Picard-Lefschetz theory

Jean-Philippe Chassé

chasseje@dms.umontreal.ca

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Abstract

Picard-Lefschetz theory is an important tool in complex geometry developed in the beginning of the XXth century in order to study the topology of smooth projective variety over $\mathbb{C}$. It has now greatly diversified, and has seen important generalizations for variety over finite fields and symplectic manifolds. In this report, we introduce the main ideas of the theory in the complex setting, and prove the Picard-Lefschetz formulas in odd dimension; these formulas form the pinnacle of the local theory. We conclude with a brief discussion on Lefschetz fibrations and symplectic Picard-Lefschetz theory.

Contents

I. Introduction

Morse theory is nowadays one of the most celebrated part of differential topology. It is based on a simple, yet powerful observation: one can recover topological data of a smooth manifold M from the critical points of a nice enough smooth function $f : M \to \mathbb{R}$.

Let us first recall some definitions in order to give a more precise statement of what has just been said.

Definition 1 *Let $f : M \to \mathbb{R}$ be a smooth function and $x \in M$ be a critical point of f, i.e. the differential df_x of f at x is 0. Then, there is a naturally-defined bilinear symmetric form on T_xM, called the Hessian of f at x, given by*

$$\mathrm{Hess}_x(f)(u, v) := U(V f)(x),$$

where U and V are extensions of u and v, respectively, on some neighborhood of x in M – in local coordinates, this is just the bilinear form defined by the matrix of second derivatives of f at x.

We say that x is a nondegenerate critical point of f if $\mathrm{Hess}_x(f)$ is a nondegenerate bilinear form. We say that f is a Morse function if all of its critical points are nondegenerate.

The next theorem resumes the main points of classical Morse theory in the case when M is compact – there are similar statement for noncompact manifolds or manifolds with boundaries, but these are beyond the illustrative scope of the theorem.

Theorem 1 (see [Mil63]) *Every Morse function $f : M \to \mathbb{R}$ defines a CW-decom -position of M such that each cell corresponds to a critical point, its dimension being given by the signature of the Hessian of f at the critical point. In particular, one can recover the homotopy type of M from any of its Morse function.*

Moreover, the set of Morse functions is open and dense in the space of smooth functions of M.

This is a very nice result, and one could be tempted to try to generalize this result to complex manifolds and holomorphic functions. Indeed, replacing real variables by complex ones makes perfect sense in the definitions given above, and thus we can talk about the complex Hessian –now $\mathbb{C}$-bilinear– of a holomorphic function $f : M \to \mathbb{C}$ at a critical point. We define complex Morse functions as holomorphic functions with only nondegenerate critical points. However, we quickly run into a problem:

Lemma 1 (Complex Morse lemma) *Let $f : M \to \mathbb{C}$ be a holomorphic function, and $z \in M$ be a nondegenerate critical point of f. Then, there exist local holomorphic coordinates $z_1, \dots, z_n$ centered at z such that*

$$f(z_1, \dots, z_n) = f(0) + \sum_{i=1}^{n} z_i^2. \tag{1}$$

We will not prove the statement here, since the proof is almost the same as for the real Morse lemma, and is not very informative for what will follow (the stubborn reader can find a proof in [Voi03]).

As a consequence of the complex Morse lemma, there are no local invariant of critical points of complex Morse functions. Going back to theorem 1, one realize that a direct generalization was always doomed: the CW-decomposition of M is determined by the signature of the Hessians, but this is a trivial invariant in the complex case. Even if this fails, can one still extract some sort of information out of these critical points?

II. The monodromy of a nondegenerate critical point

II.a. Monodromy and variation operators

Let M be a complex manifold of dimension $n \geq 2$, and $f : M \to \mathbb{C}$ be a holomorphic function with a nondegenerate critical point $z \in M$. Without loss of generality, $f(z) = 0$.

