Local exactness and C^0 Lagrangian topology

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LG&TBQ2

June 3rd, 2025

Outline

1. Definitions

2. Some questions

3. Some answers
   - A negative one
   - A few positive ones

Plan

Symplectic manifolds

Definition

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Example

A 2-dimensional symplectic manifold is a surface with an area form.

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It is said to be **Hamiltonian** if there exists a smooth function $H : [0,1] \times M \to \mathbb{R}$ such that $\psi = \varphi_H^1$, where the isotopy $\{\varphi_H^t\}$ is defined via

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We denote $\mathrm{Ham}(M) := \{\varphi_H^1 \mid H : [0,1] \times M \to \mathbb{R}\}$.

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- When $\dim M = 2$, a Lagrangian is simply a curve.
- The zero-section of $(T^*L, d\lambda_0)$, where $\lambda_0 = \sum_i p_i dq_i$.

The Hausdorff metric

Definition

Let A and B be closed subsets of a metric space M. The **Hausdorff distance** between them is

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C^0 **goal**: Understand the Hamiltonian orbit $\mathcal{L}\mathrm{Ham} := \mathrm{Ham}(M) \cdot L$ of some Lagrangian L, equipped with δ_H.

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Nearby Lagrangians

Question (A)

Let $\mathcal{W}$ be a (small) neighbourhood of some Lagrangian L. If $L' \in \mathcal{L}\mathrm{Ham}$ and $L' \subseteq \mathcal{W}$, is there a Hamiltonian isotopy from L to L' supported in $\mathcal{W}$?

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- Positive answer implies local path connectedness of $\mathcal{L}\mathrm{Ham}$.
- Hamiltonian version of the question only known for surfaces [Fathi, '80] and B^{2n} [Seyfaddini, '13].

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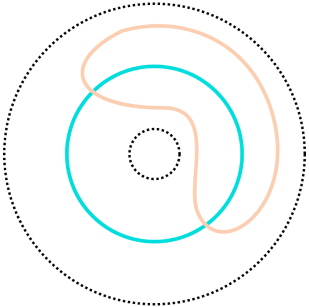

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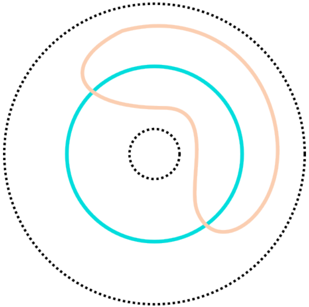

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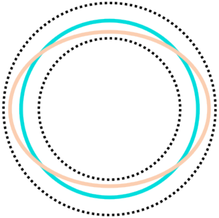

(b) $\mathcal{W}$ small

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Question (B)

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- Positive answer implies that $L \cap L' \neq \varnothing$ [Gromov, '85].

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- Weaker than Question A; equivalent if NLC holds on T^*L.
- Positive answer implies that $L \cap L' \neq \varnothing$ [Gromov, '85].
- Trivial if $H^1(L; \mathbb{R}) = 0$ or L is exact in M.

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- Hamiltonian version — the C^0 flux conjecture — of the question only known in few cases [Lalonde–McDuff–Polterovich, '97; Buhovsky, '14; Atallah-Shelukhin, in progress].

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The bad news

Theorem

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The proof relies on the classification of product tori in $\mathbb{C}^n$ [Chekanov, '96].

It uses, in a fundamental way, that these tori bound disks whose area are $\mathbb{Q}$-linearly independent.

Some nicer Lagrangians

Since $\omega|_{TL} \equiv 0$, the morphism

$$\begin{aligned} \omega : H_2(M, L; \mathbb{Z}) &\longrightarrow \mathbb{R} \\ A &\longmapsto \int_A \omega \end{aligned}$$

is well defined.

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Definition

We say that L is H-*rational* if the image of the morphism is discrete.

A positive answer to Question B

Theorem

There is a class of diffeomorphism types $\mathscr{C}$ with the following property. If $L \in \mathscr{C}$ is H-rational, then Question B has a positive answer on $\mathcal{L}\mathrm{Ham}(L)$.

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- $\mathscr{C}$ is closed under products.
- If $L \in \mathscr{C}$ and $L \to L'$ is a cover, then $L' \in \mathscr{C}$.

A positive answer to Question C

Theorem

If $L \in \mathscr{C}$ is H-rational (in M) and the NLC holds on T^*L, then $\mathcal{L}\mathrm{Ham}(L)$ is δ_H-closed in the space $\mathcal{L}(L)$ of all Lagrangians of M diffeomorphic to L.

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If $L \in \mathscr{C}$ is H-rational (in M) and the NLC holds on T^*L, then $\mathcal{L}\mathrm{Ham}(L)$ is δ_H-closed in the space $\mathcal{L}(L)$ of all Lagrangians of M diffeomorphic to L.

Remarks

In many cases, the NLC on T^*N is in fact *equivalent* to the above δ_H-closedness statement, for H-rational L of the same diffeomorphism type as N.

Back to Question A

Using the cases where we know the NLC holds [Hind, '06; Hind–Pinsonnault–Wu, '12; Rizell–Goodman–Ivrii, '16], we get the following.

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Corollary

Questions A and C have a positive answer if L is

(a) a H-rational $\mathbb{T}^2$, or

(b) any S^1, S^2, or $\mathbb{R}P^2$.

Thank you for your attention!

I will be happy to answer your questions.