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Order and Chaos in Satellite Encounters

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Abstract

In order to describe the motion of two weakly interacting satellites of a central body we suggest to use orbital elements based on the the linear theory of Kepler motion in Levi-Civita's regularizing coordinates. The basic model is the planar three-body problem with two small masses, a model in which both regular (e.g. quasi-periodic) as well as chaotic motion can occur.

This paper discusses the basics of this approach and illustrates it with a typical example. First, we will revisit Levi-Civita's regularization of the two-dimensional Kepler motion and introduce sets of orbital elements based on the differential equations of the harmonic oscillator. Then, the corresponding theory for the three-dimensional motion will be developed using a quaternion representation of Kustaanheimo-Stiefel (KS) regularization; we present it by means of an elegant new notation.

1. Introduction

We begin by summarizing the equations of motion of the three-body problem with two small masses in the form of two weakly coupled Kepler motions, valid in two or three dimensions.

Let $m_k, x_k, (k = 0, 1, 2)$ be the masses and positions of the three bodies, where we assume $x_k \in \mathbb{R}^2$ or $x_k \in \mathbb{C}$ in the planar case and $x_k \in \mathbb{R}^3$ in the spatial case. We assume the center of mass to be at rest at the origin, $\sum_{k=0}^{2} m_k x_k = 0$, and the masses satisfy the hierarchy $m_j \ll m_0, (j = 1, 2)$. The Newtonian equations of motion in inertial coordinates are

$$\begin{aligned} \ddot{x}_0 &= m_1 \frac{x_1 - x_0}{\|x_1 - x_0\|^3} + m_2 \frac{x_2 - x_0}{\|x_2 - x_0\|^3} \\ \ddot{x}_1 &= m_2 \frac{x_2 - x_1}{\|x_2 - x_1\|^3} + m_0 \frac{x_0 - x_1}{\|x_0 - x_1\|^3} \\ \ddot{x}_2 &= m_0 \frac{x_0 - x_2}{\|x_0 - x_2\|^3} + m_1 \frac{x_1 - x_2}{\|x_1 - x_2\|^3} , \end{aligned} \tag{1}$$

where dots denote differentiation with respect to time t.

We introduce relative coordinates $r_j = x_j - x_0,\ j = 1, 2$, from which the inertial coordinates may be recovered via

$$x_0 = -\frac{1}{M} \sum_{j=1}^{2} m_j r_j\,, \quad M = m_0 + m_1 + m_2\ .$$

Subtracting the first equation of (1) from the second and third equation yields the equivalent system

$$\begin{aligned} \ddot{r}_1 + (m_0 + m_1) \frac{r_1}{\|r_1\|^3} &= m_2 \left(\frac{r_2 - r_1}{\|r_2 - r_1\|^3} - \frac{r_2}{\|r_2\|^3} \right) \\ \ddot{r}_2 + (m_0 + m_2) \frac{r_2}{\|r_2\|^3} &= m_1 \left(-\frac{r_2 - r_1}{\|r_2 - r_1\|^3} - \frac{r_1}{\|r_1\|^3} \right) , \end{aligned} \tag{2}$$

which describes a system of two perturbed Kepler motions with weak coupling if the masses satisty the above hierarchy and none of the distances $\|r_1\|\,,\ \|r_2\|\,,\ \|r_2 - r_1\|$ is small.

2. Levi-Civita Regularization of Perturbed Kepler Motion

We first restrict ourselves to the two-dimensional case and take advantage of the fact that Levi-Civita's regularizing transformation [11] has the agreeable property of transforming perturbed Kepler problems into perturbed harmonic oscillators, i.e. into perturbed *linear* problems. For a recent account of regularization theory see the article [2] and other contributions in the same volume.

We will use both vector notation $x = (x_1, x_2)^T \in \mathbb{R}^2$ and complex notation $\mathbf{x} = x_1 + i\,x_2 \in \mathbb{C}$ for convenience. Consider now the perturbed Kepler problem

$$\ddot{x} + \mu\,\frac{x}{\|x\|^3} = f(x,t)\ , \tag{3}$$

where dots denote derivarives with respect to time t, and $f(x,t)$ is a small perturbation. The corresponding energy equation is obtained by integrating the dot product $\langle \dot{x}, (3)\rangle$ with respect to t:

$$\frac{1}{2}\,\|\dot{x}\|^2 - \frac{\mu}{r} = -h\ , \tag{4}$$

where $r := \|x\|$ is the distance of the moving particle from the origin and h is the energy satisfying the differential equation and initial condition

$$\frac{dh}{dt} = -\langle \dot{x}, f\rangle\ , \quad h(0) = \frac{\mu}{\|x(0)\|} - \frac{1}{2}\,\|\dot{x}(0)\|^2\ . \tag{5}$$

The **first step** of Levi-Civita's regularization consists of introducing the fictitious time τ by the differential relation $dt = r \cdot d\tau$ (differentiation with respect to τ will be denoted by primes). In view of the step to follow we write the result of transforming Equ. (3) in complex form, where $f = (f_1, f_2)^T$, $\mathbf{f} = f_1 + i\,f_2$:

$$r\,\mathbf{x}'' - r'\,\mathbf{x}' + \mu\,\mathbf{x} = r^3\,\mathbf{f}\ \in \mathbb{C}\ . \tag{6}$$

The **second step** of Levi-Civita's regularization consists of representing the complex physical coordinate $\mathbf{x}$ as the square $\mathbf{u}^2$ of a compex variable $\mathbf{u} = u_1 + i\,u_2 \in \mathbb{C}$,

$$\mathbf{x} \;=\; \mathbf{u}^2\ , \tag{7}$$

i.e. the mapping from the parametric plane to the physical plane is chosen as a conformal squaring. This implies

$$r \;=\; |\,\mathbf{x}\,| \;=\; |\,\mathbf{u}\,|^2 \;=\; \mathbf{u}\,\bar{\mathbf{u}}\;, \tag{8}$$

and differentiation of the last two equations yields

$$\mathbf{x}' = 2\,\mathbf{u}\,\mathbf{u}'\;,\quad \mathbf{x}'' = 2\,\left(\mathbf{u}\,\mathbf{u}'' + \mathbf{u}'^{\,2}\right) \in \mathbb{C}\;,\quad r' = \mathbf{u}'\,\bar{\mathbf{u}} \;+\; \mathbf{u}\,\bar{\mathbf{u}}'\;. \tag{9}$$

By substituting this into (6) we obtain

$$2\,r\,\mathbf{u}\,\mathbf{u}'' + \mathbf{u}^2\,\left(\mu - 2\,|\mathbf{u}'|^2\right) = r^3\,\mathbf{f}\;, \tag{10}$$

where the two terms $2\,r\,\mathbf{u}'^2 = 2\,\mathbf{u}'\,\bar{\mathbf{u}}\,\mathbf{u}\,\mathbf{u}'$ have cancelled out.

Remark. Obtaining initial values $\mathbf{u}(0) = \sqrt{\mathbf{x}(0)}$ requires the computation of a complex square root. This can conveniently be accomplished by means of the formula

$$\sqrt{\mathbf{x}} = \frac{\mathbf{x} + |\mathbf{x}|}{\sqrt{2\,\left(|\mathbf{x}| + \Re\,\mathbf{x}\right)}}\;, \tag{11}$$

which reflects the observation that the complex vector $\sqrt{\mathbf{x}}$ has the direction of the bisector between $\mathbf{x}$ and the real vector $|\mathbf{x}|$; it holds in the range $-\pi < \arg(\mathbf{x}) < \pi$. The alternate formula

$$\sqrt{\mathbf{x}} = \frac{\mathbf{x} - |\mathbf{x}|}{i\,\sqrt{2\,\left(|\mathbf{x}| - \Re\,\mathbf{x}\right)}}$$

holds in $0 < \arg(\mathbf{x}) < 2\pi$ and agrees with (11) in the upper half plane.