Let $\varepsilon > 0$ be such that the equality (1) holds for $\sum_i |z_i|^2 \leq 4\varepsilon^2$. At the price of a multiplication of f and the z_i's by $\varepsilon^{\pm 1}$, we can suppose that $\varepsilon = 1$. Fixing such local coordinates for the rest of the section, we will denote by B_r, $r < 2$, the closed $2n$-ball in these coordinates. Likewise, for the rest of the section, we will take $f = f|_{B_2}$.

Since there are no critical point of f on ∂B_2, $f|_{\partial B_2}$ is a submersion onto D_2, the disk of radius r in $\mathbb{C}$, with a compact domain. Therefore, by Ehresmann's theorem (see [BJ82] for example), it is a fiber bundle over D_2. Furthermore, since D_2 is contractible, the bundle is trivial. The same argument implies that f gives a (potentially nontrivial) fiber bundle $B_2 \setminus f^{-1}(0) \to D^2 \setminus \{0\}$.

Let $F_\lambda = f^{-1}(\lambda)$ be the fiber in B_2 over $\lambda \in D_2$. If $\lambda \neq 0$, i.e. if λ is a regular value of f, then F_λ is a compact complex manifold of dimension $n-1$ with boundary $\partial F_\lambda = F_\lambda \cap \partial B_2$. Consider a loop $\gamma : [0,1] \to D_2 \setminus \{0\}$ based at 1. Since the bundle $f|_{S^2} : S^2 \to D^2$ is trivial, we can take diffeomorphisms $g_t = g(-,t) : \partial F_1 \to \partial F_{\gamma(t)}$ such that $g_1 = \mathbb{1}_{F_1}$. Then, by the relative homotopy lifting property of fibrations (see [Hat09] for example), there exists a map $\Gamma : F_1 \times [0,1] \to B_2 \setminus F_0$ making the following diagram commute:

$$\begin{array}{ccc} (F_1 \times \{0\}) \cup (\partial F_1 \times [0,1]) & \xrightarrow{\iota \cup g} & B_2 \setminus F_0 \\ \big\downarrow & \overset{\Gamma}{\nearrow} & \big\downarrow f \\ F_1 \times [0,1] & \longrightarrow & D_2 \setminus \{0\} \end{array},$$

where the bottom arrow does nothing with the F_1 variable, but is equal to γ in the

$[0,1]$ one. In particular, $\Gamma_t := \Gamma(-,t)$ sends F_1 to $F_{\gamma(t)}$. Note that, up to homotopy, Γ only depends on the homotopy class of γ as a loop in $D_2 \setminus \{0\}$.

Definition 2 *The transformation $h_\gamma := \Gamma_1 : F_1 \to F_1$ is called the monodromy of γ, whilst the induced morphism $(h_\gamma)_*$ on homology (with integer coefficients) is called the monodromy operator.*

The mononodromy operator of a loop $\gamma : [0,1] \to D_2 \setminus \{0\}$ induces a morphism $H_\bullet(F_1, \partial F_1) \to H_\bullet(F_1)$ as follows. Let δ be a relative cycle of $(F_1, \partial F_1)$, i.e. $\delta \in C_\bullet(F_1)$ and $\partial\delta \in C_{\bullet-1}(\partial F_1)$; any element of $H_\bullet(F_1, \partial F_1)$ can be represented by such a chain. By construction, $h_\gamma|_{\partial F_1} = \mathbb{1}_{\partial F_1}$. Therefore, $h_\gamma\delta - \delta$ is an actual cycle of F_1, and thus defines a class $\mathrm{var}_\gamma[\delta]$ in $H_\bullet(F_1)$. A direct calculation shows that $\mathrm{var}_\gamma[\delta]$ does not depend on the relative cycle representing the homology class.

