \big(\phi^{(0)}P_m\phi^{(1)} - (\phi^{(0)}P_m\phi^{(1)})^*\big) + \big(\phi^{(0)}P_m(1-\phi^{(1)}) - (1-\phi^{(1)})P_m^*\phi^{(0)}\big),$$

where the first term is a bounded operator between the aforementioned spaces by the local argument given in the proof of the estimate (8.5.25), while the second term belongs to the class $H_{\mathrm{b}}^s\Psi_{\mathrm{b}}^{-\infty} + \Psi_{\mathrm{b}}^{-\infty}H_{\mathrm{b}}^s$ and is thus bounded on the relevant spaces as well. Here, we remark that the regularity requirements on s for $H_{\mathrm{b}}^s\Psi_{\mathrm{b}}^{-\infty}$ to map $H_{\mathrm{b}}^{(m-1)/2}$ into $H_{\mathrm{b}}^{-(m-1)/2}$ are the same as or weaker than the requirements on s from the local argument, while there are *no requirements* other than, say, $s > n/2$, in order to have $\Psi_{\mathrm{b}}^{-\infty}H_{\mathrm{b}}^s$ map $H_{\mathrm{b}}^{\sigma_1}$ into $H_{\mathrm{b}}^{\sigma_2}$ for *any* $\sigma_1, \sigma_2 \in \mathbb{R}$, $\sigma_1 \geq 0$, since $H_{\mathrm{b}}^s \cdot H_{\mathrm{b}}^{\sigma_1} \subset H_{\mathrm{b}}^{\min(s,\sigma_1)}$ gets mapped into $H_{\mathrm{b}}^\infty \subset H_{\mathrm{b}}^{\sigma_2}$ by $\Psi_{\mathrm{b}}^{-\infty}$. Similar remarks apply in the remaining arguments in this section; therefore, there is no need to analyze regularity requirements at this point, the requirements given in the statements of the respective theorems in the previous sections being sufficient.

We further point out that for the radial point estimate we need a *small* operator bound for $P_m - P_m^*$ localized near the boundary, where P_m has now coefficients which decay at ∂M, see (8.5.47); such a bound follows easily from the above localization argument, combined with the local estimate, as well.

Next, we need to show (8.5.27), i.e. adopting the notation of §8.5, we need to establish the uniform boundedness of

$$[P_{m-1}, \check{A}_t] \in \mathcal{L}(H_{\mathrm{b}}^{\sigma-1}, H_{\mathrm{b}}^{-(m-1)/2}),$$

where $\sigma = \widetilde{s} + m - 1$; recall that $\check{A}_t \in \Psi_{\mathrm{b}}^{\sigma-(m-1)/2}(M)$ uniformly. With $\phi^{(0)}$ being an element of a partition of unity, localizing in a coordinate chart as before, we need to estimate $\phi^{(0)}[P_{m-1}, \check{A}_t]$. Now, choosing $\phi^{(j+1)} \in \mathcal{C}_{\mathrm{c}}^\infty(M)$, supported in the same coordinate chart and identically one near $\operatorname{supp}\phi^{(j)}$, we have

$$\phi^{(0)}[P_{m-1}, \check{A}_t] = \phi^{(0)}[\phi^{(1)}P_{m-1}\phi^{(2)}, \phi^{(3)}\check{A}_t\phi^{(4)}]\phi^{(5)}$$

modulo controllable error terms, while the right hand side is estimated by the argument leading to (8.5.27). Here, we needed to insert $\phi^{(5)}$ in order to ensure that the commutator on the right hand side acts on functions supported in the coordinate chart. (Thus, $\phi^{(5)}$ here plays the same role as $\phi^{(2)}$ in (8.9.1).)

Lastly, we describe the analogue of (8.5.31), which relates the operator commutators (involving non-smooth operators) to the symbolic commutator computation. Again, with $\phi^{(j)} \in \mathcal{C}_{\mathrm{c}}^\infty(M)$ as before, it suffices to analyze

$$Q := \phi^{(0)}J^-\check{A}_t^*\phi^{(1)}[\phi^{(2)}P_m\phi^{(3)}, \phi^{(4)}\check{A}_t\phi^{(5)}]\phi^{(6)},$$

all other terms (involving $1 - \phi^{(j)}$ for one or several $j = 1, \ldots, 6$) being controlled by the a priori assumptions as above. The local arguments following (8.5.31) show that

$$Q = \mathrm{Op}\big(\phi^{(0)}j^-\check{a}_t H_{\phi^{(2)}p}(\phi^{(4)}\check{a}_t)\big)\phi^{(6)}$$

modulo controllable error terms, with p the principal symbol of P_m, where we used $\phi^{(j)}\phi^{(k)} =$

$\phi^{(j)}$ for $j < k$. Since $\phi^{(j)} \equiv 1$ on $\operatorname{supp} \phi^{(0)}$ for $j = 2, 4$, the latter expression is equal to

$$\mathrm{Op}\big(\phi^{(0)} j^- \check{a}_t H_p \check{a}_t\big) \phi^{(6)}.$$

At this point, one can plug in the terms of the symbolic positive commutator calculation of $\check{a}_t H_p \check{a}_t$ from Proposition 8.5.8, all of which now get multiplied by $\phi^{(0)}$. Since we have an invariant calculus for smooth operators, summing over the partition of unity (of which $\phi^{(0)}$ is a member) recovers the usual positive commutator calculation, with a non-smooth error term of the form $\mathrm{Op}(-\phi^{(0)} j^- f_t)\phi^{(6)}$ coming from each coordinate chart, see (8.5.33); but each of these error terms separately has a sign and is thus controlled by the sharp Gårding inequality.

This shows that the positive commutator argument proving the propagation of singularities generalizes to manifolds; the same then holds for the radial point and normally hyperbolic trapping estimates by completely analogous arguments. Further, the form of the tame estimates is unaffected by the partition of unity arguments; only the implicit constants change.

Chapter 9

Quasilinear wave equations

9.1 Quasilinear waves on non-trapping spacetimes

To illustrate the type of global existence result for quasilinear wave equations that we will prove in this section, we work on a domain Ω extending a part of the static model of de Sitter space beyond the cosmological horizon, as in §2.2.1, see in particular (2.2.5), and we denote by g_0 the static de Sitter metric on Ω (more precisely, the extension of the static metric to Ω). Recall that Ω is compact, since it contains its boundary at future infinity. We then have:

Theorem 9.1.1. *For $u \in C(\Omega)$, let $g(u)$ be a b-metric with $g(0) = g_0$, and in local coordinates, $g(u) = (g_{ij}(u))$ with $g_{ij} \in \mathcal{C}^\infty(\mathbb{R})$. Moreover, let*

$$q(u, du) = \sum_j u^{e_j} \prod_{l=1}^{N_j} X_{jl} u, \quad e_j + N_j \geq 2, N_j \geq 1, X_{jl} \in \mathcal{V}_{\mathrm{b}}(\Omega).$$

Fix $k > n/2 + 7$ and $\delta \in (0, 1)$. Then there exist $R, C > 0$ such that for all $f \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega^\circ; \mathbb{R})$ with $\|\tau^{-1+\delta} f\|_{H^{k-1}_{\mathrm{b}}(\Omega)} \leq C$, the equation

$$\Box_{g(u)} u = f + q(u, du) \tag{9.1.1}$$

has a unique forward solution $u = c + u'$, $c \in \mathbb{R}$, $u' \in \tau^{1-\delta} H^k_{\mathrm{b}}(\Omega; \mathbb{R})$, satisfying the bound $|c| + \|\tau^{-1+\delta} u'\|_{H^k_{\mathrm{b}}(\Omega)} \leq R$; that is, $\operatorname{supp} u \subset \{t \geq t_0\}$ *for all t_0 such that* $\operatorname{supp} f \subset \{t \geq t_0\}$.

Theorem 9.1.1 follows from Theorem 9.1.2 below which takes place in the more general geometric setting of non-trapping spacetimes, see Definition 2.5.1, and also allows for a larger class of nonlinearities. See Theorem 9.1.15 for the full statement of Theorem 9.1.1 in the more general setting, in particular for statements regarding stability and higher regularity, and the subsequent Remark 9.1.18 for more precise asymptotics. One can also consider equations on natural vector bundles; see the discussion later in the introduction. In a different direction, we can also solve *backward* problems in spaces with high decay at $\tau = 0$, see Theorem 9.1.24, where we can in fact replace $\Box_{g(u)}$ by $\Box_{g(u)} + L$ for first order operators L.

The novelty of our analysis of quasilinear wave and Klein-Gordon equations lies in combining the methods used in Chapter 5 to treat semilinear equations on static asymptotically de Sitter (and more general) spaces with the technology of pseudodifferential operators with non-smooth coefficients in the spirit of Beals and Reed [9], developed in Chapter 8, which is used to understand the regularity properties of operators like $\Box_{g(u)}$ in the above theorem. Our approach, appropriately adapted, also works in a variety of other settings, in particular on asymptotically Kerr-de Sitter spaces, where however a more delicate analysis is necessary in view of issues coming from trapping; we will obtain global well-posedness results for quasilinear wave equations on asymptotically Kerr-de Sitter spaces in §9.2, and the class of equations considered there is in fact even more general than (9.1.1) in that the metric is also allowed to depend on derivatives of u. In a different direction, asymptotically Minkowski spaces [8] should be analyzable as well using similar methods.

As in the results proved in previous chapters, the compactified picture is very powerful, as it puts equation (9.1.1) into a b-framework, where it reveals a rich microlocal structure which was already exploited in §5.2; in particular, the operator $\Box_{g(u)}$ is a perturbation of one that has radial points at the boundary. Then, as in Chapter 5, rather than solving an evolution equation for a short amount of time, controlling certain energies and iterating, we again use a global iterative procedure, where at each step we solve a linear equation, with non-smooth coefficients, of the form

$$P_{u_k} u_{k+1} \equiv (\Box_{g(u_k)} - \lambda) u_{k+1} = f + q(u_k, du_k) \tag{9.1.2}$$

globally on L^2-based b-Sobolev spaces or analogous spaces that encode partial expansions. Since the non-linearity q (as well as g) must be well-behaved relative to these, we work on high regularity spaces; recall here that $H^s(\mathbb{R}^n)$ is an algebra for $s > n/2$. Moreover, we need to prove decay (or at least non-growth) for solutions of (9.1.2) so that q can be considered a perturbation. The allowed asymptotics of solutions to the linear equation (9.1.2) are captured by the normal operator family of P_{u_k} at infinity, encoded as a compactification of the space. By virtue of the asymptotics of scalar linear waves on (approximately) static (asymptotically) de Sitter spaces, this family will for all k be a family of operators with *smooth* coefficients, thus one can use the results of Vasy [114, 111] to understand its behavior, in particular resonances, i.e. the location of the poles of the inverse Mellin transformed family and their structure, as well as stability results. Just as in the semilinear setting, we need to require the resonances to lie in the 'unphysical half-plane' $\operatorname{Im}\sigma < 0$ (a simple resonance at 0 is fine as well), since resonances in the 'physical half-plane' $\operatorname{Im}\sigma > 0$ would allow growing solutions to the equation, making the non-linearity non-perturbative and thus causing our method to fail. As in the smooth coefficient setting, we carry out the linear analysis of equations like (9.1.2) in two steps: the invertibility on high regularity spaces which however contain functions that are growing at future infinity (see Theorem 9.1.7), and the proof of decay corresponding to the location of resonances (see Theorem 9.1.8).

In the iteration scheme (9.1.2), notice that if $u_k \in H^s$ (more precisely, an H^s_{b}-based space), then the right hand side is in H^{s-1}. Now P_{u_k} has leading order coefficients in H^s and subprincipal terms with regularity H^{s-1}, and to keep the iteration running, we need that the solution operator for P_{u_k} maps H^{s-1} to H^s, the loss of (at least) one derivative being standard for hyperbolic problems. In other words, there is a delicate balance of the regularities involved, and the results of Chapter 8 provided the necessary robust regularity theory for operators like P_{u_k} on manifolds with boundary.

In order to emphasize the generality of the method, let us point out that *given an appropriate structure of the null-geodesic flow at ∞, for example radial points as above, the only obstruction to the solvability of quasilinear equations are resonances in the upper half plane.*

To set up the precise version of the above theorem, we work on a generalized static model Ω with Lorentzian b-metric g, see §2.2.2, and denote the defining function of future infinity by τ; thus, the asymptotic model at future infinity is exact static de Sitter space (extended beyond the cosmological horizon). The domain Ω is a compact submanifold with corners of the manifold M with boundary X as in (2.2.16), namely M is a neighborhood of the interior of the backward light cone from a point at future infinity of an asymptotically de Sitter-like space. See also Figure 2.4. One can in fact work on general non-trapping spacetimes, see Definition 2.5.1, provided the resonances and resonant states in the closed upper half plane are as on static de Sitter space, see equation (5.2.9), i.e. $\Box_g$ has a simple resonance at 0 with resonant space spanned by constant functions; we restrict ourselves to generalized static models merely for simplicity of the presentation. As in §5.2, we work on weighted b-Sobolev spaces $H_{\mathrm{b}}^{s,\alpha}(M) = \tau^\alpha H_{\mathrm{b}}^s(M)$, or rather on the spaces $H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}$ of restrictions of $H_{\mathrm{b}}^{s,\alpha}(M)$-functions with support in the future of the Cauchy hypersurface H_1 to Ω; that is, elements of $H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}$ are supported at H_1 and extendible at the artificial spacelike hypersurface H_2, see [64, Appendix B]. Finally, let $\mathcal{X}^{s,\alpha}$ be the space of all u which near $\tau = 0$ asymptotically look like a constant plus an $H_{\mathrm{b}}^{s,\alpha}$-function, i.e. for some $c \in \mathbb{C}$, $u' = u - c\chi(\tau) \in H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}$, where $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R})$, $\chi \equiv 1$ near 0, is a cutoff near $Y = \Omega \cap \partial M$; for such a function u, define its squared norm by

$$\|u\|_{\mathcal{X}^{s,\alpha}}^2 = |c|^2 + \|u'\|_{H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}}^2.$$

Our main theorem then is:

Theorem 9.1.2. *Let $s > n/2 + 7$ and $0 < \alpha < 1$. Assume that for $j = 0, 1$,*

$$g\colon \mathcal{X}^{s-j,\alpha} \to (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-j,\alpha})(M; S^{2\mathrm{b}}T_\Omega^* M),$$
$$q\colon \mathcal{X}^{s-j,\alpha} \times H_{\mathrm{b}}^{s-1-j,\alpha}(\Omega; {}^{\mathrm{b}}T_\Omega^* M)^{\bullet,-} \to H_{\mathrm{b}}^{s-1-j,\alpha}(\Omega)^{\bullet,-}$$

are continuous, g is locally Lipschitz, and

$$\|q(u, {}^{\mathrm{b}}du) - q(v, {}^{\mathrm{b}}dv)\|_{H_{\mathrm{b}}^{s-1-j,\alpha}(\Omega)^{\bullet,-}} \leq L_q(R)\|u - v\|_{\mathcal{X}^{s-j,\alpha}}$$

for $u, v \in \mathcal{X}^{s-j,\alpha}$ with norm $\leq R$, where $L_q\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuous and non-decreasing. Then there is a constant $C_L > 0$ so that the following holds: If $L_q(0) < C_L$, then for small $R > 0$, there is $C_f > 0$ such that for all $f \in H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-}$ with norm $\leq C_f$, there exists a unique solution $u \in \mathcal{X}^{s,\alpha}$ of the equation

$$\Box_{g(u)} u = f + q(u, {}^{\mathrm{b}}du)$$

with norm $\leq R$, and in the topology of $\mathcal{X}^{s-1,\alpha}$, u depends continuously of f.

See Theorem 9.1.11. Another case we study is $g(u) = \mu(u)g$, i.e. we only allow conformal changes of the metric; here, one can partly improve the above theorem, in particular allow non-linearities of the form $q(u, {}^{\mathrm{b}}du, \Box_{g(u)}u)$; see §9.1.3. The point of the Lipschitz assumptions on q in all these cases is to ensure that $q(u, {}^{\mathrm{b}}du)$ has a sufficient order of vanishing at $u = 0$ so that $q(u, {}^{\mathrm{b}}du)$ can be considered a perturbation of $\Box_{g(u)}$; quadratic vanishing is enough, but slightly less (simple vanishing will small Lipschitz constant near or at 0) also suffices.

Similar results hold for quasilinear Klein-Gordon equations with positive mass, where the asymptotics of solutions, hence the function spaces used, are different, namely the leading order term is now decaying; see §9.1.4 for details.

In §9.1.5 finally, we will discuss backward problems; the results there extend to the setting of Einstein's equations (after fixing a gauge) on static de Sitter and even on Kerr-de Sitter spacetimes, thus enabling constructions of dynamical black hole spacetimes in the spirit of recent work by Dafermos, Holzegel and Rodnianski [24], however the issue of constructing appropriate initial data sets is rather involved.

While all results were stated for scalar equations, corresponding results hold for operators acting on natural vector bundles, provided that all resonances lie in the unphysical half-plane $\operatorname{Im}\sigma < 0$ (with a simple resonance at 0 being fine as well): Indeed, the linear arguments go through in general for operators with scalar principal symbols; only the numerology of the needed regularities depends on estimates of the subprincipal symbol at (approximate) radial points.

Lastly, let us mention that paradifferential methods would give sharper results with respect to the regularity of the spaces in which we solve equation (9.1.1), and correspondingly we have not made any efforts here to push the regularity down. However, our entirely L^2-based method is both conceptually and technically relatively straightforward, powerful enough for our purposes, and lends itself very easily to generalizations in other contexts.

9.1.1 Analytic, geometric and dynamical assumptions on non-smooth linear problems

Since we have a calculus for non-smooth ps.d.o.s from Chapter 8 at our disposal, as well as energy estimates for non-smooth wave-type operators from §8.6, the arguments given in §5.2.1 leading to a Fredholm framework for the forward problem for wave-type operators P on non-smooth perturbations of the static model of de Sitter space, go through with only minor technical modifications. Because there are large dimension dependent losses in estimates for the adjoint of P, which acts on negative order b-Sobolev spaces (see the numerology in Proposition 8.2.9), relative to the regularity of the coefficients of P, say $\mathcal{C}^\infty + H_{\mathrm{b}}^s$ for the highest order ones, the spaces that P acts on as a Fredholm operator are roughly of the order $s - n/2$.

This can be vastly improved with a calculus for right quantizations of non-smooth symbols just like the one developed in Chapter 8 for left quantizations. Right quantizations have 'good' mapping properties on *negative* order (but lossy ones on positive order) b-Sobolev spaces. Correspondingly, all microlocal results (elliptic regularity, propagation of singularities, including at radial points) hold by the same proofs *mutatis mutandis*. Then, viewing P^* as the right quantization of a non-smooth symbol gives estimates which allow one to put P into a Fredholm framework on spaces with regularity $s - \epsilon$, $\epsilon > 0$.

Our focus here however is to prove the *invertibility* of the forward problem, whose discussion in §5.2.1 (in the smooth setting) we follow. Thus, consider a *non-trapping spacetime* (Ω, g) as in Definition 2.5.1, the main example being a generalized static model. We assume that for some $\alpha > 0$, the metric g satisfies

$$g \in \mathcal{C}^\infty(\Omega; S^{2\mathrm{b}}T^*_\Omega M) + H_{\mathrm{b}}^{s,\alpha}(\Omega; S^{2\mathrm{b}}T^*_\Omega M)^{\bullet,-},$$

where M is a neighborhood of Ω as in §2.2.1; thus, the metric g is a standard (incomplete) metric on Ω near the artificial hypersurfaces H_1 and H_2, while it is a b-metric near future infinity $Y = \Omega \cap \partial M$. We consider the operator

$$P = \Box_g + L, \quad L \in (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1,\alpha})\mathrm{Diff}_{\mathrm{b}}^1 + (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1,\alpha}),$$

thus

$$P \in (\mathcal{C}^\infty + H_{\mathrm{b}}^{s,\alpha})\mathrm{Diff}_{\mathrm{b}}^2 + (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1,\alpha})\mathrm{Diff}_{\mathrm{b}}^1 + (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1,\alpha}). \tag{9.1.3}$$

The assumption that Ω is a non-trapping spacetime in particular means that P has radial points, which are saddles of the Hamilton flow of the principal symbol of P, at the b-conormal bundle of the horizons, or more generally P is a perturbation of such an operator; see Remark 2.2.4. We use the notation $L_\pm$ for the radial set and $\widetilde{\beta}$, $\widehat{\beta}$ for the dynamical quantities at $L_\pm$ defined prior to Theorem 8.5.10, see equations (8.5.38) and (8.5.40). Moreover, the metric g is non-trapping in the sense of Proposition 2.2.3 (7). Further, we denote by $\mathfrak{t}_1$ and $\mathfrak{t}_2$ two smooth functions on M which are defining functions of H_1 and H_2, respectively, and put for δ_1, δ_2 small

$$\Omega_{\delta_1,\delta_2} := \mathfrak{t}_1^{-1}([\delta_1,\infty)) \cap \mathfrak{t}_2^{-1}([\delta_2,\infty)), \quad \Omega \equiv \Omega_{0,0},$$
$$\Omega^\circ_{\delta_1,\delta_2} := \mathfrak{t}_1^{-1}((\delta_1,\infty)) \cap \mathfrak{t}_2^{-1}((\delta_2,\infty));$$

we assume that the differentials of $\mathfrak{t}_1$ and $\mathfrak{t}_2$ have the opposite timelike character near their respective zero sets within $\Omega = \Omega_0$, more specifically, $\mathfrak{t}_1$ is future timelike, $\mathfrak{t}_2$ past timelike, and we assume that $\Omega_{\delta_1,\delta_2}$ is compact. Furthermore, we assume that the boundary defining function τ is such that $d\tau/\tau$ is timelike and past-oriented on Ω.

Recall Figure 5.1 for the setup, and Definition 2.5.1 for the full set of assumptions. Denote by $H_{\mathrm{b}}^{s,r}(\Omega_{\delta_1,\delta_2})^{\bullet,-}$ distributions which are supported ($\bullet$) at the 'artificial' boundary hypersurface $\mathfrak{t}_1^{-1}(\delta_1)$ and extendible ($-$) at $\mathfrak{t}_2^{-1}(\delta_2)$, and the other way around for $H_{\mathrm{b}}^{s,r}(\Omega_{\delta_1,\delta_2})^{-,\bullet}$. Then we have the following global energy estimate, which is entirely analogous to Lemma 4.2.1.

Lemma 9.1.3. *Suppose $s > n/2+2$. There exists $r_0 < 0$ such that for $r \leq r_0$, $-\widetilde{r} \leq r_0$, there is $C > 0$ such that for $u \in H_{\mathrm{b}}^{2,r}(\Omega_{\delta_1,\delta_2})^{\bullet,-}$, $v \in H_{\mathrm{b}}^{2,\widetilde{r}}(\Omega_{\delta_1,\delta_2})^{-,\bullet}$, one has*

$$\|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{\delta_1,\delta_2})^{\bullet,-}} \leq C\|Pu\|_{H_{\mathrm{b}}^{0,r}(\Omega_{\delta_1,\delta_2})^{\bullet,-}},$$
$$\|v\|_{H_{\mathrm{b}}^{1,\widetilde{r}}(\Omega_{\delta_1,\delta_2})^{-,\bullet}} \leq C\|P^*v\|_{H_{\mathrm{b}}^{0,\widetilde{r}}(\Omega_{\delta_1,\delta_2})^{-,\bullet}}.$$

If one replaces C by any $C' > C$, the estimates also hold for small perturbations of P in the space indicated in (9.1.3).

Proof. The proof follows the proof of Lemma 4.2.1, adapted to the non-smooth setting as in Proposition 8.6.1. □

By a duality argument and the propagation of singularities, we thus obtain solvability and higher regularity, as in Corollaries 4.1.7 and 4.2.2:

Lemma 9.1.4. *Let $0 \leq s' \leq s$ and assume $s > n/2+6$. There exists $r_0 < 0$ such that for $r \leq r_0$, there is $C > 0$ with the following property: If $f \in H_{\mathrm{b}}^{s'-1,r}(\Omega)^{\bullet,-}$, then there exists a unique $u \in H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}$ such that $Pu = f$, and u moreover satisfies*

$$\|u\|_{H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}} \leq C\|f\|_{H_{\mathrm{b}}^{s'-1,r}(\Omega)^{\bullet,-}}.$$

If one replaces C by any $C' > C$, this result also holds for small perturbations of P in the space indicated in (9.1.3).

Proof. We follow the proof of Corollary 4.1.7: Choose $\delta_1 < 0$ and $\delta_2 < 0$ small, and choose an extension

$$\widetilde{f} \in H_{\mathrm{b}}^{s'-1,r}(\Omega_{0,\delta_2})^{\bullet,-} \subset H_{\mathrm{b}}^{-1,r}(\Omega_{0,\delta_2})^{\bullet,-}$$

satisfying

$$\|\widetilde{f}\|_{H_{\mathrm{b}}^{s'-1,r}(\Omega_{0,\delta_2})^{\bullet,-}} \leq 2\|f\|_{H_{\mathrm{b}}^{s'-1,r}(\Omega)^{\bullet,-}}. \tag{9.1.4}$$

By Lemma 9.1.3, applied with $\widetilde{r} = -r$, we have

$$\|\phi\|_{H_{\mathrm{b}}^{1,\widetilde{r}}(\Omega_{0,\delta_2})^{-,\bullet}} \leq C\|P^*\phi\|_{H_{\mathrm{b}}^{0,\widetilde{r}}(\Omega_{0,\delta_2})^{-,\bullet}}$$

for $\phi \in H_{\mathrm{b}}^{2,\widetilde{r}}(\Omega_{0,\delta_2})^{-,\bullet}$. By the Hahn-Banach theorem, we conclude that there exists $\widetilde{u} \in H_{\mathrm{b}}^{0,\widetilde{r}}(\Omega_{0,\delta_2})^{\bullet,-}$ such that

$$\langle P\widetilde{u}, \phi\rangle = \langle \widetilde{u}, P^*\phi\rangle = \langle f, \phi\rangle, \quad \phi \in H_{\mathrm{b}}^{2,\widetilde{r}}(\Omega_{0,\delta_2})^{-,\bullet},$$

and

$$\|\widetilde{u}\|_{H_{\mathrm{b}}^{0,\widetilde{r}}(\Omega_{0,\delta_2})^{\bullet,-}} \leq C\|\widetilde{f}\|_{H_{\mathrm{b}}^{-1,\widetilde{r}}(\Omega_{0,\delta_2})^{\bullet,-}}. \tag{9.1.5}$$

We can view $\widetilde{u}$ as an element of $H_{\mathrm{b}}^{0,\widetilde{r}}(\Omega_{\delta_1,\delta_2})^{\bullet,-}$ with support in Ω_{0,δ_2}, similarly for $\widetilde{f}$; then $\langle P\widetilde{u}, \phi\rangle = \langle \widetilde{f}, \phi\rangle$ for all $\phi \in \dot{\mathcal{C}}_{\mathrm{c}}^{\infty}(\Omega_{\delta_1,\delta_2}^{\circ})$ (with the dot referring to infinite order of vanishing at ∂M), i.e. $P\widetilde{u} = \widetilde{f}$ as distributions on $\Omega_{\delta_1,\delta_2}^{\circ}$.

Now, $\widetilde{u}$ vanishes on $\Omega_{\delta_1,\delta_2}^{\circ} \setminus \Omega_{0,\delta_2}$, in particular is in $H_{\mathrm{b,loc}}^{s',r}$ there. Elliptic regularity and the propagation of singularities, Theorems 8.4.1, 8.5.6 and 8.5.10, imply that $\widetilde{u} \in H_{\mathrm{b,loc}}^{s',r}(\Omega_{\delta_1,\delta_2}^{\circ})$. Indeed, by Theorem 8.4.1 with $\widetilde{s} = -1$, $\widetilde{u}$ is in $H_{\mathrm{b}}^{1/2,r}$ on the elliptic set of P within $\Omega_{\delta_1,\delta_2}^{\circ}$; Theorem 8.5.6 with $\widetilde{s} = -1/2$ gives $H_{\mathrm{b}}^{1/2,r}$-control of $\widetilde{u}$ on the characteristic set away from radial points, and then an application of Theorem 8.5.10 gives $H_{\mathrm{b}}^{1/2,r}$-control of $\widetilde{u}$ on all of $\Omega_{\delta_1,\delta_2}^{\circ}$.[48] Iterating this argument gives $H_{\mathrm{b,loc}}^{s',r}(\Omega_{\delta_1,\delta_2})^{\circ}$, and we in fact get an

[48]The conditions of all theorems used here are satisfied because of $s > n/2 + 6$; if necessary, we need to make r_0 smaller, i.e. assume that $r \leq r_0$ is more negative, in order for the assumptions of Theorem 8.5.10 to be fulfilled. Strictly speaking, we in fact need to use localized estimates in the following sense: If $\widetilde{u} \in H_{\mathrm{b}}^{\widetilde{s},r}$ and $P\widetilde{u} \in H_{\mathrm{b}}^{\widetilde{s}-1/2,r}$, and if $\chi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\Omega_{\delta_1,\delta_2}^{\circ})$ is identically 1 near a point x_0, then $P\chi\widetilde{u} = \chi P\widetilde{u} + [P,\chi]\widetilde{u}$ is in $H_{\mathrm{b}}^{\widetilde{s}-1/2,r}$ in a neighborhood of x_0 and globally in $H_{\mathrm{b}}^{\widetilde{s}-1,r}$, since $[P,\chi]$ is a first order operator. By inspection of the relevant theorems, in particular (8.5.8), this regularity suffices to apply the relevant microlocal regularity

estimate

$$\|\chi\widetilde{u}\|_{H_{\mathrm{b}}^{s',r}(\Omega_{\delta_1,\delta_2})} \leq C\big(\|\widetilde{\chi}P\widetilde{u}\|_{H_{\mathrm{b}}^{s'-1,r}(\Omega_{\delta_1,\delta_2})} + \|\widetilde{\chi}\widetilde{u}\|_{H_{\mathrm{b}}^{0,r}(\Omega_{\delta_1,\delta_2})}\big)$$

for appropriate $\chi, \widetilde{\chi} \in \mathcal{C}_{\mathrm{c}}^\infty(\Omega^\circ_{\delta_1,\delta_2})$, $\widetilde{\chi} \equiv 1$ on $\operatorname{supp}\chi$. In view of the support properties of $\widetilde{u}$, an appropriate choice of χ and $\widetilde{\chi}$ gives that the restriction of $\widetilde{u}$ to Ω is an element of $H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}$, with norm bounded by the $H_{\mathrm{b}}^{s'-1,r}(\Omega)^{\bullet,-}$-norm of f in view of (9.1.5) and (9.1.4).