The **third step** of Levi-Civita's regularization process produces linear differential equations for the unperturbed problem $\mathbf{f} = 0$ by combining Equ. (10) with the energy relation. By using $\dot{\mathbf{x}} = \frac{1}{r}\cdot 2\,\mathbf{u}\mathbf{u}'$ Equ. (4) becomes

$$\mu - 2\;|\mathbf{u}'|^{\;2} \;=\; r\,h\;; \tag{12}$$

therefore the perturbed Kepler problem (3) is equivalent with

$$\begin{aligned} 2\,\mathbf{u}'' \;+\; h\cdot\mathbf{u} \;&=\; r\,\bar{\mathbf{u}}\,\mathbf{f} \quad \text{where} \quad \mathbf{x} = \mathbf{u}^2 \;\in \mathbb{C} \\ h' \;&=\; -\langle x', f\rangle\;, \end{aligned} \tag{13}$$

as is seen by substituting (12) into (10) and dividing by $r\,\mathbf{u}$, using (8). Also, Equ. (5) for h has been added in order to obtain a complete system of differential equations for the dependent variables $\mathbf{u} \in \mathbb{C},\ h \in \mathbb{R}$.

The following cases are of particular interest:

1. $f = 0 \Longrightarrow\ h = h(0) = \text{const.}$ Equ. (13) describes a harmonic (linear) oscillator in two dimensions.

2. f has a potential V, $f = -\operatorname{grad} V \Longrightarrow h(x) = h(0) + V(x) - V(0)$. Equ. (13) describes a perturbed harmonic oscillator with varying frequency.

3. $f = O(\varepsilon),\ \varepsilon \to 0\ \Longrightarrow h(x) = h(0) + O(\varepsilon)$. Equ. (13) describes a perturbed harmonic oscillator with slowly varying frequency.

3. Regular Elements

We will now take advantage of the linear structure of the unperturbed version $f = 0$ of Equs. (13). Consider, as a model problem, the perturbed harmonic oscillator

$$\frac{d^2u}{d\tau^2} + \omega^2\, u = F\ , \tag{14}$$

where F is small, and ω is slowly varying. First, we transform the perturbed oscillator (14) to constant frequency by introducing the new independent variable E according to the differential relation

$$dE = \omega\, d\tau\ , \quad \frac{d}{d\tau} = \omega\, \frac{d}{dE}\ , \quad \frac{d^2}{d\tau^2} = \omega^2\, \frac{d^2}{dE^2} + \omega'\, \frac{d}{dE}\ , \tag{15}$$

where primes – in this section – denote derivatives with respect to E ($2E$ is the *eccentric anomaly* of the osculating Kepler motion). Equ. (14) now becomes

$$\frac{d^2u}{dE^2} + u = G \qquad \text{with} \qquad G = \frac{F - \omega'\, du/dE}{\omega^2}\ . \tag{16}$$

We now discuss two ways of introducing regular elements to Equ. (16):

3.1. Variation of the constant.

With the notation $v := du/dE$ Equ. (16) may be written as the vector differential equation

$$\begin{pmatrix} u \\ v \end{pmatrix}' = A \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ G \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{17}$$

Departing from the matrix solution

$$U(E) = \begin{pmatrix} u(E) \\ v(E) \end{pmatrix} = \begin{pmatrix} \cos E & \sin E \\ -\sin E & \cos E \end{pmatrix} \tag{18}$$

satisfying the unperturbed equation $U' = A\,U$, the method of varying the constant consists of seeking a solution of (17) of the form

$$\begin{pmatrix} u(E) \\ v(E) \end{pmatrix} = U(E) \begin{pmatrix} \alpha(E) \\ \beta(E) \end{pmatrix} , \tag{19}$$

where $\alpha(E), \beta(E)$ are the (orbital) elements. Substituting this into (17) and solving for the derivatives of the elements yields

$$\begin{aligned} \frac{d\vec{\alpha}}{dE} &= -\vec{G} \cdot \sin E \\ \frac{d\vec{\beta}}{dE} &= \vec{G} \cdot \cos E . \end{aligned} \tag{20}$$

Here we have used vector symbols in order to indicate that Equs. (20) not only hold for scalars $\alpha, \beta, G \in \mathbb{R}$, but also for vectors $\vec{\alpha}, \vec{\beta}, \vec{G} \in \mathbb{R}^n$.

3.2. Singular-value decomposition.

This set of elements is based on the original perturbation problem (14), now written for vectors $\vec{u}, \vec{F} \in \mathbb{R}^n$, here with $n = 2$,

$$\frac{d^2\vec{u}}{d\tau^2} + \omega^2\,\vec{u} = \vec{F} . \tag{21}$$

As it is often done, we define the *osculating orbit* at the fixed value $\tau = \tau_0$ as the orbit determined by Equ. (21) for $\tau \geq \tau_0$ if ω and $\vec{F}$ are fixed at the

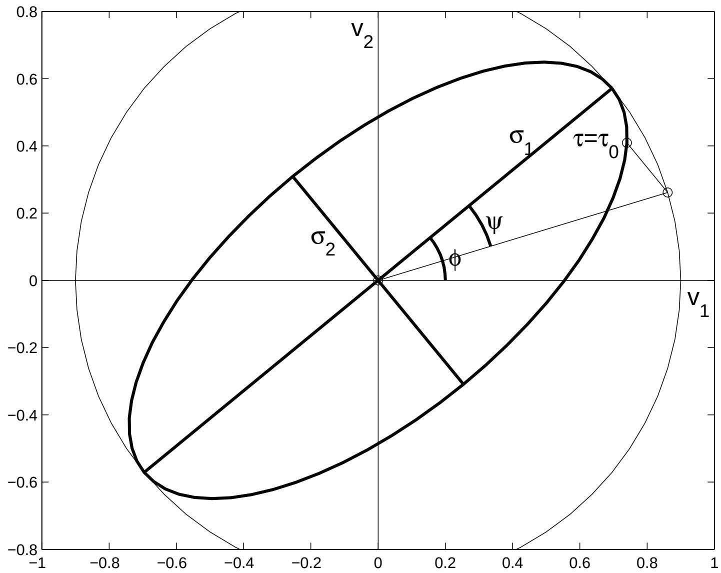


Figure 1: The harmonic elements $\sigma_1, \sigma_2, \varphi, \psi$. The axes are v_1, v_2

values $\omega_0 = \omega(\tau_0)$, $\vec{F}_0 = \vec{F}(\tau_0)$ for all $\tau \geq \tau_0$. It is convenient to shift the origin by introducing the new coordinate $\vec{v}$ according to $\vec{u} = \vec{v} + \vec{F}_0/\omega_0^2$; $\vec{v}$ satisfies

$$\frac{d^2\vec{v}}{d\tau^2} + \omega_0^2\,\vec{v} = 0 \tag{22}$$

for $\tau \geq \tau_0$. Any four quantities uniquely charactrizing the solution of (22) may be used as orbital elements, e.g. the initial values $\vec{v}_0 = \vec{v}(\tau_0)$, $\vec{v}\,'_0 = \vec{v}\,'(\tau_0)$ at $\tau = \tau_0$. With these initial values, the solution of (22) is

$$\vec{v}(\tau) = \vec{v}_0\,\cos(\omega_0\tilde{\tau}) + \frac{1}{\omega_0}\,\vec{v}\,'_0\,\sin(\omega_0\tilde{\tau})\,, \quad \tilde{\tau} = \tau - \tau_0\,, \tag{23}$$

or, by representing $\vec{v}(\tilde{\tau}) = (v_1(\tilde{\tau}),\, v_2(\tilde{\tau}))^T$ in components and using matrix notation:

$$\begin{pmatrix} v_1(\tilde{\tau}) \\ v_2(\tilde{\tau}) \end{pmatrix} = M \begin{pmatrix} \cos(\omega_0\tilde{\tau}) \\ \sin(\omega_0\tilde{\tau}) \end{pmatrix}, \quad M = \begin{pmatrix} v_{10} & v'_{10}/\omega_0 \\ v_{20} & v'_{20}/\omega_0 \end{pmatrix} \tag{24}$$

with $v_{10}, v_{20}, v'_{10}, v'_{20}$ being the components of $\vec{v}_0$ and $\vec{v}\,'_0$, respectively.