Definition 3 *The ensuing group homomorphism*

$$\mathrm{var}_\gamma \colon H_\bullet(F_1, \partial F_1) \longrightarrow H_\bullet(F_1)$$

is called the variation operator of γ

Note that we have the relations

$$(h_\gamma)_* = \mathbb{1} + (\mathrm{var}_\gamma) j \quad \text{and} \quad (h_\gamma^{(r)})_* = \mathbb{1} + j(\mathrm{var}_\gamma), \tag{2}$$

where $h_\gamma^{(r)}$ is just h_γ, but seen as a relative map $(F_1, \delta F_1) \to (F_1, \partial F_1)$, and $j : H_\bullet(F_1) \to H_\bullet(F_1, \partial F_1)$ is the canonical map. Furthermore, these operators are well-behaved with respect to concatenation of loops:

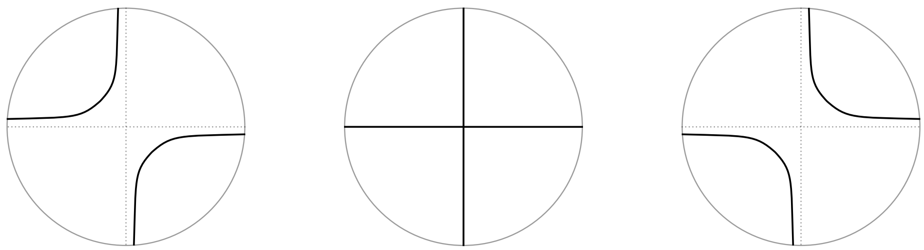
$$\begin{gathered}(h_{\gamma_1\gamma_2})_* = (h_{\gamma_2})_*(h_{\gamma_1})_*, \qquad (h^{(r)}_{\gamma_1\gamma_2})_* = (h^{(r)}_{\gamma_2})_*(h^{(r)}_{\gamma_1})_* \\ \mathrm{var}_{\gamma_1\gamma_2} = \mathrm{var}_{\gamma_1} + \mathrm{var}_{\gamma_2} + \mathrm{var}_{\gamma_2}\, j\, \mathrm{var}_{\gamma_1}\,.\end{gathered} \tag{3}$$

The first two formulas follow from the observation that $h_{\gamma_1\gamma_2}$ may be chosen to be the concatenation of h_{γ_1} and h_{γ_2}, which is homotopic relative to ∂F_1 to the composition of the two; the last formula then follows directly.

These three operators will precisely be what we will be interested in for the rest of this section. However, before continuing our investigation, let us look at an explicit example.

II.b. The case $n = 2$: An illustrative example

All this abstract nonsense can somewhat be visualized when $n = 2$. Indeed, making the change of variable $x = z_1 + iz_2$ and $y = z_1 - iz_2$, the equation $z_1^2 + z_2^2 = \lambda$ becomes $xy = \lambda$. Therefore, in the real 2-ball in these coordinates, the fiber take on



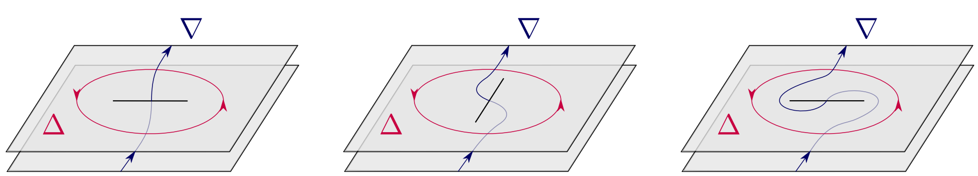
(a) $\lambda < 0$ (b) $\lambda = 0$ (c) $\lambda > 0$

Figure 1: Real pictures of F_λ.

a familiar shape, as shown in Figure 1. Intuitively, one should think of opposing pair of points in the real picture as being the intersection of a circle in $\mathbb{C}^2$ with $\mathbb{R}^2 \subseteq \mathbb{C}^2$. Therefore, the nonsingular fibers are cylinders, and the singular one is a cone.

To make matters a bit more rigorous, identify F_λ with the Riemann surface associated with the holomorphic function $w = \sqrt{\lambda - z^2}$ over D_2. For $\lambda \neq 0$, this surface is obtained by making a cut in two copies of D_2 along the line from $-\sqrt{\lambda}$ to $\sqrt{\lambda}$, and gluing to two disks along the boundary created by the cut. Topologically, this just corresponds to gluing compact cylinders along one of their border, which just produces another cylinder. Furthermore, as λ goes to 0, the edge along which we glue becomes a point; the singular fiber is thus a cone. In particular, we get

$$H_k(F_1) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad H_k(F_1, \partial F_1) = \begin{cases} \mathbb{Z} & \text{if } k = 1, 2 \\ 0 & \text{otherwise} \end{cases}.$$



(a) $t = 0$ (b) $t = \frac{1}{2}$ (c) $t = 1$

Figure 2: Image of Δ and ∇ under Γ_t.