To prove uniqueness, suppose $u \in H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}$ satisfies $Pu = 0$, then, viewing u as a distribution on $\Omega^\circ_{\delta_1,0}$ with support in Ω, elliptic regularity and the propagation of singularities, applied as above, give $u \in H_{\mathrm{b,loc}}^{s,r}(\Omega^\circ_{\delta_1,0}) \subset H_{\mathrm{b,loc}}^{2,r}(\Omega^\circ_{\delta_1,0})$; hence, for any $\widetilde{\delta} > 0$, Lemma 9.1.3 applied to $u' = u|_{\Omega_{0,\widetilde{\delta}}} \in H_{\mathrm{b}}^{2,r}(\Omega_{0,\widetilde{\delta}})^{\bullet,-}$ gives $u' = 0$, thus, since $\widetilde{\delta} > 0$ is arbitrary, $u = 0$. □

Corollary 9.1.5. *(Cf. Corollary 4.2.3.) Let $0 \leq s' \leq s$ and assume $s > n/2 + 6$. There exists $r_0 < 0$ such that for $r \leq r_0$, there is $C > 0$ with the following property: If $u \in H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}$ is such that $Pu \in H_{\mathrm{b}}^{s'-1,r}(\Omega)^{\bullet,-}$, then*

$$\|u\|_{H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}} \leq C\|Pu\|_{H_{\mathrm{b}}^{s'-1,r}(\Omega)^{\bullet,-}}.$$

If one replaces C by any $C' > C$, this result also holds for small perturbations of P in the space indicated in (9.1.3).

Proof. Let $u' \in H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}$ be the solution of $Pu' = Pu$ given by the existence part Lemma 9.1.4, then $P(u - u') = 0$, and the uniqueness part implies $u = u'$. □

We also obtain the following propagation of singularities type result:

Corollary 9.1.6. *(Cf. Proposition 4.1.10.) Let $0 \leq s'' \leq s' \leq s$ and assume $s > n/2 + 6$; moreover, let $r \in \mathbb{R}$ be such that $s'' - 1 + \inf_{L_\pm}(\widehat{\beta} - r\widetilde{\beta}) > 0$. Then there is $C > 0$ such that the following holds: Any $u \in H_{\mathrm{b}}^{s'',r}(\Omega)^{\bullet,-}$ with $Pu \in H_{\mathrm{b}}^{s'-1,r}(\Omega)^{\bullet,-}$ in fact satisfies $u \in H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}$, and obeys the estimate*

$$\|u\|_{H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}} \leq C(\|Pu\|_{H_{\mathrm{b}}^{s'-1,r}(\Omega)^{\bullet,-}} + \|u\|_{H_{\mathrm{b}}^{s'',r}(\Omega)^{\bullet,-}}).$$

If one replaces C by any $C' > C$, this result also holds for small perturbations of P in the space indicated in (9.1.3).

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results and deduce microlocal $H_{\mathrm{b}}^{\widetilde{s}+1/2,r}$-regularity of $\widetilde{u}$.

Proof. As in the proof of Lemma 9.1.4, working on $\Omega_{\delta_1,0}$ for $\delta_1 < 0$ small, we obtain $u \in H^{s',r}_{\mathrm{b,loc}}$ by iteratively using elliptic regularity, real principal type propagation and the propagation near radial points; the latter, applied in the first step with $\widetilde{s} = s'' - 1/2$, is the reason for the condition on s''. Thus, $u \in H^{s',r}_{\mathrm{b}}(\Omega_{0,\widetilde{\delta}})^{\bullet,-}$ for $\widetilde{\delta} > 0$. From here, arguing as in the proof of Proposition 4.1.10, we obtain the desired conclusion. □

Let us rephrase Lemma 9.1.4 and Corollary 9.1.5 as an invertibility statement:

Theorem 9.1.7. *(Cf. Theorem 4.2.4.) Let $0 \leq s' \leq s$ and assume $s > n/2 + 6$. There exists $r_0 < 0$ with the following property: Let $r \leq r_0$ and define the spaces*

$$\mathcal{X}^{s,r} = \{u \in H^{s,r}_{\mathrm{b}}(\Omega)^{\bullet,-} : Pu \in H^{s-1,r}_{\mathrm{b}}(\Omega)^{\bullet,-}\}, \quad \mathcal{Y}^{s,r} = H^{s,r}_{\mathrm{b}}(\Omega)^{\bullet,-}.$$

Then $P \colon \mathcal{X}^{s,r} \to \mathcal{Y}^{s-1,r}$ is a continuous, invertible map with continuous inverse.

Moreover, the operator norm of the inverse, as a map from $H^{s-1,r}_{\mathrm{b}}(\Omega)^{\bullet,-}$ to $H^{s,r}_{\mathrm{b}}(\Omega)^{\bullet,-}$, of small perturbations of P in the space indicated in (9.1.3) is uniformly bounded.

We can now apply the arguments of §5.2.1, see also [114] for the dilation-invariant case, to obtain more precise asymptotics of solutions u to $Pu = f$ using the knowledge of poles of the inverse of the Mellin transformed normal operator family $\widehat{P}(\sigma)$, where the normal operator $N(P)$ of P is defined just as in the smooth setting by 'freezing' the coefficients of P at the boundary ∂M. This makes sense in our setting since the coefficients of P are continuous; also, the coefficients of $N(P)$ are then smooth, since all non-smooth contributions to P vanish at the boundary.

Theorem 9.1.8. *(Cf. Theorem 5.2.3.) Let $s > n/2 + 6$, $0 < \alpha < 1$, and assume $g \in \mathcal{C}^\infty(\Omega; S^{2\mathrm{b}}T^*_\Omega M) + H^{s,\alpha}_{\mathrm{b}}(\Omega; S^{2\mathrm{b}}T^*_\Omega M)$. Let*

$$P = \Box_g + L, \quad L \in (\mathcal{C}^\infty + H^{s-1,\alpha}_{\mathrm{b}})\mathrm{Diff}^1_{\mathrm{b}} + (\mathcal{C}^\infty + H^{s-1,\alpha}_{\mathrm{b}}).$$

Further, let $\mathfrak{t}_1$ and $\Omega \subset M$ and the metric g be as above. Let σ_j be the poles of $\widehat{P}^{-1}(\sigma)$, of which there are only finitely many in any half space $\operatorname{Im}\sigma \geq -C$. Let $r \in \mathbb{R}$ be such that $r \neq \operatorname{Im}\sigma_j$ and $r \leq -\operatorname{Im}\sigma_j + \alpha$ for all j, and let $r_0 \in \mathbb{R}$. Moreover, let $1 \leq s_0 \leq s' \leq s$, and suppose that

$$s' - 2 + \inf_{L_\pm}(\widehat{\beta} - r\widetilde{\beta}) > 0.$$

Finally, let $\phi \in \mathcal{C}^\infty(\mathbb{R})$ be such that $\operatorname{supp}\phi \subset (0, \infty)$ and $\phi \circ \mathfrak{t}_1 \equiv 1$ near $\partial M \cap \Omega$.

Then any solution $u \in H_{\mathrm{b}}^{s_0,r_0}(\Omega)^{\bullet,-}$ *of* $Pu = f$ *with* $f \in H_{\mathrm{b}}^{s'-1,r}(\Omega)^{\bullet,-}$ *satisfies*

$$u - \sum_j x^{i\sigma_j}(\phi \circ \mathfrak{t}_1)a_j = u' \in H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}$$

for some $a_j \in \mathcal{C}^\infty(\partial M \cap \Omega)$*, where the sum is understood over the finite set of* j *such that* $-\operatorname{Im}\sigma_j < r < -\operatorname{Im}\sigma_j + \alpha$*.*

The result is stable under small perturbations of P *in the space indicated in assumption (9.1.3) in the sense that, even though the* σ_j *might change, all* $\mathcal{C}^\infty$*-seminorms of the expansion terms* a_j *and the* $H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}$*-norm of the remainder term* u' *are bounded by* $C(\|u\|_{H_{\mathrm{b}}^{s_0,r_0}(\Omega)^{\bullet,-}} + \|f\|_{H_{\mathrm{b}}^{s'-1,r}(\Omega)^{\bullet,-}})$ *for some uniform constant* C *(depending on which norm we are bounding).*

Proof. By making r_0 smaller (i.e. more negative) if necessary, we may assume that $r_0 \leq r$ and

$$s_0 - 1 + \inf_{L_\pm}(\widehat{\beta} - r_0\widetilde{\beta}) > 0.$$

First, assume $\sigma_* := \min_j\{-\operatorname{Im}\sigma_j\} > r$. Then $u \in H_{\mathrm{b}}^{s_0,r_0}(\Omega)^{\bullet,-}$ and $Pu = f \in H_{\mathrm{b}}^{s'-1,r}(\Omega)^{\bullet,-}$ imply $u \in H_{\mathrm{b}}^{s',r_0}(\Omega)^{\bullet,-}$ by Corollary 9.1.6. Since

$$P - N(P) \in (x\mathcal{C}^\infty + H_{\mathrm{b}}^{s,\alpha})\mathrm{Diff}_{\mathrm{b}}^2 + (x\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1,\alpha})\mathrm{Diff}_{\mathrm{b}}^1 + (x\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1,\alpha}),$$

we thus obtain $\widetilde{f} := (P - N(P))u \in H_{\mathrm{b}}^{s'-2,r_0+\alpha}(\Omega)^{\bullet,-}$, where we use $s \geq s' - 2$ and $s - 1 \geq s' - 1$; hence

$$N(P)u = f - \widetilde{f} \in H_{\mathrm{b}}^{s'-2,r'}(\Omega)^{\bullet,-}$$

with $r' = \min(r, r_0 + \alpha)$. Applying[49] [114, Lemma 3.1] gives $u \in H_{\mathrm{b}}^{s'-1,r'}(\Omega)^{\bullet,-}$ in view of the absence of poles of $\widehat{P}(\sigma)$ in $\operatorname{Im}\sigma \geq -r$; but then $Pu \in H_{\mathrm{b}}^{s'-1,r}(\Omega)^{\bullet,-}$ implies $u \in H_{\mathrm{b}}^{s',r'}(\Omega)^{\bullet,-}$, again by Corollary 9.1.6, where we use

$$(s' - 1) - 1 + \inf(\widehat{\beta} - r'\widetilde{\beta}) \geq s' - 2 + \inf(\widehat{\beta} - r\widetilde{\beta}) > 0.$$

If $r' = r$, we are done; otherwise, we iterate, replacing r_0 by $r_0 + \alpha$, and obtain $u \in H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}$ after finitely many steps.

[49]This requires $s' \geq 1$ in view of the supported/extendible spaces that we are using here; see also the proof of Theorem 5.2.3.

If there are σ_j with $-\operatorname{Im}\sigma_j < r$, then, assuming that $\sigma_* - \alpha < r_0 < \sigma_*$, in fact that r_0 is arbitrarily close to σ_*, as we may by the first part of the proof, the application of [114, Lemma 3.1] gives a partial expansion u_1 of u with remainder $u' \in H_{\mathrm{b}}^{s'-1,r'}(\Omega)^{\bullet,-}$, where $r' = \min(r, r_0 + \alpha)$. Now $N(P)u_1 = 0$, and u_1 is a sum of terms of the form $a_j x^{i\sigma_j}$ with $\operatorname{Im}\sigma_j \leq -\sigma_*$ and $a_j \in \mathcal{C}^\infty(\partial M \cap \Omega)$, in particular $u_1 \in H_{\mathrm{b}}^{\infty,r_0}(\Omega)^{\bullet,-}$; thus

$$(P - N(P))u_1 \in H_{\mathrm{b}}^{\infty,r_0+1}(\Omega)^{\bullet,-} + H_{\mathrm{b}}^{s-1,\sigma_*+\alpha}(\Omega)^{\bullet,-} \subset H_{\mathrm{b}}^{s-1,\sigma_*+\alpha}(\Omega)^{\bullet,-}, \tag{9.1.6}$$

where the two terms correspond to the coefficients of $P - N(P)$ being sums of $x\mathcal{C}^\infty$- and $H_{\mathrm{b}}^{s-1,\alpha}$-functions. Therefore,

$$Pu' = Pu - N(P)u_1 - (P - N(P))u_1 \in H_{\mathrm{b}}^{s'-1,r}(\Omega)^{\bullet,-}, \tag{9.1.7}$$

which by Corollary 9.1.6 implies $u' \in H_{\mathrm{b}}^{s',r'}(\Omega)^{\bullet,-}$, finishing the proof in the case that $r' = r$, i.e. $r < \sigma_* + \alpha$. If $r = \sigma_* + \alpha$, we need one more iterative step to establish the improvement in the weight of u': We use $u' \in H_{\mathrm{b}}^{s',r'}$ to deduce

$$N(P)u = f - (P - N(P))u \in H_{\mathrm{b}}^{s'-1,r} + H_{\mathrm{b}}^{s'-2,r'+\alpha} + H_{\mathrm{b}}^{s-1,\sigma_*+\alpha} \subset H_{\mathrm{b}}^{s'-2,\sigma_*+\alpha},$$

where we use $(P - N(P))u' \in H_{\mathrm{b}}^{s'-2,r'+\alpha}$ and (9.1.6). Hence [114, Lemma 3.1] implies that the partial expansion $u = u_1 + u'$ in fact holds with $u' \in H_{\mathrm{b}}^{s'-1,r}$, and then Corollary 9.1.6 and (9.1.7) imply $u' \in H_{\mathrm{b}}^{s',r}$, finishing the proof in the case $r = \sigma_* + \alpha$. □

Remark 9.1.9. In the smooth setting, one can use the partial expansion u_1 to obtain better information on $\widetilde{f}$ for a next step in the iteration. This however relies on the fact that $P - N(P) \in x\mathrm{Diff}_{\mathrm{b}}^2$ there (see the proof of Theorem 5.2.3); here, however, we also have terms in the space $H_{\mathrm{b}}^{s-1,\alpha}\mathrm{Diff}_{\mathrm{b}}^2$ in $P - N(P)$, and H_{b}^{s-1}-functions do not have a Taylor expansion at $x = 0$, hence the above iteration scheme does not yield additional information after the first step in which one gets a non-trivial part u_1 of the expansion of u. If however we encode more precise asymptotics in the function space in which g lies, then $P - N(P)$ has a partial polyhomogeneous expansion which can be used to obtain more precise asymptotics for u. See also Remark 9.1.18.

Combining Theorem 9.1.8 with Theorem 9.1.7 gives us a forward solution operator for P which, provided we understand the poles of $\widehat{P}(\sigma)^{-1}$, will be the key tool in our discussion of quasilinear wave equations on non-trapping spacetimes in the next section.

9.1.2 Solving quasilinear wave equations

We are now prepared to discuss existence, uniqueness and asymptotics of solutions to quasilinear wave and Klein-Gordon equations for complex- and/or real-valued functions on the static model of de Sitter space, in fact on the domain Ω described in the previous section, with small data, i.e. small forcing. Keep in mind though that the methods work in greater generality, as explained at the beginning of this section. In particular, we will prove Theorems 9.1.1 and 9.1.2.

We stick to the scalar case here for simplicity, rather than considering wave equations on natural vector bundles. We remark however that we understand resonances for instance for differential forms rather precisely, see §7.4.1. The general statement is that as long as there is no resonance or only a simple resonance at 0 in the closed upper half plane (with the non-linearity annihilating it), the arguments presented in this section go through. Likewise, we can work on the more general class of static asymptotically de Sitter spaces, since the normal operator, hence the resonances are the same as on exact static de Sitter space, and in fact on much more general spacetimes, namely non-trapping spacetimes in the sense of Definition 2.5.1, provided the above resonance condition as well as all assumptions in §9.1.1 are satisfied; examples of the latter kind include perturbations (even of the asymptotic model) of asymptotically de Sitter spaces. See Remark 9.1.12. The results on spacetimes with normally hyperbolic trapping in §9.2 will be formulated in this type of generality.

Let us from now on denote by g_{dS} the static de Sitter metric. We start with a discussion of quasilinear wave equations.

Definition 9.1.10. For $s, \alpha \in \mathbb{R}$, define the Hilbert space

$$\mathcal{X}^{s,\alpha} := \mathbb{C} \oplus H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}$$

with norm $\|(c,v)\|^2_{\mathcal{X}^{s,\alpha}} = |c|^2 + \|v\|^2_{H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}}$. We will identify an element $(c,v) \in \mathcal{X}^{s,\alpha}$ with the distribution $(\phi \circ \mathfrak{t}_1)c + v$, where ϕ and $\mathfrak{t}_1$ are as in the statement of Theorem 9.1.8.

Theorem 9.1.11. *Let $s > n/2 + 7$ and $0 < \alpha < 1$. Assume that for $j = 0, 1$,*

$$g \colon \mathcal{X}^{s-j,\alpha} \to (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-j,\alpha})(\Omega; S^{2\,\mathrm{b}}T^*_\Omega M),$$
$$q \colon \mathcal{X}^{s-j,\alpha} \times H_{\mathrm{b}}^{s-l-j,\alpha}(\Omega; {}^{\mathrm{b}}T^*_\Omega M)^{\bullet,-} \to H_{\mathrm{b}}^{s-1-j,\alpha}(\Omega)^{\bullet,-}$$

are continuous, g is Lipschitz near 0, and

$$\|q(u,{}^{\mathrm{b}}du)-q(v,{}^{\mathrm{b}}dv)\|_{H_{\mathrm{b}}^{s-1-j,\alpha}(\Omega)^{\bullet,-}}\leq L_q(R)\|u-v\|_{\mathcal{X}^{s-j,\alpha}} \tag{9.1.8}$$

for $u,v\in\mathcal{X}^{s-j,\alpha}$ with norm $\leq R$, then there is a constant $C_L>0$ so that the following holds: If $L_q(0)<C_L$, then for small $R>0$, there is $C_f>0$ such that for all $f\in H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-}$ with norm $\leq C_f$, there exists a unique *solution $u\in\mathcal{X}^{s,\alpha}$ of the equation*

$$\Box_{g(u)}u=f+q(u,{}^{\mathrm{b}}du) \tag{9.1.9}$$

with norm $\leq R$, and in the topology of $\mathcal{X}^{s-1,\alpha}$, u depends continuously on f.

Remark 9.1.12. Note that the poles of the meromorphic family $\widehat{\Box_{g(u)}}^{-1}$ depend continuously on u (see [114]), and the simple pole at 0, corresponding to constant functions being annihilated by $N(\Box_{g(u)})$, is preserved under perturbations. This will be crucial in the proof, and it also shows that we may allow the metric $g(0)$ to be a perturbation (in the b-sense) of g_{dS}, rather than exact g_{dS}, without any additional work. (As mentioned before, working on general non-trapping spacetimes requires the understanding of the resonances, which goes beyond the perturbative regime with which this remark deals.)

Remark 9.1.13. Of course, we require all sections $g(u)$ of $S^2\,{}^{\mathrm{b}}T^*_\Omega M$ to take values in symmetric 2-tensors with real coefficients. If we assume that q and f are real-valued, we may therefore work in the *real* Hilbert space

$$\mathcal{X}^{s,\alpha}_{\mathbb{R}}:=\mathbb{R}\oplus H_{\mathrm{b}}^{s,\alpha}(\Omega;\mathbb{R})^{\bullet,-} \tag{9.1.10}$$

and find the solution u there. This remark also applies to all theorems later in this section.

Proof of Theorem 9.1.11. To not overburden the notation, we will occasionally write $H_{\mathrm{b}}^{\sigma,\rho}$ in place of $H_{\mathrm{b}}^{\sigma,\rho}(\Omega)^{\bullet,-}$ if the context is clear.

By assumption on g, there exists R_S such that for $u\in\mathcal{X}^{s,\alpha}$ with $\|u\|_{\mathcal{X}^{s,\alpha}}\leq R_S$, the domain Ω equipped with the metric $g(u)$ is still a non-trapping spacetime, i.e. the operator $\Box_{g(u)}$ satisfies the assumptions listed at the beginning of §9.1.1. Hence, Theorem 9.1.7 is applicable, giving a continuous forward solution operator $S_{g(u)}$ on sufficiently weighted b-Sobolev spaces. For such u, the normal operator $N(\Box_{g(u)})$ is a small perturbation of $N(\Box_{g_{\mathrm{dS}}})$ in $\mathrm{Diff}^2(Y)$, and since further $s-2-\alpha>0$, we can apply Theorem 9.1.8 to conclude that the solution operator in fact maps

$$S_{g(u)}\colon H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-} \to \mathcal{X}^{s,\alpha}$$

continuously, with uniformly bounded operator norm

$$\|S_{g(u)}\| \leq C_S, \quad \|u\|_{\mathcal{X}^{s,\alpha}} \leq R_S. \tag{9.1.11}$$

Let $C_L := C_S^{-1}$, and assume that $L_q(0) < C_L$, then $L_q(R_q) < C_L$ for $R_q > 0$ small. Put $R := \min(R_S, R_q)$ and $C_f = R(C_S^{-1} - L_q(R))$; let $f \in H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-}$ have norm $\leq C_f$. Define $u_0 := 0$ and iteratively $u_{k+1} \in \mathcal{X}^{s,\alpha}$ by solving

$$\Box_{g(u_k)} u_{k+1} = f + q(u_k, {}^{\mathrm{b}}du_k), \tag{9.1.12}$$

i.e. $u_{k+1} = S_{g(u_k)}\big(f + q(u_k, {}^{\mathrm{b}}du_k)\big)$. For u_{k+1} to be well-defined, we need to check that $\|u_k\|_{\mathcal{X}^{s,\alpha}} \leq R$ for all k. For $k = 0$, this is clear; for $k > 0$, we deduce from (9.1.11) and (9.1.8) that

$$\begin{aligned} \|u_{k+1}\|_{\mathcal{X}^{s,\alpha}} &\leq C_S\big(\|f\|_{H_{\mathrm{b}}^{s-1,\alpha}} + L_q(R)\|u_k\|_{\mathcal{X}^{s,\alpha}}\big) \\ &\leq C_S\big(R(C_S^{-1} - L_q(R)) + L_q(R)R\big) = R. \end{aligned}$$

We aim to show that the sequence $(u_k)_k$ is in fact Cauchy in $\mathcal{X}^{s-1,\alpha}$. First, we observe that for $u \in \mathcal{X}^{s-1,\alpha}$, we have

$$\Box_{g(u)} = g^{ij}(u)\,{}^{\mathrm{b}}D_i\,{}^{\mathrm{b}}D_j + \widetilde{g}^j(u, {}^{\mathrm{b}}du)\,{}^{\mathrm{b}}D_j$$

with $g^{ij}(u) \in \mathcal{C}^\infty + H_{\mathrm{b}}^{s-1,\alpha}$, $\widetilde{g}^j(u, {}^{\mathrm{b}}du) \in \mathcal{C}^\infty + H_{\mathrm{b}}^{s-2,\alpha}$; using the explicit formula for the inverse of a metric, Corollary 8.2.10 and Lemma 8.3.2, we deduce from the Lipschitz assumption on g that

$$g^{ij}\colon \mathcal{X}^{s-1,\alpha} \to \mathcal{C}^\infty + H_{\mathrm{b}}^{s-1,\alpha}, \quad \widetilde{g}^j\colon \mathcal{X}^{s-1,\alpha} \to \mathcal{C}^\infty + H_{\mathrm{b}}^{s-2,\alpha}$$

are Lipschitz as well; hence, for some constant $C_g(R)$, we obtain

$$\|\Box_{g(u)} - \Box_{g(v)}\|_{\mathcal{L}(\mathcal{X}^{s,\alpha}, H_{\mathrm{b}}^{s-2,\alpha})} \leq C_g(R)\|u - v\|_{\mathcal{X}^{s-1,\alpha}}$$

for $u, v \in \mathcal{X}^{s-1,\alpha}$ with $\mathcal{X}^{s,\alpha}$-norms $\leq R$. Therefore, we get the following estimate for the difference of two solution operators $S_{g(u)}$ and $S_{g(v)}$, $u, v \in \mathcal{X}^{s,\alpha}$, with a loss of 2 derivatives relative to the elliptic setting, using a 'resolvent identity:'

$$\begin{aligned} \|S_{g(u)}-S_{g(v)}\|_{\mathcal{L}(H_{\mathrm{b}}^{s-1,\alpha},\mathcal{X}^{s-1,\alpha})} &= \|S_{g(u)}(\Box_{g(v)}-\Box_{g(u)})S_{g(v)}\|_{\mathcal{L}(H_{\mathrm{b}}^{s-1,\alpha},\mathcal{X}^{s-1,\alpha})} \\ &\leq C_S^2\|\Box_{g(u)}-\Box_{g(v)}\|_{\mathcal{L}(\mathcal{X}^{s,\alpha},H_{\mathrm{b}}^{s-2,\alpha})} \leq C_S^2 C_g(R)\|u-v\|_{\mathcal{X}^{s-1,\alpha}}. \end{aligned} \tag{9.1.13}$$

Here, we assumed C_S is such that $\|S_{g(u)}\|_{\mathcal{L}(H_{\mathrm{b}}^{s-2,\alpha},\mathcal{X}^{s-1,\alpha})} \leq C_S$ for small $u \in \mathcal{X}^{s,\alpha}$, which is where we use that $s-1 > n/2+6$. Returning to the goal of proving that $(u_k)_k$ is Cauchy in $\mathcal{X}^{s-1,\alpha}$, we estimate

$$\begin{aligned} \|u_{k+1}-u_k\|_{\mathcal{X}^{s-1,\alpha}} &\leq \big\|(S_{g(u_k)}-S_{g(u_{k-1})})\big(f+q(u_{k-1},{}^{\mathrm{b}}du_{k-1})\big)\big\|_{\mathcal{X}^{s-1,\alpha}} \\ &\qquad + \|S_{g(u_k)}(q(u_k,{}^{\mathrm{b}}du_k)-q(u_{k-1},{}^{\mathrm{b}}du_{k-1}))\|_{\mathcal{X}^{s-1,\alpha}} \\ &\leq C_S\big(L_q(R)+C_S C_g(R)(C_f+L_q(R)R)\big)\|u_k-u_{k-1}\|_{\mathcal{X}^{s-1,\alpha}}. \end{aligned}$$

Since $C_S L_q(0) < 1$, the constant on the right hand side is less than 1 for small $R > 0$, recalling that $C_f = C_f(R) \to 0$ as $R \to 0$. Therefore, $(u_k)_k$ converges exponentially fast to a limit $u \in \mathcal{X}^{s-1,\alpha}$ as $k \to \infty$. Since $\{u_k\}$ is bounded in the Hilbert space $\mathcal{X}^{s,\alpha}$, it in fact has a weakly convergent subsequence in $\mathcal{X}^{s,\alpha}$, and the limit is necessarily equal to u, so $u \in \mathcal{X}^{s,\alpha}$. This easily implies the weak convergence of the full sequence $u_k \rightharpoonup u$ in $\mathcal{X}^{s,\alpha}$.

We can prove uniqueness and stability in one stroke: Suppose that $u_1, u_2 \in \mathcal{X}^{s,\alpha}$ have norm $\leq R$ and satisfy

$$\Box_{g(u_j)}u_j = f_j + q(u_j, {}^{\mathrm{b}}du_j), \quad j = 1, 2,$$

where the $f_j \in H_{\mathrm{b}}^{s-1,\alpha}$, $j = 1, 2$, have norm $\leq C_f$. Then the estimate (9.1.13) yields

$$\begin{aligned} \|u_1-u_2\|_{\mathcal{X}^{s-1,\alpha}} \leq C_S\big(&\|f_1-f_2\|_{H_{\mathrm{b}}^{s-2,\alpha}} + \\ &(L_q(R)+C_S C_g(R)(C_f+L_q(R)R))\|u_1-u_2\|_{\mathcal{X}^{s-1,\alpha}}\big). \end{aligned}$$

Arguing as before, the second term on the right can be absorbed into the left hand side for small $R > 0$. Hence

$$\|u_1-u_2\|_{\mathcal{X}^{s-1,\alpha}} \leq C'\|f_1-f_2\|_{H_{\mathrm{b}}^{s-2,\alpha}},$$

as desired. □

Remark 9.1.14. In the case that $g(u) \equiv g$ is constant, see Theorem 5.2.6 for a discussion of the corresponding semilinear equations. There, one in particular obtains more precise asymptotics in the case of polynomial non-linearities, see Theorem 5.2.17; see also Remark 9.1.9.

We next turn to a special case of Theorem 9.1.11 which is very natural and allows for a stronger conclusion.

Theorem 9.1.15. *Let $s > n/2 + 7$ and $0 < \alpha < 1$. Let $N, N' \in \mathbb{N}$, and suppose $c_k \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$, $g_k \in (\mathcal{C}^\infty + H_{\mathrm{b}}^s)(\Omega; S^{2\mathrm{b}}T^*_\Omega M)$ for $1 \leq k \leq N$; define the map $g\colon \mathcal{X}^{s,\alpha}_{\mathbb{R}} \to (\mathcal{C}^\infty + H_{\mathrm{b}}^{s,\alpha})(\Omega; S^{2\mathrm{b}}T^*_\Omega M)$ by*

$$g(u) = \sum_{k=1}^N c_k(u) g_k,$$

and assume $g(0) = g_{\mathrm{dS}}$. Moreover, define

$$q(u, {}^{\mathrm{b}}du) = \sum_{j=1}^{N'} u^{e_j} \prod_{k=1}^{N_j} X_{jk}u, \quad e_j + N_j \geq 2, N_j \geq 1, X_{jk} \in (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1})\mathcal{V}_{\mathrm{b}}.$$

Then for small $R > 0$, there exists $C_f > 0$ such that for all $f \in H_{\mathrm{b}}^{s-1,\alpha}(\Omega;\mathbb{R})^{\bullet,-}$ with norm $\leq C_f$, the equation

$$\Box_{g(u)} u = f + q(u, {}^{\mathrm{b}}du) \tag{9.1.14}$$

has a unique solution $u \in \mathcal{X}^{s,\alpha}_{\mathbb{R}}$, with norm $\leq R$, and in the topology of $\mathcal{X}^{s-1,\alpha}_{\mathbb{R}}$, u depends continuously on f. If one in fact has $f \in H_{\mathrm{b}}^{s'-1,\alpha}(\Omega;\mathbb{R})^{\bullet,-}$ for some $s' \in (s, \infty]$, then $u \in \mathcal{X}^{s',\alpha}_{\mathbb{R}}$.

The initial metric $g(0)$ can be more general; see Remark 9.1.12.

Remark 9.1.16. One could, for instance, choose the metrics g_k such that at every point $p \in M$, the linear space $S^{2\mathrm{b}}T_pM$ is spanned by the $g_k(p)$, and in a similar manner the b-vector fields X_{jk}.

Remark 9.1.17. The point of the last part of the theorem is that even though a priori the radius of the ball which is the set of $f \in H_{\mathrm{b}}^{s'-1,\alpha}(\Omega)^{\bullet,-}$ for which one has solvability in $\mathcal{X}^{s',\alpha}$ according to Theorem 9.1.11 could shrink to 0 as $s' \to \infty$, this does not happen in the setting of Theorem 9.1.15. We use a straightforward approach to proving this by differentiating the PDE; a much more robust way is to use Nash-Moser iteration, as we will do in §9.2.