The osculating orbit (24) is an ellipse centered at $\vec{v} = 0$, or $\vec{u} = \vec{F}_0/\omega_0^2$. A more natural choice of orbital elements than $\vec{v}_0, \vec{v}'_0$ are four geometric parameters of the ellipse (24). We suggest to use the singular-value decomposition (SVD)

$$M = U\,S\,V^T \tag{25}$$

of the matrix M in (24), where U, V are orthogonal and S is diagonal,

$$U = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \; V = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}, \; S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \tag{26}$$

with nonnegative singular values $\sigma_1 \geq \sigma_2 \geq 0$. The two quantities σ_1, σ_2 (the semi-axes of the osculating ellipse) and the two angles φ, ψ will be referred to as the *harmonic elements* of the perturbed oscillator (21). The geometric meaning of the angles φ, ψ is shown in Figure 1: φ is the angle of rotation of the axes of the ellipse with respect to fixed axes, ψ is related to the position of the moving point on the ellipse corresponding to $\tau = \tau_0$.

4. Weakly Coupled Harmonic Oscillators

In order to apply one of the proposed sets of elements for describing coorbital motion we formulate the equations of motion (2) of the weakly coupled Kepler motions in terms of the Levi-Civita coordinates of Section 2. In this way the unperturbed problem will be defined by *linear* differential equations. Using the symbols $\mu_j = m_0 + m_j$, $j = 1, 2$ as well as complex notation $\mathbf{r}_1, \mathbf{r}_2 \in \mathbb{C}$ and the abbreviations $\mathbf{f}_1, \mathbf{f}_2 \in \mathbb{C}$ for the right-hand sides, Equ. (2) reads as

$$\begin{aligned} \ddot{\mathbf{r}}_1 + \mu_1 \frac{\mathbf{r}_1}{|\mathbf{r}_1|^3} &= \mathbf{f}_1 \\ \ddot{\mathbf{r}}_2 + \mu_2 \frac{\mathbf{r}_2}{|\mathbf{r}_2|^3} &= \mathbf{f}_2\,. \end{aligned} \tag{27}$$

For $j = 1, 2$ we will introduce the individual fictitious times τ_j and Levi-Civita's complex coordinates $\mathbf{u}_j$ as well as the derivatives $\mathbf{v}_j = 2\,d\mathbf{u}_j/d\tau_j$ (the factor 2 is for convenience), and the energies h_j. According to (13), (8) we obtain for $j = 1, 2$

$$\begin{aligned} \frac{d\,\mathbf{u}_j}{d\tau_j} &= \frac{\mathbf{v}_j}{2} \\ \frac{d\,\mathbf{v}_j}{d\tau_j} &= -\,h_j\,\mathbf{u}_j + |\mathbf{u}_j|^2\ \bar{\mathbf{u}}_j\ \mathbf{f}_j \\ \frac{d\,h_j}{d\tau_j} &= -\,\Re\left(\bar{\mathbf{u}}_j\,\bar{\mathbf{v}}_j\ \mathbf{f}_j\right) \\ \frac{d\,t}{d\tau_j} &= |\,\mathbf{u}_j\,|^2\ . \end{aligned} \tag{28}$$

The inconvenience of two individual fictitious times τ_1, τ_2 is easily circumvented by going back to physical time t as independent variable in both oscillators $j = 1, 2$ (by using the last equations of (28)):

$$\begin{aligned} \frac{d\,\mathbf{u}_j}{d\,t} &= \frac{\mathbf{v}_j}{2\ |\,\mathbf{u}_j\,|^2} \\ \frac{d\,\mathbf{v}_j}{d\,t} &= -\,\frac{h_j}{\bar{\mathbf{u}}_j} + \bar{\mathbf{u}}_j\ \mathbf{f}_j\ , \quad j = 1, 2 \\ \frac{d\,h_j}{d\,t} &= -\,\Re\left(\frac{\bar{\mathbf{v}}_j}{\mathbf{u}_j}\ \mathbf{f}_j\right)\ . \end{aligned} \tag{29}$$

In the near-circular case we have $|\mathbf{u}_j| \approx$ const, and the osculating elements of Section 3.2 can be used.

Consider, as an example, the planar motion of two small satellites m_1, m_2 under the influence of the large central body m_0. We assume the satellites to be initially on nearly identical circular osculating orbits of radii $\rho_1 \approx \rho_2$ with velocities $s_j = \sqrt{M/\rho_j}$, $j = 1, 2$, where $M = m_0 + m_1 + m_2$ is the total mass of the system. Assuming opposite initial postions of the satellites with respect to the central body yields the position and velocity vectors

$$r_1 = (\rho_1, 0)^T, \quad r_2 = (-\rho_2, 0)^T, \quad \dot{r}_1 = (0, s_1)^T, \quad \dot{r}_2 = (0, -s_2)^T\ ,$$

to be used as initial conditions for the equations of motion (2). The choice $m_0 = 777$, $m_1 = 1$, $m_2 = 2$, $\rho_1 = 1.01$, $\rho_2 = 1.00$, which is the basis of the orbits shown in Figure 2, causes the considered system to display the typical behaviour of coorbital motion. The mass ratios and geometry are such that a few close encounters of the satellites with the well known orbit exchanges occur before the system suddenly breaks up.

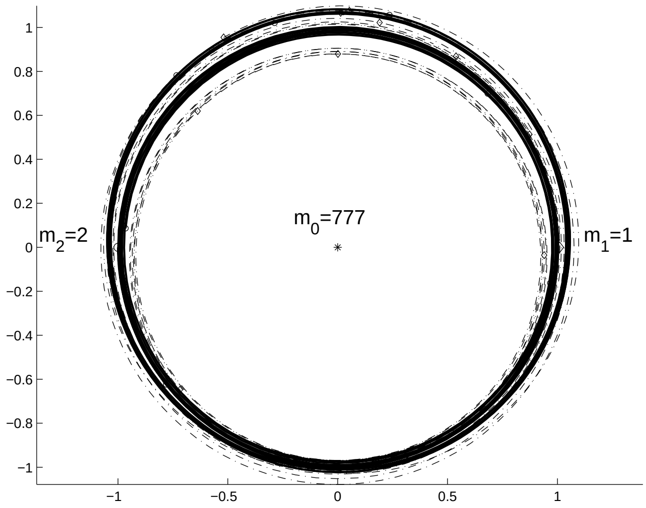


Figure 2: A few revolutions of the satellites $m_1 = 1$ (diamonds, dashdotted) and $m_2 = 2$ (circles, solid line) about the central body $m_0 = 777$, shown in heliocentric coordinates. Initial distances $\rho_1 = 1.01, \rho_2 = 1.00$, opposite starting points, circular initial velocities

Whereas Figure 2 hardly shows any structure, the behaviour of the harmonic elements clearly reflects the dynamics of the orbits of the two satellites. In Figure 3 the semi-axes σ_1, σ_2 of the ellipses (24) associated with the two satellites are plotted versus time, the thin lines corresponding to the smaller satellite, m_1. The wiggles correspond to the near-Keplerian revolutions around the central body when the satellites are far appart; the motion is quite orderly. If the satellites have a close encounter the harmonic elements change dramatically, corresponding to the transition into new near-Keplerian orbits. This process repeats in a more or less regular way, with an increasing tendency towards chaos, however. When deviations from this pattern have sufficiently accumulated – here after 5 close encounters of the satellites – the orderly motion ceases: *Order and chaos in satellite encounters.* The reader is referred to the wide literature on this topic, beginning with [8] and [4]; a more extended list of references may be found in [18].

