Polynomial processes in stochastic portofolio theory

Christa Cuchiero

University of Vienna

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Introduction and overview

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  - the development and analysis of tractable models which allow for relative arbitrage with respect to the market portfolio and match empirically observed facts, e.g, the dynamics and shape of the capital distribution curves;
  - to understand various aspects of relative arbitrages, in particular the properties of portfolios generating them (e.g., so-called functionally generated and/or long only portfolios, their implementation on different time horizons, etc.)

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- A lot of research has been conducted in this field, in particular by Adrian Banner, Daniel Fernholz, Robert Fernholz, Irina Goia, Tomoyuki Ichiba, Ioannis Karatzas, Kostas Kardaras, Soumik Pal, Radka Pickova, Johannes Ruf, Mykhaylo Shkolnikov etc.

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Motivation

- Model classes used in SPT:
  - Diverse models (Fernholz (2002)) (No single company is allowed to dominate the entire market in terms of relative capitalization)
  - Rank based models (Atlas and hybrid Atlas model) (Ichiba, Papathanakos, Banner, Karatzas, Fernholz (2011,2013))
  - Volatility stabilized models (Fernholz &Karatzas (2005))
  - Generalized volatility stabilized models (Pickova (2014))
- The most tractable class in view of pricing and the implementation of optimal relative arbitrages are volatility stabilized models which however have some drawbacks, for instance that the asset prices are uncorrelated.

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Question

High dimensional realistic modeling of say 500 stocks with the aim to

- preserve tractability in view of calibration and relative arbitrage;
- incorporate correlations;
- match the dynamics of the ranked marked weights.

Goal of this talk

Introduce market weight and asset price models based on polynomial processes

- of which volatility stabilized models are a specific example,
- whose structural properties fit the transformation between market capitalizations and market weights particularly well.
- to treat high-dimensionality by exploiting certain structural features of these models (extension by introducing polynomial measure valued processes).

The financial market model - Setting

- Filtered probability space: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ for some finite time horizon T and some right continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$.
- (Undiscounted) asset capitalizations are modeled by an $\mathbb{R}^d_{++}$-valued semimartingale S.
- Multiplicative modeling framework: each component of S is strictly positive $\Rightarrow$ we can write $S^i = S^i_0 \mathcal{E}(R^i)$ with $R_0 = 0$ and $\Delta R^i > -1$ and R is interpreted as the process of returns.
- A portfolio π is a predictable process with values in $\left\{ z \in \mathbb{R}^d \,\middle|\, \sum_{i=1}^d z^i = 1 \right\}$ such that $(\pi^1, \dots, \pi^d)^\top$ is R-integrable. Each π^i_t represents the proportion of current wealth invested at time t in the i^{th} asset for $i \in \{1, \dots, d\}$.
- The wealth process $V^{v,\pi}_t$ corresponding to initial wealth $V^{v,\pi}_0 = v \in \mathbb{R}_{++}$ and a portfolio π is given by $V^{v,\pi}_t = v\mathcal{E}\left(\int_0^T \sum_{i=1}^d \pi^i_t \frac{dS^i_t}{S^i_{t-}}\right) = v\mathcal{E}\left(\pi \bullet R\right)$.
- A portfolio π is called (multiplicatively) admissible if $V^{1,\pi} > 0$ and $V^{1,\pi}_- > 0$ a.s.

The market portfolio and the relative wealth process

Let us now draw our attention to one particular portfolio, namely the market portfolio, denoted by $\mu = (\mu^1, \dots, \mu^d)$.

- It invests in all assets in proportion to their relative weights, i.e.,

$$\mu^i = \frac{S^i}{\sum_{i=1}^d S^i} = \frac{S^i}{\bar{S}}, \quad i \in \{1, \dots, d\}$$

  where $\bar{S} = \sum_{i=1}^d S^i$ and generates wealth $V_t^{v,\mu} = v\frac{\bar{S}_t}{\bar{S}_0}$.

- μ takes values in Δ^d, which denotes the unit simplex, i.e.,

$$\Delta^d = \{z \in [0,1]^d \mid \sum_{i=1}^d z^i = 1\}.$$

- For $q \in \mathbb{R}_{++}$ and a portfolio π the relative wealth process with respect to the market portfolio is given by

$$Y_t^{q,\pi} = \frac{V_t^{q,\pi}}{V_t^{1,\mu}}, \quad Y_0^{q,\pi} = q.$$

Relative arbitrage with respect to the market portfolio

Definition

An admissible portfolio π constitutes a relative arbitrage opportunity with respect to the market portfolio over the time horizon $[0, T]$ if

$$P\left[Y_T^{1,\pi} \geq 1\right] = 1 \text{ and } P\left[Y_T^{1,\pi} > 1\right] > 0$$

and a strong relative arbitrage opportunity if

$$P\left[Y_T^{1,\pi} > 1\right] = 1$$

holds true.