Remark 9.1.18. If f has more decay, say $f \in H_{\mathrm{b}}^{\infty,\infty}$, it is relatively straightforward to show that the solution u in fact has an asymptotic expansion to any fixed order, assuming f is small in an appropriate space. Indeed, for such a statement, one only needs to replace the spaces $\mathcal{X}^{s,\alpha}$ by similar spaces which now encode more precise partial asymptotic expansions, as in §5.2.4, and prove the persistence of such spaces under taking reciprocals, compositions with smooth functions etc. See also Remark 9.1.9.

For the proof, we need one more definition:

Definition 9.1.19. (Cf. [9, Definition 1.1].) For $s' > s$, $\alpha \in \mathbb{R}$ and $\Gamma \subset {}^{\mathrm{b}}S^*M$, let

$$H_{\mathrm{b}}^{s,\alpha;s',\Gamma} := \{u \in H_{\mathrm{b}}^{s,\alpha} \colon \mathrm{WF}_{\mathrm{b}}^{s',\alpha}(u) \cap \Gamma = \emptyset\}.$$

Proof of Theorem 9.1.15. The map g satisfies the requirements of Theorem 9.1.11 by Proposition 8.3.8, and q satisfies (9.1.8) with $L_q(0) = 0$, thus Theorem 9.1.11 implies the existence and uniqueness of solutions in $\mathcal{X}^{s,\alpha}$ with small norm as well as their stability in the topology of $\mathcal{X}^{s-1,\alpha}$. The uniqueness of u in all of $\mathcal{X}_{\mathbb{R}}^{s,\alpha}$, in fact in $H_{\mathrm{b,loc}}^{s}(\Omega^\circ)$, follows from local uniqueness for quasilinear symmetric hyperbolic systems, see e.g. [108, §16.3].

It remains to establish the higher regularity statement; by an iterative argument, it suffices to prove the following: If $s' > s$, $u \in \mathcal{X}_{\mathbb{R}}^{s'-1/2,\alpha}$, $\|u\|_{\mathcal{X}^{s,\alpha}} \leq R$, and u solves (9.1.14) with $f \in H_{\mathrm{b}}^{s'-1,\alpha}$, then $u \in \mathcal{X}_{\mathbb{R}}^{s',\alpha}$. We only assume that the $\mathcal{X}^{s,\alpha}$-norm of u is small – the reason for this assumption is that it ensures that $\Box_{g(u)}$ fits into our framework. *We will use the summation convention for the remainder of the proof.* Equation (9.1.14) in local coordinates reads

$$\big(g^{ij}(u){}^{\mathrm{b}}\partial_{ij}^2 + h^j(u, {}^{\mathrm{b}}\partial u){}^{\mathrm{b}}\partial_j\big)u = f + q(u, {}^{\mathrm{b}}\partial u), \tag{9.1.15}$$

where $g^{ij}(v)$, $h^j(v;z)$ and $q(v;z)$ are $\mathcal{C}^\infty$-functions of v and z. As is standard procedure to obtain higher regularity (and exploited in a similar setting by Beals and Reed [9, §4]), we will differentiate this equation with respect to certain b-vector field V: After differentiating and collecting/rewriting terms, one obtains an equation like (9.1.15) for Vu, where only the coefficients of first order terms are changed, and without q and with a different forcing term; one can then appeal to the regularity theory for the equation for Vu, which is thus again a wave equation with lower order terms. Concretely, suppose $\widetilde{\Sigma} \subset \Sigma$ is a closed subset of the characteristic set of $\Box_{g(u)}$, consisting of bicharacteristic strips and contained in the coordinate patch we are working in; we want to propagate $\mathcal{X}^{s',\alpha}$-regularity of u into $\widetilde{\Sigma}$, assuming we have this regularity on backward/forward bicharacteristics from $\widetilde{\Sigma}$ or in a punctured neighborhood of $\widetilde{\Sigma}$. With $\pi\colon {}^{\mathrm{b}}S^*M \to M$ denoting the projection to the base, choose $\chi, \chi_0 \in \mathcal{C}^\infty_{\mathrm{c}}(\overline{\mathbb{R}^n_+})$ so that χ is identically 1 near $\pi(\widetilde{\Sigma})$ and χ_0 is identically 1 on $\operatorname{supp}\chi$. Let $V_0 \in \mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ be a constant coefficient b-vector field which is non-characteristic (in the b-sense) on $\widetilde{\Sigma}$, which is possible if $\widetilde{\Sigma}$ is sufficiently small, and put $V = \chi_0 V_0$. Applying V to (9.1.15), we obtain, suppressing the arguments $u, {}^{\mathrm{b}}\partial u$,

$$
\begin{aligned}
\big(g^{ij}\,{}^{\mathrm{b}}\partial^2_{ij} + [h^j + (\partial_{z_j}h^k)\,{}^{\mathrm{b}}\partial_k u - \partial_{z_j}q]\,{}^{\mathrm{b}}\partial_j\big)Vu + (g^{ij})'Vu\,{}^{\mathrm{b}}\partial^2_{ij}u + g^{ij}[V, {}^{\mathrm{b}}\partial^2_{ij}]u \\
= Vf + (\partial_v q)Vu + (\partial_{z_j}q)[V, {}^{\mathrm{b}}\partial_j]u \\
- (\partial_v h^j)Vu\,{}^{\mathrm{b}}\partial_j u - h^j[V, {}^{\mathrm{b}}\partial_j]u - (\partial_{z_j}h^k)[V, {}^{\mathrm{b}}\partial_k]u =: f_1.
\end{aligned}
$$

Since V_0 annihilates constants, $Vu \in H_{\mathrm{b}}^{s'-3/2,\alpha}$ locally near $\pi(\widetilde{\Sigma})$. Similarly, $[V, {}^{\mathrm{b}}\partial_j]u \in H_{\mathrm{b}}^{s'-3/2,\alpha}$ locally near $\pi(\widetilde{\Sigma})$, and $h^j(u, {}^{\mathrm{b}}\partial u) \in \mathcal{C}^\infty + H_{\mathrm{b}}^{s'-3/2,\alpha}$, $q(u, {}^{\mathrm{b}}\partial u) \in H_{\mathrm{b}}^{s'-3/2,\alpha}$, similarly for derivatives of h^j and q; lastly, $Vf \in H_{\mathrm{b}}^{s'-2,\alpha}$, thus $f_1 \in H_{\mathrm{b}}^{s'-2,\alpha}$ locally near $\pi(\widetilde{\Sigma})$. We need to analyze the last two terms on the left hand side: Since V is non-characteristic on $\operatorname{supp}\chi \supset \pi(\widetilde{\Sigma})$, we can write

$$
{}^{\mathrm{b}}\partial_j = (1-\chi)\,{}^{\mathrm{b}}\partial_j + Q_j V + \widetilde{R}_j, \quad Q_j \in \Psi^0_{\mathrm{b}}, \widetilde{R}_j \in \Psi^1_{\mathrm{b}}, \mathrm{WF}'_{\mathrm{b}}(\widetilde{R}_j) \cap \widetilde{\Sigma} = \emptyset;
$$

put $R_j := (1-\chi)\,{}^{\mathrm{b}}\partial_j + \widetilde{R}_j = {}^{\mathrm{b}}\partial_j - Q_j V$. Note that R_j annihilates constants. We can then write

$$
{}^{\mathrm{b}}\partial^2_{ij}u = {}^{\mathrm{b}}\partial_i Q_j Vu + {}^{\mathrm{b}}\partial_i R_j u,
$$

and the second term is in $H_{\mathrm{b}}^{\infty,\alpha}$ microlocally near $\widetilde{\Sigma}$. Thus, we have

$$
(g^{ij})'Vu\,{}^{\mathrm{b}}\partial^2_{ij}u = \big((g^{ij})'Vu\,{}^{\mathrm{b}}\partial_i Q_j\big)Vu + (g^{ij})'Vu\,{}^{\mathrm{b}}\partial_i R_j u;
$$

the second term on the right is a product of a function in $H_{\mathrm{b}}^{s'-3/2,\alpha}$ with ${}^{\mathrm{b}}\partial_i R_j u$, the latter a priori being an element of $H_{\mathrm{b}}^{s'-5/2,\alpha;\infty,\widetilde{\Sigma}}$; we will prove below in Lemma 9.1.20 that this product is an element of $H_{\mathrm{b}}^{s'-5/2,\alpha;s'-3/2,\widetilde{\Sigma}}$. Moreover, $[V, {}^{\mathrm{b}}\partial^2_{ij}]$ is a second order b-differential operator, vanishing on constants, with coefficients vanishing near $\pi(\widetilde{\Sigma})$; this implies $g^{ij}[V, {}^{\mathrm{b}}\partial^2_{ij}]u \in H_{\mathrm{b}}^{s'-5/2,\alpha;\infty,\widetilde{\Sigma}}$. We conclude that

$$P_1(Vu) = f_2 \in H_{\mathrm{b}}^{s'-5/2,\alpha;s'-2,\widetilde{\Sigma}}, \tag{9.1.16}$$

where

$$P_1 = \Box_{g(u)} + \widetilde{P}, \quad \widetilde{P} = [(\partial_{z_j} h^k)^{\mathrm{b}}\partial_k u - \partial_{z_j} q]^{\mathrm{b}}\partial_j + (g^{ij})' V u\, {}^{\mathrm{b}}\partial_i Q_j.$$

Since we are assuming $u \in \mathcal{X}^{s'-1/2,\alpha}$, and moreover $\widetilde{P}$ is an element of $H_{\mathrm{b}}^{s'-3/2,\alpha}\Psi_{\mathrm{b}}^1$ near $\pi(\widetilde{\Sigma})$, we see that, a forteriori,

$$P_1 \in (\mathcal{C}^\infty + H_{\mathrm{b}}^{s'-1,\alpha})\mathrm{Diff}_{\mathrm{b}}^2 + (\mathcal{C}^\infty + H_{\mathrm{b}}^{s'-2,\alpha})\Psi_{\mathrm{b}}^1.$$

Hence, we can propagate $H_{\mathrm{b}}^{s'-1,\alpha}$-regularity of Vu into $\widetilde{\Sigma}$ by Theorems 8.5.6 and 8.5.10; recall that these two theorems only deal with the propagation of regularity which is $1/2$ more than than the a priori regularity of Vu, which is $H_{\mathrm{b}}^{s'-3/2,\alpha}$. The point here is that real principal type propagation only depends on the principal symbol of P_1, which is the same as the principal symbol of $\Box_{g(u)}$, and the propagation of $H_{\mathrm{b}}^{s'-1,\alpha}$-regularity near radial points works for arbitrary $H_{\mathrm{b}}^{s'-2,\alpha}\Psi_{\mathrm{b}}^1$-perturbations of $\Box_{g(u)}$; see Remark 8.5.11. Therefore, writing $u = c + u'$ with $u' \in H_{\mathrm{b}}^{s'-1/2,\alpha}$ a priori, we obtain $u' \in H_{\mathrm{b}}^{s',\alpha}$ microlocally near $\widetilde{\Sigma}$ by standard elliptic regularity, since V is non-characteristic on $\widetilde{\Sigma}$. Away from the characteristic set of $\Box_{g(u)}$, which is the same as the characteristic set of P_1, we simply use $P_1 V u \in H_{\mathrm{b}}^{s'-5/2,\alpha}$ and elliptic regularity for $P_1 V$ to deduce that $u' \in H_{\mathrm{b}}^{s'+1/2,\alpha}$ there; we stress the importance of only using local rather than microlocal regularity information of $P_1 V u$, since the proof of Theorem 8.4.1, giving elliptic regularity for Vu solving $P_1(Vu) = f$, only works with local assumptions on f, see Remark 8.4.2. For this application of elliptic regularity, we choose V in such a way that it is non-characteristic on a set disjoint from Σ. Putting all such pieces of regularity information together by choosing finitely many such sets $\widetilde{\Sigma}$, we obtain $u' \in H_{\mathrm{b,loc}}^{s',\alpha}(\Omega)^{\bullet,-}$.

We can make this is a global rather than local statement by extending Ω to the slightly larger domain Ω_{0,δ_2}, $\delta_2 < 0$, solving the quasilinear PDE there, and restricting back to Ω; thus $u' \in H_{\mathrm{b}}^{s',\alpha}(\Omega)^{\bullet,-}$. □

To finish the proof, we need the following lemma, which we prove using ideas from [9, Theorem 1.3].

Lemma 9.1.20. *Let $\alpha \in \mathbb{R}$ and $s > n/2 + 1$. Then, in the notation of Definition 9.1.19, for $u \in H_{\mathrm{b}}^s$ and $v \in H_{\mathrm{b}}^{s-1,\alpha;s,\Gamma}$, we have $uv \in H_{\mathrm{b}}^{s-1,\alpha;s,\Gamma}$.*

Proof. Without loss, we may assume $\alpha = 0$. By Corollary 8.2.10, $uv \in H_{\mathrm{b}}^{s-1}$, and we must prove the microlocal regularity of uv. Using a partition of unity, it suffices to assume $\Gamma = (\overline{\mathbb{R}^n_+})_z \times K$ for a conic set $K \subset \mathbb{R}^n_\zeta \setminus o$; moreover, since the complement of the wave front set is open, we can assume that K is open. By assumption, we can then write

$$|\widehat{u}(\zeta)| = \frac{u_0(\zeta)}{\langle\zeta\rangle^s}, u_0 \in L^2, \quad |\widehat{v}(\zeta)| = \left(\frac{\chi_K(\zeta)}{\langle\zeta\rangle^s} + \frac{\chi_{K^c}(\zeta)}{\langle\zeta\rangle^{s-1}}\right) v_0(\zeta), v_0 \in L^2,$$

where χ_K denotes the characteristic function of K, and K^c the complement of K. Now, let $K_0 \subset K$ be closed and conic. Then

$$\chi_{K_0}(\zeta)|\widehat{uv}(\zeta)|\langle\zeta\rangle^s \leq \int \frac{\chi_{K_0}(\zeta)\langle\zeta\rangle^s}{\langle\zeta-\xi\rangle^s}\left(\frac{\chi_K(\xi)}{\langle\xi\rangle^s} + \frac{\chi_{K^c}(\xi)}{\langle\xi\rangle^{s-1}}\right) u_0(\zeta-\xi) v_0(\xi)\, d\xi$$

We show that this is an element of L^2, thus finishing the proof: We have

$$\frac{\langle\zeta\rangle^s}{\langle\zeta-\xi\rangle^s\langle\xi\rangle^s} \in L^\infty_\zeta L^2_\xi,$$

and on the support of $\chi_{K_0}(\zeta)\chi_{K^c}(\xi)$, we have $|\zeta-\xi| \geq c|\zeta|$, $c > 0$, thus

$$\frac{\chi_{K_0}(\zeta)\chi_{K^c}(\xi)\langle\zeta\rangle^s}{\langle\zeta-\xi\rangle^s\langle\xi\rangle^{s-1}} \lesssim \frac{1}{\langle\xi\rangle^{s-1}} \in L^\infty_\zeta L^2_\xi,$$

since $s > n/2 + 1$. □

9.1.3 Conformal changes of the metric

Reconsidering the proof of Theorem 9.1.11, one *cannot* bound

$$\|(S_{g(u)} - S_{g(v)})\|_{\mathcal{L}(H_{\mathrm{b}}^{s-1,\alpha},\mathcal{X}^{s,\alpha})} \lesssim \|u-v\|_{\mathcal{X}^{s,\alpha}}$$

in general,[50] which however would immediately give uniqueness and stability of solutions to (9.1.9) in the space $\mathcal{X}^{s,\alpha}$. But there is a situation where we do have good control on $S_{g(u)} - S_{g(v)}$ as an operator from $H_{\mathrm{b}}^{s-1,\alpha}$ to $\mathcal{X}^{s,\alpha}$, namely when $\Box_{g(u)}$ and $\Box_{g(v)}$ have the same characteristic set, since in this case, in (9.1.13) the composition of $\Box_{g(v)} - \Box_{g(u)}$ with $S_{g(v)}$ loses no derivative (ignoring issues coming from the limited regularity of $g(u), g(v)$ for the moment – they will turn out to be irrelevant). This situation arises if $g(u) = \mu(u)g(0)$ for $\mu(u) \in \mathcal{C}^\infty(M) + H_{\mathrm{b}}^s(M)$; that this is in fact the only possibility is shown by a pointwise application of the following lemma.

Lemma 9.1.21. *Let $d \geq 1$, and assume g, g' are bilinear forms on $\mathbb{R}^{1+d}$ with signature $(1, d)$ such that the zero sets of the associated quadratic forms q, q' coincide. Then $g = \mu g'$ for some $\mu \in \mathbb{R}^\times$.*

Proof. By a linear change of coordinates, we may assume that g' is the Minkowski bilinear form on $\mathbb{R}^{1+d}$. Let g_{ij}, $0 \leq i, j \leq d$, be the components of g, and let us write vectors in $\mathbb{R}^{1+d}$ as $(x_1, x') \in \mathbb{R} \times \mathbb{R}^d$. Since $g'(1, 0) \neq 0$, we have $g(1, 0) = g_{00} \neq 0$. Dividing g by $\mu := g_{00}$, we may assume $g_{00} = 1$; we now show that $g = g'$. For all $x' \in \mathbb{R}^d$, $|x'| = 1$ (Euclidean norm!), we have $q(1, x') = 0$ and $q(1, -x') = 0$, hence $q(1, x') - q(1, -x') = 0$, in coordinates

$$4 \sum_{i \geq 1} g_{0i} x'_i = 0, \quad |x'| = 1,$$

and thus $g_{0i} = 0$ for all $i \geq 1$. Now let $\widetilde{q}(x') := q(0, x')$ and $\widetilde{q}'(x') := q'(0, x')$, then

$$\widetilde{q}(x') = -1 \iff q(1, x') = 0 \iff q'(1, x') = 0 \iff \widetilde{q}'(x') = -1,$$

thus by scaling $\widetilde{q} \equiv \widetilde{q}'$ on $\mathbb{R}^d$, hence by polarization $g_{ij} = g'_{ij}$ for $1 \leq i, j \leq d$, and the proof is complete. $\square$

In this restricted setting, we have the following well-posedness result; notice that the topology in which we have stability is stronger than in Theorem 9.1.11, and we also allow

[50] Indeed, consider a similar situation for scalar first order operators $P_a := \partial_t - a\partial_x$, $a \in \mathbb{R}$, on $[0, 1]_t \times \mathbb{R}_x$. The forward solution operator S_a is constructed by integrating the forcing term along the bicharacteristics $s \mapsto (s, x_0 - as)$ of P_a, and it is easy to see that $S_a \in \mathcal{L}(L^2, L^2)$. However, $S_a - S_b$ is constructed using the difference of integrals of the forcing f along two different bicharacteristics, which one can naturally only bound using df, i.e. one only obtains the estimate $\|(S_a - S_b)f\|_{L^2} \lesssim |a - b| \|f\|_{H^1}$, which is an estimate with a loss of 2 derivatives, similar to (9.1.13). The core of the problem is that there is no estimate of the form $\|f(\cdot + a) - f\|_{L^2} \lesssim |a| \|f\|_{L^2}$, although such an estimate holds if the norm on the right is replaced by the H^1-norm.

more general non-linearities q.

Theorem 9.1.22. *Let $s > n/2 + 6$, $0 < \alpha < 1$. Let $g_0 = g_{\mathrm{dS}}$ (but see Remark 9.1.12), and let $\mu\colon \mathcal{X}^{s,\alpha} \to \mathcal{X}^{s,0}_{\mathbb{R}}$ be a continuous map with $\mu(0) = 1$ and*

$$\|\mu(u) - \mu(v)\|_{\mathcal{X}^{s,0}} \leq L_\mu(R)\|u - v\|_{\mathcal{X}^{s,\alpha}} \tag{9.1.17}$$

for all $u, v \in \mathcal{X}^{s,\alpha}$ with norms $\leq R$, where $L_\mu\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuous and non-decreasing. Put $g(u) := \mu(u)g_0$.

(1) Let

$$q\colon \mathcal{X}^{s,\alpha} \times H_{\mathrm{b}}^{s-1,\alpha}(\Omega; {}^{\mathrm{b}}T^*_\Omega M)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-}$$

be continuous with $q(0) = 0$, satisfying

$$\|q(u, {}^{\mathrm{b}}du) - q(v, {}^{\mathrm{b}}dv)\|_{H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-}} \leq L_q(R)\|u - v\|_{\mathcal{X}^{s,\alpha}}$$

for all $u, v \in \mathcal{X}^{s,\alpha}$ with norms $\leq R$, where $L_q\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuous and non-decreasing. Then there is a constant $C_L > 0$ so that the following holds: If $L_q(0) < C_L$, then for small $R > 0$, there is $C_f > 0$ such that for all $f \in H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-}$ with norm $\leq C_f$, there exists a unique solution $u \in \mathcal{X}^{s,\alpha}$ of the equation

$$\Box_{g(u)} u = f + q(u, {}^{\mathrm{b}}du) \tag{9.1.18}$$

with norm $\leq R$, which depends continuously on f.

(2) More generally, if

$$q\colon \mathcal{X}^{s,\alpha} \times H_{\mathrm{b}}^{s-1,\alpha}(\Omega; {}^{\mathrm{b}}T^*_\Omega M)^{\bullet,-} \times H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-}$$

is continuous with $q(0) = 0$ and satisfies

$$\begin{aligned}\|q(u_1, {}^{\mathrm{b}}du_1, w_1) - q(u_2, {}^{\mathrm{b}}du_2, w_2)\|_{H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-}} \\ \leq L_q(R)\big(\|u_1 - u_2\|_{\mathcal{X}^{s,\alpha}} + \|w_1 - w_2\|_{H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-}}\big)\end{aligned}$$

for all $u_j \in \mathcal{X}^{s,\alpha}$, $w_j \in H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-}$ with $\|u_j\| + \|w_j\| \leq R$, then there is a constant $C_L > 0$ such that the following holds: If $L_q(0) < C_L$, then for small $R > 0$, there *is* $C_f > 0$ *such that for all* $f \in H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-}$ *with norm* $\leq C_f$*, there exists a unique solution* $u \in \mathcal{X}^{s,\alpha}$ *of the equation*

$$\Box_{g(u)} u = f + q(u, {}^{\mathrm{b}}du, \Box_{g(u)} u) \tag{9.1.19}$$

with $\|u\|_{\mathcal{X}^{s,\alpha}} + \|\Box_{g_0} u\|_{H_{\mathrm{b}}^{s-1,\alpha}} \leq R$*, which depends continuously on* f*.*

Proof. First, note that $N(\Box_{g(u)}) = \mu(u)|_Y N(\Box_{g_0})$, which is a constant multiple of $N(\Box_{g_0})$ by the definition of the space $\mathcal{X}^{s,\alpha}$. Thus, as in the proof of Theorem 9.1.11, there exists $R_S > 0$ such that

$$S_{g(u)} \colon H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-} \to \mathcal{X}^{s,\alpha}$$

is continuous with uniformly bounded operator norm

$$\|S_{g(u)}\| \leq C_S;$$

for $\|u\|_{\mathcal{X}^{s,\alpha}} \leq R_S$; let us also assume that

$$|\mu(u)| \geq c_0 > 0, \quad \|u\|_{\mathcal{X}^{s,\alpha}} \leq R_S. \tag{9.1.20}$$

We now prove the first half of the theorem. Let $C_L := C_S^{-1}$, and assume that $L_q(0) < C_L$, then $L_q(R_q) < C_L$ for $R_q > 0$ small. Put $\widetilde{R} := \min(R_S, R_q)$; let $0 < R \leq \widetilde{R}$, to be specified later, and put and $C_f(R) = R(C_S^{-1} - L_q(R))$; let $f \in H_{\mathrm{b}}^{s-1,\alpha}(\Omega)^{\bullet,-}$ have norm $\leq C_f(R)$. Let $B(R)$ denote the metric ball of radius R in $\mathcal{X}^{s,\alpha}$, and define $T \colon B(R) \to B(R)$,

$$Tu := S_{g(u)}\big(f + q(u, {}^{\mathrm{b}}du)\big).$$

By the choice of R, C_L and C_f, T is well-defined by the same estimate as in the proof of Theorem 9.1.11. The crucial new feature here is that for R sufficiently small, T is in fact a contraction. This follows once we prove the existence of a constant $C_i > 0$ such that for $u, v \in \mathcal{X}^{s,\alpha}$ with norms $\leq R$, we have

$$\|S_{g(u)} - S_{g(v)}\|_{\mathcal{L}(H_{\mathrm{b}}^{s-1,\alpha}, \mathcal{X}^{s,\alpha})} \leq C_S C_i L_\mu(R) \|u - v\|_{\mathcal{X}^{s,\alpha}}, \tag{9.1.21}$$

which is an estimate on spaces with regularity improved by one relative to the estimate

(9.1.13). Indeed, assuming (9.1.21), we obtain

$$\begin{aligned}
&\|Tu - Tv\|_{\mathcal{X}^{s,\alpha}} \\
&\quad \leq \Big\| S_{g(u)}\big(q(u, {}^{\mathrm{b}}du) - q(v, {}^{\mathrm{b}}dv)\big)\Big\|_{\mathcal{X}^{s,\alpha}} + \|(S_{g(u)} - S_{g(v)})(f + q(v, {}^{\mathrm{b}}dv))\|_{\mathcal{X}^{s,\alpha}} \\
&\quad \leq \big(C_S L_q(R) + C_S C_i L_\mu(R)(C_f(R) + L_q(R)R)\big)\|u - v\|_{\mathcal{X}^{s,\alpha}};
\end{aligned}$$

and since $C_S L_q(R) \leq C_S L_q(\widetilde{R}) < \theta < 1$ for $R \leq \widetilde{R}$, we can choose R so small that

$$C_S C_i L_\mu(R)(C_f(R) + L_q(R)R) \leq \theta - C_S L_q(R), \tag{9.1.22}$$

where we use that $C_f(R) \to 0$ as $R \to 0$. With this choice of R, T is a contraction, thus has a unique fixed point $u \in \mathcal{X}^{s,\alpha}$ which solves the PDE (9.1.18).

Continuing to assume (9.1.21), let us prove the continuous dependence of the solution u on f. For this, let us assume that $u_j \in \mathcal{X}^{s,\alpha}$, $j = 1, 2$, solves

$$\Box_{g(u_j)} u_j = f_j + q(u_j, {}^{\mathrm{b}}du_j),$$

where $f_j \in H_{\mathrm{b}}^{s-1,\alpha}$ has norm $\leq C_f$. Then, as in the proof of Theorem 9.1.11,

$$\begin{aligned}
\|u_1 - u_2\|_{\mathcal{X}^{s,\alpha}} \leq C_S\big(&\|f_1 - f_2\|_{H_{\mathrm{b}}^{s-1,\alpha}} \\
&+ (L_q(R) + C_i L_\mu(R)(C_f + L_q(R)R))\|u_1 - u_2\|_{\mathcal{X}^{s,\alpha}}\big).
\end{aligned}$$

Because of (9.1.22), the prefactor of $\|u_1 - u_2\|$ on the right hand side is $\leq \theta < 1$, hence we conclude

$$\|u_1 - u_2\|_{\mathcal{X}^{s,\alpha}} \leq \frac{C_S}{1-\theta}\|f_1 - f_2\|_{H_{\mathrm{b}}^{s-1,\alpha}},$$

as desired.

We now prove the crucial estimate (9.1.21) by using the identity in (9.1.13), as follows: By definition of $\Box$, we have

$$\Box_{g(u)} = \Box_{\mu(v) g_0 \frac{\mu(u)}{\mu(v)}} = \frac{\mu(v)}{\mu(u)}\Box_{g(v)} + E_{u,v}, \tag{9.1.23}$$

where $E_{u,v} \in H_{\mathrm{b}}^{s-1,\alpha}\mathcal{V}_{\mathrm{b}}$ satisfies the estimate[51]

$$\|E_{u,v}\|_{H_{\mathrm{b}}^{s-1,\alpha}\mathcal{V}_{\mathrm{b}}} \leq C \left\| {}^{\mathrm{b}}d\left(\frac{\mu(v)}{\mu(u)}\right)\right\|_{H_{\mathrm{b}}^{s-1}},$$

where the constant C is uniform for $\|u\|_{\mathcal{X}^{s,\alpha}}, \|v\|_{\mathcal{X}^{s,\alpha}} \leq R$. Thus,

$$\begin{aligned}
&\|(\Box_{g(v)} - \Box_{g(u)})S_{g(v)}\|_{\mathcal{L}(H_{\mathrm{b}}^{s-1,\alpha})} \\
&\quad \leq \left\|1 - \frac{\mu(v)}{\mu(u)}\right\|_{\mathcal{L}(H_{\mathrm{b}}^{s-1,\alpha})} + \|E_{u,v}\|_{\mathcal{L}(\mathcal{X}^{s,\alpha},H_{\mathrm{b}}^{s-1,\alpha})}\|S_{g(v)}\|_{\mathcal{L}(H_{\mathrm{b}}^{s-1,\alpha},\mathcal{X}^{s,\alpha})} \\
&\quad \leq \left\|1 - \frac{\mu(v)}{\mu(u)}\right\|_{\mathcal{X}^{s-1,0}} + CC_S\left\| {}^{\mathrm{b}}d\left(\frac{\mu(v)}{\mu(u)}\right)\right\|_{H_{\mathrm{b}}^{s-1}}.
\end{aligned}$$

Now,

$$\begin{aligned}
\left\|1 - \frac{\mu(v)}{\mu(u)}\right\|_{\mathcal{X}^{s-1,0}} &\leq C'\left\|\frac{1}{\mu(u)}\right\|_{\mathcal{X}^{s-1,0}} \|\mu(u) - \mu(v)\|_{\mathcal{X}^{s-1,0}} \\
&\leq C_i' L_\mu(R)\|u - v\|_{\mathcal{X}^{s,\alpha}},
\end{aligned} \tag{9.1.24}$$

where

$$C_i' := C' \sup_{\|w\|_{\mathcal{X}^{s,\alpha}} \leq R} \left\|\frac{1}{\mu(w)}\right\|_{\mathcal{X}^{s,0}} < \infty$$

by assumption (9.1.20) and Lemma 8.3.2. Likewise, since ${}^{\mathrm{b}}d(\mu(v)/\mu(u)) = {}^{\mathrm{b}}d\big(\mu(v)/\mu(u) - 1\big)$,

$$\left\| {}^{\mathrm{b}}d\left(\frac{\mu(v)}{\mu(u)}\right)\right\|_{H_{\mathrm{b}}^{s-1}} \leq \left\|1 - \frac{\mu(v)}{\mu(u)}\right\|_{\mathcal{X}^{s,0}} \leq C_i' L_\mu(R)\|u - v\|_{\mathcal{X}^{s,\alpha}};$$

therefore,

$$\|(\Box_{g(v)} - \Box_{g(u)})S_{g(v)}\|_{\mathcal{L}(H_{\mathrm{b}}^{s-1,\alpha})} \leq C_i L_\mu(R)\|u - v\|_{\mathcal{X}^{s,\alpha}} \tag{9.1.25}$$

for $C_i = C_i'(1 + CC_S)$, and with $\|S_{g(u)}\|_{\mathcal{L}(H_{\mathrm{b}}^{s-1,\alpha},\mathcal{X}^{s,\alpha})} \leq C_S$ and the identity in (9.1.13), we finally obtain the estimate (9.1.21).