The derivation of the perturbation equations for the harmonic elements and the development of a perturbation theory based on these elements will not be discussed here.

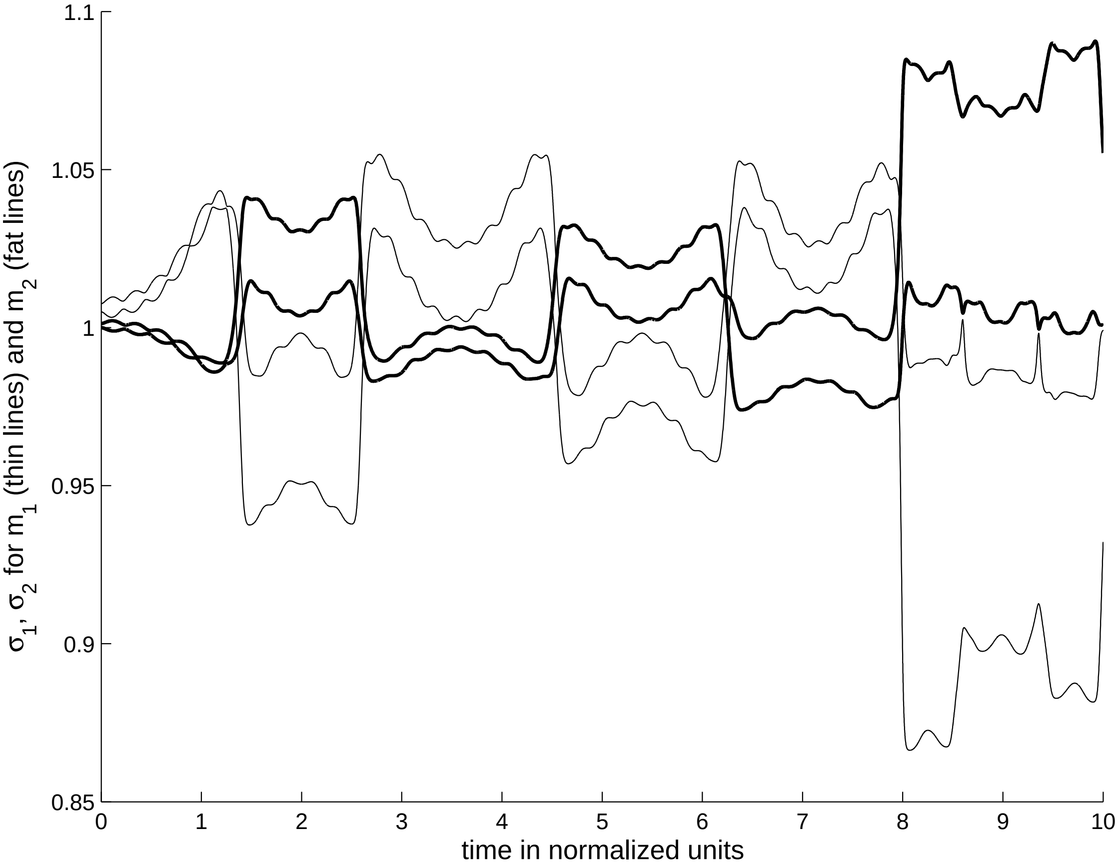


Figure 3: The harmonic elements σ_1, σ_2 of the orbits of Figure 2. The fat lines refer to the larger satellite $m_2 = 2$

5. Quaternion Algebra

In the remaining sections we indicate how the ideas discussed above may be generalized to three-dimensional motion. The essential step is to replace Levi-Civita's regularization with the Kustaanheimo-Stiefel (KS) regularization, described in the original papers [7] and [6], and extensively discussed in the comprehensive text [10]. The relevant mapping from the 3-sphere onto the 2-sphere was discovered already in 1931 by Heinz Hopf [5] and is referred to in topology as the Hopf mapping.

Both the Levi-Civita and the Kustaanheimo-Stiefel regularization share the property of "linearizing" the equations of motion of the two-body problem. Quaternion algebra, introduced by W. R. Hamilton in 1856 [3], was originally rejected [10, p. 286] as a tool for regularizing the three-dimensional Kepler motion. Here we will present a new elegant way of extending the Levi-Civita regularization to three dimensions by means of quaternions. Similar techniques were used earlier by M. D. Vivarelli [12] and J. Vrbik [13, 14, 15].

5.1. Basics.

Quaternion algebra is a generalization of the algebra of complex numbers obtained by using three independent "imaginary" units i, j, k. As for the single imaginary unit i in the algebra of complex numbers, the rules

$$i^2 = j^2 = k^2 = -1$$

are postulated, together with the non-commutative multiplication rules

$$i\,j = -j\,i = k\,, \quad j\,k = -k\,j = i\,, \quad k\,i = -i\,k = j\,.$$

Given real numbers $u_l \in \mathbb{R}\,,\ l = 0, 1, 2, 3$, the object

$$\mathbf{u} = u_0 + i\,u_1 + j\,u_2 + k\,u_3 \tag{30}$$

is called a *quaternion* $\mathbf{u} \in \mathbb{U}$, where $\mathbb{U}$ denotes the set of all quaternions (in the remaining sections bold-face characters denote quaternions). The sum $iu_1 + ju_2 + ku_3$ is called the *quaternion part* of $\mathbf{u}$, whereas u_0 is naturally referred to as its real part. The above multiplication rules and vector space addition define the *quaternion algebra.* Multiplication is generally non-commutative; however, any quaternion commutes with a real:

$$c\,\mathbf{u} = \mathbf{u}\,c\,, \quad c \in \mathbb{R}\,, \ \mathbf{u} \in \mathbb{U}\,, \tag{31}$$

and for any three quaternions $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{U}$ the associative law holds:

$$(\mathbf{u}\,\mathbf{v})\,\mathbf{w} = \mathbf{u}\,(\mathbf{v}\,\mathbf{w})\,. \tag{32}$$

The quaternion $\mathbf{u}$ may naturally be associated with the corresponding vector $u = (u_0, u_1, u_2, u_3) \in \mathbb{R}^4$. For later reference we introduce notation for 3-vectors in two important particular cases: $\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ for the vector associated with the *pure quaternion* $\mathbf{u} = i\,u_1 + j\,u_2 + k\,u_3$, and $\underline{u} = (u_0, u_1, u_2)$ for the vector associated with the quaternion with a vanishing fourth component, $\mathbf{u} = u_0 + i\,u_1 + j\,u_2$.

For convenience we also introduce the vector $\vec{\imath} = (i, j, k)$; the quaternion $\mathbf{u}$ may then formally be written as $\mathbf{u} = u_0 + \langle \vec{\imath}, \vec{u} \rangle$. For the two quaternion products of $\mathbf{u}$ with $\mathbf{v} = v_0 + \langle \vec{\imath}, \vec{v} \rangle$ we then obtain the concise expressions

$$\begin{aligned} \mathbf{u}\,\mathbf{v} &= u_0 v_0 - \langle \vec{u}, \vec{v} \rangle + \langle \vec{\imath},\, u_0\,\vec{v} + v_0\,\vec{u} + \vec{u} \times \vec{v} \rangle \\ \mathbf{v}\,\mathbf{u} &= u_0 v_0 - \langle \vec{u}, \vec{v} \rangle + \langle \vec{\imath},\, u_0\,\vec{v} + v_0\,\vec{u} - \vec{u} \times \vec{v} \rangle\,, \end{aligned} \tag{33}$$

where $\times$ denotes the vector product. Note that the non-commutativity shows only in the sign of the term with the vector product.