Consider the loop $\gamma(t) = e^{2\pi i t}$ in $D_2 \setminus \{0\}$; since $[\gamma]$ generate $\pi_1(D_2 \setminus \{0\})$, it follows from properties (3) that we only need to look at this loop in order to understand the monotonicity operator. We can then take its lift $\Gamma_t : F_1 \to F_{\gamma(t)}$ to be

$$\Gamma_t(z_1, z_2) = e^{\pi i t \chi(|z_1|^2 + |z_2|^2)}(z_1, z_2),$$

where $\chi : [0, +\infty) \to [0, 1]$ is a smooth map such that $\chi|_{[0,2]} \equiv 1$ and $\chi|_{[3,+\infty)} \equiv 0$. Taking representatives Δ and ∇ of the generator of $H_1(F_1)$ and $H_1(F_1, \partial F_1)$, respectively, such that their intersection product if $\nabla \cdot \Delta = 1$, we can see the effect of Γ_t on them.

Indeed, as Figure 2 reveals (disks have been changed to squares for ease of visualization), Δ stays unchanged, whilst ∇ gets a twist in the opposite direction of Δ. We can write this more properly in homological terms:

$$(h_\gamma)_*(\Delta) = \Delta \quad \text{and} \quad (h_\gamma^{(r)})_*(\nabla) = \nabla - j(\Delta),$$

thus

$$\operatorname{var}_\gamma(\nabla) = -\Delta,$$

where we have made the slight abuse of notation of using the same symbol for the homology class and its representative. The variation operator is of course trivial in other degree, since $H_0(F_1, \partial F_1) = 0$ and $H_2(F_1) = 0$. This turns out to be quite representative of the general behavior, but we still need more work in order to prove it.

II.c. The vanishing cycle and the Picard-Lefschetz formulas

Note that the $(n-1)$-sphere (of radius 1) embeds in the fiber F_λ, $\lambda \in D_2 \setminus \{0\}$, as the set

$$S(\lambda) := \left\{ (z_1, \ldots, z_n) \in B_2 \;\middle|\; z_j = \sqrt{|\lambda|} e^{\frac{i}{2}\arg(\lambda)} x_j,\ x_j \in \mathbb{R},\ \sum_{j=1}^{n} x_j^2 = 1 \right\}.$$

Furthermore, the embedding depends smoothly on $\lambda \in D_2 \setminus [-2, 0]$.

Definition 4 *The homology class $\Delta \in H_{n-1}(F_1)$ represented by $S(1) \subseteq F_1$ is called the vanishing cycle.*

The next lemma shows that the nomenclature is well-chosen: Δ is precisely the homology class of F_λ that vanishes as $\lambda \to 0$. Going back to the case $n = 2$, one should have the image in mind of a nonsingular fiber as a cylinder getting pinched along a parallel as we approach the singular fiber.

Lemma 2 *The embedding $S^{n-1} \hookrightarrow F_\lambda$ extends to a diffeomorphism from a disk subbundle of TS^{n-1} onto F_λ.*

Proof:. We will only prove the case $\lambda = 1$, as the general case is very similar.