Note that the existence of relative arbitrages achieved with portfolios depends only on the market weight process μ.

Part I

Polynomial models in SPT - Theoretical part

Polynomial processes

- $\mathcal{P}_m$: finite dimensional vector space of polynomials up to degree $m \geq 0$ on D where D is a closed subset of $\mathbb{R}^n$,

$$\mathcal{P}_m := \left\{ D \ni x \mapsto \sum_{|\mathbf{k}|=0}^{m} \alpha_{\mathbf{k}} x^{\mathbf{k}}, \,\middle|\, \alpha_{\mathbf{k}} \in \mathbb{R} \right\},$$

where we use multi-index notation $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, $|\mathbf{k}| = k_1 + \cdots + k_n$ and $x^{\mathbf{k}} = x_1^{k_1} \cdots x_n^{k_n}$.

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Definition

We call an D-valued time-homogeneous Markov process *m*-polynomial if for all $k \in \{0, \ldots, m\}$, all $f \in \mathcal{P}_k$, initial values $x \in D$ and $t \in [0, T]$,

$$x \mapsto E_x[f(X_t)] \in \mathcal{P}_k.$$

Polynomial processes - Characterization

Theorem (C., Keller-Ressel, Teichmann (2012))

Let $m \geq 2$. Then for a Markovian Itô-semimartingale X with state space D and $E_x[\int \|\xi\|^m K(X_t, d\xi)] < \infty$ for all $x \in D$, $t \in [0,T]$ the following are equivalent:

1. *X is a polynomial process.*
2. *The differential characteristics of X denoted by $(b(X_t), c(X_t), K(X_t, d\xi))_{t \in [0,T]}$ (with respect to the "truncation" function $\chi(\xi) = \xi$) satisfy*

$$\begin{aligned} b_i(x) &\in \mathcal{P}_1 \quad i \in \{1, \ldots, n\}, \\ c_{ij}(x) + \int \xi_i \xi_j K(x, d\xi) &\in \mathcal{P}_2 \quad i,j \in \{1, \ldots, n\}, \\ \int \xi^{\mathbf{k}} K(x, d\xi) &\in \mathcal{P}_{|\mathbf{k}|} \quad |\mathbf{k}| = 3, \ldots. \end{aligned}$$

Polynomial processes - Tractability

- Expectations of polynomials of X_t can be computed via matrix exponentials, more precisely for every $k \in \mathbb{N}$, there exists a linear map A on $\mathcal{P}_k$, such that for all $t \geq 0$, the semigroup (P_t) restricted to $\mathcal{P}_k$ can be written as
$$P_t|_{\mathcal{P}_k} = e^{tA}.$$
$\Rightarrow$ Easy and efficient computation of moments without knowing the the probability distribution or characteristic function.
- Pathwise estimation techniques of the integrated covariance are particular well suited to estimate parameters in high dimensional situations.

Volatility stabilized market models - Introduction

Volatility stabilized market models (R. Fernholz and I. Karatzas (2005)) are examples of polynomial processes. These models are remarkable since

- the the wealth process of the market portfolio corresponds to a specific Black Scholes model;
- the individual stocks are going all over the place and reflect the fact that log prices of smaller stocks tend to have greater volatility than the log prices of larger ones, in particular in these models we have $c_{ii}^{\log S} = \frac{1}{\mu^i}$ and no correlation $c_{ij}^{\log S} = 0$.
- they exhibit a constant positive excess growth rate $\gamma_*^{\mu} = \frac{1}{2}\sum_{i=1}^{d} \frac{c_{ii}^{\mu}}{\mu^i} = \frac{d-1}{2}$. Such a positive excess growth rate allows for strong relative arbitrage with long only portfolios on sufficiently long time horizons via functionally generated portfolios.
- Strong relative arbitrage with long only portfolios on arbitrary time horizons is also possible.

The model

- The dynamics of the asset prices in volatility stabilized market models are defined through

$$dS^i_t = S^i_t\left(\frac{1+\alpha}{2\mu^i_t}dt + \frac{1}{\sqrt{\mu^i_t}}dW^i_t\right), \quad S^i_0 = s^i, \quad i \in \{1,\dots,d\},$$

where $\alpha \geq 0$ and $(W^1,\dots,W^d)$ is a standard Brownian motion.

- Recalling $\bar{S} = \sum_{i=1}^d S^i$ and $\mu^i = \frac{S^i}{\bar{S}}$, we can rewrite

$$dS^i_t = \frac{1+\alpha}{2}\bar{S}_t dt + \sqrt{S^i_t \bar{S}_t} dW^i,$$

from which the polynomial property is easily seen.