We proceed to prove the second half of the theorem along the lines of the proof of Theorem 5.2.6. We work on the space

$$\mathcal{Y}^{s,\alpha} := \{u \in \mathcal{X}^{s,\alpha} \colon \Box_{g_0}u \in H_{\mathrm{b}}^{s-1,\alpha}\}, \quad \|u\|_{\mathcal{Y}^{s,\alpha}} = \|u\|_{\mathcal{X}^{s,\alpha}} + \|\Box_{g_0}u\|_{H_{\mathrm{b}}^{s-1,\alpha}}, \tag{9.1.26}$$

[51] To define a norm of an element $E \in H_{\mathrm{b}}^{\sigma,\rho}\mathcal{V}_{\mathrm{b}}(M)$, use a partition of unity on M to reduce this task to a local one, and as the norm of $E \in H_{\mathrm{b}}^{\sigma,\rho}\mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$, take the sum of the $H_{\mathrm{b}}^{\sigma,\rho}$-norms of the coefficients of E.

which is complete, see Remark 4.2.5. The idea is that all operators $\Box_{g(u)}$ are (pointwise) multiples of each other modulo first order operators, thus $\Box_{g_0}$ is as good as any other such operator, and therefore $\Box_{g_0}$ in the third argument of the non-linearity q acts as a first order operator on the successive approximations $T^k(0)$ in the iteration scheme implicit in the application of the Banach fixed point theorem used above to solve equation (9.1.18). Thus, let $B(R)$ denote the metric ball of radius $R \leq R_S$ in $\mathcal{Y}^{s,\alpha}$, and define $T\colon B(R) \to \mathcal{Y}^{s,\alpha}$,

$$Tu := S_{g(u)}\big(f + q(u)\big)$$

where we write $q(u) := q(u, {}^{\mathrm{b}}du, \Box_{g(u)}u)$ to simplify the notation. We will prove that for $R > 0$ small enough, the image of T is contained in $B(R)$. We first estimate for $u \in B(R)$ and $w \in \mathcal{Y}^{s,\alpha}$, using (9.1.23) and an estimate similar to (9.1.24) (with $v = 0$):

$$\begin{aligned}\|\Box_{g(u)}w\|_{H_{\mathrm{b}}^{s-1,\alpha}} &\leq \|\Box_{g(0)}w\|_{H_{\mathrm{b}}^{s-1,\alpha}} + \|(\Box_{g(u)} - \Box_{g(0)})w\|_{H_{\mathrm{b}}^{s-1,\alpha}} \\ &\leq \|w\|_{\mathcal{Y}^{s,\alpha}} + \widetilde{C}_i\|u\|_{\mathcal{X}^{s,\alpha}}\|w\|_{\mathcal{Y}^{s,\alpha}} \leq (1 + \widetilde{C}_i R)\|w\|_{\mathcal{Y}^{s,\alpha}}\end{aligned}$$

for some constant $\widetilde{C}_i > 0$. For convenience, we choose $R \leq \widetilde{C}_i^{-1}$, thus

$$\|\Box_{g(u)}w\|_{H_{\mathrm{b}}^{s-1,\alpha}} \leq 2\|w\|_{\mathcal{Y}^{s,\alpha}}, \quad w \in \mathcal{Y}^{s,\alpha}.$$

Using this, we obtain for $u, v \in B(R)$:

$$\begin{aligned}\|\Box_{g(u)}u - \Box_{g(v)}v\|_{H_{\mathrm{b}}^{s-1,\alpha}} &\leq \|\Box_{g(u)}(u - v)\|_{H_{\mathrm{b}}^{s-1,\alpha}} + \|(\Box_{g(u)} - \Box_{g(v)})v\|_{H_{\mathrm{b}}^{s-1,\alpha}} \\ &\leq 2\|u - v\|_{\mathcal{Y}^{s,\alpha}} + \left\|\left(\Big(1 - \frac{\mu(u)}{\mu(v)}\Big)\Box_{g(u)} - E_{v,u}\right)v\right\|_{H_{\mathrm{b}}^{s-1,\alpha}} \\ &\leq 2\|u - v\|_{\mathcal{Y}^{s,\alpha}} + C'L_\mu(R)\|u - v\|_{\mathcal{X}^{s,\alpha}}\left(\|\Box_{g(u)}v\|_{H_{\mathrm{b}}^{s-1,\alpha}} + \|v\|_{\mathcal{X}^{s,\alpha}}\right) \\ &\leq (2 + 3C'L_\mu(R)R)\|u - v\|_{\mathcal{Y}^{s,\alpha}} \leq 3\|u - v\|_{\mathcal{Y}^{s,\alpha}}\end{aligned}$$

for sufficiently small R, where $C' = C_i'(1 + C)$. Thus, with $L_q'(R) := 3L_q(R)$, we have

$$\|q(u) - q(v)\|_{H_{\mathrm{b}}^{s-1,\alpha}} \leq L_q'(R)\|u - v\|_{\mathcal{Y}^{s,\alpha}}$$

for $u, v \in \mathcal{Y}^{s,\alpha}$ with norm $\leq R$.

We can now analyze the map T: First, for $u \in B(R)$ and $f \in H_{\mathrm{b}}^{s-1,\alpha}$, $\|f\| \leq C_f$, we have, recalling (9.1.25), here applied with $v = 0$,

$$\|Tu\|_{\mathcal{X}^{s,\alpha}} \leq C_S(C_f + L'_q(R)R)$$

and

$$\begin{aligned}\|\Box_{g(0)}Tu\|_{H_{\mathrm{b}}^{s-1,\alpha}} &\leq \|(\Box_{g(0)} - \Box_{g(u)})S_{g(u)}(f + q(u))\|_{H_{\mathrm{b}}^{s-1,\alpha}} \\ &\quad + \|f + q(u)\|_{H_{\mathrm{b}}^{s-1,\alpha}} \\ &\leq (1 + C_i L_\mu(R)R)(C_f + L'_q(R)R).\end{aligned}$$

Thus, if $L'_q(0) < (1 + C_S)^{-1}$, then

$$C_f(R) := R\big((1 + C_S + C_i L_\mu(R)R)^{-1} - L'_q(R)\big)$$

is positive for small enough $R > 0$. We conclude that for $f \in H_{\mathrm{b}}^{s-1,\alpha}$ with norm $\leq C_f(R)$, the map T indeed maps $B(R)$ into itself. We next have to check that T is in fact a contraction on $B(R)$, where we choose R even smaller if necessary. As in the proof of the first half of the theorem, we can arrange

$$\|Tu - Tv\|_{\mathcal{X}^{s,\alpha}} \leq \theta\|u - v\|_{\mathcal{Y}^{s,\alpha}}, \quad u, v \in B(R) \tag{9.1.27}$$

for some fixed $\theta < 1$. Moreover, for $u, v \in B(R)$,

$$\begin{aligned}\|\Box_{g(0)}(Tu - Tv)\|_{H_{\mathrm{b}}^{s-1,\alpha}} &\leq \|\Box_{g(0)}S_{g(u)}(q(u) - q(v))\|_{H_{\mathrm{b}}^{s-1,\alpha}} \\ &\quad + \|\Box_{g(0)}(S_{g(u)} - S_{g(v)})(f + q(v))\|_{H_{\mathrm{b}}^{s-1,\alpha}}.\end{aligned} \tag{9.1.28}$$

The first term on the right can be estimated by

$$\begin{aligned}\|q(u) - q(v)\|_{H_{\mathrm{b}}^{s-1,\alpha}} &+ \|(\Box_{g(u)} - \Box_{g(0)})S_{g(u)}(q(u) - q(v))\|_{H_{\mathrm{b}}^{s-1,\alpha}} \\ &\leq L'_q(R)(1 + C_i L_\mu(R)R)\|u - v\|_{\mathcal{Y}^{s,\alpha}}.\end{aligned}$$

For the second term on the right hand side of (9.1.28), we use the algebraic identity

$$\Box_{g(0)}(S_{g(u)} - S_{g(v)}) = (I + (\Box_{g(0)} - \Box_{g(u)})S_{g(u)})(\Box_{g(v)} - \Box_{g(u)})S_{g(v)},$$

which gives

$$\|\Box_{g(0)}(S_{g(u)} - S_{g(v)})\|_{\mathcal{L}(\mathcal{X}^{s-1,\alpha})} \leq (1 + C_i L_\mu(R) R) C_i L_\mu(R) \|u - v\|_{\mathcal{Y}^{s,\alpha}}.$$

Plugging this into equation (9.1.28), we obtain

$$\|\Box_{g(0)}(Tu - Tv)\|_{H_{\mathrm{b}}^{s-1,\alpha}} \leq C'(R) \|u - v\|_{\mathcal{Y}^{s,\alpha}}$$

with

$$C'(R) = (1 + C_i L_\mu(R) R)\big(L'_q(R) + C_i L_\mu(R)(C_f(R) + L'_q(R) R)\big).$$

Now if $L'_q(0)$ is sufficiently small, then since the second summand of the second factor of $C'(R)$ tends to 0 as $R \to 0$, we can choose R so small that $C'(R) < 1 - \theta$, and we finally get with (9.1.27):

$$\|Tu - Tv\|_{\mathcal{Y}^{s,\alpha}} \leq \theta' \|u - v\|_{\mathcal{Y}^{s,\alpha}}, \quad u, v \in B(R),$$

for some $\theta' < 1$, which proves that T is a contraction on $B(R)$, thus has a unique fixed point, which solves the PDE (9.1.19). The continuous dependence on f is shown as in the proof of the first half of the theorem. □

Remark 9.1.23. The space $\mathcal{Y}^{s,\alpha}$ introduced in the proof of the second part, see equation (9.1.26), which the solution u of equation (9.1.19) belongs to, is a coisotropic space similar to the ones used in [114] and §5.2.1, with the difference being that here $\Box_{g_0}$ is allowed to have non-smooth coefficients. It still is a natural space in the sense that the space of elements of the form $c(\phi \circ \mathfrak{t}_1) + w$, $c \in \mathbb{C}, w \in \dot{\mathcal{C}}_{\mathrm{c}}^\infty$, is dense. Indeed, since $\Box_{g_0}$ annihilates constants, it suffices to check that $\dot{\mathcal{C}}_{\mathrm{c}}^\infty$ is dense in $\mathcal{Y}_0^{s,\alpha} := \{u \in H_{\mathrm{b}}^{s,\alpha} : \Box_{g_0} u \in H_{\mathrm{b}}^{s-1,\alpha}\}$. Let J_ϵ be a mollifier as in Lemma 8.5.5. Given $u \in \mathcal{Y}_0^{s,\alpha}$, put $u_\epsilon := J_\epsilon u$. Then $u_\epsilon \to u$ in $H_{\mathrm{b}}^{s,\alpha}$, and

$$\Box_{g_0} u_\epsilon = J_\epsilon \Box_{g_0} u + [\Box_{g_0}, J_\epsilon] u;$$

the first term converges to $\Box_{g_0} u$ in $H_{\mathrm{b}}^{s-1,\alpha}$. To analyze the second term, observe that we have

$$\Box_{g_0} J_\epsilon - J_\epsilon \Box_{g_0} = \Box_{g_0}(J_\epsilon - I) + (I - J_\epsilon)\Box_{g_0} \to 0 \text{ strongly in } \mathcal{L}(H_{\mathrm{b}}^{s+1,\alpha}, H_{\mathrm{b}}^{s-1,\alpha}),$$

and since $H_{\mathrm{b}}^{s+1,\alpha} \subset H_{\mathrm{b}}^{s,\alpha}$ is dense, it suffices to show that $[\Box_{g_0}, J_\epsilon]$ is a bounded family in

$\mathcal{L}(H_{\mathrm{b}}^{s,\alpha}, H_{\mathrm{b}}^{s-1,\alpha})$. Write

$$\Box_{g_0} = Q_1 + Q_2 + E, \quad Q_1 \in \mathrm{Diff}_{\mathrm{b}}^2, Q_2 \in H_{\mathrm{b}}^{s,\alpha}\mathrm{Diff}_{\mathrm{b}}^2, E \in (\mathcal{C}^\infty + H_{\mathrm{b}}^{s-1,\alpha})\mathrm{Diff}_{\mathrm{b}}^1.$$

Then $[Q_1, J_\epsilon]$ and $[E, J_\epsilon]$ are bounded in $\mathcal{L}(H_{\mathrm{b}}^{s,\alpha}, H_{\mathrm{b}}^{s-1,\alpha})$. Now $Q_2 J_\epsilon$ can be expanded into a leading order term Q'_ϵ and a remainder $R_{1,\epsilon}$ which is uniformly bounded in $H_{\mathrm{b}}^s \Psi_{\mathrm{b}}^1$; but also $J_\epsilon Q_2$ has an expansion by Theorem 8.2.12 (2a) (with $k = k' = 1$) into the same leading order term Q'_ϵ and a remainder $R_{2,\epsilon}$ which is uniformly bounded in $\Psi_{\mathrm{b}}^{1;0} H_{\mathrm{b}}^{s-1}$. Hence $[Q_2, J_\epsilon] = R_{1,\epsilon} - R_{2,\epsilon}$ is bounded in $\mathcal{L}(H_{\mathrm{b}}^{s,\alpha}, H_{\mathrm{b}}^{s-1,\alpha})$ by Proposition 8.2.9, finishing the argument.

9.1.4 Solving quasilinear Klein-Gordon equations

One has corresponding results to the theorems in the previous two sections for quasilinear Klein-Gordon equations, i.e. for Theorems 9.1.11, 9.1.15 and 9.1.22 with $\Box$ replaced by $\Box - m^2$; only the function spaces need to be adapted to the situation at hand, as follows: Denote $P := \Box_{g_{\mathrm{dS}}} - m^2$ and let $(\sigma_j)_{j \in \mathbb{N}}$ be the sequence of poles of $\widehat{P}(\sigma)^{-1}$, with multiplicity, sorted by increasing $-\operatorname{Im}\sigma_j$; see equation (5.2.9) for the explicit formula. However, keep in mind that everything we do works in greater generality, see the discussion at the beginning of §9.1.2; we stick to the case of exact de Sitter space here for clarity. Let us assume that the 'mass' $m \in \mathbb{C}$ is such that $\operatorname{Im}\sigma_1 < 0$. A major new feature of Klein-Gordon equations as compared to wave equations is that non-linearities like $q(u) = u^p$ can be dealt with, more generally

$$q(u, {}^{\mathrm{b}}du) = \sum_j u^{e_j} \prod_{l=1}^{N_j} X_{jl} u, \quad e_j + N_r \geq 2, X_{jl} \in \mathcal{V}_{\mathrm{b}}.$$

See Theorem 5.2.6 for the related discussion of semilinear equations. We give an (incomplete) short list of possible scenarios and the relevant function spaces; for concreteness, we work on exact de Sitter space, but our methods work in much greater generality.

(1) If $\operatorname{Im}\sigma_1 \neq \operatorname{Im}\sigma_2$, as is e.g. the case for small mass $m^2 < (n-1)^2/4$, let $\alpha_0 = \min(1, \operatorname{Im}\sigma_1 - \operatorname{Im}\sigma_2)$, and for $-\operatorname{Im}\sigma_1 < \alpha < -\operatorname{Im}\sigma_1 + \alpha_0$, put

$$\mathcal{X}^{s,\alpha} := \mathbb{C}(\tau^{i\sigma_1}) \oplus H_{\mathrm{b}}^{s,\alpha}.$$

We can then solve quasilinear equations of the form explained above with forcing in $H_{\mathrm{b}}^{s-1,\alpha}$ and get one term, $c\tau^{i\sigma_1}$, in the expansion of the solution. Notice that if the mass is real and small, then all σ_j are purely imaginary, hence the term in the expansion is real as well if all data are, which is necessary for an analogue of Theorem 9.1.15 to hold.

(2) If $\operatorname{Im}\sigma_1 - \operatorname{Im}\sigma_2 < 1$, e.g. if $m^2 \geq n(n-2)/4$, let $\alpha_0 := \min(1, \operatorname{Im}\sigma_1 - \operatorname{Im}\sigma_3)$, and for $-\operatorname{Im}\sigma_2 < \alpha < -\operatorname{Im}\sigma_1 + \alpha_0$, put

$$\mathcal{X}^{s,\alpha} := \mathbb{C}(\tau^{i\sigma_1}) \oplus \mathbb{C}(\tau^{i\sigma_2}) \oplus H_{\mathrm{b}}^{s,\alpha}, \quad \sigma_2 \neq \sigma_1,$$
$$\mathcal{X}^{s,\alpha} := \mathbb{C}(\tau^{i\sigma_1}) \oplus \mathbb{C}(\tau^{i\sigma_1}\log\tau) \oplus H_{\mathrm{b}}^{s,\alpha}, \quad \sigma_2 = \sigma_1,$$

then we can solve equations as above with forcing in $H_{\mathrm{b}}^{s-1,\alpha}$ and obtain two terms in the expansion. For masses $m^2 > (n-1)^2/4$, we have $\operatorname{Im}\sigma_1 = \operatorname{Im}\sigma_2 =: -\sigma$ and $\operatorname{Re}\sigma_1 = -\operatorname{Re}\sigma_2 =: \rho$, hence the terms in the expansion for real data are a linear combination of $\tau^\sigma\cos(\rho\tau)$ and $\tau^\sigma\sin(\rho\tau)$.

(3) If the forcing decays more slowly than $\tau^{i\sigma_1}$, then with $0 < \alpha < -\operatorname{Im}\sigma_1$, we can work on the space

$$\mathcal{X}^{s,\alpha} := H_{\mathrm{b}}^{s,\alpha},$$

with forcing in $H_{\mathrm{b}}^{s-1,\alpha}$.

To prove the higher regularity statement in Theorem 9.1.15 for quasilinear Klein-Gordon equations, one first obtains higher regularity $H_{\mathrm{b}}^{s',\alpha}$ with $0 \leq \alpha < -\operatorname{Im}\sigma_1$ and then, if the amount of decay of the forcing is high enough to allow for it, applies Theorem 9.1.8 to obtain a partial expansion of u.

In the third setting, the assumption that the mass m is independent of the solution u can easily be relaxed: Namely, assuming that $m = m(u)$ or $m = m(u, {}^{\mathrm{b}}du)$ with continuous (or Lipschitz) dependence on $u \in \mathcal{X}^{s,\alpha}$, the poles of the inverse of the normal operator family of $\Box_{g(u)} - m(u)^2$ depend continuously on u, hence for small u, there is still no pole with imaginary part $\geq -\alpha$, therefore the solution operator produces an element of $H_{\mathrm{b}}^{s,\alpha}$ for small u; thus, well-posedness results analogous to Theorems 9.1.11 and 9.1.22 continue to hold in this setting. If the forcing in fact does decay faster than $\tau^{i\sigma_1}$, these results can be improved in many cases: Once one has the solution $u \in H_{\mathrm{b}}^{s,\alpha}$, in particular the mass $m(u)$ is now fixed, one can apply Theorem 9.1.8 to obtain a partial expansion of u.

9.1.5 Backward problems

We briefly indicate how our methods also apply to backward problems on static patches of (asymptotically) de Sitter spaces; see Figure 9.1 for an exemplary setup.

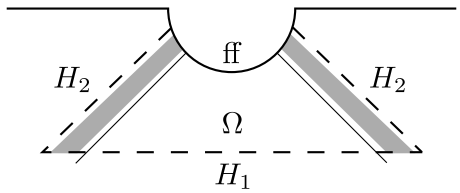

Figure 9.1: Setup for a backward problem on static de Sitter space: We work on spaces with high decay, consisting of functions supported at H_2 and extendible at H_1 (notice the switch compared to the forward problem). In the situation shown, we prescribe initial data at H_2 or, put differently, forcing in the shaded region.

We only state an analogue of Theorem 9.1.15, but remark that analogues of Theorems 9.1.11 and 9.1.22 also hold. For simplicity, we again only work on static de Sitter spaces. We use the notation from §9.1.1.

Theorem 9.1.24. *Let $s > n/2+6$, $N, N' \in \mathbb{N}$, and suppose $c_k \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$, $g_k \in (\mathcal{C}^\infty + H_{\mathrm{b}}^s)(\Omega; S^{2\mathrm{b}}T^*_\Omega M)$ for $1 \leq k \leq N$; for $r \in \mathbb{R}$, define the map*

$$g\colon H_{\mathrm{b}}^{s,r}(\Omega)^{-,\bullet} \to (\mathcal{C}^\infty + H_{\mathrm{b}}^{s,r})(\Omega; S^{2\mathrm{b}}T^*_\Omega M), \quad g(u) = \sum_{k=1}^N c_k(u) g_k,$$

and assume $g(0) = g_{\mathrm{dS}}$. Moreover, define

$$q(u, {}^{\mathrm{b}}du) = \sum_{j=0}^{N'} u^{e_j} \prod_{k=1}^{N_j} X_{jk} u, \quad e_j + N_j \geq 2, X_{jk} \in \mathcal{V}_{\mathrm{b}}(M),$$

and let further $L \in \mathrm{Diff}_{\mathrm{b}}^1$ with real coefficients. Then there is $r_* \in \mathbb{R}$ such that for all $r > r_*$, the following holds: For small $R > 0$, there exists $C_f > 0$ such that for all $f \in H_{\mathrm{b}}^{s-1,r}(\Omega;\mathbb{R})^{-,\bullet}$ with norm $\leq C_f$, the equation

$$(\Box_{g(u)} + L)u = f + q(u, {}^{\mathrm{b}}du)$$

has a unique solution $u \in H_{\mathrm{b}}^{s,r}(\Omega;\mathbb{R})^{-,\bullet}$ with norm $\leq R$, and in the topology of $H_{\mathrm{b}}^{s-1,r}(\Omega)^{-,\bullet}$, u depends continuously on f. If one in fact has $f \in H_{\mathrm{b}}^{s'-1,r}(\Omega;\mathbb{R})^{-,\bullet}$ for some $s' \in (s,\infty]$, *then* $u \in H_{\mathrm{b}}^{s',r}(\Omega;\mathbb{R})^{-,\bullet}$.

Remark 9.1.25. Notice that the structure of lower order terms is completely irrelevant here! One could in fact let L depend on u in a Lipschitz fashion and still have well-posedness.

Proof of Theorem 9.1.24. Let $r_0 < 0$ as given by Lemma 9.1.3, and suppose $r > -r_0$. As in the proof of Lemma 9.1.4, we obtain for $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{-,\bullet}$ with $\|u\| \leq R$, $R > 0$ sufficiently small, a *backward* solution operator

$$S_{g(u)} \colon H_{\mathrm{b}}^{-1,r}(\Omega)^{-,\bullet} \to H_{\mathrm{b}}^{0,r}(\Omega)^{-,\bullet}$$

for $\Box_{g(u)} + L$, with uniformly bounded operator norm. Now, if we take $r > r_*$ with $r_* \geq -r_0$ sufficiently large, $S_{g(u)}$ restricts to an operator

$$S_{g(u)} \colon H_{\mathrm{b}}^{s-1,r}(\Omega)^{-,\bullet} \to H_{\mathrm{b}}^{s,r}(\Omega)^{-,\bullet}.$$

Indeed, given $v \in H_{\mathrm{b}}^{0,r}(\Omega)^{-,\bullet}$ solving $\Box_{g(u)} v \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{-,\bullet}$, we apply the propagation near radial points, Theorem 8.5.10, this time propagating regularity *away from* the boundary, and the real principal type propagation and elliptic regularity iteratively to prove $v \in H_{\mathrm{b}}^{s,r}(\Omega)^{-,\bullet}$; the last application of the radial points result requires that r be larger than an s-dependent quantity, hence the condition on r_* in the statement of the theorem. From here, a Picard iteration argument, namely considering

$$u \mapsto S_{g(u)}(f + q(u, {}^{\mathrm{b}}du)),$$

gives existence and well-posedness. The higher regularity statement is proved as in the proof of Theorem 9.1.15. □

A slightly more elaborate version of this theorem, applied to the Einstein vacuum equations, should enable us to construct vacuum asymptotically de Sitter spacetimes as done in the Kerr setting in [24]. In fact, apart from constructing appropriate initial data, this should work in the Kerr-de Sitter setting as well, yielding the existence of dynamical vacuum black holes in de Sitter spacetimes; the point here is that for the backward problem, one works in decaying spaces, where one has non-trapping estimates as proved in §3.3.2 in the smooth setting, and in §8.5.5 in the non-smooth setting. We point out however that the authors of [24] consider a *characteristic* problem, whereas our analysis, without further modifications, would require initial data on *spacelike* hypersurfaces placed beyond the horizons, which makes the construction of initial data much more difficult.

9.2 Quasilinear waves on spacetimes with normally hyperbolic trapping

We next consider quasilinear wave equations on non-trapping spacetimes *with normally hyperbolic trapping*, see Definition 2.5.1, for which infinity has a structure generalizing that of Kerr-de Sitter space, see §2.4. An important feature is that, as in perturbations of Kerr-de Sitter space, the trapped geodesics form a normally hyperbolic invariant manifold. We prove the global existence and decay of solutions; this means decay to constants for the actual wave equation. The main new tool introduced in this section as compared to §9.1 is a Nash-Moser iteration necessitated by the loss of derivatives in the linear estimates at the normally hyperbolic trapping. To our knowledge, this is the first global result for the forward problem for a quasilinear wave equation on either a Kerr or a Kerr-de Sitter background. We remark, however, that Dafermos, Holzegel and Rodnianski [24] have constructed backward solutions for Einstein's equations on the Kerr background; for backward constructions the trapping does not cause difficulties. For concreteness, we state our results first in the special case of Kerr-de Sitter space, but it is important to keep in mind that the setting is more general.

By adding an 'ideal boundary' at infinity in the standard description of Kerr-de Sitter space, the region of Kerr-de Sitter space we are interested in can be considered a (non-compact) 4-dimensional manifold M with boundary; the boundary $X = \partial M$ is the boundary of M at future infinity. See §2.4 for the setup in the language of b-geometry. The spacetime M is equipped with a Lorentzian b-metric g_0 depending on three parameters $\Lambda > 0$ (the cosmological constant), $M_\bullet > 0$ (the black hole mass) and a (the angular momentum), though we usually drop this in the notation. We continue to assume throughout this section that Λ, $M_\bullet$ and a are such that the non-degeneracy condition [114, (6.2)] holds, which in particular ensures that the cosmological horizon lies outside the black hole event horizon. As discussed in §§2.3 and 2.4, this Lorentzian metric has a specific global dynamical structure, captured by Definition 2.5.1.

In order to set up our problem, see Figure 9.2 for an illustration, we again consider a submanifold with corners $\Omega \subset M$, which is bounded at future infinity by $\Omega \cap \partial M$, and

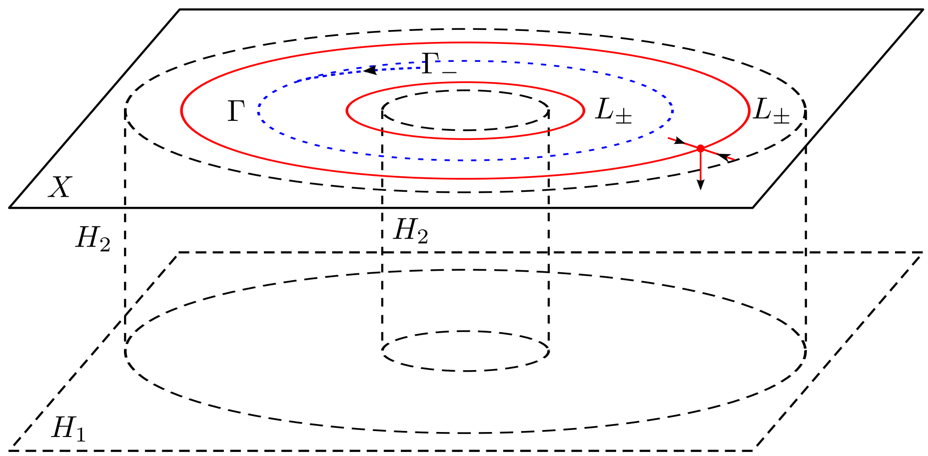

Figure 9.2: Setup for the discussion of the forward problem on Kerr-de Sitter space. Indicated are the ideal boundary X, the Cauchy hypersurface H_1 and the hypersurface H_2, which has two connected components which lie beyond the cosmological horizon and beyond the black hole event horizon, respectively. The horizons at X themselves are the projections to the base of the (generalized) radial sets $L_\pm$, discussed below, each of which has two components, corresponding to the two horizons. The projection to the base of the bicharacteristic flow is indicated near a point on L_+; near L_-, the directions of the flow-lines are reversed. Lastly, Γ is the trapped set, and the projection of a trapped trajectory approaching Γ within $\Gamma_- = \Gamma_-^+ \cup \Gamma_-^-$, discussed below, is indicated.

beyond the horizons by artificial spacelike hypersurfaces H_1 and H_2 (intersected with Ω), which are the level sets of two functions $\mathfrak{t}_j$, $j = 1, 2$, with forward, resp. backward, time-like differentials near their respective 0-set H_j, which are linearly independent at their joint 0-set; the domain $\Omega = \mathfrak{t}_1^{-1}([0,\infty)) \cap \mathfrak{t}_2^{-1}([0,\infty))$ is compact. As usual, we are interested in solving the forward problem for wave-like equations in Ω, i.e. imposing vanishing Cauchy data at H_1, which we assume is disjoint from X; initial value problems with general Cauchy data can always be converted into an equation of this type.

The wave equations we consider include those of the form

$$\Box_{g(u,{}^{\mathrm{b}}du)} u = f + q(u, {}^{\mathrm{b}}du),$$

where $g(0,0) = g_0$, and for each $p \in M$, $g_p(v_0, v)\colon \mathbb{R} \oplus {}^{\mathrm{b}}T_p^*M \to S^{2\mathrm{b}}T_p^*M$, depending smoothly on $p \in M$, and

$$q(u, {}^{\mathrm{b}}du) = \sum_{j=1}^{N'} a_j u^{e_j} \prod_{k=1}^{N_j} X_{jk} u, \quad e_j, N_j \in \mathbb{N}_0,\ N_j + e_j \geq 2,$$

with

$$a_j \in \mathcal{C}^\infty(M),\ X_{jk} \in \mathcal{V}_{\mathrm{b}}(M). \tag{9.2.1}$$

Here, a_j is only relevant if $N_j = 0$.