Writing $z_j = u_j + iv_j$, for $u_j, v_j \in \mathbb{R}$, we have

$$\begin{aligned} F_1 &= \left\{ \sum_{j=1}^n z_j^2 = 1,\ \sum_{j=1}^n |z_j|^2 \leq 4 \right\} \\ &= \left\{ \sum_{j=1}^n (u_j^2 - v_j^2) = 1,\ \sum_{j=1}^n u_j v_j = 0,\ \sum_{j=1}^n (u_j^2 + v_j^2) \leq 4 \right\} \\ &= \left\{ \sum_{j=1}^n x_j^2 = 1,\ \sum_{j=1}^n x_j y_j = 0,\ \sum_{j=1}^n y_j^2 \leq \frac{3}{2} \right\}, \end{aligned}$$

where $x_k = u_k / \sqrt{\sum_j u_j^2}$ and $y_k = v_k$. But the last set naturally identifies with the radius $\sqrt{\frac{3}{2}}$ disk subbundle of $TS^{n-1} \subseteq \mathbb{R}^{2n}$; the inverse morphism

$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \longmapsto \left(x_1 \sqrt{1 + \sum_j y_j^2} + iy_1, \ldots, x_n \sqrt{1 + \sum_j y_j^2} + iy_n \right)$$

is thus the diffeomorphism we were looking for. □

Corollary 1 *We have,*

$$H_k(F_1) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H_k(F_1, \partial F_1) = \begin{cases} \mathbb{Z} & \text{if } k = n-1, 2(n-1) \\ 0 & \text{otherwise} \end{cases}.$$

In particular, the variation operator is zero on all degree except $n-1$ for any path in $D_2 \setminus \{0\}$.

PROOF:. The calculation of the homology of F_1 follows from the fact that a disk bundle deformation retracts onto the image of the zero section, which is naturally identified with S^{n-1}. The one on the relative homology of $(F_1, \partial F_1)$ then follows from Poincaré-Lefschetz duality (see [Hat09] for example). □

As suggested by the illustrations in Section II.b, the variation operator can be nontrivial in degree $n-1$; but what is the precise relation? In order to know, let us fix the generator ∇ of $H_{n-1}(F_1, \partial F_1)$ such that $\nabla \cdot \Delta = 1$, where $\cdot$ denotes the intersection product. Furthermore, by the relations (3), we only need to look at the case $\gamma(t) = e^{2\pi it}$, so we will omit it from the notation.

Theorem 2

$$\operatorname{var}(\nabla) = (-1)^{\frac{n(n+1)}{2}} \Delta$$

Using the fact that $H_{n-1}(F_1) = \mathbb{Z}\cdot\Delta$ and $H_{n-1}(F_1, \partial F_1) = \mathbb{Z}\cdot\nabla$, and relations (2), we thus get formulas for the monodromy and variation operators:

Corollary 2 (Picard-Lefschetz formulas) *For any $a \in H_{n-1}(F_1, \partial F_1)$ and $b \in H_{n-1}(F_1)$, we have*

$$\begin{aligned} \operatorname{var}(a) &= (-1)^{\frac{n(n+1)}{2}}(a\cdot\Delta)\Delta \\ h_*^{(r)}(a) &= a + (-1)^{\frac{n(n+1)}{2}}(a\cdot\Delta)j(\Delta) \\ h_*(b) &= b + (-1)^{\frac{n(n+1)}{2}}(b\cdot\Delta)\Delta \end{aligned}$$

Although the full proof of theorem 2 is beyond the scope of this report, we can give a neat quick proof when n is odd; this will hopefully convince the skeptical reader if Section II.b was not enough. However, before going in, we need a simple lemma.

Lemma 3 *The self-intersection number of Δ is given by*

$$\begin{aligned} \Delta\cdot\Delta &= (-1)^{\frac{(n-1)(n-2)}{2}}(1+(-1)^{n-1}) \\ &= \begin{cases} 0 & \textit{if } n \equiv 0 \mod 2 \\ 2 & \textit{if } n \equiv 1 \mod 4 \\ -2 & \textit{if } n \equiv 3 \mod 4 \end{cases}. \end{aligned}$$

Proof:. Remember that the self-intersection number of a manifold in its tangent space is given by the Euler characteristic, which is $1+(-1)^{n-1}$ for the $(n-1)$-sphere.