The *conjugate* $\bar{\mathbf{u}}$ of the quaternion $\mathbf{u}$ is defined as

$$\bar{\mathbf{u}} = u_0 - i\,u_1 - j\,u_2 - k\,u_3\,; \tag{34}$$

then the *modulus* $|\mathbf{u}|$ of $\mathbf{u}$ is obtained from

$$|\mathbf{u}|^2 = \mathbf{u}\,\bar{\mathbf{u}} = \bar{\mathbf{u}}\,\mathbf{u} = \sum_{l=0}^{3} u_l^2\,. \tag{35}$$

As transposition of a product of matrices, conjugation of a quaternion product reverses the order of its factors:

$$\overline{\mathbf{u}\,\mathbf{v}} = \bar{\mathbf{v}}\,\bar{\mathbf{u}}\,. \tag{36}$$

5.2. Rotations in Three Dimensions.

This is a short digression in order to demonstrate the elegance of quaternion representations in three-dimensional geometry. Let $\vec{x}$ be a vector of the Euclidean 3-space $\mathbb{R}^3$, and consider the right-handed rotation about the unit vector $\vec{a} = (a_1, a_2, a_3)^T$, $|\vec{a}| = 1$ through the angle ω. One way of representing the mapping

$$\vec{x} \in \mathbb{R}^3 \mapsto \vec{y} = T\,\vec{x} \tag{37}$$

is to use Cayley's parametrization

$$T = \frac{I \cdot \cos\frac{\omega}{2} + S(\vec{a}) \cdot \sin\frac{\omega}{2}}{I \cdot \cos\frac{\omega}{2} - S(\vec{a}) \cdot \sin\frac{\omega}{2}} \tag{38}$$

of the orthogonal matrix T. Here I is the unit matrix, and matrix "division" may be interpreted as multiplication with the inverse of the denominator (if it exists) from the left *or* from the right. The skew symmetric matrix

$$S(\vec{a}) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

is the *vector product matrix* associated with $\vec{a}$, i.e. for every column vector $\vec{x} \in \mathbb{R}^3$ we have $S(\vec{a})\,\vec{x} = \vec{a} \times \vec{x}$.

A *proof* of (38) may be obtained by considering the relation

$$\cos\frac{\omega}{2} \cdot (\vec{y} - \vec{x}) = \sin\frac{\omega}{2} \cdot \vec{a} \times (\vec{y} + \vec{x})\,, \tag{39}$$

which is equivalent with (37), (38) and may be obtained by multiplying (37) from the left by the denominator of (38). Equ. (39), in turn, exactly reflects the geometry of the rotation under consideration, as is easily deduced from Figure 4, in particular from the triangle CMB.

The mapping (37) with T from (38) will now be written as a relation between the pure quaternions $\mathbf{x} = \langle \vec{\imath}, \vec{x} \rangle$, $\mathbf{y} = \langle \vec{\imath}, \vec{y} \rangle$.

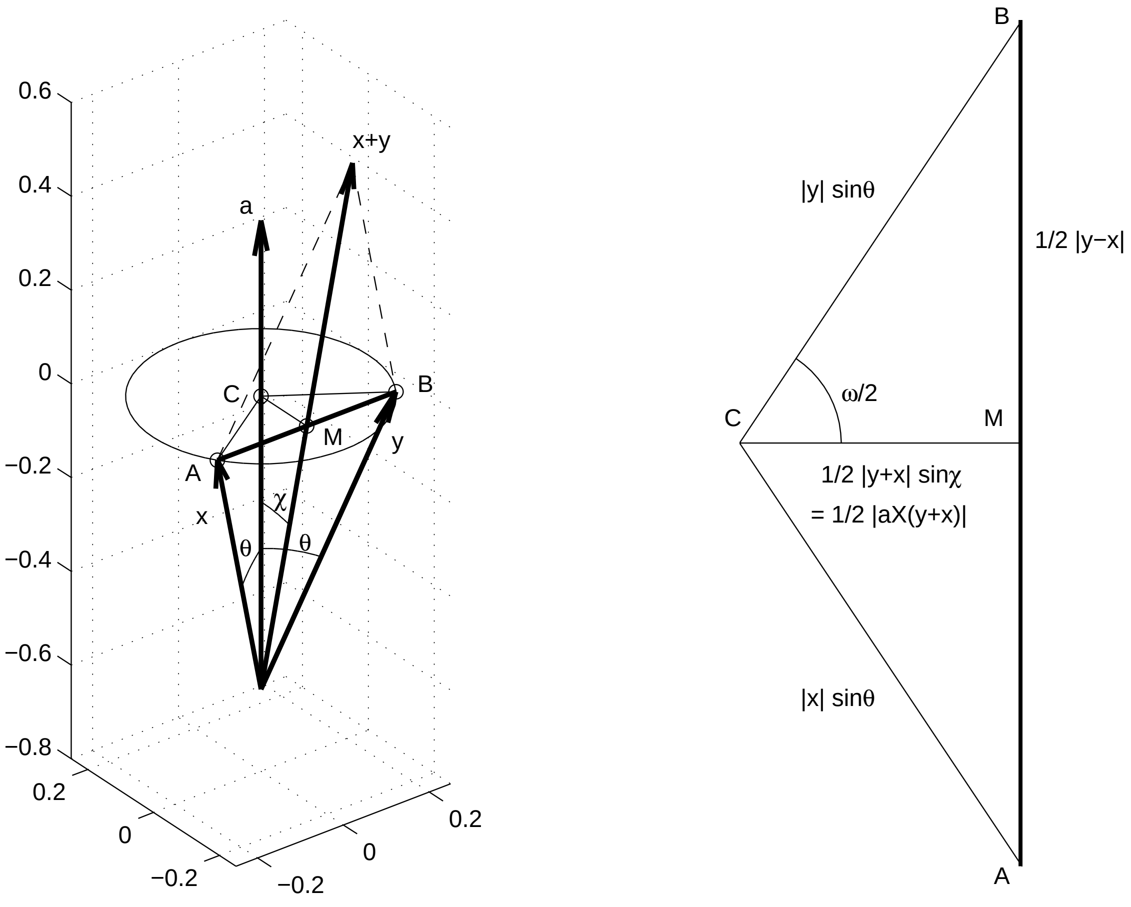


Figure 4: Geometric proof of Equ. (39) for the rotation about the axis $\vec{a}$, $|\vec{a}| = 1$ through the angle ω. Equ. (39) follows from the geometry of the triangle CMB

Theorem 1. Let $\vec{a} \in \mathbb{R}^3$ be a unit vector and define the unit quaternion

$$\mathbf{r} := \cos\frac{\omega}{2} + \langle \vec{\imath}, \vec{a} \rangle \sin\frac{\omega}{2}\,.$$

Furthermore, let $\vec{x} \in \mathbb{R}^3$ be an arbitrary vector and $\mathbf{x} = \langle \vec{\imath}, \vec{x} \rangle$ the associated pure quaternion. Then the map

$$\mathbf{x} \mapsto \mathbf{y} = \mathbf{r}\,\mathbf{x}\,\mathbf{r}^{-1}$$

describes the right-handed rotation of $\vec{x}$ about the axis $\vec{a}$ through the angle ω. □

Remark. Since $\mathbf{r}$ is a unit quaternion we have $\mathbf{r}^{-1} = \bar{\mathbf{r}}$.