Our central result in the form which is easiest to state, without reference to the natural Sobolev spaces, is:

Theorem 9.2.1. *On Kerr-de Sitter space with angular momentum $|a| \ll M_\bullet$, for $\alpha > 0$ sufficiently small and $f \in \mathcal{C}_c^\infty(\Omega^\circ)$ with sufficiently small H^{14}-norm, the wave equation $\Box_{g(u,{}^{\mathrm{b}}du)} u = f + q(u, {}^{\mathrm{b}}du)$, with q as above with $N_j \geq 1$ for all j, has a unique smooth (in Ω°) global forward solution of the form $u = u_0 + \widetilde{u}$, $x^{-\alpha}\widetilde{u}$ bounded, $u_0 = c\chi$, $\chi \in \mathcal{C}^\infty(\Omega)$ identically 1 near $\Omega \cap \partial M$.*

Further, the analogous conclusion holds for the Klein-Gordon operator $\Box - m^2$ with $m > 0$ sufficiently small, without the presence of the u_0 term, i.e. for $\alpha > 0$, $m > 0$ sufficiently small, if $f \in \mathcal{C}_c^\infty(\Omega^\circ)$ has sufficiently small H^{14}-norm, $(\Box_{g(u,{}^{\mathrm{b}}du)} - m^2)u = f + q(u, {}^{\mathrm{b}}du)$ has a unique smooth global forward solution $u \in x^\alpha L^\infty(\Omega)$. In fact, for Klein-Gordon equations one can also obtain a leading term, analogously to u_0, which now has the form $cx^{i\sigma_1}\chi$, σ_1 the resonance of $\Box_{g(0)} - m^2$ with the largest imaginary part; thus $\operatorname{Im}\sigma_1 < 0$, so this is a decaying solution.

The *only* reason the assumption $|a| \ll M_\bullet$ is made is due to the possible presence (to the extent that we do not disprove it here) of resonances in $\operatorname{Im}\sigma \geq 0$, apart from the 0-resonance with constants as the resonant state, for larger a. Below, in §9.2.1, we give a general result in a form that makes it clear that this is the only remaining item to check – indeed, this even holds in natural vector bundle settings.

The natural global regularity assumptions are expressed in terms of weighted b-Sobolev spaces on Ω. We then relax (9.2.1) to

$$a_j \in \mathcal{C}^\infty(M) + H_{\mathrm{b}}^\infty(M),\ X_{jk} \in (\mathcal{C}^\infty + H_{\mathrm{b}}^\infty)\mathcal{V}_{\mathrm{b}}(M), \tag{9.2.2}$$

in our assumptions. (This is an invariant assumption, see §2.1.2.) Generalizing the forcing as well, and making the conclusion more precise, the more natural version of Theorem 9.2.1 is, with further generalization given in Theorems 9.2.3 and 9.2.4:

Theorem 9.2.2. *On Kerr-de Sitter space with angular momentum $|a| \ll M_\bullet$, for $\alpha > 0$ sufficiently small and $f \in H_{\mathrm{b}}^{\infty,\alpha}$ with sufficiently small $H_{\mathrm{b}}^{14,\alpha}$-norm, the wave equation $\Box_{g(u,{}^{\mathrm{b}}du)} u = f + q(u, {}^{\mathrm{b}}du)$, with q as above with $N_j \geq 1$ for all j, has a unique, smooth in Ω°, global forward solution of the form $u = u_0 + \widetilde{u}$, $\widetilde{u} \in H_{\mathrm{b}}^{\infty,\alpha}$, $u_0 = c\chi$, $\chi \in \mathcal{C}^\infty(\Omega)$ identically 1 near $\Omega \cap \partial M$.*

Further, the analogous conclusion holds for the Klein-Gordon equation $\Box - m^2$ with $m > 0$ sufficiently small, without the presence of the u_0 term, i.e. for $\alpha > 0$, $m > 0$ sufficiently small, if $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega)$ has sufficiently small $H_{\mathrm{b}}^{14,\alpha}$-norm, $(\Box_{g(u,{}^{\mathrm{b}}du)} - m^2)u = f + q(u, {}^{\mathrm{b}}du)$ has a unique, smooth in Ω°, global forward solution $u \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega)$.

For the proofs, we refer to Corollaries 9.2.17 and 9.2.20, which are special cases of Theorems 9.2.14 and 9.2.19. For any finite amount of regularity of the solution, our arguments only require a finite number of derivatives: Indeed, for sufficiently large $s_0, C \in \mathbb{R}$ and for $s \geq s_0$, it is sufficient to assume $f \in H_{\mathrm{b}}^{Cs,\alpha}$, with small $H_{\mathrm{b}}^{14,\alpha}$-norm, to ensure the existence of a unique global forward solution u with $H_{\mathrm{b}}^{s,\alpha}$-regularity, i.e. with $\widetilde{u} \in H_{\mathrm{b}}^{s,\alpha}$ in the case of wave equations, $u \in H_{\mathrm{b}}^{s,\alpha}$ in the case of Klein-Gordon equations; see Remark 9.2.16 for details.

In the next section, §9.2.1, we explain the ingredients of the proof of Theorem 9.2.2, and we also state natural generalizations.

9.2.1 Overview of the proof and the more general results

As pointed out at the end of §3.3.4, the study of b-differential operators such as wave operators associated with a Lorentzian b-metric relies on high frequency regularity and normal operator analysis, and both aspects have been treated both in non-trapping as well as in trapping situations in Chapter 5, specifically §§5.2 and 5.3, and Chapters 6 and 9. In particular, recall that for the scalar wave operator $\Box_g$, the only resonance with non-negative imaginary part is 0, with the kernel of $\widehat{\Box_g}(\sigma)$ one dimensional, consisting of constants. Since strips above the bands of resonances caused by the trapping [44] can only have finitely many resonances, there is $r > 0$ such that in $\operatorname{Im} \sigma \geq -r$ the only resonance is 0; then $H_{\mathrm{b}}^{s,r} \oplus \mathbb{C}$ is a space on which we have uniqueness and existence for the forward the forward problem for $\Box_g$, as described in §§5.2 and 5.3. For the Klein-Gordon equation with $m > 0$ small, the $m = 0$ resonance at 0 moves to $\sigma_1 = \sigma_1(m)$ inside $\operatorname{Im}\sigma < 0$, see [40] and Lemma 5.3.3. Thus, one can either work with $H_{\mathrm{b}}^{s,r'}$ where r' is sufficiently small (depending on m), or with $H_{\mathrm{b}}^{s,r} \oplus \mathbb{C}$, though with $\mathbb{C}$ now identified with $c\tau^{i\sigma_1}\chi$, with τ a defining function of future infinity as usual.

We now discuss the nonlinear terms. As in Chapter 8 and §9.1, the basic point is that $H_{\mathrm{b}}^{s,0}$ is an algebra if $s > n/2$, and thus for such s, products of elements of $H_{\mathrm{b}}^{s,r}$ possess even more decay if $r > 0$, but they become more growing if $r < 0$. Thus, one is forced to work with $r \geq 0$.

First, with the simplest semilinear equation, with no derivatives in the non-linearity q (so $N_j \geq 2$ is replaced by $N_j = 0$), the regularity losses due to the normally hyperbolic trapping are in principle sufficiently small to allow for a contraction mapping principle based argument. However, for the actual wave equation on Kerr-de Sitter space, the 0-resonance prohibits this, as the iteration maps outside the space $H_{\mathrm{b}}^{s,r} \oplus \mathbb{C}$; see also Remark 5.2.11. Thus, it is the semilinear Klein-Gordon equation that is well-behaved from this perspective, and this was solved in §5.3. On the other hand, if derivatives are allowed, with an at least quadratic behavior in ${}^{\mathrm{b}}du$, then the non-linearity annihilates the 0-resonance. However, since the normally hyperbolic estimate loses $1 + \epsilon$ derivatives, as opposed to the usual real principal type/radial point loss of one derivative, the solution operator for $\Box_g$ will not map $q(u, {}^{\mathrm{b}}du)$ back into the desired Sobolev space, preventing a nonlinear analysis based on the contraction mapping principle.

The Nash-Moser iteration is designed to deal with just such a situation. In this chapter we adapt the iteration to our requirements, and in particular show that semilinear equations of the kind just described are in fact solvable. Our arguments rely in particular strongly on the tame estimates for linear problems with non-smooth coefficients proved in §§8.7 and 8.8. Here we remark that Klainerman's early work on global solvability involved the Nash-Moser scheme [68], though this was later removed by Klainerman and Ponce [71]. In the present situation the loss of derivatives seems much more serious, however, due to the trapping, and is unclear whether the solution scheme can be made more 'classical.'

Quasilinear versions of the above non-trapping scenario were studied in §9.1 on perturbations of static de Sitter space; the key ingredient in dealing with quasilinear equations is to allow operators with coefficients with regularity the same kind as what one is proving for the solutions, in this case $H_{\mathrm{b}}^{s,r}$-regularity. All of the smooth linear ingredients (microlocal elliptic regularity, propagation of singularities, radial points) have their analogue for

$H_{\mathrm{b}}^{s,r}$ coefficients if s is sufficiently large. In our Kerr-de Sitter situation, there is normally hyperbolic trapping. However, notice that as we work in decaying Sobolev spaces modulo constants, $\Box_{g(u)}$ differs from a Kerr-de Sitter operator with smooth coefficients, $\Box_{g(c)}$, by one with *decaying coefficients*. This means that one can combine the smooth coefficient normally hyperbolic theory, as in the work of Dyatlov [42], with a tame estimate in $H_{\mathrm{b}}^{s,r}$ with $r < 0$; the sign of r here is a crucial gain since for $r < 0$ the propagation estimates through normally hyperbolic trapped sets behave in exactly the same way as real principal type estimates, as we have seen in §§3.3.2 and 8.5.5. In combination this provides the required tame estimates for Kerr-de Sitter wave equations, and Nash-Moser iteration completes the proof of the main theorem.

We emphasize that our treatment of these quasilinear equations is systematic and general. Thus, quasilinear equations which at $X = \partial M$ are modelled on a finite dimensional family $L = L(v_0)$, $v_0 \in \mathbb{C}^d$ small corresponding to the zero resonances (thus the family is 0-dimensional without 0-resonances!), of smooth b-differential operators on a vector bundle with scalar principal symbol which has the bicharacteristic dynamics of a non-trapping spacetime with normally hyperbolic trapping, as described by Definition 2.5.1, fits into it, *provided two conditions hold for the normal operator*:

(1) First, *the resonances for the model* $L(v_0)$ have negative imaginary part, or if they have 0 imaginary part, the non-linearity annihilates them.

(2) Second, the *normally hyperbolic trapping estimates* of Dyatlov [42] hold for $\widehat{L}(\sigma)$ (as $|\operatorname{Re}\sigma| \to \infty$) in $\operatorname{Im}\sigma > -r_0$ for some $r_0 > 0$. In the semiclassical rescaling, with $\sigma = h^{-1}z$, $h = |\sigma|^{-1}$, this is a statement about $L_{h,z} = h^m \widehat{L}(h^{-1}z)$, $\operatorname{Im} z > -r_0 h$. This indeed is the case if $L_{h,z}$ satisfies that *at* Γ its skew-adjoint part, $\frac{1}{2i}(L_{h,z} - L_{h,z}^*) \in h\mathrm{Diff}_h^1(X)$, for $z \in \mathbb{R}$ has semiclassical principal symbol bounded above by $h\nu_{\min}/2$ for some $\epsilon > 0$, where $\nu_{\min}$ is the minimal expansion rate in the normal directions at Γ; see [42, Theorem 1] and the remark below it (which allows the non-trivial skew-adjoint part, denoted by Q there, microlocally at Γ).

Further, the differential operator needs to be second order, with principal symbol a Lorentzian dual metric near the Cauchy hypersurfaces if the latter are used; otherwise the order m of the operator is irrelevant. It is important to point out that in view of the decay of the solutions either to 0 if there are no real resonance, or to the space of resonant states corresponding to real resonances, the conditions must be checked for at most a finite dimensional family of elements of the 'smooth' algebra $\Psi_b(M)$, and moreover there is no need to prove tame estimates, deal with rough coefficients, etc., for this point, and one is in a dilation invariant setting, i.e. can simply Mellin transform the problem. Thus, in principle, solving wave-type equations on more complicated bundles is reduced to analyzing these two aspects of the associated linear model operator at infinity. In Chapters 6 and 7, we have shown how this can be done for differential forms and more general tensors. Concretely, we have the following two theorems:

Theorem 9.2.3. *Let M be a Kerr-de Sitter space with angular momentum $|a| < \frac{\sqrt{3}}{2} M_\bullet$ that satisfies [114, (6.13)], E a vector bundle over it with a positive definite metric k on E, and let $L_{g(u,{}^b du)} \in \mathrm{Diff}_b^2(M;E)$ have principal symbol $G = g^{-1}(u, {}^b du)$ (times the identity), and suppose that $L_0 = L_{g(0,0)}$ satisfies that*

(1) the large parameter principal symbol of $\frac{1}{2i|\sigma|}(L_0 - L_0^*)$, with the adjoint taken relative to $k\,|dg|$, at the trapped set Γ is $< \nu_{\min}/2$ as an endomorphism of E,

(2) $\widehat{L}_0(\sigma)$ has no resonances in $\operatorname{Im}\sigma \geq 0$.

Then for $\alpha > 0$ sufficiently small, there exists $d > 0$, given in (9.2.34), such that the following holds: If $f \in H_b^{\infty,\alpha}(\Omega)$ has a sufficiently small H_b^{2d}-norm, then the equation $L_{g(u,{}^b du)} u = f + q(u, {}^b du)$ has a unique, smooth in M°, global forward solution $u \in H_b^{\infty,\alpha}(\Omega)$.

(This condition on $\Lambda, M_\bullet$ and a ensures non-trapping classical dynamics for the null-geodesic flow.) In condition (1), one can in fact take k to be a *pseudodifferential inner product*, as we have shown in Chapter 6. In particular, the conditions at Γ for the theorem hold if $|a| \ll M_\bullet$, $E = {}^b\Lambda^* M$, $L_{g(u,{}^b du)} = \Box_{g(u,{}^b du)}$ the differential form d'Alembertian, or indeed if $L_{g(u,{}^b du)} - \Box_{g(u,{}^b du)}$ is a 0-th order operator; see Theorem 6.4.8. Thus, in this case the only assumption in the theorem remaining to be checked is the second one, concerning resonances.

Theorem 9.2.4. *Let M be a Kerr-de Sitter space with angular momentum $|a| < \frac{\sqrt{3}}{2} M_\bullet$ that satisfies [114, (6.13)], E a vector bundle over it with a positive definite metric k on E, and let $L_{g(u,{}^b du)} \in \mathrm{Diff}_b^2(M;E)$ have principal symbol $G = g^{-1}(u, {}^b du)$ (times the identity). Suppose that $L_0 = L_{g(0,0)}$ is such that $\widehat{L}_0(\sigma)$ has a simple resonance at 0, with resonant states spanned by $u_{0,1}, \ldots, u_{0,d}$, and no other resonances in $\operatorname{Im}\sigma \geq 0$. Consider the family $\widehat{L}_{g(u_0,{}^b du_0)}(\sigma)$, $u_0 \in \operatorname{span}\{u_{0,1}, \ldots, u_{0,d}\}$ with small enough norm. Suppose that*

(1) this family only has a resonance at 0 *in* $\operatorname{Im}\sigma \geq 0$*, and the corresponding resonant states are given by* $\operatorname{span}\{u_{0,1}, \ldots, u_{0,d}\}$*,*

(2) Γ *is uniformly normally hyperbolic for* $\widehat{L}_{g(u_0, {}^{\mathrm{b}}du_0)}(\sigma)$ *for* u_0 *of small norm,*

(3) the large parameter principal symbol of $\frac{1}{2i|\sigma|}(L_0 - L_0^*)$*, with the adjoint taken relative to* $k\,|dg|$ *(or a* Ψ*-inner product as above), at the trapped set* Γ *is* $< \nu_{\min}/2$*,*

(4) $q(u_0, {}^{\mathrm{b}}du_0) = 0$ *for* $u_0 \in \operatorname{span}\{u_{0,1}, \ldots, u_{0,d}\}$*.*

Then for $\alpha > 0$ *sufficiently small, there exists* $d > 0$*, given in* (9.2.34)*, such that the following holds: If* $f \in H_{\mathrm{b}}^{\infty,\alpha}$ *has a sufficiently small* $H_{\mathrm{b}}^{2d,\alpha}$*-norm, then the equation* $L_{g(u,{}^{\mathrm{b}}du)}u = f + q(u, {}^{\mathrm{b}}du)$ *has a unique, smooth in* M°*, global forward solution of the form* $u = u_0 + \widetilde{u}$*,* $\widetilde{u} \in H_{\mathrm{b}}^{\infty,\alpha}$*,* $u_0 = \chi \sum_{j=1}^d c_j u_{0,j}$*,* $\chi \in \mathcal{C}^\infty(M)$ *identically* 1 *near* ∂M*.*

Here 'uniformly normally hyperbolic' in the theorem means that one has a smooth family $\Gamma = \Gamma_{u_0}$ of trapped sets, with a smooth family of stable/unstable manifolds, with uniform bounds (within this family) on the normal expansion rates for the flow, which ensures that the normally hyperbolic estimates are uniform within the family (for small u_0); see the discussion around (9.2.4) for details.

Again, the conditions at Γ for the theorem hold if $|a| \ll M_\bullet$, $E = {}^{\mathrm{b}}\Lambda^* M$, if $L_{g(u,{}^{\mathrm{b}}du)} - \Box_{g(u,{}^{\mathrm{b}}du)}$ is a 0-th order operator, $\Box_{g(u,{}^{\mathrm{b}}du)}$ the differential form d'Alembertian; on Kerr-de Sitter spaces with $|a| \ll M_\bullet$, we computed the space of 0-resonances in Theorem 7.5.1. See Remark 7.5.3 for an example of a quasilinear wave equation that can be solved by means of the method presented here. The uniform normal hyperbolicity condition at Γ holds if $|a| < \frac{\sqrt{3}}{2} M_\bullet$, since the hyperbolicity of Γ was shown in this generality in [114].

The plan of the rest of this section is the following. In §9.2.2, we adapt Dyatlov's analysis at normally hyperbolic trapping given in [42] to our needs. Then, beginning in §9.2.3, we solve our quasilinear equations by first showing that the microlocal results of §8.8 combine with the high energy estimates for the relevant normal operators following from the discussion of §9.2.2 to give tame estimates for the forward propagator in, and then showing in §9.2.4 that the Nash-Moser iteration indeed allows for solving our wave equations. In §9.2.5, we then explain the changes required for quasilinear Klein-Gordon equations. Finally, in §9.2.6 we show how our methods apply in the general settings of Theorems 9.2.3 and 9.2.4.

9.2.2 Trapping estimates at normally hyperbolic trapping

Complementing the results proved in §§3.3.2 and 8.5.5 on negatively weighted spaces, we recall results of Dyatlov from [44, 42] on semiclassical estimates for smooth operators at normally hyperbolic trapping, which via the Mellin transform correspond to estimates on non-negatively weighted spaces. Here we present the results in the semiclassical setting, then in §9.2.3 we relate this to the solvability of linear equations with Sobolev coefficients in Theorem 9.2.9 and Theorem 9.2.10. The advantage of Dyatlov's framework for us, especially as espoused in [42], is the explicit size of the spectral gap, which was also shown by Nonnenmacher and Zworski [95], the explicit inclusion of a subprincipal term of the correct sign (which was crucial in our analysis in Chapter 6), and the relative ease with which the parameter dependence can be analyzed.

We first recall Dyatlov's semiclassical setting for

$$\widetilde{P}_0 = \widetilde{P}_0(h), \widetilde{Q}_0 = \widetilde{Q}_0(h) \in \Psi_\hbar^m(X),$$

both formally self-adjoint, with $\widetilde{Q}_0$ having non-negative principal symbol, $\widetilde{P}_0 - i\widetilde{Q}_0$ elliptic in the standard sense. In fact, Dyatlov states the results in the special case $m = 0$, but by ellipticity of $\widetilde{P}_0 - i\widetilde{Q}_0$ in the standard sense, it is straightforward to allow general m; see also the remark [42, Bottom of p. 2]. The main assumption, see [42, p. 3], then is that $\widetilde{P}_0$ has normally hyperbolic trapping semiclassically at $\widetilde{\Gamma} \subset T^*X$ compact,[52] with all bicharacteristics of $\widetilde{P}_0$, except those in the stable ($-$) and unstable ($+$) submanifolds $\widetilde{\Gamma}_\pm$, entering the elliptic set of $\widetilde{Q}_0$ in the forward (the exception being for only the $-$ sign), resp. backward ($+$) direction, and $\gamma < \nu_{\min}/2$, where $\nu_{\min} > 0$ is the minimal normal expansion rate of the flow at $\widetilde{\Gamma}$, discussed above and in (9.2.4). If $\widetilde{Q}_0$ is microlocally in $h\Psi_\hbar(X)$ near $\widetilde{\Gamma}$, with $h^{-1}\widetilde{Q}_0$ having a non-negative principal symbol there, Dyatlov shows that there is $h_0 > 0$ such that for $\operatorname{Im} z > -\gamma$,

$$\|v\|_{H_\hbar^s} \lesssim h^{-2} \|(\widetilde{P}_0 - i\widetilde{Q}_0 - hz)v\|_{H_\hbar^{s-m}}, \quad h < h_0. \tag{9.2.3}$$

In view of $\widetilde{\Gamma}$ lying in a compact subset of T^*X, the order s is irrelevant in the sense that the estimate for one value of s implies that for all other via elliptic estimates; thus, one may just take $s = 0$, and even replace $s - m$ by 0, in which case this is an L^2-estimate, as

[52] Our $\widetilde{\Gamma}$ is the intersection of Dyatlov's K with the semiclassical characteristic set of P, and similarly our $\widetilde{\Gamma}_\pm$ are the intersection of Dyatlov's $\Gamma_\pm$ with the characteristic set of P.

stated by Dyatlov.

Suppose now that one has a family of operators $\widetilde{P}_0(\omega)$ depending on another parameter, ω, in a compact space S, with $\widetilde{P}_0, \widetilde{Q}_0$ depending continuously on ω, with values in $\Psi_\hbar^m(X)$, satisfying all of the assumptions listed above. Suppose moreover that this family satisfies the normally hyperbolic assumptions with $\widetilde{\Gamma}, \widetilde{\Gamma}_\pm$ continuously depending on ω in the $\mathcal{C}^\infty$ topology, and uniform bounds for the normal expansion rates in the sense that both ν and the constant C' in

$$\sup_{\rho\in\Gamma} \|de^{\mp tH_p}(\rho)|_{\mathcal{V}_\pm}\| \leq C'e^{-\nu t},\ t \geq 0, \tag{9.2.4}$$

with $\mathcal{V}_\pm$ the unstable and stable normal tangent bundles at Γ, can be chosen uniformly (cf. [44, Equation (5.1)]); $\nu_{\min}$ is then the sup of these possible choices of ν. (Note that since the trapped set dynamics involves arbitrarily large times, it is *not* automatically stable, unlike the dynamics away from the trapped set.) In this case the implied constant C in (9.2.5), as well as h_0, is uniform in ω. Note that r-normal hyperbolicity for every r implies the local (hence global, in view of compactness) uniformity of the normal dynamics by structural stability; see [124, §1] and [44, §5.2].

To see this uniformity in C, we first point out that in [44, Lemma 5.1] the construction of $\phi_\pm$ can be done continuously with values in $\mathcal{C}^\infty$ in this case. Then in the proof of (9.2.5) given in [42], we only need to observe that the direct estimates provided are certainly uniform in this case for families $\widetilde{P}_0, \widetilde{Q}_0$, and furthermore for the main argument, using semiclassical defect measures, one can pass to an L^2-bounded subsequence u_j such that $(\widetilde{P}_0(\omega_j) - i\widetilde{Q}_0(\omega_j) - \lambda_j)u_j = O(h^2)$, with $\omega_j \to \omega$ for some $\omega \in S$ in addition to $h^{-1}\lambda_j$ converging to some $\widetilde{\lambda}$. Concretely, all of Dyatlov's results in [42, §2] are based on elliptic or (positive) commutator identities or estimates which are uniform in this setting. In particular, [42, Lemma 2.3] is valid with $P_j = P(\omega_j) \to P$, $W_j = W(\omega_j) \to W$ with convergence in $\Psi_\hbar(X)$. (This uses that one can take $A_j(h_j)$ in Definition 2.1, with $A_j \to A$, since the difference between $A_j(h_j)$ and $A(h_j)$ is bounded by a constant times the squared L^2-norm of u_j times the operator norm bound of $A_j(h_j) - A(h_j)$, with the latter going to 0.) Then with $\Theta_{+,j}$ in place of Θ_+, one still gets Lemma 3.1, which means that Lemma 3.2 still holds with ϕ_+ (the limiting $\phi_{+,j}$) using Lemma 2.3. Then the displayed equation above [42, Equation (3.9)] still holds with the limiting $\widetilde{P}_0 = \widetilde{P}_0(\omega)$, again by Lemma 2.3, and then one can finish the argument as Dyatlov did. With this modification, one obtains the desired uniformity. This in particular allows one to apply (9.2.5) even if $\widetilde{P}_0$ and $\widetilde{Q}_0$ depend on z

(in a manner consistent with the other requirements), which can also be dealt with more directly using Dyatlov's model form [44, Lemma 4.3]. It also allows for uniform estimates for families depending on a small parameter in $\mathbb{C}$, denoted by v_0 below, needed in §9.2.3 and the subsequent sections.

Allowing $\widetilde{P}_0$ and $\widetilde{Q}_0$ depending on z means, in particular, that we can replace the requirement on $h^{-1}\widetilde{Q}_0$ by the principal symbol of $h^{-1}\widetilde{Q}_0$ being $> -\beta$, $\beta < \nu_{\min}/2$, and drop z, so one has

$$\|v\|_{H^s_\hbar} \lesssim h^{-2}\|(\widetilde{P}_0 - i\widetilde{Q}_0)v\|_{H^{s-m}_\hbar},\ h < h_0. \tag{9.2.5}$$

At this point it is convenient to rewrite this estimate, removing $\widetilde{Q}_0$ from the right hand side at the cost (or benefit!) of making it microlocal. An alternative would be using the gluing result of Datchev and Vasy [32], which is closely related in approach. *From here on it is convenient to change the conventions and not require that $\widetilde{P}_0$ is formally self-adjoint (though it is at the principal symbol level, namely it has a real principal symbol); translating back into the previous notation, one would replace $\widetilde{P}_0$ by its (formally) self-adjoint part, and absorb its skew-adjoint part into $\widetilde{Q}_0$.* Namely, we have

Theorem 9.2.5. *Suppose $\widetilde{P}_0$ satisfies the above assumptions, in particular the semiclassical principal symbol of $\frac{1}{2ih}(\widetilde{P}_0 - \widetilde{P}_0^*)$ being $< \beta < \nu_{\min}/2$ at $\widetilde{\Gamma}$.[53] With $\widetilde{B}_j$ analogous to Proposition 8.8.5, with wave front set sufficiently close to $\widetilde{\Gamma}$, we have, for sufficiently small $h > 0$ and for all N and s_0,*

$$\|\widetilde{B}_0 u\|_{H^s_\hbar} \lesssim h^{-2}\|\widetilde{B}_1\widetilde{P}_0 u\|_{H^{s-m+1}_\hbar} + h^{-1}\|\widetilde{B}_2 u\|_{H^s_\hbar} + h^N\|u\|_{H^{s_0}_\hbar}. \tag{9.2.6}$$

Note that the differential orders are actually irrelevant here due to wave front set conditions.

Proof. Take $\widetilde{Q}_0 \in \Psi^0_\hbar(X)$ with non-negative principal symbol such that $\mathrm{WF}'_\hbar(\widetilde{Q}_0)$ is disjoint from $\mathrm{WF}'_\hbar(\widetilde{B}_0)$, and so that all backward bicharacteristics from points not in $\widetilde{\Gamma}_+$, as well as forward bicharacteristics from points not in $\widetilde{\Gamma}_-$, reach the elliptic set of $\widetilde{Q}_0$, and with $\widetilde{B}_1$ elliptic on the complement of the elliptic set of $\widetilde{Q}_0$. Let $\widetilde{B}_3 \in \Psi^0_\hbar(X)$ to be such that $\mathrm{WF}'_\hbar(I - \widetilde{B}_3)$ is disjoint from $\mathrm{WF}'_\hbar(\widetilde{B}_0)$ but $\mathrm{WF}'_\hbar(\widetilde{Q}_0) \cap \mathrm{WF}'_\hbar(\widetilde{B}_3) = \emptyset$. Let $\widetilde{A}_+ \in \Psi^0_\hbar(X)$

[53] The apparent sign change here as compared to before comes from the fact that for formally self-adjoint $\widetilde{P}_0, \widetilde{Q}_0$, one has $\frac{1}{2ih}\big((\widetilde{P}_0 - i\widetilde{Q}_0) - (\widetilde{P}_0 - i\widetilde{Q}_0)^*\big) = -h^{-1}\widetilde{Q}_0$; notice the minus sign on the right hand side.

have wave front set near $\widetilde{\Gamma}_+$, with

$$\mathrm{WF}'_\hbar(I - \widetilde{A}_+) \cap \mathrm{WF}'_\hbar(\widetilde{B}_3) \cap \widetilde{\Gamma}_+ = \emptyset$$

and with

$$\mathrm{WF}'_\hbar(\widetilde{A}_+) \cap \mathrm{WF}'_\hbar(I - \widetilde{B}_3) \cap \widetilde{\Gamma}_- = \emptyset,$$

and with no backward bicharacteristic from $\mathrm{WF}'_\hbar(\widetilde{B}_0)$ reaching

$$\mathrm{WF}'_\hbar(\widetilde{A}_+) \cap \mathrm{WF}'_\hbar(I - \widetilde{B}_3) \cap \widetilde{\Gamma}_+.$$

Take $\widetilde{Q}_1$ elliptic on $\widetilde{\Gamma}$, with $\mathrm{WF}'_\hbar(\widetilde{Q}_1) \cap \mathrm{WF}'_\hbar(I - \widetilde{B}_3) = \emptyset$, again with non-negative principal symbol, with no backward bicharacteristic from $\mathrm{WF}'_\hbar(\widetilde{Q}_1)$ reaching

$$\mathrm{WF}'_\hbar(\widetilde{A}_+) \cap \mathrm{WF}'_\hbar(I - \widetilde{B}_3).$$

Thus, all backward and forward bicharacteristics of $\widetilde{P}_0$ reach the elliptic set of $\widetilde{Q}_1$ or $\widetilde{Q}_0$. See Figure 9.3 for the setup.