The additional sign comes from the fact that the orientation of TS^{n-1} induced by the embedding in $\mathbb{R}^{2n}$ differs from the one induced by the complex structure. Indeed, at $(1, 0, \ldots, 0)$ – it does not matter at which point we look at the change of orientation, might as well take a nice one – the base $(u_2, \ldots, u_n, v_2, \ldots, v_n)$ is positively oriented in the orientation coming from $\mathbb{R}^{2n}$, whilst it is $(u_2, v_2, \ldots, u_n, v_n)$ which is positively oriented in the complex orientation. Getting from one oriented base to the other requires a permutation of sign $(-1)^{\frac{(n-1)(n-2)}{2}}$, which gives the right formula. □

Proof of theorem 2 in odd dimension: Consider the lift $F_1 \to F_{\gamma(t)}$ of $\gamma(t) = e^{2\pi it}$ given by

$$\Omega_t(z_1, \ldots, z_n) = e^{\pi it}(z_1, \ldots, z_n).$$

The map Ω_1 clearly does not fix ∂F_1, but it is homotopic to one since all lift are. Therefore, it can be used in order to calculate h_*. But $\Omega_1 = -1$, which is just the reflection of $S(1)$ about its center. Hence,

$$h_*(\Delta) = (\Omega_1)_*(\Delta) = -\Delta,$$

since $-1 : S^{n-1} \to S^{n-1}$ has degree $(-1)^n$ (once again, see [Hat09]).

On the other hand, by lemma 3, we have

$$j(\Delta) \cdot \Delta = \Delta \cdot \Delta = 2(-1)^{\frac{(n-1)(n-2)}{2}}.$$

Since the pairing $H_{n-1}(F_1, \partial F_1) \otimes H_{n-1}(F_1) \to \mathbb{Z}$ induced by the intersection product is nondegenerate, as all the homology groups are free by corollary 1 – and that we have chosen generators of $H_{n-1}(F_1, \partial F_1)$ and $H_{n-1}(F_1)$ such that $\nabla \cdot \Delta = 1$ – this implies that

$$j(\Delta) = 2(-1)^{\frac{(n-1)(n-2)}{2}} \nabla.$$

This is the part that really requires to be in odd dimension. Indeed, in even dimension, we get $j(\Delta) = 0$, and cannot conclude anything from that.

Let $m \in \mathbb{Z}$ be such that $\operatorname{var}(\nabla) = m\Delta$. Putting the two last bit together, we get

$$\begin{aligned} -\Delta &= h_*(\Delta) \\ &= \Delta + \operatorname{var}(j(\Delta)) \\ &= \Delta + 2(-1)^{\frac{(n-1)(n-2)}{2}} \operatorname{var}(\nabla) \\ &= \left(1 + 2(-1)^{\frac{(n-1)(n-2)}{2}} m\right) \Delta, \end{aligned}$$

where we have used relations (2) in order to get from the first to the second line. This of course implies that $m = (-1)^{\frac{n(n+1)}{2}}$, just as desired. $\square$

Remark: The local information of an isolated nondegenerate critical point can be used in order to study degenerate critical points. Indeed, suppose that $f : X \to \mathbb{C}$, where $X \subseteq \mathbb{C}^n$ is an quasi-affine smooth variety, is a regular function with a unique critical point. One can then show that, for any linear function $g : \mathbb{C}^n \to \mathbb{C}$ and any $\lambda > 0$ small enough, $f + \lambda g$ is a complex Morse function with critical values in D_2. Therefore, it suffices to study these nondegenerate critical points, and their behavior as $\lambda \to 0$, in order to understand the one critical point of f. This is the road taken in the second chapter of [AGZV88], but we will not go down it any further.

III. From local to global: A rough picture

III.a. Lefschetz pencils and fibrations

After all that talk about Morse theory in the introduction, one would be tempted to ask whether it is possible to recover global topological information from the monodromy of the critical points of a complex Morse function. The answer is mostly yes, but it is not as direct as in the real case. We give here a rough picture on how to do it.