Proof. Show the equivalent statement $\mathbf{y}\,\mathbf{r} = \mathbf{r}\,\mathbf{x}$ by means of the multiplication rules (33) and Equ. (39). □

6. The KS Transformation in Quaternions

In this section we will revisit KS regularization and present a new, elegant derivation of it, using quaternion algebra and an unconventional "conjugate" $\mathbf{u}^\star$ referred to as the *star conjugate* of the of the quaternion $\mathbf{u} = u_0 + i\,u_1 + j\,u_2 + k\,u_3$:

$$\mathbf{u}^\star := u_0 + i\,u_1 + j\,u_2 - k\,u_3\,. \tag{40}$$

The star conjugate of $\mathbf{u}$ may be expressed in terms the conventional conjugate $\bar{\mathbf{u}}$ as

$$\mathbf{u}^\star = k\,\bar{\mathbf{u}}\,k^{-1} = -k\,\bar{\mathbf{u}}\,k\,;$$

however, it turns out that the definition (40) leads to a particularly elegant treatment of KS regularization. The following elementary properties are easily verified:

$$\begin{aligned} (\mathbf{u}^\star)^\star &= \mathbf{u}\,. \\ |\,\mathbf{u}^\star|^2 &= |\mathbf{u}|^2 \\ (\mathbf{u}\,\mathbf{v})^\star &= \mathbf{v}^\star\,\mathbf{u}^\star \end{aligned} \tag{41}$$

Consider now the mapping

$$\mathbf{u} \in \mathbb{U} \;\longmapsto\; \mathbf{x} = \mathbf{u}\,\mathbf{u}^\star\,. \tag{42}$$

Star conjugation immediately yields $\mathbf{x}^\star = (\mathbf{u}^\star)^\star\,\mathbf{u}^\star = \mathbf{x}$; hence $\mathbf{x}$ is a quaternion of the form $\mathbf{x} = x_0 + i\,x_1 + j\,x_2$ which may be associated with the vector $\underline{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$. From $\mathbf{u} = u_0 + i\,u_1 + j\,u_2 + k\,u_3$ we obtain

$$\begin{aligned} x_0 &= u_0^2 - u_1^2 - u_2^2 + u_3^2 \\ x_1 &= 2(u_0\,u_1 - u_2\,u_3) \\ x_2 &= 2(u_0\,u_2 + u_1\,u_3)\ , \end{aligned} \tag{43}$$

which is exactly the KS transformation in its classical form or – up to a permutation of the indices – the Hopf map. Therefore we have

Theorem 2: The KS transformation which maps $u = (u_0, u_1, u_2, u_3) \in \mathbb{R}^4$ to $\underline{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ is given by the quaternion relation

$$\mathbf{x} = \mathbf{u}\,\mathbf{u}^\star\,,$$

where $\mathbf{u} = u_0 + i\,u_1 + j\,u_2 + k\,u_3\,,\ \mathbf{x} = x_0 + i\,x_1 + j\,x_2$. □

Corollary: The norms of the vectors u and $\underline{x}$ satisfy $\|\underline{x}\| = \|u\|^2$. □

Proof: By appropriately combining the two conjugations and using the rules (31), (32), (35), (36), (41) we obtain

$$\|\underline{x}\|^2 = \mathbf{x}\,\bar{\mathbf{x}} = \mathbf{u}\,(\mathbf{u}^\star\,\bar{\mathbf{u}}^\star)\,\bar{\mathbf{u}} = |\mathbf{u}^\star|^2\,|\mathbf{u}\,|^2 = |\mathbf{u}\,|^4 = \|u\|^4\ ,$$

from where the statement follows. □

As a side step, we will briefly discuss two topics not directly related to our primary objective, perturbation theory. Both the inverse map and the Birkhoff transformation in three dimensions allow for an elegant treatment in terms of quaternions.

6.1. The Inverse Map.

Being given a quaternion $\mathbf{x} = x_0 + i\,x_1 + j\,x_2$ with vanishing fourth component, $x_3 = 0$, we want to find all quaternions $\mathbf{u}$ such that $\mathbf{u}\,\mathbf{u}^\star = \mathbf{x}$. We propose the following solution in two steps:

First step: Find a particular solution $\mathbf{u} = \mathbf{v} = \mathbf{v}^\star = v_0 + i\,v_1 + j\,v_2$ which has also a vanishing fourth component. Since $\mathbf{v}\,\mathbf{v}^\star = \mathbf{v}^2$ we may use Equ. (11), which was developed for the complex square root, also for the square root of a quaternion:

$$\mathbf{v} = \frac{\mathbf{x} + |\mathbf{x}|}{\sqrt{2\,(|\mathbf{x}| + x_0)}}\,.$$

Clearly, $\mathbf{v}$ has a vanishing fourth component.

Second step: The entire family of solutions (geometrically a circle in $\mathbb{R}^4$), parametrized by the angle φ, is given by

$$\mathbf{u} = \mathbf{v} \cdot e^{k\,\varphi} = \mathbf{v}\,(\cos\varphi + k\sin\varphi)\,.$$

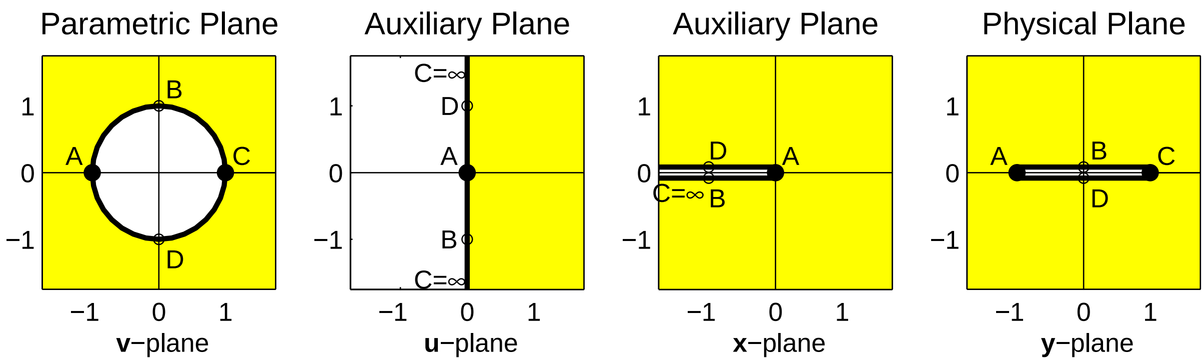
Proof. $\mathbf{u}\,\mathbf{u}^\star = \mathbf{v}\,e^{k\,\varphi}\,e^{-k\,\varphi}\,\mathbf{v}^\star = \mathbf{v}\,\mathbf{v}^\star = \mathbf{x}\,.$ □

6.2. The Birkhoff Transformation.

This regularizing transformation was proposed in 1915 by George David Birkhoff [1] in order to regularize all singularities of the planar restricted three-body problem with a single transformation. In 1965 E. Stiefel and this author [9] published a generalization of Birkhoff's transformation to three dimensions, using the KS transformation. Later these ideas resulted in the publications [16] and [17].

Here we will first revisit the classical Birkhoff transformation (the same conformal map is known in aerodynamics as the Joukowsky transformation) and represent it as the composition of three conformal mappings; this will then readily generalize to the spatial situation by means of quaternions.

Consider a rotating physical plane parametrized by the complex variable $\mathbf{y} \in \mathbb{C}$; for convenience we assume the fixed primaries of the restricted three-body problem to be situated at the points A, C given by the complex posititons $\mathbf{y} = -1$ and $\mathbf{y} = 1$, respectively (see Figure 5). The complex variable of the parametric plane will be denoted by $\mathbf{v}$ and will be normalized in such a way that the primaries are mapped to $\mathbf{v} = -1$ or $\mathbf{v} = 1$, respectively.



$$\mathbf{v} \mapsto \mathbf{u} = \frac{\mathbf{v}+1}{\mathbf{v}-1} \qquad \mathbf{u} \mapsto \mathbf{x} = \mathbf{u}^2 \qquad \mathbf{x} \mapsto \mathbf{y} = \frac{\mathbf{x}+1}{\mathbf{x}-1}$$

Figure 5: The sequence of conformal maps generating the Birkhoff transformation

The key observation is that Levi-Civita's conformal map (7), $\mathbf{u} \mapsto \mathbf{x} = \mathbf{u}^2$, not only regularizes collisions at $\mathbf{x} = 0$ but also analogous singularities at $\mathbf{x} = \infty$. This is seen by closing the complex planes to become Riemann spheres (by adding the point at infinity) and using inversions $\mathbf{x} = 1/\tilde{\mathbf{x}}$, $\mathbf{u} = 1/\tilde{\mathbf{u}}$.