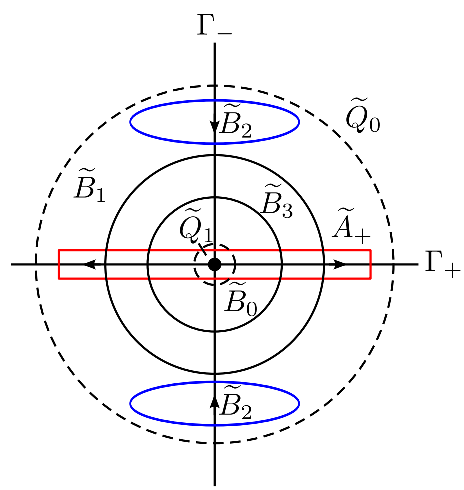

Figure 9.3: Setup for the proof of the microlocalized normally hyperbolic trapping estimate (9.2.6): Indicated are the backward and forward trapped sets Γ_+ and Γ_-, respectively, which intersect at Γ (large dot). We use complex absorbing potentials $\widetilde{Q}_0$ (with $\mathrm{WF}'_\hbar(\widetilde{Q}_0)$ outside the large dashed circle) and $\widetilde{Q}_1$ (with $\mathrm{WF}'_\hbar(\widetilde{Q}_1)$ inside the small dashed circle). We obtain an estimate for $\widetilde{B}_0 u$ by combining (9.2.5) with microlocal propagation from the elliptic set of $\widetilde{B}_2$.

Then

$$(\widetilde{P}_0 - i\widetilde{Q}_0)\widetilde{B}_3 u = \widetilde{B}_3\widetilde{P}_0 u + \widetilde{A}_+[\widetilde{P}_0, \widetilde{B}_3]u + (I - \widetilde{A}_+)[\widetilde{P}_0, \widetilde{B}_3]u - i\widetilde{Q}_0\widetilde{B}_3 u,$$

so

$$\begin{aligned}\widetilde{B}_0 u &= \widetilde{B}_0\widetilde{B}_3 u + \widetilde{B}_0(I - \widetilde{B}_3)u \\ &= \widetilde{B}_0(\widetilde{P}_0 - i\widetilde{Q}_0)^{-1}\widetilde{B}_3\widetilde{P}_0 u + \widetilde{B}_0(\widetilde{P}_0 - i\widetilde{Q}_0)^{-1}\widetilde{A}_+[\widetilde{P}_0, \widetilde{B}_3]u \\ &\quad + \widetilde{B}_0(\widetilde{P}_0 - i\widetilde{Q}_0)^{-1}(I - \widetilde{A}_+)[\widetilde{P}_0, \widetilde{B}_3]u \\ &\quad - i\widetilde{B}_0(\widetilde{P}_0 - i\widetilde{Q}_0)^{-1}\widetilde{Q}_0\widetilde{B}_3 u + \widetilde{B}_0(I - \widetilde{B}_3)u,\end{aligned} \tag{9.2.7}$$

and by (9.2.5), for $h < h_0$,

$$\|(\widetilde{P}_0 - i\widetilde{Q}_0)^{-1}\widetilde{B}_3\widetilde{P}_0 u\|_{H^s_\hbar} \lesssim h^{-2}\|\widetilde{B}_3\widetilde{P}_0 u\|_{H^{s-m}_\hbar}.$$

Now, $\widetilde{Q}_0\widetilde{B}_3, \widetilde{B}_0(I - \widetilde{B}_3) \in h^\infty\Psi^{-\infty}_\hbar(X)$, so the corresponding terms in (9.2.7) can be absorbed into $h^N\|u\|_{H^{s_0}_\hbar}$. On the other hand, since $\mathrm{WF}'_\hbar((I - \widetilde{A}_+)[\widetilde{P}_0, \widetilde{B}_3])$ is disjoint from $\widetilde{\Gamma}_+$, the backward bicharacteristics from it reach the elliptic set of $\widetilde{B}_2$, and so we have the microlocal real principal type estimate for u:

$$\|(I - \widetilde{A}_+)[\widetilde{P}_0, \widetilde{B}_3]u\|_{H^{s-m}_\hbar} \lesssim h\|\widetilde{B}_2 u\|_{H^{s-1}_\hbar} + \|\widetilde{B}_1\widetilde{P}_0 u\|_{H^{s-m}_\hbar}$$

as $(I - \widetilde{A}_+)[\widetilde{P}_0, \widetilde{B}_3] \in h\Psi^{m-1}_\hbar(X)$, so by (9.2.5),

$$\|(\widetilde{P}_0 - i\widetilde{Q}_0)^{-1}(I - \widetilde{A}_+)[\widetilde{P}_0, \widetilde{B}_3]u\|_{H^s_\hbar} \lesssim h^{-1}\|\widetilde{B}_2 u\|_{H^{s-1}_\hbar} + h^{-2}\|\widetilde{B}_1\widetilde{P}_0 u\|_{H^{s-m}_\hbar}.$$

Thus, (9.2.6) follows if we can prove an estimate for $\|\widetilde{B}_0(\widetilde{P}_0 - i\widetilde{Q}_0)^{-1}\widetilde{A}_+[\widetilde{P}_0, \widetilde{B}_3]u\|_{H^s_\hbar}$. Now, $\mathrm{WF}'_\hbar(\widetilde{A}_+[\widetilde{P}_0, \widetilde{B}_3]) \cap \widetilde{\Gamma}_- = \emptyset$ by arrangement. In order to microlocalize, we now introduce a nontrapping model, $\widetilde{P}_0 - i(\widetilde{Q}_0 + \widetilde{Q}_1)$. We claim that

$$v = \big(\widetilde{P}_0 - i(\widetilde{Q}_0 + \widetilde{Q}_1)\big)^{-1}\widetilde{A}_+[\widetilde{P}_0, \widetilde{B}_3]u - (\widetilde{P}_0 - i\widetilde{Q}_0)^{-1}\widetilde{A}_+[\widetilde{P}_0, \widetilde{B}_3]u$$

satisfies

$$\|v\|_{H^{s'}_\hbar} \lesssim h^N\|u\|_{H^{s_0}_\hbar} \tag{9.2.8}$$

for all s', N. Notice that for any s'' one certainly has

$$\|v\|_{H_\hbar^{s''}} \lesssim h^{-1}\|u\|_{H_\hbar^{s''-1}}$$

by (9.2.5) plus its non-trapping analogue. To see (9.2.8), notice that

$$(\widetilde{P}_0 - i\widetilde{Q}_0)v = i\widetilde{Q}_1\bigl(\widetilde{P}_0 - i(\widetilde{Q}_0 + \widetilde{Q}_1)\bigr)^{-1}\widetilde{A}_+[\widetilde{P}_0, \widetilde{B}_3]u,$$

so by (9.2.5), with s_0 replaced by any s_0' (since s_0 was arbitrary), and for any N,

$$\|v\|_{H_\hbar^{s'}} \lesssim h^{-2}\|\widetilde{Q}_1\bigl(\widetilde{P}_0 - i(\widetilde{Q}_0 + \widetilde{Q}_1)\bigr)^{-1}\widetilde{A}_+[\widetilde{P}_0, \widetilde{B}_3]u\|_{H_\hbar^{s'-m}} \lesssim h^N\|u\|_{H_\hbar^{s_0}},$$

since $\widetilde{P}_0 - i(\widetilde{Q}_0 + \widetilde{Q}_1)$ is non-trapping, hence $\bigl(\widetilde{P}_0 - i(\widetilde{Q}_0 + \widetilde{Q}_1)\bigr)^{-1}$ propagates semiclassical wave front sets along forward bicharacteristics, and no backward bicharacteristic from $\mathrm{WF}_\hbar'(\widetilde{Q}_1)$ can reach $\mathrm{WF}_\hbar'(\widetilde{A}_+[\widetilde{P}_0, \widetilde{B}_3]) \subset \mathrm{WF}_\hbar'(\widetilde{A}_+) \cap \mathrm{WF}_\hbar'(I - \widetilde{B}_3)$, proving the claim. Then, since backward bicharacteristics from $\mathrm{WF}_\hbar'(\widetilde{B}_0)$ do not encounter $\mathrm{WF}_\hbar'(\widetilde{A}_+[\widetilde{P}_0, \widetilde{B}_3]) \cap \widetilde{\Gamma}_+$ before reaching the elliptic set of $\widetilde{Q}_0$ or $\widetilde{Q}_1$, we conclude that

$$\begin{aligned}
&\|\widetilde{B}_0(\widetilde{P}_0 - i\widetilde{Q}_0)^{-1}\widetilde{A}_+[\widetilde{P}_0, \widetilde{B}_3]u\|_{H_\hbar^s} \\
&\qquad \le \|\widetilde{B}_0(\widetilde{P}_0 - i\widetilde{Q}_0 - i\widetilde{Q}_1)^{-1}\widetilde{A}_+[\widetilde{P}_0, \widetilde{B}_3]u\|_{H_\hbar^s} + \|\widetilde{B}_0 v\|_{H_\hbar^s} \\
&\qquad \lesssim h\|\widetilde{B}_2 u\|_{H_\hbar^s} + \|\widetilde{B}_1\widetilde{P}_0 u\|_{H_\hbar^{s-m+1}} + h^N\|u\|_{H_\hbar^{s_0}}.
\end{aligned}$$

This proves (9.2.6), and thus the theorem. □

9.2.3 Forward solution operators

We now generalize the setting considered in §9.1.1 for the study of quasilinear equations on static asymptotically de Sitter spaces to allow for normally hyperbolic trapping.

Thus, working on a compact manifold M with boundary X, we assume that the operator P is of the form $P = P_0 + \widetilde{P}$, continuously depending on a small parameter $v = v_0 + \widetilde{v} \in \mathcal{X}^{\widetilde{s},\alpha}$, with the space $\mathcal{X}^{\widetilde{s},\alpha}$ introduced in Definition 9.1.10 and recalled below, and we assume

$$\begin{aligned}
P_0 = P_0(v_0) &= \Box_{g(v_0)} + L(v_0) \in \mathrm{Diff}_\mathrm{b}^2(M), \\
L(v_0) &\in \mathrm{Diff}_\mathrm{b}^1(M), \qquad L(0) - L(0)^* \in \mathrm{Diff}_\mathrm{b}^0(M), \\
\widetilde{P} = \widetilde{P}(v) &\in H_\mathrm{b}^{\widetilde{s},\alpha}\mathrm{Diff}_\mathrm{b}^2(M)
\end{aligned} \tag{9.2.9}$$

for a smooth b-metric g on M that continuously depends on one real parameter; here, $\alpha > 0$. The main example to keep in mind for the remainder of the section is the wave operator on an (asymptotically) Kerr-de Sitter space or a $H_{\mathrm{b}}^{\widetilde{s}+1,\alpha}$-perturbation thereof. We assume that $\Omega \subset M$ is a submanifold with corners, which, equipped with the metric $g(v_0)$, is a non-trapping spacetime with normally hyperbolic trapping, see Definition 2.5.1, for small v_0, and is bounded by a Cauchy hypersurface H_1, an artificial hypersurface (possibly with several connected components) H_2 as in Figure 9.2, and at future infinity by $\Omega \cap \partial M$, with defining function τ which has $d\tau/\tau$ past timelike in Ω. The metric $g(v_0)$ then has normally hyperbolic trapping at $\Gamma \subset {}^{\mathrm{b}}T_X^*M \setminus {}^{\mathrm{b}}T^*X$, in the sense of Definition 2.3.1, which we assume to be uniform in v_0, as described in §9.2.2; furthermore, at the (approximate) radial sets $L_\pm$, the notation and numerology for Hamilton derivatives of fiber infinity and future infinity is taken to be as in §3.3.1. Lastly, and most importantly, the metric $g(v_0)$ is non-trapping in the sense of non-trapping spacetimes with normally hyperbolic trapping, i.e. the flow satisfies Proposition 2.3.2 (3').

Recall the space $H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$ of distributions which are supported ($\bullet$) at the 'artificial' boundary hypersurface H_1 and extendible ($-$) at H_2, and the other way around for $H_{\mathrm{b}}^{s,r}(\Omega)^{-,\bullet}$. The space $\mathcal{X}^{\widetilde{s},\alpha}$ is then defined as

$$\mathcal{X}^{\widetilde{s},\alpha} = \mathbb{C} \oplus H_{\mathrm{b}}^{\widetilde{s},\alpha}(\Omega)^{\bullet,-}.$$

We then have global energy estimates for the operator P, provided $\widetilde{s} > n/2+2$, as in Lemma 9.1.3; recall that this only relies on the timelike nature of the boundary defining functions of H_1 and H_2 and the timelike nature of $d\tau/\tau$.

Let us stress that we assume the parameter v to be *small* so that in particular the skew-adjoint part of $P_0(v_0)$ is small and does not affect the radial point and normally hyperbolic trapping estimates which are used in what follows; the general case without symmetry assumptions on $P_0(0)$ will be discussed in §9.2.6. Using a duality argument and the tame estimates for elliptic regularity and the propagation of singularities (real principal type, radial points, normally hyperbolic trapping) given in Propositions 8.8.1, 8.8.3, 8.8.4 and 8.8.5, we thus obtain solvability and higher regularity:

Lemma 9.2.6. *(Cf. Lemma 9.1.4.) Let $0 \le s \le \widetilde{s}$ and assume $\widetilde{s} > n/2+6$, $s_0 > n/2+1/2$. There exists $r_0 < 0$ such that for $r \le r_0$, there is $C > 0$ with the following property: If $f \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}$, then there exists a unique $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$ such that $Pu = f$, and u*

moreover satisfies

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}} \lesssim \|f\|_{H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}} + \|f\|_{H_{\mathrm{b}}^{s_0,r}(\Omega)^{\bullet,-}}\|v\|_{\mathcal{X}^{\widetilde{s},\alpha}}. \tag{9.2.10}$$

Here, the implicit constant depends only on s and $\|v\|_{\mathcal{X}^{n/2+6+\epsilon,\alpha}}$ for $\epsilon > 0$.

Proof. The proof proceeds as the proof of Lemma 9.1.4. The tame estimate (9.2.10) in particular is obtained by iterative use of the aforementioned microlocal regularity estimates; the given bound for s_0 comes from an inspection of the norms in these estimates which correspond to the terms called u_*^ℓ in (8.8.1). □

We deduce tame analogues of Corollaries 9.1.5 and 9.1.6:

Corollary 9.2.7. *Let $0 \leq s \leq \widetilde{s}$ and assume $\widetilde{s} > n/2+6$, $s_0 > n/2+1/2$. There exists $r_0 < 0$ such that for $r \leq r_0$, there is $C > 0$ with the following property: If $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$ is such that $Pu \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}$, then the estimate (9.2.10) holds.*

Corollary 9.2.8. *Let $s_0 > n/2+1/2$, $s_0 \leq s' \leq s \leq \widetilde{s}$, and assume $\widetilde{s} > n/2+6$; moreover, let $r < 0$. Then there is $C > 0$ such that the following holds: Any $u \in H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}$ with $Pu \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}$ in fact satisfies $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$, and obeys the estimate*

$$\begin{aligned}\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}} \lesssim{}& \|Pu\|_{H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}} + \|u\|_{H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}} \\ &+ (\|Pu\|_{H_{\mathrm{b}}^{s_0,r}(\Omega)^{\bullet,-}} + \|u\|_{H_{\mathrm{b}}^{s_0+1,r}(\Omega)^{\bullet,-}})\|v\|_{\mathcal{X}^{\widetilde{s},\alpha}}.\end{aligned}$$

Proof. The proof of the two corollaries is as in the referenced corollaries from the non-trapping discussion. For the radial point estimate involved in the proof of Corollary 9.2.8, we need the additional assumption $s'-1+\sup_{L_\pm}(r\widetilde{\beta}) > 0$, which however is automatically satisfied since $s' \geq 1$ and the sup is negative for $r < 0$. □

We now note that the Mellin transformed normal operator $\widehat{N}(P)(\sigma)$ of P, with σ the Mellin-dual of τ, satisfies global large parameter estimates corresponding to the semiclassical microlocal estimates of Theorem 9.2.5. Now recall from §3.3.4 that if $P_0 = P_0(v_0) \in \Psi_{\mathrm{b}}^m(M)$, then $N(P_0)$ is dilation invariant on $[0,\infty) \times X$, and its conjugate by the Mellin transform is $\widehat{P}_0 = \widehat{N}(P_0)$, whose rescaling $\widetilde{P}_0 = |\sigma|^{-m}\widehat{P}_0$ is an element of $\Psi_\hbar^m(X)$. Further, with P_0 having normally hyperbolic trapping in the b-sense (with the convention changed regarding formal self-adjointness, as stated before Theorem 9.2.5), $\widetilde{P}_0$ is normally hyperbolic in the semiclassical sense, see [42]. Fix a smooth b-density on M near X, identified with $[0,\epsilon_0)\times X$; we require this to be of the product form $\frac{|dx|}{x}\nu$, ν a smooth density on X; we compute adjoints with respect to this density. Then for any $B \in \Psi_{\mathrm{b}}^m(M)$, $\widehat{B^*}(\sigma) = (\widehat{B}(\overline{\sigma}))^*$, see also (3.3.36). In particular, if $B = B^*$, then $\widehat{B}(\sigma) = (\widehat{B}(\sigma))^*$ for $\sigma \in \mathbb{R}$. Relaxing (9.2.9) momentarily, we then assume that

$$\frac{1}{2i}(P_0 - P_0^*) \in \Psi_{\mathrm{b}}^{m-1}(M), \qquad \sigma_{\mathrm{b},m-1}\Big(\frac{1}{2i}(P_0 - P_0^*)\Big)\Big|_\Gamma < |\sigma|^{m-1}\nu_{\min}/2, \tag{9.2.11}$$

with $\nu_{\min}$ the minimal normal expansion rate for the Hamilton flow of the principal symbol of P_0 at $\Gamma \subset {}^{\mathrm{b}}T^*_X M$, as above; note that σ is elliptic on Γ. This gives that for $\sigma \in \mathbb{R}$, $\widehat{P}_0(\sigma) - \widehat{P}_0(\sigma)^*$ is order $m-1$ in the large parameter pseudodifferential algebra, so, defining $z = \sigma/|\sigma|$, the semiclassical version gives

$$\widetilde{P}_0 - \widetilde{P}_0^* \in h\Psi_\hbar^{m-1}(X),\ z \in \mathbb{R},$$

with

$$\sigma_{\hbar,m-1}\Big(\frac{1}{2ih}(\widetilde{P}_0 - \widetilde{P}_0^*)\Big)\Big|_{\widetilde{\Gamma}} < \nu_{\min}/2,\ z \in \mathbb{R},$$

where $\widetilde{\Gamma}$ is the image of Γ under the semiclassical identification. In particular, there is $\gamma_\Gamma > 0$ and $\beta_\Gamma < \nu_{\min}/2$ such that if $|\operatorname{Im} z| < h\gamma_\Gamma$ then

$$\sigma_{\hbar,m-1}\Big(\frac{1}{2ih}(\widetilde{P}_0 - \widetilde{P}_0^*)\Big)\Big|_{\widetilde{\Gamma}} < \beta_\Gamma. \tag{9.2.12}$$

With this background, under our assumptions on the dynamics, propagating estimates from the radial points towards H_2, in particular through $\widetilde{\Gamma}$, and using the uniformity in parameters described above Theorem 9.2.5, we have:

Theorem 9.2.9. *Let $C_0 > 0$. Suppose $P_0 = P_0(v_0)$ satisfies (9.2.11) at Γ, $\widetilde{P}_0$ is the semiclassical rescaling of $\widehat{P}_0 = \widehat{N}(P_0)$, $s > 1/2 + \sup(\widetilde{\beta})\gamma$ (with $\widetilde{\beta}$ coming from the radial point numerology, see (3.3.8)), $s > 1$, $\gamma < \gamma_\Gamma$, $\gamma_\Gamma > 0$ as in (9.2.12). Then there is $h_0 > 0$ such that for $h < h_0$, $|\operatorname{Im} z| < h\gamma$,*

$$\|u\|_{H_\hbar^s} \lesssim h^{-2}\|\widetilde{P}_0 u\|_{H_\hbar^{s-m+1}}, \tag{9.2.13}$$

with the implied constant and h_0 uniform in v_0 with $|v_0| \le C_0$.

Proof. This is immediate from piecing together the semiclassical propagation estimates from radial points (which is where $s > 1/2 + \sup(\widetilde{\beta})\gamma$ is used, see also [114, Propositions 2.3 and 2.4] and the corresponding statement in the b-setting given in Proposition 3.3.8) through $\widetilde{\Gamma}$, using Theorem 9.2.5, which is where $\gamma < \gamma_\Gamma$ is used and where h^{-2}, rather than h^{-1}, is obtained for the right hand side, to $H_2 \cap X$, which is where $s > 1$ is used.

An alternative proof would be using Dyatlov's setting [42] directly, together with the gluing of Datchev and Vasy [32], exactly as described in [114, Theorem 2.17]. □

Going back to the operator $P_0(v_0)$ satisfying the conditions stated at the beginning of this section, and under the additional assumption of uniform normal hyperbolicity as explained above, we can now obtain partial expansions of solutions to $Pu = f$ at infinity, i.e. at X:

Theorem 9.2.10. *(Cf. Theorem 9.1.8.) Let $0 < \alpha < \min(1, \gamma_\Gamma)$. Suppose P has a simple rank 1 resonance at 0 with resonant state 1, and that all other resonances have imaginary part less than $-\alpha$. Let $\widetilde{s} > n/2 + 6$, $s_0 > \max(n/2 + 1/2, 1 + \sup(\widetilde{\beta})\alpha)$,*[54] *and assume $s_0 \leq s \leq \widetilde{s} - 4$. Let $0 \neq r \leq \alpha$. Then any solution $u \in H_{\mathrm{b}}^{s+4,r_0}(\Omega)^{\bullet,-}$ of $Pu = f$ with $f \in H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}$ satisfies $u \in \mathcal{X}^{s',r}$ with $s' = s + 4$ for $r < 0$ and $s' = s$ for $r > 0$, and the following tame estimate holds:*

$$\begin{aligned}\|u\|_{\mathcal{X}^{s',r}} \lesssim \|f\|_{H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}} + \|u\|_{H_{\mathrm{b}}^{s+4,r_0}(\Omega)^{\bullet,-}} \\ + (\|f\|_{H_{\mathrm{b}}^{s_0,r}(\Omega)^{\bullet,-}} + \|u\|_{H_{\mathrm{b}}^{s_0+1,r_0}(\Omega)^{\bullet,-}})\|v\|_{\mathcal{X}^{\widetilde{s},\alpha}}.\end{aligned}$$

Proof. The proof works in the same way as the proofs of Theorems 5.2.3 and 9.1.8, using an iterative argument that consists of rewriting $Pu = f$ as $N(P)u = f - (P - N(P))u$ and employing a contour deformation argument, see also [114, Lemma 3.1] (which uses high-energy estimates for the inverse normal operator family $\widehat{P}(\sigma)^{-1}$ and the location of resonances, i.e. of the poles of this family), to improve on the decay of u by α in each step, but losing an order of differentiability as we are treating $P - N(P)$ as an error term; using tame microlocal regularity for the equation $Pu = f$, Corollary 9.2.8, one can regain this loss. We obtain $u \in H_{\mathrm{b}}^{s+1,r}$ after a finite number of iterations in case $r < 0$,[55] and $u \in H_{\mathrm{b}}^{s+4,r_0}$ for all $r_0 < 0$ in case $r > 0$.

[54] In particular, if we merely assume $s_0 > n/2+1/2$, then the full condition on s_0 holds if we choose $\alpha > 0$ sufficiently small.

[55] In particular, this holds under the weaker conditions $s + 1 \leq \widetilde{s}$, $\alpha \leq 1$.

Assuming we are in the latter case, the next step of the iteration gives a partial expansion $u = c + u'$ with $c \in \mathbb{C}$ (identified, as before, with $c\chi$, where χ is a smooth cutoff near the boundary) and $u' \in H_{\mathrm{b}}^{s+2,r'}$ for any r' satisfying $r' \leq r$ and $r' < \alpha$; here, we need $0 < \alpha < \gamma_\Gamma$ so that the normally hyperbolic trapping estimate (9.2.13) holds with $\gamma > \alpha$, with loss of two derivatives. If $r = \alpha$, we can use this information to deduce

$$N(P)u = f - (P - N(P))u = f - \widetilde{f}, \quad \widetilde{f} \in H_{\mathrm{b}}^{\widetilde{s},\alpha} + H_{\mathrm{b}}^{s,r'+\alpha} \subset H_{\mathrm{b}}^{s,r},$$

which implies that the expansion $u = c + u'$ in fact holds with the membership $u' \in H_{\mathrm{b}}^{s,r}$; notice the improvement in the weight. Therefore, $u \in \mathcal{X}^{s,r}$, finishing the proof. □

Pipelining this result with the existence of solutions, Lemma 9.2.6, we therefore obtain:

Theorem 9.2.11. *Under the assumptions of Theorem 9.2.10 with $r > 0$ and $s > n/2 + 2$, define the space*

$$\mathcal{Y}^{s,r} = \{u \in \mathcal{X}^{s,r} \colon Pu \in H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}\}.$$

Then the operator $P\colon \mathcal{Y}^{s,r} \to H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}$ has a continuous inverse S that satisfies the tame estimate

$$\|Sf\|_{\mathcal{X}^{s,r}} \leq C(s, \|v\|_{\mathcal{X}^{s_0,\alpha}})(\|f\|_{H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}} + \|f\|_{H_{\mathrm{b}}^{s_0,r}(\Omega)^{\bullet,-}}\|v\|_{\mathcal{X}^{s+4,\alpha}}). \tag{9.2.14}$$

9.2.4 Solving quasilinear wave equations

We continue to work in the setting of the previous section. With the tame forward solution operator constructed in Theorem 9.2.11 in our hands, we are now in a position to use a Nash-Moser implicit function theorem to solve quasilinear wave equations. We use the following simple form of Nash-Moser, given in [99]:

Theorem 9.2.12. *Let $(B^s, |\cdot|_s)$ and $(\mathbb{B}^s, \|\cdot\|_s)$ be Banach spaces for $s \geq 0$ with $B^s \subset B^t$ and indeed $|v|_t \leq |v|_s$ for $s \geq t$, likewise for $\mathbb{B}^*$ and $\|\cdot\|_*$; put $B^\infty = \bigcap_s B^s$ and similarly $\mathbb{B}^\infty = \bigcap_s \mathbb{B}^s$. Assume there are smoothing operators $(S_\theta)_{\theta>1}\colon B^\infty \to B^\infty$ satisfying for every $v \in B^\infty$, $\theta > 1$ and $s, t \geq 0$:*

$$\begin{aligned} |S_\theta v|_s &\leq C_{s,t}\theta^{s-t}|v|_t \text{ if } s \geq t, \\ |v - S_\theta v|_s &\leq C_{s,t}\theta^{s-t}|v|_t \text{ if } s \leq t. \end{aligned} \tag{9.2.15}$$

Let $\phi\colon B^\infty \to \mathbb{B}^\infty$ *be a* C^2 *map, and assume that there exist* $u_0 \in B^\infty$, $d \in \mathbb{N}$, $\delta > 0$ *and constants* C_1, C_2 *and* $(C_s)_{s\geq d}$ *such that for any* $u, v, w \in B^\infty$,

$$|u-u_0|_{3d} < \delta \Rightarrow \begin{cases} \forall s \geq d, \quad \|\phi(u)\|_s \leq C_s(1+|u|_{s+d}), \\ \|\phi'(u)v\|_{2d} \leq C_1|v|_{3d}, \\ \|\phi''(u)(v,w)\|_{2d} \leq C_2|v|_{3d}|w|_{3d}. \end{cases} \tag{9.2.16}$$

Moreover, assume that for every $u \in B^\infty$ *with* $|u-u_0|_{3d} < \delta$ *there exists an operator* $\psi(u)\colon \mathbb{B}^\infty \to B^\infty$ *satisfying*

$$\phi'(u)\psi(u)h = h$$

and the tame estimate

$$|\psi(u)h|_s \leq C_s(\|h\|_{s+d} + |u|_{s+d}\|h\|_{2d}), \quad s \geq d, \tag{9.2.17}$$

for all $h \in \mathbb{B}^\infty$. *Then if* $\|\phi(u_0)\|_{2d}$ *is sufficiently small depending on* $\delta, |u_0|_D$ *and* $(C_s)_{s\leq D}$, *where* $D = 16d^2 + 43d + 24$, *there exists* $u \in B^\infty$ *such that* $\phi(u) = 0$.

To apply this in our setting, we let $B^s = \mathcal{X}^{s,\alpha}(\Omega) = \mathbb{C}\oplus H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}$ and $\mathbb{B}^s = H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}$ with the corresponding norms; $\phi(u)$ will be the quasilinear equation, with implicit dependence on the forcing term. We now construct the smoothing operators S_θ; we may assume, using a partition of unity, that Ω is the closure of an open subset of $\overline{\mathbb{R}^n_+}$, say $\Omega = \Omega(1)$, where we let $\Omega(x_0) = \{x \leq x_0, |y| \leq 1\}$. Then there are bounded extension and restriction operators

$$E\colon H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n_+}), \quad R\colon H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n_+}) \to H_{\mathrm{b}}^{s,\alpha}(\Omega)^{-,-},$$

for $s \geq 0$; the operator E can be constructed such that $\operatorname{supp} Ev \subset \{x \leq 1\}$ for $v \in H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}$. If we then define for $\theta > 1$ and $v = (c,u) \in \mathcal{X}^{s,\alpha}$:

$$S_\theta^1 v = (c, RS'_\theta Ev),$$

where S'_θ is a smoothing operator on $\overline{\mathbb{R}^n_+}$ with properties as in (9.2.15), then S_θ^1 satisfies (9.2.15) in view of RE being the identity on $H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}$ if the norms on the left hand side are understood to be $H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n_+})$-norms. However, note that S_θ^1 does not map $\mathcal{X}^{\infty,\alpha}$ into itself, since smoothing operators such as S'_θ enlarge supports; we will thus need to modify S_θ^1 below to obtain the operators S_θ. In order to construct S'_θ on weighted b-Sobolev spaces

$H_{\mathrm{b}}^{s,\alpha}$, it suffices by conjugation by the weight to construct it on the unweighted spaces H_{b}^{s}; then, by a logarithmic change of coordinates, we only need to construct the smoothing operator $\widetilde{S}_\theta$ on the standard Sobolev spaces $H^s(\mathbb{R}^n)$, which we will do in Lemma 9.2.13 below. In order to deal with the issue of S^1_θ enlarging supports, we will define $\widetilde{S}_\theta$ such that

$$v \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n_{x',y'}),\ \operatorname{supp} v \subset \{x' \leq 0\} \ \Rightarrow\ \operatorname{supp} \widetilde{S}_\theta v \subset \{x' \leq \theta^{-1/2}\}.$$

In particular, when one undoes the logarithmic change of coordinates, this implies

$$S^1_\theta \colon \mathcal{X}^{s,\alpha}(\Omega(1)) \to \mathcal{X}^{s,\alpha}\left(\Omega\big(\exp(\theta^{-1/2})\big)\right);$$

more generally, with D_λ denoting dilations $D_\lambda(x,y) = (\lambda x, y)$ on $\overline{\mathbb{R}^n_+}$, we have

$$S^\lambda_\theta := (D_\lambda^{-1})^* S^1_\theta (D_\lambda)^* \colon \mathcal{X}^{s,\alpha}(\Omega(\lambda)) \to \mathcal{X}^{s,\alpha}\left(\Omega\big(\lambda\exp(\theta^{-1/2})\big)\right), \quad \lambda > 0, \tag{9.2.18}$$

with the operator norm independent of λ near 1. Now, in our application of Theorem 9.2.12, we will have

$$\phi \colon \mathcal{X}^{\infty,\alpha}(\Omega(x_0)) \to H_{\mathrm{b}}^{\infty,\alpha}(\Omega(x_0))^{\bullet,-} \text{ for all } x_0 \text{ near } 1,$$

and correspondingly we will have forward solution operators ψ going in the reverse direction, with all relevant constants being uniform in x_0. Looking at the proof of Theorem 9.2.12 in [99], one only uses the smoothing operator S_{θ_k} with $\theta_k = \theta_0^{(5/4)^k}$ in the k-th step of the iteration, with θ_0 chosen sufficiently large; in our situation, where we have (9.2.18), we can therefore use the smoothing operator

$$S_{\theta_k} := S^{\lambda_k}_{\theta_k}, \quad \lambda_k = \exp\left(\sum_{j=0}^{k-1} \theta_j^{-1/2}\right)$$

in the k-th iteration step. Note that, for θ_0 large, we have

$$1 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_\infty = \exp\left(\sum_{j=0}^{\infty} \theta_j^{-1/2}\right) \leq 1 + 2\theta_0^{-1/2}.$$

The solution u to $\phi(u) = 0$, obtained as a limit of an iterative scheme (see [99, Lemma 1]), therefore is an element of $\mathcal{X}^{s,\alpha}(\Omega(\lambda_\infty))$. Taking the hyperbolic nature of the PDE $\phi(u) = 0$ into account once more, it will then, in our concrete setting, be easy to conclude that in fact $u \in \mathcal{X}^{s,\alpha}(\Omega)$.