Suppose that $M \subseteq \mathbb{P}^N = \mathbb{P}^N_{\mathbb{C}}$ is a smooth projective variety. Suppose we are given two hyperplanes H_0 and H_1 of $\mathbb{P}^N$ with defining polynomials F_0 and F_1, respectively. Then, for each $[t:s] \in \mathbb{P}^1$, one can define

$$M_{[t:s]} := V(tF_0 + sF_1) \cap M.$$

If H_0 and H_1 are in general position, it is quite easy to see that $M = \cup_{[t:s]\in\mathbb{P}^1} M_{[t:s]}$, and that each $M_{[t:s]}$ has at most one ordinary double point as singularity. In that case, we say that $\{M_{[t:s]}\}_{[t:s]\in\mathbb{P}^1}$ is a *Lefschetz pencil* in M – these can more generally defined via linear systems of divisors, but we will not need such generality here.

The *base locus* of a Lefschetz pencil $\{M_{[t:s]}\}_{[t:s]\in\mathbb{P}^1}$ is

$$B := \bigcap_{[t:s]\in\mathbb{P}^1} M_{[t,s]} = H_0 \cap H_1 \cap M.$$

By transversality arguments, for H_0 and H_1 in general position, B is a smooth subvariety of M of codimension 2. In fact, the map sending a point $z \in M \setminus B$ to the point $[t:s] \in \mathbb{P}^1$ such that $z \in M_{[t:s]}$ actually defines a rational map $f : M \dashrightarrow \mathbb{P}^1$ with indeterminacy locus B. By blowing up M along B, one gets a regular map $\tilde{f} : \tilde{M} \to \mathbb{P}^1$, called a *Lefschetz fibration*.

Note that a Lefschetz fibration is neither a fiber bundle nor a fibration in the sense of homotopy theory; one should think more of it as some sort of generalization of a branched cover. Actually, the fact that $M_{[t:s]}$ only has at most one ordinary double point as singularity means that $\tilde{f}$ has at most one critical point per fiber, and that such a point admits local coordinates of the form (1). Therefore, one should think of $\tilde{f}$ as a meromorphic Morse function.

We will not go any further than this when speaking about Lefschetz fibrations, as setting up the appropriate theory would go beyond the scope of this report. However, it is important to know that one can then extract a lot of topological information of M from a Lefschetz fibration, e.g. one can use this construction to prove Lefschetz's theorem on hyperplane sections (see [Voi03] for example).

III.b. Dehn twists and symplectic topology

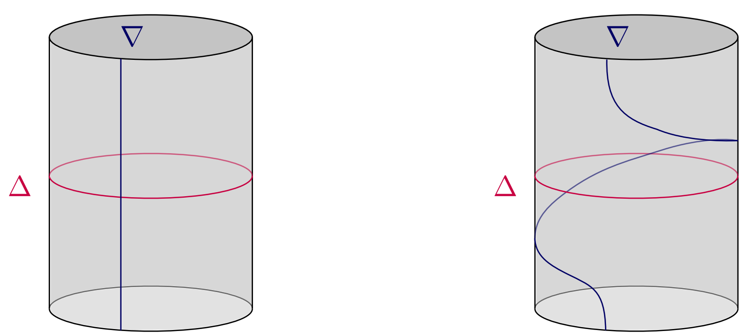
In this section, we briefly explain how symplectic topology offers a natural setting in which to talk about monodromy. In order to get to the meat of the subject, we will assume that the reader knows all basic definitions of symplectic topology; they can easily be found in any introductory reference (see [MS98] for example). We will however note that any smooth projective variety is naturally a symplectic manfinold, as it inherits the Fubini-Study form from $\mathbb{P}^N$.

The story begins by a very short note of Arnol'd [Arn95], where he makes a very interesting observation: in the local model

$$\begin{aligned} f\colon \mathbb{C}^n &\longrightarrow \mathbb{C} \\ (z_1,\dots,z_n) &\longmapsto z_1^2+\cdots+z_n^2 \end{aligned},$$

the embedding $S^{n-1} \hookrightarrow F_\lambda$ can be extended to a symplectomorphism $(T^*S^{n-1},\omega_{\text{can}}) \xrightarrow{\sim} (F_\lambda,\omega_0)$, where ω_{can} is the canonical symplectic form on the cotangent bundle, and ω_0 is the restriction to F_λ of the standard symplectic form on $\mathbb{C}^n$. In particular, the vanishing cycle is represented by a Lagrangian sphere.