Taking advantage of this fact, we first map the $\mathbf{v}$-sphere to an auxiliary $\mathbf{u}$-sphere by the Möbius transformation

$$\mathbf{v} \longmapsto \mathbf{u} = \frac{\mathbf{v}+1}{\mathbf{v}-1} = 1 + \frac{2}{\mathbf{v}-1}\,, \tag{44}$$

which takes the primaries A, C to the points $\mathbf{u} = 0$, $\mathbf{u} = \infty$, respectively. The Levi-Civita map (7) will leave these points invariant while regularizing collisions at A or C. Finally, the Möbius transformation

$$\mathbf{x} \longmapsto \mathbf{y} = \frac{\mathbf{x}+1}{\mathbf{x}-1} = 1 + \frac{2}{\mathbf{x}-1} \tag{45}$$

maps A, C to $\mathbf{y} = -1$ and $\mathbf{y} = 1$, respectively. The composition of the maps (44), (7), (45) yields

$$\mathbf{y} = \frac{\left(\dfrac{\mathbf{v}+1}{\mathbf{v}-1}\right)^2 + 1}{\left(\dfrac{\mathbf{v}+1}{\mathbf{v}-1}\right)^2 - 1} \qquad \text{or} \qquad \mathbf{y} = \frac{1}{2}\left(\mathbf{v} + \frac{1}{\mathbf{v}}\right)\,, \tag{46}$$

the well known map used by Birkhoff.

In the spatial case we choose $\mathbf{v}, \mathbf{u}, \mathbf{x}, \mathbf{y} \in \mathbb{U}$ to be quaternions, $\mathbf{x} = \mathbf{x}^\star$, $\mathbf{y} = \mathbf{y}^\star$ being quaternions with vanishing fourth components, associated with 3-vectors $\underline{x}$, $\underline{y}$. Then the mappings (44), (45), being shifted inversions in 4 or 3 dimensions, are both conformal maps, in fact the only conformal maps existing in those dimensions, except for the translations, magnifications, and rotations. Composing these with the KS or Hopf map (42), $\mathbf{u} \mapsto \mathbf{x} = \mathbf{u}\,\mathbf{u}^\star$, yields

$$\mathbf{y} = 1 \; + \; (\mathbf{v}^\star - 1)\,(\mathbf{v} + \mathbf{v}^\star)^{-1}\,(\mathbf{v} - 1) \tag{47}$$

after a few lines of careful noncommutative algebra. This is easily split up into components by means of the inversion formula $1/\mathbf{v} = \bar{\mathbf{v}}/|\mathbf{v}|^2$; it agrees with the results given in [9] up to the sign of v_3. Both transformations regularize; the discrepancy is due to the different definition of the orientation in the inversions.

7. Perturbation Theory in Three Dimensions

In order to regularize the perturbed three-dimensional Kepler motion by means of the KS transformation it is necessary to look at the properties of the map (42) under differentiation.

7.1. Differentiation.

The transformation (42) is a mapping from $\mathbb{R}^4$ to $\mathbb{R}^3$; it therefore leaves one degree of freedom in the parametric space undetermined. In KS theory [7], [10], this freedom is taken advantage of by trying to inherit as much as possible of the conformality properties of the Levi-Civita map, but other approaches exist [15]. By imposing the "bilinear relation"

$$u_3\,du_0 - u_2\,du_1 + u_1\,du_2 - u_0\,du_3 = 0 \tag{48}$$

between the vector $u = (u_0, u_1, u_2, u_3)$ and its differential du on orbits the tangential map of (42) becomes a linear map with an orthogonal (but non-normalized) matrix.

This property has a simple consequence on the differentiation of the quaternion representation (42) of the KS transformation. Considering the noncommutative multiplication of quaternions, the differential of Equ. (42) becomes

$$d\,\mathbf{x} = d\,\mathbf{u} \cdot \mathbf{u}^\star + \mathbf{u} \cdot d\,\mathbf{u}^\star\,, \tag{49}$$

whereas (48) takes the form of a commutator relation,

$$\frac{1}{2}\;(-d\,\mathbf{u} \cdot \mathbf{u}^\star + \mathbf{u} \cdot d\,\mathbf{u}^\star) = 0\,. \tag{50}$$

Combining (49) with the relation (50) yields the elegant result

$$d\,\mathbf{x} = 2\;\mathbf{u} \cdot d\,\mathbf{u}^\star\,, \tag{51}$$

i.e. the bilinear relation (48) of KS theory is eqivalent with the requirement that the tangential map of $\mathbf{u} \mapsto \mathbf{u}\,\mathbf{u}^\star$ behaves like in a commutative algebra.

7.2. KS Regularization.

The procedure of Section 2 for regularizing the planar case now carries over almost identically to the spatial case; care must be taken to preserve the order of the factors in quaternion products. Changing the order is only permitted if one of the factors is real. Let $\mathbf{x} = x_0 + i\,x_1 + j\,x_2 \in \mathbb{U}$ be the quaternion associated with the vector $\underline{x} = (x_0, x_1, x_2)$; then the perturbed Kepler problem (3) is given by

$$\ddot{\mathbf{x}} + \mu\,\frac{\mathbf{x}}{r^3} = \mathbf{f}(\mathbf{x}, t) \in \mathbb{U}\,, \quad r = |\mathbf{x}|\,, \tag{52}$$

where $\mathbf{f}(\mathbf{x}, t) = f_0(\mathbf{x}, t) + i\,f_1(\mathbf{x}, t) + j\,f_2(\mathbf{x}, t) = \mathbf{f}^\star(\mathbf{x}, t)$ is the quaternion associated with the perturbation $\underline{f}(\underline{x}, t) \in \mathbb{R}^3$. The energy equation (4) becomes

$$\frac{1}{2}\;|\dot{\mathbf{x}}|^2 - \frac{\mu}{r} = -h\;, \tag{53}$$

whereas the result of the **first step**, i.e. introducing τ by $dt = r \cdot d\tau$ exactly agrees with Equ. (6),

$$r\;\mathbf{x}'' - \;r'\;\mathbf{x}' + \mu\;\mathbf{x} = r^3\,\mathbf{f}\;\;\in \mathbb{U}\;. \tag{54}$$

The relations (7), (8), (9) needed in the **second step** read as

$$\mathbf{x} = \mathbf{u}\,\mathbf{u}^\star\,, \qquad\qquad r = \mathbf{u}\,\bar{\mathbf{u}} \tag{55}$$

and

$$\mathbf{x}' = 2\,\mathbf{u}\,\mathbf{u}^{\star\,\prime}\,, \qquad \mathbf{x}'' = 2\,\mathbf{u}\,\mathbf{u}^{\star\,\prime\prime} + 2\,\mathbf{u}'\,\mathbf{u}^{\star\,\prime}\,, \qquad r' = \mathbf{u}'\,\bar{\mathbf{u}} + \mathbf{u}\,\bar{\mathbf{u}}'\,. \tag{56}$$

The energy equation (53) needed in the **third step** becomes

$$\frac{1}{2\,r^2}\,|\mathbf{x}'|^2 - \frac{\mu}{r} = -h \qquad \text{or} \qquad \frac{4}{2\,r^2}\,\mathbf{u}\,(\,\mathbf{u}^{\star\,\prime}\,\bar{\mathbf{u}}^{\star\,\prime}\,)\,\bar{\mathbf{u}} - \frac{\mu}{r} = -h$$

which results in the relation

$$\mu - 2\,|\mathbf{u}'|^{\,2} \;=\; r\,h\,, \tag{57}$$

in complete agreement with (12) found for the planar case.