We now construct the smoothing operators on $\mathbb{R}^n$; the first step of the argument follows the Appendix of [99].

Lemma 9.2.13. *There is a family $(\widetilde{S}_\theta)_{\theta>1}$ of operators on $H^\infty(\mathbb{R}^n)$ satisfying*

$$\|\widetilde{S}_\theta v\|_s \leq C_{s,t}\theta^{s-t}\|v\|_t \text{ if } s \geq t \geq 0, \tag{9.2.19}$$

$$\|v - \widetilde{S}_\theta v\|_s \leq C_{s,t}\theta^{s-t}\|v\|_t \text{ if } 0 \leq s \leq t, \tag{9.2.20}$$

$$\operatorname{supp} \widetilde{S}_\theta v \subseteq \{x_1 \leq \theta^{-1/2}\} \tag{9.2.21}$$

for all $v \in H^\infty(\mathbb{R}^n)$ with $\operatorname{supp} v \subseteq H := \{x_1 \leq 0\}$. Here $\|\cdot\|_s$ denotes the $H^s(\mathbb{R}^n)$-norm, and we write $x = (x_1, x') \in \mathbb{R}^n$.

Proof. Choose $\chi = \chi_1(x_1)\chi_2(x') \in S(\mathbb{R}^n)$ with $\chi_1 \in S(\mathbb{R}), \chi_2 \in S(\mathbb{R}^{n-1})$ so that the Fourier transform $\widehat{\chi}$ is identically 1 near 0; put $\chi_\theta(z) = \theta^n\chi(\theta z)$ and define the operator $C_\theta v = \chi_\theta * v$. Then $(C_\theta v)\widehat{\ } = \widehat{\chi_\theta}\widehat{v}$ with $\widehat{\chi_\theta}(\xi) = \widehat{\chi}(\xi/\theta)$, therefore (9.2.19) holds for C_θ in place of $\widetilde{S}_\theta$ with constants $C'_{s,t}$ since $\widehat{\chi}$ decays super-polynomially, and (9.2.20) holds for C_θ in place of $\widetilde{S}_\theta$ with constants $C'_{s,t}$ since $1 - \widehat{\chi}(\xi)$ vanishes at $\xi = 0$ with all derivatives.

Next, let $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ be a smooth function depending only on x_1, i.e. $\psi = \psi(x_1)$, so that $\psi(x_1) \equiv 1$ for $x_1 \in (-\infty, 1/2]$, $\psi(x_1) \equiv 0$ for $x_1 \in [1, \infty)$, and $0 \leq \psi \leq 1$. Put $\psi_\theta(x_1, x') = \psi(\theta x_1, x')$, and define

$$\widetilde{S}_\theta v := \psi_{\theta^{1/2}} C_\theta v.$$

Condition (9.2.21) is satisfied by the support assumption on ψ. Let $\varphi = 1 - \psi$ and $\varphi_\theta = 1 - \psi_\theta$. To prove the other two conditions, we use the estimate

$$\|\varphi_{\theta^{1/2}} C_\theta v\|_s \leq C''_{s,N}\theta^{-N}\|v\|_{L^2}, \quad \operatorname{supp} v \subset H,\ s, N \geq 0, \tag{9.2.22}$$

which we will establish below. Taking this for granted, we obtain for v with $\operatorname{supp} v \subset H$:

$$\|\widetilde{S}_\theta v\|_s \leq \|C_\theta v\|_s + \|\varphi_{\theta^{1/2}} C_\theta v\|_s \leq C'_{s,t}\theta^{s-t}\|v\|_t + C''_{s,0}\|v\|_0$$

for $s \geq t \geq 0$, which is the estimate (9.2.19); and (9.2.20) follows from

$$\|v - \widetilde{S}_\theta v\|_s \leq \|v - C_\theta v\|_s + \|\varphi_{\theta^{1/2}} C_\theta v\|_s \leq C'_{s,t}\theta^{s-t}\|v\|_t + C''_{s,t-s}\theta^{s-t}\|v\|_0$$

for $0 \leq s \leq t$.

We now prove (9.2.22) for $s \in \mathbb{N}_0$. For multiindices $\alpha = (\alpha_1, \alpha')$ with $|\alpha| \leq s$, we have for v with $\operatorname{supp} v \subset H$ and for $(x_1, x') \in \operatorname{supp} \varphi_{\theta^{1/2}} C_\theta v$, which in particular implies $x_1 \geq 1/(2\theta^{1/2})$:

$$\begin{aligned}\partial^\alpha(\varphi_{\theta^{1/2}} C_\theta v)(x_1, x') &= \sum_{j=0}^{\alpha_1} \binom{\alpha_1}{j} \theta^{(\alpha_1 - j)/2} \varphi^{(\alpha_1 - j)}(\theta^{1/2} x_1) \\ &\quad \times \iint_{y_1 \geq 1/(2\theta^{1/2})} \theta^{n+j+|\alpha'|} \chi_1^{(j)}(\theta y_1) \chi_2^{(\alpha')}(\theta y') v(x_1 - y_1, x' - y')\, dy_1\, dy',\end{aligned}$$

thus

$$\|\partial^\alpha(\varphi_{\theta^{1/2}} C_\theta v)\|_{L^2} \leq C_s \theta^{n+s} \|\check{\chi}_\theta\|_{L^1} \|v\|_{L^2},$$

where

$$\check{\chi}_\theta(x_1, x') = \begin{cases} 0, & x_1 < 1/(2\theta^{1/2}), \\ \sum_{j=0}^{\alpha_1} |\chi_1^{(j)}(\theta x_1) \chi_2^{(\alpha')}(\theta x')| & \text{otherwise.} \end{cases}$$

But $\|\check{\chi}_\theta\|_{L^1} \leq C_{N,s} \theta^{-N}$ for all N: Indeed, this reduces to the statement that for a fixed $\chi_0 \in S(\mathbb{R})$, one has

$$\int_{1/(2\theta^{1/2})}^\infty |\chi_0(\theta x)|\, dx \leq C_N \int_{\theta^{-1/2}}^\infty (\theta x)^{-2N+1}\, dx = C'_N \theta^{-N}.$$

Hence, we obtain (9.2.22), and the proof is complete. □

We now combine Theorem 9.2.11, giving the existence of tame forward solution operators, with Theorem 9.2.12, in the extended form described above, to solve quasilinear wave equations. We use the space $\mathcal{X}^{s,\alpha}_{\mathbb{R}}$ of real-valued elements of $\mathcal{X}^{s,\alpha}$.

Theorem 9.2.14. *Let $N \in \mathbb{N}$ and $c_k \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$, $g_k \in (\mathcal{C}^\infty + H_{\mathrm{b}}^\infty)(M; S^{2\mathrm{b}}TM)$ for $1 \leq k \leq N$; define the map $g\colon \mathcal{X}^{s,\alpha} \to (\mathcal{C}^\infty + H_{\mathrm{b}}^{s,\alpha})(M; S^{2\mathrm{b}}TM)$ by $g(u) = \sum_{k=1}^N c_k(u) g_k$ and assume that $\Box_{g(0)}$ satisfies the assumptions of §9.2.3 and of Theorem 9.2.11. Moreover, let $N' \in \mathbb{N}$ and define*

$$q(u, {}^{\mathrm{b}}du) = \sum_{j=1}^{N'} u^{e_j} \prod_{k=1}^{N_j} X_{jk} u, \tag{9.2.23}$$

where

$$e_j, N_j \in \mathbb{N}_0, N_j \geq 1, N_j + e_j \geq 2, X_{jk} \in (\mathcal{C}^\infty + H_{\mathrm{b}}^\infty)\mathcal{V}_{\mathrm{b}}.$$

Then there exists $C_f > 0$ such that for all forcing terms $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega;\mathbb{R})^{\bullet,-}$ satisfying $\|f\|_{H_{\mathrm{b}}^{\max(12,n+5),\alpha}(\Omega)^{\bullet,-}} \leq C_f$, the equation

$$\Box_{g(u)} u = f + q(u, {}^{\mathrm{b}}du) \tag{9.2.24}$$

has a unique solution $u \in \mathcal{X}_{\mathbb{R}}^{\infty,\alpha}$.

If more generally $g(u, {}^{\mathrm{b}}du) = \sum_{k=1}^{N} c_k(u, X_1 u, \ldots, X_L u)$, where $X_1, \ldots, X_L \in \mathcal{V}_{\mathrm{b}}(M)$ and $c_k \in \mathcal{C}^\infty(\mathbb{R}^{1+L};\mathbb{R})$, then there exists $C_f > 0$ such that for all forcing terms $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega;\mathbb{R})^{\bullet,-}$ satisfying $\|f\|_{H_{\mathrm{b}}^{\max(14,n+5),\alpha}(\Omega)^{\bullet,-}} \leq C_f$, the equation

$$\Box_{g(u,{}^{\mathrm{b}}du)} u = f + q(u, {}^{\mathrm{b}}du) \tag{9.2.25}$$

has a unique solution $u \in \mathcal{X}_{\mathbb{R}}^{\infty,\alpha}$.

Proof. We write $|\cdot|_s$ for the $\mathcal{X}^{s,\alpha}$-norm and $\|\cdot\|_s$ for the $H_{\mathrm{b}}^{s,\alpha}$-norm. For brevity, we do not specify the underlying set, which, in the notation of §9.2.3, is $\mathfrak{t}_1^{-1}([-\lambda,\infty)) \cap \mathfrak{t}_2^{-1}([0,\infty))$ for varying $\lambda \geq 0$. We define the map

$$\phi(u; f) = \Box_{g(u)} u - q(u, {}^{\mathrm{b}}du) - f$$

and check that it satisfies the conditions of Theorem 9.2.12 with $u_0 = 0$. From the definition of $\Box_{g(u)}$ and the tame estimates for products, reciprocals and compositions, Corollary 8.7.2 and Propositions 8.7.4 and 8.7.7, we obtain

$$\|\phi(u; f)\|_s \leq \|f\|_s + C(|u|_{s_0+2})(1 + |u|_{s+2}), \quad s \geq s_0 > n/2 + 1,$$

thus the first estimate of (9.2.16) for $3d \geq s_0 + 2$, $d \geq s_0$, $d \geq 2$. Next, we have $\phi'(u; f)v = \big(\Box_{g(u)} + L(u, {}^{\mathrm{b}}du)\big)v$, where the first order b-differential operator L is of the form

$$L = \sum_{|\beta| \leq 1} \Bigg(\sum_{1 \leq |\alpha| \leq 2} a_{\alpha\beta}(u, {}^{\mathrm{b}}du) {}^{\mathrm{b}}D^\alpha u \Bigg) {}^{\mathrm{b}}D^\beta v + \sum_{|\beta|=1} a_\beta(u, {}^{\mathrm{b}}du) u \, {}^{\mathrm{b}}D^\beta v, \tag{9.2.26}$$

with the second sum capturing one term of the linearization of terms $u^{e_j} X_{j1} u$ in q (i.e. terms for which $N_j = 1$). In particular,

$$\phi'(u; f) = P_0(u_0) + \widetilde{P}(u, {}^{\mathrm{b}}Du, {}^{\mathrm{b}}D^2 u), \tag{9.2.27}$$

where $P_0 \in \mathrm{Diff}_\mathrm{b}^2$ and $\widetilde{P} \in H_\mathrm{b}^{s-2,\alpha}\mathrm{Diff}_\mathrm{b}^2$ for $u \in \mathcal{X}^{s,\alpha}$. Therefore,

$$\|\phi'(u;f)v\|_s \leq C(|u|_{s+2})|v|_{s+2}, \quad s > n/2+1,$$

which gives the second estimate of (9.2.16) for $2d > n/2+1$ and $3d \geq 2d+2$. Next, we observe that $\phi''(u;f)(v,w)$ is bilinear in v, w, involves up to two b-derivatives of each v and w, and the coefficients depend on up to two b-derivatives of u, thus

$$\|\phi''(u;f)(v,w)\|_s \leq C(|u|_{s+2})|v|_{s+2}|w|_{s+2}, \quad s > n/2+1,$$

which gives the third estimate of (9.2.16) for $3d > n/2+3$, $3d \geq 2d+2$. In summary, we obtain (9.2.16) for integer $d > n/2+1$.

Finally, we determine d so that we have the tame estimate (9.2.17): Given $u \in \mathcal{X}^{s+6,\alpha}$, we can write $\phi'(u;f)$ as in (9.2.27), with $P_0 \in \mathrm{Diff}_\mathrm{b}^2$ and $\widetilde{P} \in H_\mathrm{b}^{s+4,\alpha}\mathrm{Diff}_\mathrm{b}^2$; hence, by Theorem 9.2.11, we obtain a solution operator

$$\begin{aligned} \psi(u;f)\colon H_\mathrm{b}^{s+3,\alpha} &\to \mathcal{X}^{s,\alpha}, \\ |\psi(u;f)v|_s &\leq C(s,|u|_{s_0})(\|v\|_{s+3} + \|v\|_{s_0}|u|_{s+6}), \end{aligned} \tag{9.2.28}$$

where $s, s_0 > n/2+2$, provided $|u|_{s_0}$ is small enough so that all dynamical and geometric hypotheses hold for $\phi'(u;f)$. Notice that the subprincipal term of $\phi'(u;f)$ can differ from that of $\Box_{g(0)}$ by terms of the form $a(u_0)u_0{}^\mathrm{b}D^\beta$, $a \in \mathcal{C}^\infty$, $|\beta| = 1$, see (9.2.26); however, since such terms eliminate constants, the simple rank 1 resonance at 0 with resonant state 1 does not change; and moreover such terms are *small* because of the factor u_0, hence high energy estimates still hold in a (possibly slightly smaller) strip in the analytic continuation, see the remark below [42, Theorem 1]. Since s_0 is independent of s, we have (9.2.28) for all $s > n/2+2$, in particular $\psi(u;f)\colon H_\mathrm{b}^{\infty,\alpha} \to \mathcal{X}^{\infty,\alpha}$. Now, (9.2.28) implies that (9.2.17) holds for $d > n/2+2$, $d \geq 6$, so we need to control $\max(12, n+5)$ derivatives of f.

Thus, we can apply Nash-Moser iteration, Theorem 9.2.12, to obtain a solution $u \in \mathcal{X}^{s,\alpha}$ of the PDE (9.2.24), with the caveat that u is a priori supported on a space slightly larger than Ω. However, local uniqueness for quasilinear hyperbolic equations, see e.g. [108, §16.3], implies that u in fact is supported in Ω, and that u is the unique solution of (9.2.24), finishing the proof of the first part.

The proof of the second part proceeds in the same way, only we need that $d \geq 7$, which makes the control of the stronger $H_{\mathrm{b}}^{\max(14,n+5)}$-norm of f necessary. $\square$

Remark 9.2.15. In the asymptotically de Sitter setting considered in §9.1, the above theorem extends Theorem 9.1.15 (at the cost of requiring the control of more derivatives) since we allow the dependence of the metric $g(u, {}^{\mathrm{b}}du)$ on ${}^{\mathrm{b}}du$ as well.

Remark 9.2.16. An inspection of the proof of the abstract Nash-Moser theorem 9.2.12 in [99] shows that there are constants C and s_0, depending only on the 'loss of derivatives' d, such that the following holds: In order to obtain a solution $u \in \mathcal{X}^{s,\alpha}$ for some finite $s \geq s_0$, it is sufficient to take $f \in H_{\mathrm{b}}^{Cs,\alpha}$, still assuming the norm of f in the space indicated in the statement of Theorem 9.2.14 to be small.

Theorem 9.2.14 immediately implies the following result on Kerr-de Sitter space:

Corollary 9.2.17. *Under the assumptions of Theorem 9.2.14, the quasilinear wave equation (9.2.24), resp. (9.2.25), on a 4-dimensional asymptotically Kerr-de Sitter space with $|a| \ll M_\bullet$ has a unique global smooth (i.e. conormal, in the space $\mathcal{X}^{\infty,\alpha}$) solution if the $H_{\mathrm{b}}^{12,\alpha}(\Omega)^{\bullet,-}$-norm, resp. $H_{\mathrm{b}}^{14,\alpha}(\Omega)^{\bullet,-}$-norm, of the forcing term $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega)^{\bullet,-}$ is sufficiently small.*

Proof. For a verification of the dynamical assumptions for asymptotically Kerr-de Sitter spaces, we refer the reader to [114, §6]; the resonances on the other hand were computed by Dyatlov [40]. $\square$

9.2.5 Solving quasilinear Klein-Gordon equations

The only difference between wave and Klein-Gordon equations with mass m (which is to be distinguished from the black hole mass $M_\bullet$) is that the resonance of the Klein-Gordon operator $\Box - m^2$ with largest imaginary part, which gives the leading order asymptotics, is no longer at 0 for $m \neq 0$. Thus, if we sort the resonances $\sigma_1, \sigma_2, \ldots$ of $\Box - m^2$ with multiplicity by decreasing imaginary part, assume

$$0 < -\operatorname{Im}\sigma_1 < r < -\operatorname{Im}\sigma_2,$$

and moreover that the high energy estimates for the normal operator family of $\Box - m^2$ hold in $\operatorname{Im}\sigma \geq -r$, the only change in the statement of Theorem 9.2.10 for Klein-Gordon operators is that the conclusion now is $u \in \mathcal{X}_{\sigma_1}^{s-3,r}$, where $\mathcal{X}_{\sigma_1}^{s-3,r} = \mathbb{C} \oplus H_{\mathrm{b}}^{s-3,r}(\Omega)^{\bullet,-}$, with

(c, u') identified with $cx^{i\sigma_1}\chi + u'$ for a smooth cutoff χ near the boundary.[56] We thus obtain the following adapted version of Theorem 9.2.11:

Theorem 9.2.18. *In the notation of §9.2.4, under the above assumptions and for $s > n/2 + 2$, define the space*

$$\mathcal{Y}^{s,r}_{\sigma_1} = \{u \in \mathcal{X}^{s,r}_{\sigma_1} : Pu \in H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}\}.$$

Then the operator $P\colon \mathcal{Y}^{s,r} \to H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}$ has a continuous inverse S that satisfies the tame estimate

$$\|Sf\|_{\mathcal{X}^{s,r}_{\sigma_1}} \leq C(s, \|v\|_{\mathcal{X}^{s_0,\alpha}_{\sigma_1}})(\|f\|_{H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}} + \|f\|_{H_{\mathrm{b}}^{s_0,r}(\Omega)^{\bullet,-}}\|v\|_{\mathcal{X}^{s+4,\alpha}_{\sigma_1}}). \tag{9.2.29}$$

This immediately gives:

Theorem 9.2.19. *Under the above assumptions and the assumption $\alpha < -2\operatorname{Im}\sigma_1$, let $N, N' \in \mathbb{N}$ and $c_k \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$, $g_k \in (\mathcal{C}^\infty + H_{\mathrm{b}}^\infty)(M; S^2\,{}^{\mathrm{b}}TM)$ for $1 \leq k \leq N$; define the map $g\colon \mathcal{X}^{s,\alpha}_{\sigma_1} \to (\mathcal{C}^\infty + H_{\mathrm{b}}^{s,\alpha})(M; S^2\,{}^{\mathrm{b}}TM)$ by $g(u) = \sum_{k=1}^N c_k(u)g_k$ and assume that $\Box_{g(0)}$ satisfies the assumptions of §9.2.3 and of Theorem 9.2.18. Moreover, define*

$$q(u, {}^{\mathrm{b}}du) = \sum_{j=1}^{N'} a_j u^{e_j} \prod_{k=1}^{N_j} X_{jk}u,$$

where

$$e_j, N_j \in \mathbb{N}_0, e_j + N_j \geq 2, a_j \in \mathcal{C}^\infty, X_{jk} \in (\mathcal{C}^\infty + H_{\mathrm{b}}^\infty)\mathcal{V}_{\mathrm{b}}.$$

Then there exists $C_f > 0$ such that for all forcing terms $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega;\mathbb{R})^{\bullet,-}$ satisfying $\|f\|_{H_{\mathrm{b}}^{\max(12,n+5),\alpha}(\Omega)^{\bullet,-}} \leq C_f$, the equation

$$(\Box_{g(u)} - m^2)u = f + q(u, {}^{\mathrm{b}}du) \tag{9.2.30}$$

has a unique solution $u \in \mathcal{X}^{\infty,\alpha}_{\sigma_1,\mathbb{R}}$.

If more generally $g(u, {}^{\mathrm{b}}du) = \sum_{k=1}^N c_k(u, X_1u, \ldots, X_Lu)$, where $X_1, \ldots, X_L \in \mathcal{V}_{\mathrm{b}}(M)$ and $c_k \in \mathcal{C}^\infty(\mathbb{R}^{1+L};\mathbb{R})$, then there exists $C_f > 0$ such that for all forcing terms $f \in$

[56] There are more cases of potential interest: If $r < -\operatorname{Im}\sigma_1$, we obtain $u \in H_{\mathrm{b}}^{s-3,r}(\Omega)^{\bullet,-}$; if $r < 0$, the statement of Theorem 9.2.10 is unchanged; and if $\operatorname{Im}\sigma_1$ and $\operatorname{Im}\sigma_2$ are close enough together (including the case that σ_1 is a double resonance), one gets two terms in the expansion of u. For brevity, we only explain one scenario here. See also the related discussion in §9.1.4.

$H_{\mathrm{b}}^{\infty,\alpha}(\Omega;\mathbb{R})^{\bullet,-}$ *satisfying* $\|f\|_{H_{\mathrm{b}}^{\max(14,n+5),\alpha}(\Omega)^{\bullet,-}} \leq C_f$, *the equation*

$$(\Box_{g(u,{}^{\mathrm{b}}du)} - m^2)u = f + q(u, {}^{\mathrm{b}}du) \tag{9.2.31}$$

has a unique solution $u \in \mathcal{X}_{\mathbb{R}}^{\infty,\alpha}$.

Together with Theorem 9.2.14, this proves Theorem 9.2.2.

Proof of Theorem 9.2.19. The proof proceeds as the proof of Theorem 9.2.14. Notice that we allow the nonlinear term q to be more general, the point being that firstly, any at least quadratic expression in $(u, {}^{\mathrm{b}}du)$ with $u \in \mathcal{X}_{\sigma_1}^{s,\alpha}$ gives an element of $H_{\mathrm{b}}^{s,\alpha}$, and secondly, every element in $\mathcal{X}_{\sigma_1}^{s,\alpha}$ vanishes at the boundary, thus the normal operator family of the linearization of $\Box_{g(u)} - m^2 - q(u, {}^{\mathrm{b}}du) - f$ at any $u \in \mathcal{X}_{\sigma_1}^{s,\alpha}$ is equal to the normal operator family of $\Box_{g(0)} - m^2$, for which one has high energy estimates by assumption. □

By Lemma 5.3.3, the assumptions of Theorem 9.2.19 are satisfied on asymptotically Kerr-de Sitter spaces as long as the mass parameter m is small:

Corollary 9.2.20. *Under the assumptions of Theorem 9.2.19 and for* a *and* $m > 0$ *sufficiently small, the quasilinear Klein-Gordon equation* (9.2.30)*, resp.* (9.2.31)*, on a 4-dimensional asymptotically Kerr-de Sitter space with angular momentum* a *has a unique global smooth (i.e. conormal, in the space* $\mathcal{X}_{\sigma_1,\mathbb{R}}^{\infty,\alpha}$*) solution if the* $H_{\mathrm{b}}^{12,\alpha}(\Omega)^{\bullet,-}$*-norm, resp.* $H_{\mathrm{b}}^{14,\alpha}(\Omega)^{\bullet,-}$*-norm, of the forcing term* $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega)^{\bullet,-}$ *is sufficiently small.*

9.2.6 Proofs of the general statements

Finally, following the same arguments as used in the previous section, we indicate how to prove the general Theorems 9.2.3 and 9.2.4 stated in the introduction to the present section. We continue to use, but need to generalize the setting considered in §9.2.3: Namely, generalizing (9.2.9), we now allow L to be any first order b-differential operator, and correspondingly need information on the skew-adjoint part of P_0; concretely, we define $\widehat{\beta}$ at the (generalized) radial sets $L_\pm$, using the same notation as in (3.3.11), by

$$\sigma_{\mathrm{b},1}\Big(\frac{1}{2i}(P_0 - P_0^*)\Big)\Big|_{L_\pm} = \pm\widehat{\beta}\beta_0\rho. \tag{9.2.32}$$

Moreover, at the trapped set $\Gamma = \Gamma^- \cup \Gamma^+$, we assume that

$$\mathsf{e}_1|_\Gamma < \nu_{\min}/2, \quad \mathsf{e}_1 = |\sigma|^{-1}\sigma_{\mathrm{b},1}\Big(\frac{1}{2i}(P_0 - P_0^*)\Big), \tag{9.2.33}$$

with $\nu_{\min}$ the minimal normal expansion rate for the Hamilton flow of the principal symbol of P_0, and σ the Mellin dual variable of x after an identification of a collar neighborhood of X in M with $[0,\epsilon')_x \times X$; note that σ is elliptic on Γ. Let r_{thr} be the threshold weight for the first part of Theorem 8.8.5, i.e. $r_{\mathrm{thr}} = -\sup \mathsf{e}_1/c_\partial$ with c_∂ as defined in (8.5.53).

Then Corollary 9.2.8 holds in the current, more general setting, provided we assume $r < r_{\mathrm{thr}}$ and $s' > 1 + \sup_{L_\pm}(r\widetilde{\beta} - \widehat{\beta})$. Likewise, we obtain the high energy estimates of Theorem 9.2.5 under the assumption $s > 1/2 + \sup_{L_\pm}(\gamma\widetilde{\beta} - \widehat{\beta})$.

In order to generalize Theorem 9.2.10, we first choose $0 < r_+ < 1$ such that

$$(\mathsf{e}_1 + r_+ c_\partial)|_\Gamma < \nu_{\min}/2,$$

which holds for sufficiently small r_+ in view of (9.2.33) by the compactness of Γ in ${}^{\mathrm{b}}S^*M$. We moreover assume that there are no (nonzero) resonances in $\operatorname{Im}\sigma > -r_+$ in the case of Theorem 9.2.3 (Theorem 9.2.4), and we assume further that $0 < \alpha < r_+$. Then in the proof of Theorem 9.2.10, ignoring the issue of threshold regularities at radial sets momentarily, we can use the contour shifting argument without loss of derivatives up to, but excluding, the weight r_{thr}, corresponding to the contour of integration $\operatorname{Im}\sigma = -r_{\mathrm{thr}}$. Shifting the contour further down, we cannot use the non-smooth real principal type estimate at Γ anymore and thus lose 2 derivatives at each step; the total number of additional steps needed to shift the contour down to $\operatorname{Im}\sigma = -\alpha$ is easily seen to be at most

$$N = \max\left(0, \left\lceil \frac{\alpha - r_{\mathrm{thr}}}{\alpha} \right\rceil + 1\right),$$

hence in order to have the final conclusion that u has an expansion with remainder in $H_{\mathrm{b}}^{s,\alpha}$, we need to assume that u initially is known to have regularity $H_{\mathrm{b}}^{s+2N,r_0}$ for any $r_0 \in \mathbb{R}$, which in turn requires $\widetilde{s} \geq s + 2N$ and $f \in H_{\mathrm{b}}^{s+2N-1,r_0}$ for the first, lossless, part of the argument to work. Taking the regularity requirements at the radial sets into account, we further need to assume $s \geq s_0 > \max(n + 1/2, 1 + \sup(r\widetilde{\beta} - \widehat{\beta}))$. Under these assumptions, the proof of Theorem 9.2.10 applies, mutatis mutandis, to our current situation, and we obtain a tame solution operator as in Theorem 9.2.11, which now loses $2N - 1$ derivatives.

Thus, we can prove Theorems 9.2.3 and 9.2.4 using the same arguments which we used in the proof of Theorem 9.2.14; the 'loss of derivatives' parameter d now needs to satisfy the conditions

$$d \geq 2N + 3, \quad d > n/2 + 6, \quad d > 1 + \sup(r\widetilde{\beta} - \widehat{\beta}), \tag{9.2.34}$$

with the first condition being the actual loss of derivatives, the second one coming from $s > n/2 + 6$ certainly being a high enough regularity for $\widetilde{s} = s + 2N$ to be $> n/2 + 6$, which is required for the application of the non-smooth microlocal regularity results, and the last condition being the threshold regularity (for the non-smooth estimates) at the radial sets. We remark that the first condition could be made independent of N once one proves a general b-estimate on slightly decaying b-spaces analogous to the semiclassical estimate (9.2.13): Indeed, such an estimate would then be used in the contour shifting argument to regain lost derivatives in exactly the same fashion as the general b-estimate at radial sets eliminated the loss in the non-trapping setting of Theorem 5.2.3.