Furthermore, in T^*S^{n-1}, the monodromy can be realized by a well-known symplectomorphism: the $(2n-2)$-dimensional Dehn twist. We will only describe the time-1 map Γ_1, but know that one may take $\{\Gamma_t\}_{t\in[0,1]}$ to be a path of symplectomorphisms – this is at least plausible since all fibers are symplectomorphic to T^*S^{n-1}. Let $\psi : S^{n-1} \to S^{n-1}$ be the antipodal map, i.e. multiplication by -1. Like any diffeomorphism, it lifts to a symplectomorphism between cotangent bundles $\Psi : T^*S^{n-1} \to T^*S^{n-1}$. Consider the function $H(q,p) = \frac{|p|^2}{2}$, where the norm is taken in the standard metric. It is well known that the Hamiltonian diffeomorphism φ_H^t generated by H is just the geodesic flow under the canonical identification $T^*S^{n-1} \cong TS^{n-1}$. Thus, φ_H^t parallel transports a covector of length ℓ along the geodesic corresponding to it at a distance ℓt. In particular, φ_H^π acts of covector of length 1 just like Ψ. The symplectomorphism $\Gamma_1 = A\varphi_H^\pi$ is the $(2n-2)$-dimensional Dehn twist; it acts like the antipodal map on the zero section, twists the fibers, but leaves the extremities of the unit-1 disk subbundle fixed (see Figure 3). Note that H may be modified outside of the unit disk subbundle so as to make sure that Γ_1 extends to a compactly supported symplectomorphism on T^*S^{n-1}.



(a) D^*S^1 (b) $\Gamma_1(D^*S^1)$

Figure 3: The Dehn twist and its effect on a fiber and the zero section.

Sadly however, one cannot in general take local coordinates on a smooth projective variety – or more generally on a Kähler manifold – of the form of lemma 1 which are both holomorphic and symplectic, as it would imply that the manifold is flat in that neighborhood. Therefore, one is led to make a choice between looking at monodromy from the complex side, or from the symplectic side. However, Arnol'd's observation implies that the symplectic side is the right one. Indeed, although monodromy may be realized as a family of symplectomorphisms, it cannot be realized as a family of biholomorphisms as the complex isomorphism type of the fiber $F_{\gamma(t)}$ might change with t. For example, this is the case in the Lefschetz fibration coming from a pencil of cubic curves, seen as a Lefschetz pencil in $\mathbb{P}^2$ by embedding it in $\mathbb{P}^9$ using the Veronese embedding, since the nonsingular fibers will be elliptic curves with different parameters.

In fact, by the Weinstein neighborhood theorem, as soon as one has an embedded Lagrangian sphere in a symplectic manifold, one has a neighborhood symplectomorphic to a disk subbundle of T^*S^{n-1}, and thus has a generalized Dehn twist. This simple fact has led, mainly through the work of Seidel, to a greater understanding of many symplectic manifolds. For example, he proved [Sei03] that the Dehn twist induces a long exact sequence on what is called symplectic Floer homology, an important algebraic invariants of Lagrangian submanifolds of exact symplectic manifolds. This long exact sequence actually comes from an exact triangle in the famous Fukaya A_∞-category, which has deep links with algebraic geometry via the homological mirror symmetry conjecture (see [Sei08] for all details).

We conclude by noting that, since the work of Donaldson, Lefschetz pencils and Lefschetz fibrations have also taken an important place in symplectic topology. One can generalize these notions, by so-called topological Lefschetz pencils and fibrations, and use them to probe the symplectic topology of certain manifold. For example, this has been used [Don98] to prove the existence of symplectic hypersurfaces representing certain homology classes in a vast family of symplectic manifolds. Also worthy of mention, is the role that these topological Lefschetz pencils have taken in the study of 4-manifolds, where they are equivalent to the existence of symplectic forms (more details in [Gom01]).

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