Substitution of (55) and (56) into (54) yields the lengthy formula

$$(\mathbf{u}\,\bar{\mathbf{u}})\,(2\,\mathbf{u}\,\mathbf{u}^{\star\,\prime\prime} + 2\,\mathbf{u}'\,\mathbf{u}^{\star\,\prime}) \;-\; (\mathbf{u}'\,\bar{\mathbf{u}} + \mathbf{u}\,\bar{\mathbf{u}}')\cdot 2\,\mathbf{u}\,\mathbf{u}^{\star\,\prime} \;+\; \mu\,\mathbf{u}\,\mathbf{u}^{\star} \;=\; r^3\,\mathbf{f}\,, \tag{58}$$

which is considerably simplified by observing that the second and third term – after applying the distributive law – compensate:

$$2\,(\mathbf{u}\,\bar{\mathbf{u}})\,\mathbf{u}'\,\mathbf{u}^{\star\,\prime} \;-\; 2\,\mathbf{u}'\,(\bar{\mathbf{u}}\,\mathbf{u})\,\mathbf{u}^{\star\,\prime} \;=\; 0\,.$$

Furthermore, the fourth term of (58) reduces to

$$-2\,\mathbf{u}\,\bar{\mathbf{u}}'\cdot\mathbf{u}\,\mathbf{u}^{\star\,\prime} \;=\; -\,2\,\mathbf{u}\,(\bar{\mathbf{u}}'\,\mathbf{u}')\,\mathbf{u}^{\star} \;=\; -\,2\,\mathbf{u}\,|\,\mathbf{u}'|^2\,\mathbf{u}^{\star} \;=\; (r\,h - \mu)\,\mathbf{u}\,\mathbf{u}^{\star}$$

by using (57). Therefore, (58) becomes

$$2\,r\,\mathbf{u}\,\mathbf{u}^{\star\,\prime\prime} \;+\; r\,h\,\mathbf{u}\,\mathbf{u}^{\star} \;=\; r^2\,\mathbf{u}\,\bar{\mathbf{u}}\,\mathbf{f}\,,$$

and, finally, left-division by $r\,\mathbf{u}$ and star-conjugation yields

$$2\,\mathbf{u}'' \;+\; h\,\mathbf{u} \;=\; r\,\mathbf{f}\,\bar{\mathbf{u}}^{\star}\,, \tag{59}$$

a differential equation in perfect agreement with (13) for the planar case; however, it takes more than an educated guess to get the correct right-hand side.

7.3. Osculating Elements.

The considerations of Section 3.2., i.e. the introducing osculating *harmonic* elements by means of the singular-value decomposition, readily carries over to the spatial case. Consider, as in Equ. (22), the vector-valued harmonic oscillator

$$\frac{d^2\vec{v}}{d\tau^2} + \omega_0^2\,\vec{v} = 0\,, \tag{60}$$

with initial values $\vec{v}(\tau_0) = \vec{v}_0$, $\vec{v}\,'(\tau_0) = \vec{v}\,'_0$ at some time $\tau = \tau_0$, now with $\vec{v} \in \mathbb{R}^4$. As in Equ. (24), we write the solution in matrix form as

$$\begin{pmatrix} v_0(\tau) \\ v_1(\tau) \\ v_2(\tau) \\ v_3(\tau) \end{pmatrix} = M \begin{pmatrix} \cos(\omega_0(\tau-\tau_0)) \\ \sin(\omega_0(\tau-\tau_0)) \end{pmatrix}, \quad M = \begin{pmatrix} v_{00} & v'_{00}/\omega_0 \\ v_{10} & v'_{10}/\omega_0 \\ v_{20} & v'_{20}/\omega_0 \\ v_{30} & v'_{30}/\omega_0 \end{pmatrix} \tag{61}$$

with $v_{00}, v_{10}, v_{20}, v_{30}, v'_{00}, v'_{10}, v'_{20}, v'_{30}$ being the components of $\vec{v}_0$ and $\vec{v}\,'_0$, respectively. The SVD of M becomes

$$M = U\,S\,V^T \tag{62}$$

with an orthogonal matrix $U \in \mathbb{R}^{4\times4}$ having 5 essential degrees of freedom and matrices

$$S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$$

with 3 more degrees of freedom, totally 8, as expected.

In the preceding text we have presented a unified theory of regularization of the perturbed Kepler motion. Quaternion algebra allows for an elegant treatment of the spatial case in a way completely analogous to the way the planar case is traditionally handeled by means of complex numbers. As a consequence of the linearity of the regularized equations of the perturbed Kepler motion, the problem of satellite encounters reduces to a linear perturbation problem, the problem of coupled harmonic oscillators. Orbital elements based on the oscillators may lead to a simpified discussion of ordered and chaotic behaviour in repeated satellite encounters. This has been demonstrated by means of an instructive example.

References

1. George David Birkhoff: *The restricted problem of three bodies.* Rendiconti del Circolo Matematico di Palermo **39** (1915), 1.

2. Alessandra Celletti: *The Levi-Civita, KS and radial-inversion regularizing transformations.* In: D. Benest and C. Frœschlé (eds.): Singularities in Gravitational Systems. Applications to Chaotic Transport in the Solar System. Lecture Notes in Physics, Springer 2002, 25 - 48.

3. William Rowan Hamilton: *Memorandum respending a new system of Roots of Unity.* Philosophical Magazine **12**, 4 (1856), 446.

4. Michel Hénon and Jean-Marc Petit: *Series expansions for encounter-type solutions of Hill's problem.* Celest. Mech. **38** (1986), 67-100.

5. Heinz Hopf: *Über die Abbildung der dreidimensionalen Sphäre auf die Kugelfläche.* Math. Ann. **104**, (1931). Reprinted in Selecta Heinz Hopf, p. 38-63, Springer-Verlag Berlin Heidelberg New York, 1964.

6. Paul Kustaanheimo: *Spinor regularization of the Kepler motion.* Ann. Univ. Turku, Ser. AI **73** (1964).

7. Paul Kustaanheimo and Eduard L. Stiefel: *Perturbation theory of Kepler motion based on spinor regularization.* J. Reine Angew. Math. **218** (1965), 204-219.

8. Franz Spirig and Jörg Waldvogel: *The three-body problem with two small masses: A singular-perturbation approach to the problem of Saturn's coorbiting satellites.* In: V. Szebehely (ed.), Stability of the Solar System and its Minor Natural and Artificial Bodies, Reidel 1985, 53-63.

9. Eduard L. Stiefel et Jörg Waldvogel: *Généralisation de la régularisation de Birkhoff pour le mouvement du mobile dans l'espace à trois dimensions.* C.R. Acad. Sc. Paris **260** (1965), 805.

10. Eduard L. Stiefel and Gerhard Scheifele: *Linear and Regular Celestial Mechanics.* Springer-Verlag Berlin Heidelberg New York, 1971, 301 pp.

11. Tullio Levi-Civita: *Sur la régularisation du problème des trois corps.* Acta Math. **42** (1920), 99-144.

12. Maria Dina Vivarelli: *The KS transformation revisited.* Meccanica **29** (1994), 15-26.

13. Jan Vrbik: *Celestial mechanics via quaternions.* Can. J. Phys. **72** (1994), 141-146.

14. Jan Vrbik: *Perturbed Kepler problem in quaternionic form.* J. Phys. A **28** (1995), 193-198.

15. Jan Vrbik: *Novel analysis of tadpole and horseshoe orbits.* Celestial Mechanics and Dynamical Astronomy **69** (1998), 283-291.

16. Jörg Waldvogel: *Die Verallgemeinerung der Birkhoff-Regularisierung für das räumliche Dreikörperproblem.* Bulletin Astronomique, Série 3, Tome II, Fasc. 2 (1967), 295-341.

17. Jörg Waldvogel: *The restricted elliptic three-body problem.* In: E. Stiefel et. al., Methods of Regularization for Computing Orbits in Celestial Mechanics, NASA Contractor Report NASA CR **769** (June 1967), 88-115.

18. Jörg Waldvogel: *Long-term evolution of coorbital motion.* In: B.A. Steves and A.E.Roy (eds.), The Dynamics of Small Bodies in the Solar System: A Major Key to Solar System Studies, NATO ASI Series C, Plenum 1999, 257-276.