Appendix A

Perturbation theory for resonances

We present some general results on the behavior of resonances for a semiclassical operator under perturbations. Our arguments are essentially standard and well-known in related contexts, see e.g. [54, 66], but in the non-elliptic Fredholm framework formulated in [114], they require some care; thus, we give a self-contained treatment adapted to our needs here: The main purpose of this appendix is to fully justify the arguments in the proof of Lemma 5.3.3 and in §7.5.

We will reduce the perturbation of resonances for analytic Fredholm families in §A.2 to a large extent to a finite-dimensional problem; therefore, we begin our discussion by studying parameter-dependent holomorphic matrix-valued functions in §A.1.

A.1 Families of holomorphic matrix-valued functions

Let $U \subset \mathbb{C}$ be open, connected and non-empty, let A be a neighborhood of the origin in $\mathbb{R}^L$, and let $r \in \{0, 1, 2, \ldots, \infty, \omega\}$. We consider a family

$$P(\sigma; a) \in C^r\big(A_a; \mathcal{O}(U_\sigma; M(N, \mathbb{C}))\big) \tag{A.1.1}$$

of holomorphic $N \times N$ matrix-valued functions with C^r dependence on the parameter a. We assume that $P(\sigma; 0)$ is invertible for some $\sigma \in U$, hence $P(\sigma; 0)^{-1}$ is meromorphic, and therefore so is $P(\sigma; a)^{-1}$ for $|a| < \delta$, $\delta > 0$ small. *For the remainder of this section, we will assume that a satisfies this smallness assumption.* As usual, we call the poles of $P(\sigma; a)^{-1}$ *resonances*. Our interest here lies in understanding the dependence of the order and the rank of resonances on the parameter a. Recall here that the order of $P(\sigma;a)^{-1}$ at σ_* is

$$\operatorname{ord}_{\sigma_*} P(\sigma;a)^{-1} = \min\bigl\{\ell \in \mathbb{N}_0 \colon (\sigma-\sigma_*)^\ell P(\sigma;a)^{-1} \in \mathcal{O}(\sigma_*)\bigr\}, \tag{A.1.2}$$

with $\mathcal{O}(\sigma_*)$ denoting the space of germs holomorphic functions at σ_*; i.e. $\operatorname{ord}_{\sigma_*} P(\sigma;a)^{-1}$ is the most negative power in the Laurent series expansion of $P(\sigma;a)^{-1}$ around $\sigma=\sigma_*$. On the other hand, we define the singular range [82, §5] of $P(\sigma;a)^{-1}$ at $\sigma=\sigma_*$ to be

$$\operatorname{sing\,ran}_{\sigma_*} P(\sigma;a)^{-1} = \left\{ u(\sigma) = \sum_{j=1}^{\operatorname{ord}(\sigma_*;a)} u_j(\sigma-\sigma_*)^{-j} \colon P(\sigma;a)u(\sigma) \in \mathcal{O}(\sigma_*) \right\},$$

and then

$$\operatorname{rank}_{\sigma_*} P(\sigma;a)^{-1} = \dim\bigl(\operatorname{sing\,ran}_{\sigma_*} P(\sigma;a)^{-1}\bigr). \tag{A.1.3}$$

Hence, $\operatorname{ord}_{\sigma_*} P(\sigma;a)^{-1} = 0$ (and thus $\operatorname{rank}_{\sigma_*} P(\sigma;a)^{-1} = 0$) if and only if $P(\sigma;a)^{-1}$ is regular at $\sigma=\sigma_*$. We introduce the notation

$$\operatorname*{p.\,p.}_{\sigma_*}\left(\sum_{j=-\infty}^{\infty} u_j(\sigma-\sigma_*)^j\right) := \sum_{j=-\infty}^{-1} u_j(\sigma-\sigma_*)^j$$

for the principal part of a Laurent series. We first study $P(\sigma;a)^{-1}$ for fixed a, so for now, *we suppress the parameter a in the notation.*

Lemma A.1.1. *For all $\sigma_* \in U$, we have* $\operatorname{rank}_{\sigma_*} P(\sigma)^{-1} \geq \operatorname{ord}_{\sigma_*} P(\sigma)^{-1}$.

Proof. This is true if σ_* is not a resonance, so let us assume σ_* is a resonance, and let $d = \operatorname{ord}_{\sigma_*} P(\sigma)^{-1}$. Then

$$P(\sigma)^{-1} = (\sigma-\sigma_*)^{-d} P_{-d} + (\sigma-\sigma_*)^{-d+1}\mathcal{O}(\sigma_*)$$

with $P_{-d} \in M(N,\mathbb{C})$ non-zero. Take $v \in \mathbb{C}^N$ such that $P_{-d}v \neq 0$, and put

$$u_j(\sigma) = \operatorname*{p.\,p.}_{\sigma_*}\bigl[(\sigma-\sigma_*)^j P(\sigma)^{-1} v\bigr] \neq 0, \quad j = 0,1,\ldots,d-1.$$

Then $P(\sigma)u(\sigma) = (\sigma-\sigma_*)^j (v + \mathcal{O}(\sigma_*))$ is holomorphic indeed, so $u_j \in \operatorname{sing\,ran}_{\sigma_*} P(\sigma)^{-1}$, which implies the statement of the lemma. □

An analytically more convenient description of the rank is the following:

Lemma A.1.2. *Suppose σ_* is a pole of $P(\sigma)^{-1}$. Then*

$$\operatorname{rank}_{\sigma_*} P(\sigma)^{-1} = \operatorname{tr}\Big(\frac{1}{2\pi i}\oint_{\sigma_*} P(\sigma)^{-1}\partial_\sigma P(\sigma)\,d\sigma\Big), \tag{A.1.4}$$

where $\oint_{\sigma_*}$ is the line integral along a small circle around σ_*.

Proof. We give a proof that directly generalizes to the case that $P(\sigma)$ is an analytic Fredholm family (with meromorphic inverse). First, with $k = \dim\ker P(\sigma_*)$, we can multiply $P(\sigma)$ from the left by an invertible constant coefficient matrix C_1 so that

$$C_1 P(\sigma_*) = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix},$$

where we split $\mathbb{C}^N$ into a subspace which is complementary to $\ker P(\sigma_*)$ (first summand) and $\ker P(\sigma_*) \cong \mathbb{C}^k$ (second summand). The $(1,1)$ block of $C_1 P(\sigma)$ will remain invertible for σ near σ_*, and we can then choose holomorphic matrix-valued functions $C_1(\sigma), C_2(\sigma)$, invertible near σ_*, such that

$$C_1(\sigma)P(\sigma)C_2(\sigma) = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & P_{22}(\sigma) \end{pmatrix}, \tag{A.1.5}$$

with $P_{22}(\sigma) \in \mathcal{O}(\sigma_*; M(k,\mathbb{C}))$ invertible in a punctured neighborhood of σ_*. Now, $P_{22}(\sigma)^{-1}$ near $\sigma = \sigma_*$ is a $k \times k$ matrix of meromorphic functions. Suppose that its (i,j)-entry has a pole of order equal to $d := \operatorname{ord}_{\sigma_*} P_{22}(\sigma)^{-1}$. By multiplying $P_{22}(\sigma)^{-1}$ from the left and the right by holomorphic matrices $\widetilde{C}_1(\sigma)$ and $\widetilde{C}_2(\sigma)$, invertible near σ_*, we can move this entry to the $(1,1)$-position (within the $k \times k$ block), make it equal to $(\sigma-\sigma_*)^{-d}$, and then eliminate the remaining entries of the resulting matrix in the first row and first column. Continuing this process with the lower $(k-1)\times(k-1)$ block of the resulting matrix and iterating, we can arrange

$$\big(\widetilde{C}_1(\sigma)P_{22}(\sigma)\widetilde{C}_2(\sigma)\big)^{-1} = D(\sigma)^{-1}, \quad D(\sigma) := \operatorname{diag}\big((\sigma-\sigma_*)^{e_1}, \ldots, (\sigma-\sigma_*)^{e_k}\big),$$

with integers $e_1 = d \geq e_2 \geq \cdots \geq e_k$, where $e_k \geq 0$ since $P_{22}(\sigma)$ is holomorphic. (We have simply computed the Smith normal form of $P_{22}(\sigma)^{-1}$, where the principal ideal domain is the ring of germs of meromorphic functions at σ_*.) Therefore, updating $C_1(\sigma)$ and $C_2(\sigma)$

in (A.1.5) correspondingly, we have arranged

$$\widetilde{P}(\sigma) := C_1(\sigma)P(\sigma)C_2(\sigma) = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & D(\sigma) \end{pmatrix}. \tag{A.1.6}$$

Clearly, $\operatorname{rank}_{\sigma_*} \widetilde{P}(\sigma)^{-1} = \operatorname{rank}_{\sigma_*} D(\sigma)^{-1} = \sum_{j=1}^{k} e_j$, which equals the trace of

$$\frac{1}{2\pi i} \oint_{\sigma_*} \widetilde{P}(\sigma)^{-1}\partial_\sigma \widetilde{P}(\sigma)\, d\sigma = \begin{pmatrix} 0 & 0 \\ 0 & \operatorname{diag}(e_1, \ldots, e_k) \end{pmatrix},$$

which proves the lemma for $\widetilde{P}(\sigma)$ in place of $P(\sigma)$. To prove the lemma for $P(\sigma)$, we show that both quantities in (A.1.4) are invariant when passing from $P(\sigma)$ to $C_1(\sigma)P(\sigma)C_2(\sigma)$. For the right hand side, this follows from the cyclicity of the trace, which gives[57]

$$\operatorname{tr} \oint_{\sigma_*} C_2^{-1}P^{-1}C_1^{-1}(\partial_\sigma C_1)PC_2\, d\sigma = \operatorname{tr} \oint_{\sigma_*} C_1^{-1}\partial_\sigma C_1\, d\sigma = 0$$

by analyticity, similarly when differentiating C_2 instead of C_1, and lastly

$$\operatorname{tr} \oint_{\sigma_*} C_2^{-1}P^{-1}C_1^{-1}C_1(\partial_\sigma P)C_2\, d\sigma = \operatorname{tr} \oint_{\sigma_*} P^{-1}\partial_\sigma P\, d\sigma.$$

For the left hand side, we note that $\operatorname{sing\,ran}_{\sigma_*}(C_1PC_2)^{-1} = \operatorname{sing\,ran}_{\sigma_*}(PC_2)^{-1}$, and the map

$$\operatorname{sing\,ran}_{\sigma_*}(PC_2)^{-1} \ni u(\sigma) \mapsto \underset{\sigma_*}{\mathrm{p.\,p.}}\, C_2(\sigma)u(\sigma) \in \operatorname{sing\,ran}_{\sigma_*} P^{-1}$$

is invertible with inverse $v(\sigma) \mapsto \mathrm{p.\,p.}_{\sigma_*}\, C_2(\sigma)^{-1}v(\sigma)$, since

$$\underset{\sigma_*}{\mathrm{p.\,p.}}\big(C_2(\sigma)^{-1}\big[\underset{\sigma_*}{\mathrm{p.\,p.}}\, C_2(\sigma)u(\sigma)\big]\big) = \underset{\sigma_*}{\mathrm{p.\,p.}}\big(C_2(\sigma)^{-1}\big[C_2(\sigma)u(\sigma) - \mathcal{O}(\sigma_*)\big]\big) = u(\sigma).$$

This establishes $\operatorname{sing\,ran}_{\sigma_*}(PC_2)^{-1} \cong \operatorname{sing\,ran}_{\sigma_*} P^{-1}$ and thus finishes the proof of the lemma. □

We remark that pulling the trace inside the integral in (A.1.4) shows that $\operatorname{rank}_{\sigma_*} P(\sigma)^{-1}$ equals the multiplicity of the zero of $\det P(\sigma)$ at σ_*.

[57] In the case of meromorphic Fredholm families, one needs to replace $P(\sigma)^{-1}$ by its principal part, which is a meromorphic family of operators acting between fixed *finite-dimensional* spaces, in order to pull the trace in and out of the integral.

From the proof of Lemma A.1.2, we can read off a simple criterion for a resonance to be simple, i.e. for its order to be equal to 1:

Lemma A.1.3. *Let* σ_* *be a pole of* $P(\sigma)^{-1}$. *Then* $\operatorname{rank}_{\sigma_*} P(\sigma)^{-1} = \dim\ker P(\sigma_*)$ *is a necessary and sufficient condition for* $\operatorname{ord}_{\sigma_*} P(\sigma)^{-1} = 1$.

Proof. Modifying $P(\sigma)$ as in (A.1.6) and noting that $\operatorname{ord}_{\sigma_*} P^{-1} = \operatorname{ord}_{\sigma_*}(C_1 P C_2)^{-1}$ for invertible holomorphic C_1, C_2, which follows directly from the definition, as well as

$$\dim\ker P(\sigma_*) = \dim\ker C_1(\sigma_*)P(\sigma_*)C_2(\sigma_*),$$

we may assume

$$P(\sigma) = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & D(\sigma) \end{pmatrix}, \quad D(\sigma) = \operatorname{diag}((\sigma-\sigma_*)^{e_1}, \ldots, (\sigma-\sigma_*)^{e_k}),$$

with $k = \dim\ker P(\sigma_*)$ and $e_1 \geq \cdots \geq e_k \geq 1$ as before. But then

$$\operatorname{rank}_{\sigma_*} P^{-1} = \sum_{j=1}^k e_j \geq k,$$

with equality if and only if $e_j = 1$ for all $1 \leq j \leq k$, which is evidently equivalent to $P(\sigma)^{-1}$ having a simple pole at σ_*. □

We next study the parameter dependence of $P(\sigma; a)^{-1}$. Denote by $\operatorname{Res}(a)$ the set of resonances of $P(\sigma; a)^{-1}$.

Lemma A.1.4. *Suppose* $V \subset U$ *is open and has compact closure in* U, *with* $\partial V \cap \operatorname{Res}(0) = \emptyset$. *Then the total rank*

$$\sum_{\sigma_* \in \operatorname{Res}(a)} \operatorname{rank}_{\sigma_*} P(\sigma; a)^{-1}$$

is constant for small a.

Proof. Since the poles of $P(\sigma; 0)^{-1}$ are discrete, we can enumerate them in V, so $V \cap \operatorname{Res}(0) = \{\sigma_1, \ldots, \sigma_M\}$ with $M < \infty$. For $j = 1, \ldots, M$, choose $r_j > 0$ such that the closed ball $\overline{B}_{r_j}(\sigma_j) \subset \mathbb{C}$ is contained in V and does not contain any σ_k for $k \neq j$. Then on the compact set $K := \overline{V} \setminus \bigcup_{j=1}^M B_{r_j}(\sigma_j)$, the operator $P(\sigma; 0)$ is invertible, hence $P(\sigma; a)$ is invertible for $\sigma \in K$ for small a as well. Hence, $V \cap \operatorname{Res}(a) \subset \bigcup_{j=1}^{M} B_{r_j}(\sigma_j)$, and the total rank then equals

$$\sum_{j=1}^{M} \operatorname{tr}\Big(\frac{1}{2\pi i}\oint_{|\sigma-\sigma_j|=r_j} P(\sigma;a)^{-1}\partial_\sigma P(\sigma;a)\,d\sigma\Big),$$

which on the one hand depends continuously on a and on the other hand is an integer by Lemma A.1.2. Hence, it is constant for small a, as desired. □

For a rank 1 resonance, we can track the location of the resonance. We first note that if σ_* is a resonance of *order* 1, then the Taylor and Laurent expansions of P and P^{-1} are

$$P(\sigma) = P_0 + (\sigma-\sigma_*)P_1 + \cdots, \quad P(\sigma)^{-1} = (\sigma-\sigma_*)^{-1}A_{-1} + A_0 + \cdots.$$

Now, $P(\sigma)P(\sigma)^{-1} = \mathrm{id}$ implies $P_0 A_{-1} = 0$, hence $\operatorname{ran} A_{-1} \subset \ker P_0$, while $P(\sigma)^{-1}P(\sigma) = \mathrm{id}$ yields $A_{-1}P_1 + A_0 P_0 = \mathrm{id}$, so we have

$$A_{-1}P_1 = \mathrm{id} \quad \text{on } \ker P_0, \tag{A.1.7}$$

which implies $\ker P_0 \subset \operatorname{ran} A_{-1}$. Therefore, $\operatorname{ran} A_{-1} = \ker P_0$.

Remark A.1.5. If we know $\ker P_0 = \operatorname{span}\{\phi_1,\ldots,\phi_k\}$ explicitly in the order 1 case, the above arguments imply that

$$A_{-1} = \sum_{j=1}^{k} \langle\cdot,\psi_j\rangle\phi_j$$

for some $\psi_j \in \ker P_0^*$. Then (A.1.7) implies $\langle P_1\phi_\ell,\psi_j\rangle = \delta_{\ell j}$ for $1 \le j,\ell \le k$, which allows us to find the ψ_j provided we know $\ker P_0^*$ explicitly. See also the discussion prior to Lemma 7.4.4.

We can now prove:

Lemma A.1.6. *Suppose $P(\sigma;a)$ as in (A.1.1) has a resonance for $a=0$ at σ_* with rank 1. Then for sufficiently small $r>0$ and small a, there is exactly one resonance $\sigma_*(a)$ of $P(\sigma;a)$ in $|\sigma-\sigma_*|<r$, and the resonance is simple; moreover, $\sigma_*(a)$ is a C^r-function of $a\in A$, and we can find a C^r-family $u(a)\in\mathbb{C}^N$ such that $\ker P(\sigma_*(a);a) = \operatorname{span}\{u(a)\}$.*

Proof. Pick $r>0$ so that σ_* is the only resonance of $P(\sigma;0)$ in $B_r(\sigma_*)$. The total rank of resonances in $|\sigma-\sigma_*|<r$ remains equal to 1 for small a by the previous lemma, proving

the existence of a unique rank 1 resonance $\sigma_*(a)$ in $B_r(\sigma_*)$. The location of $\sigma_*(a)$ is then given by

$$\sigma_*(a) = \frac{1}{2\pi i} \int_{|\sigma-\sigma_*|=r} \sigma \frac{\partial_\sigma \det P(\sigma; a)}{\det P(\sigma; a)}\, d\sigma,$$

which depends in a C^r manner on a if $P(\sigma; a)$ does. To prove the last statement, fix a non-zero $u \in \ker P_0$, then

$$u(a) := \Big(\frac{1}{2\pi i} \oint_{|\sigma-\sigma_*|=r} P(\sigma; a)^{-1} \partial_\sigma P(\sigma; a)\, d\sigma\Big) u$$

is C^r in a, lies in $\ker P(\sigma_*(a); a)$ by (A.1.7) and equals $u \neq 0$ for $a = 0$, hence remains non-zero for small a and is therefore a basis of $\ker P(\sigma_*(a); a)$ indeed. □

Remark A.1.7. In a similar vein, one obtains the following statement: Suppose $P(\sigma; a)^{-1}$ has a single pole in $|\sigma - \sigma_*| < r$, located at $\sigma_*(a)$ and of order 1. Then $\sigma_*(a)$ is a C^r function of a, and $\ker P(\sigma_*(a); a)$ depends in a C^r manner on a: Indeed, one can find a basis of $\ker P(\sigma_*(a); a)$ with each basis vector a C^r-function of a.

A.2 Families of analytic Fredholm families

We now turn to the study of the parameter dependence of analytic families of operators which are Fredholm in a non-elliptic Fredholm framework as described in [114]. For concreteness, our functional analytic setup is directly related to [114], so we assume:

(1) A, the parameter space, is a neighborhood of the origin in $\mathbb{R}^L$, and $U \subset \mathbb{C}$ is open, connected and non-empty;

(2) for some fixed $r \in \{0, 1, \ldots, \infty, \omega\}$, we have a family $P(\sigma; a) \in C^r\big(A_a; \mathcal{O}(U; \Psi^m(X))\big)$ of operators, and for any fixed $a \in A$, the principal symbol $\sigma_m(P(\sigma; a))$ is independent of σ; here X is a compact manifold without boundary.

(3) There exists $s_0 \in \mathbb{R}$ such that for all $s > s_0$, we have an estimate

$$\|u\|_{H^s} \leq C(\|P(\sigma; a)u\|_{H^{s-m+1}} + \|u\|_{H^{s_0}}), \tag{A.2.1}$$

where the constant $C = C(s)$ is independent of a, and a similar estimate for the adjoint,

$$\|v\|_{H^{-s+m-1}} \leq C(\|P(\sigma; a)^* v\|_{H^{-s}} + \|u\|_{H^N}), \tag{A.2.2}$$

for any $N < -s + m - 1$, with $C = C(s, N)$ independent of a. (Both estimates are understood in the sense that if the terms on the right hand side are finite, then so is the left hand side, and the inequality holds.)

(4) For $a = 0$, the inverse $P(\sigma; 0)^{-1}$ exists for some $\sigma \in U$.

By a standard functional analytic argument [64, Proof of Theorem 26.1.7], the estimates (A.2.1) and (A.2.2) imply that $P(\sigma; a)\colon \mathcal{X}^s(a) \to H^{s-m+1}(X)$ is Fredholm for $s > s_0$, where we define

$$\mathcal{X}^s(a) := \{u \in H^s(X)\colon P(\sigma; a)u \in H^{s-m+1}(X)\},$$

which is independent of σ by assumption (2). Therefore, $P(\sigma; a) \in \mathcal{L}(\mathcal{X}^s(a), H^{s-m+1}(X))$ is an analytic family of Fredholm operators, and its inverse family is meromorphic for $a = 0$ by assumption (4). In [114, §2.7], it is demonstrated that then $P(\sigma; a)^{-1}$ is a meromorphic family of operators $H^{s-m+1}(X) \to \mathcal{X}^s(a)$ for small a; moreover, $P(\sigma; a)^{-1} \in \mathcal{L}(H^{s-m+1}(X), H^s(X))$ exists in an open subset of $U_\sigma \times A_a$, in which it depends continuously on $(\sigma; a)$ in the weak operator topology, and in fact in the norm topology of $\mathcal{L}(H^{s-m+1+\epsilon}(X), H^{s-\epsilon}(X))$, $\epsilon > 0$.

Our goal is to prove results analogous to Lemmas A.1.4 and A.1.6 for the resonances of $P(\sigma; a)$. Observe here that for any fixed a, we can define order and rank of resonances as in (A.1.2) and (A.1.3), and Lemmas A.1.1, A.1.2 and A.1.3 remain valid for $P(\sigma; a)$ by the same proofs, *mutatis mutandis*. In particular, formula (A.1.4) for the rank makes sense, since the integrand is a σ-dependent family of operators in $\mathcal{L}(H^s(X))$, as $\partial_\sigma P(\sigma; a) \in \Psi^{m-1}(X)$ maps $H^s(X)$ continuously into $H^{s-m+1}(X)$ which gets mapped back into $H^s(X)$ by $P(\sigma; a)^{-1}$, and the integral gives a finite rank, thus trace class, operator on $H^s(X)$. The problem to be overcome is that the dependence on a in this space of operators is continuous only in a very weak topology.

Now, suppose $\sigma_* \in U$ is a pole of $P(\sigma; 0)^{-1}$. By (A.2.1), we have $\dim\ker P(\sigma_*; 0) = k < \infty$ and elements of the kernel of $P(\sigma_*; 0)$ in $H^s(X)$, $s > s_0$, are automatically $\mathcal{C}^\infty$, so

$$\ker P(\sigma_*; 0) = \operatorname{span}\{\phi_1, \dots, \phi_k\}$$

with $\phi_j \in \mathcal{C}^\infty(X)$. Furthermore, $\operatorname{ran}_{\mathcal{C}^\infty(X)} P(\sigma_*; 0) \subset \mathcal{C}^\infty(X)$ is closed and has codimension k. Thus, we can choose a complementary subspace $\mathcal{Y}_2 \subset \mathcal{C}^\infty(X)$ of $\operatorname{ran}_{\mathcal{C}^\infty(X)} P(\sigma_*; 0)$, say

$$\mathcal{Y}_2 = \operatorname{span}\{\psi_1, \ldots, \psi_k\}.$$

We define the operator

$$R = \sum_{j=1}^k \langle \cdot, \phi_j \rangle \psi_j \colon \mathscr{D}'(X) \to \mathcal{C}^\infty(X)$$

which is thus an element of $\Psi^{-\infty}(X)$. The operator family

$$\widetilde{P}(\sigma; a) := P(\sigma; a) + R$$

then satisfies assumptions (2) and (3), and $\widetilde{P}(\sigma_*; 0)$ is invertible, as it still has index 0 and trivial kernel by construction. We conclude that $\widetilde{P}(\sigma; a)$ is invertible for $(\sigma; a)$ near $(\sigma_*; 0)$, with $\widetilde{P}(\sigma; a)^{-1}$ depending continuously on $(\sigma; a)$ in $\mathcal{L}(H^{s-m+1+\epsilon}, H^{s-\epsilon})$ for $s > s_0$, $\epsilon > 0$. We will henceforth assume that $(\sigma; a)$ are sufficiently close to $(\sigma_*; 0)$ so that $\widetilde{P}(\sigma; a)$ is invertible. Note that $\sigma_m(\widetilde{P}(\sigma; a)) = \sigma_m(P(\sigma; a))$, so the spaces $\mathcal{X}^s(a)$ are the same for P and $\widetilde{P}$. Writing

$$P(\sigma; a)^{-1} = \widetilde{P}(\sigma; a)^{-1} Q(\sigma; a)^{-1}, \quad Q(\sigma; a) = \operatorname{id} - R\widetilde{P}(\sigma; a)^{-1},$$

we see that $P(\sigma; a)^{-1} \in \mathcal{L}(H^{s-m+1}, \mathcal{X}^s(a))$ exists if and only if $Q(\sigma; a)$ is invertible. Let us fix $s > s_0$, and let

$$\mathcal{Y}_1 = \operatorname{ran}_{\mathcal{X}^s(0)} P(\sigma_*; 0),$$

which is a closed subspace of $H^{s-m+1}(X)$ complementary to $\mathcal{Y}_2$. In the decomposition $H^{s-m+1}(X) = \mathcal{Y}_1 \oplus \mathcal{Y}_2$, we then have

$$Q(\sigma; a) = \begin{pmatrix} \operatorname{id} & Q_{12}(\sigma; a) \\ 0 & Q_{22}(\sigma; a) \end{pmatrix},$$

where $Q_{12} = -R\widetilde{P}^{-1} \in \mathcal{L}(\mathcal{Y}_2, \mathcal{Y}_1)$, and $Q_{22} = \operatorname{id} - R\widetilde{P}^{-1} \in \mathcal{L}(\mathcal{Y}_2)$ is an operator on the finite-dimensional space $\mathcal{Y}_2$ whose invertibility is equivalent to that of $Q(\sigma; a)$ (and thus of $P(\sigma; a)$). Notice that $Q_{22}(\sigma_*; 0) = 0$ is not invertible, while $Q_{22}(\sigma; 0)$ is invertible for σ in a punctured neighborhood of σ_*, since this is true for $P(\sigma; 0)$. Moreover, since R is smoothing, the continuous dependence of $\widetilde{P}^{-1}(\sigma;a)$ on the parameter implies that $Q_{22}(\sigma;a)$ depends continuously on a as well (and analytically on σ).

From the contour integral expression for the rank, we see that if $\sigma_*(a)$ is a pole of $Q_{22}(\sigma;a)^{-1}$, then

$$\operatorname{rank}_{\sigma_*(a)} P(\sigma;a)^{-1} = \operatorname{rank}_{\sigma_*(a)} Q_{22}(\sigma;a)^{-1};$$

the explicit expression for the inverse of $Q(\sigma;a)$ in terms of $Q_{22}(\sigma;a)$ yields the equality

$$\operatorname{ord}_{\sigma_*(a)} P(\sigma;a)^{-1} = \operatorname{ord}_{\sigma_*(a)} Q_{22}(\sigma;a)^{-1};$$

and moreover the map

$$\Phi\colon \ker Q_{22}(\sigma_*(a);a) \ni v \mapsto \begin{pmatrix} -Q_{12}(\sigma_*(a);a) \\ \mathrm{id} \end{pmatrix} \widetilde{P}(\sigma_*(a);a)^{-1} v \in \ker P(\sigma_*(a);a) \tag{A.2.3}$$

is an isomorphism. Since Lemma A.1.4 applies to $Q_{22}(\sigma;a)$, it therefore implies the analogous statement for $P(\sigma;a)$.

In order to obtain the analogue of Lemma A.1.6, we need to show the C^r-dependence of $Q_{22}(\sigma;a) \in \mathcal{L}(\mathcal{Y}_2)$ on a, which follows if we can show the C^r-dependence of $\widetilde{P}(\sigma;a)^{-1} \in \mathcal{O}(U_\sigma;\mathcal{L}(H^{s_1},H^{s_2}))$ on a (in the operator norm topology) for $s_1,s_2 \in \mathbb{R}$ which we are free to choose. Since $\ker Q_{22} \subset \mathcal{Y}_1 \subset \mathcal{C}^\infty(X)$, this would also imply the C^r-dependence of $\ker P(\sigma_*(a);a) \subset \mathcal{C}^\infty(X)$ on a (via the map Φ in (A.2.3)) in the setting of Lemma A.1.6. Now for $r=1$ and $a \in A$, $b \in \mathbb{R}^L$, we have

$$\begin{aligned} s^{-1}\big(&\widetilde{P}(\sigma;a+sb)^{-1} - \widetilde{P}(\sigma;a)^{-1}\big) \\ &= -\widetilde{P}(\sigma;a+sb)^{-1} \circ s^{-1}\big(\widetilde{P}(\sigma;a+sb) - \widetilde{P}(\sigma;a)\big) \circ \widetilde{P}(\sigma;a)^{-1}, \end{aligned}$$

and as $s \to 0$, the last term on the right hand side is constant in $\mathcal{L}(H^{s-m+1},H^s)$, the second term converges in $C^0(U;\mathcal{L}(H^s,H^{s-m}))$ to $D_2\widetilde{P}(\sigma;a)(b)$, and the first term converges in $C^0(U;\mathcal{L}(H^{s-m},H^{s-1-\epsilon}))$ to $\widetilde{P}(\sigma;a)^{-1}$ if $s > s_0+1$; here, all spaces of linear operators are equipped with the norm topology. Therefore, for all $s > s_0+1$, the above difference quotient converges in $C^0(U;\mathcal{L}(H^{s-m+1},H^{s-1-\epsilon}))$ to $-\widetilde{P}(\sigma;a)^{-1}D_2\widetilde{P}(\sigma;a)(b)\widetilde{P}(\sigma;a)^{-1}$, which itself depends continuously on a as an element of $C^0(U;\mathcal{L}(H^{s-m+1},H^{s-1-\epsilon}))$. This establishes

$$\widetilde{P}(\sigma;a)^{-1} \in C^1\big(A_a;C^0(U_\sigma;\mathcal{L}(H^{s-m+1},H^{s-1-\epsilon}))\big).$$

An analogous argument (with additional losses of derivatives) shows that we can in fact replace the continuity in σ by analyticity. In an entirely analogous manner, losing additional derivatives, we obtain the corresponding regularity statement for $r > 1$.

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