- We here consider weak solutions to such SDEs or equivalently solutions to the corresponding martingale problem.

Polynomial property of the model

Proposition

The volatility stabilized model $(S^1, \ldots, S^d)$ satisfies the following properties:

1. $(S^1, \ldots, S^d)$ *is a polynomial diffusion process on* $\mathbb{R}^d_{++}$ *whose differential characteristics are of the form*

$$b^S_{i,t} = \frac{1+\alpha}{2} \sum_{j=1}^{d} S^j_t, \quad c^S_{ii,t} = S^i_t \left(\sum_{j=1}^{d} S^j_t \right), \quad c^S_{ij,t} = 0.$$

2. *The dynamics of the wealth process* $\bar{S}$ *of the market portfolio are described by a Black Scholes model of the form*

$$d\bar{S}_t = \bar{S}_t \left(\frac{d(1+\alpha)}{2} dt + dB_t \right),$$

for some Brownian motion B*, which is named stabilization by volatility (and drift in the case* $\alpha > 0$*).*

The market weights process

Proposition (c.f. I.Goia, S.Pal (2009))

The dynamics of the market weights $(\mu^1, \ldots, \mu^d)$ can be described by a multivariate Jacobi process of the form

$$d\mu_t^i = \left(\frac{1+\alpha}{2} - \frac{d(1+\alpha)}{2}\mu_t^i\right) dt + \sqrt{\mu_t^i}(1-\mu_t^i)dZ_t^i - \sum_{i \neq j} \mu_t^i \sqrt{\mu_t^j} dZ_t^j,$$

where Z denotes a d-dimensional standard Brownian motion. In particular $(\mu^1, \ldots, \mu^d)$ is a polynomial diffusion process with respect to its natural filtration (made right continuous), with state space $\mathring{\Delta}^d$ and differential characteristics of the form

$$b_{i,t}^\mu = \frac{1+\alpha}{2} - \mu_t^i \frac{d(1+\alpha)}{2}, \quad c_{ii,t}^\mu = \mu_t^i(1-\mu_t^i), \quad c_{ij,t}^\mu = -\mu_t^i \mu_t^j.$$

Polynomial market weight and asset price models - Definition

Definition

Consider a process $(\mu, S) \in D \subseteq \Delta^d \times \mathbb{R}_+^d$ such that $\mu^i = \frac{S^i}{S}$ for all $i \in \{1, \ldots, d\}$.

1. We call μ a polynomial market weight model if μ is a polynomial process on Δ^d with respect to its natural filtration (made right continuous).
2. We call (μ, S) a polynomial market weight and asset price model if additionally to (1) the joint process (μ, S) is a polynomial process.

Polynomial market weight and asset price model

Theorem (C. 2016)

Consider an Itô diffusion process $(\mu, S) \in D \subseteq \Delta^d \times \mathbb{R}^d_+$ *such that* $\mu^i = \frac{S^i}{\bar{S}}$ *for all* $i \in \{1, \ldots, d\}$*. Then the following assertions are equivalent:*

1. *The process* (μ, S) *is a polynomial market weight and asset price model.*
2. μ *and* $\bar{S}$ *are independent polynomial processes on* Δ^d *and* $\mathbb{R}_{++}$ *with differential characteristics of the form*

$$b^{\mu}_{i,t} = \beta^{\mu}_i + \sum_{j=1}^{d} B^{\mu}_{ij}\mu^j_t, \quad c^{\mu}_{ii,t} = \sum_{i \neq j} \gamma^{\mu}_{ij}\mu^i_t\mu^j_t, \quad c^{\mu}_{ij,t} = -\gamma^{\mu}_{ij}\mu^i_t\mu^j_t$$

$$b^{\bar{S}}_t = \kappa + \lambda \bar{S}_t \quad c^{\bar{S}}_t = \sigma^2 \bar{S}^2_t + \varphi \bar{S}_t,$$

for parameters $\kappa, \lambda, \sigma, \varphi \in \mathbb{R}$, $\beta^{\mu} \in \mathbb{R}^d$, $B^{\mu}, \gamma \in \mathbb{R}^{d \times d}$ *satisfying certain admissibility conditions.*

Polynomial market weight and asset price model

Corollary (C. 2016)

Under the assumptions of the above theorem the following assertions are equivalent:

1. *The process (μ, S) is a polynomial market weight and asset price model such that the characteristics of S do not depend on μ.*
2. *μ and $\bar{S}$ are independent polynomial processes on Δ^d and $\mathbb{R}_{++}$ where the differential characteristics of μ are as above and $\bar{S}$ is a Black & Scholes model of the form*

$$d\bar{S}_t = \lambda \bar{S}_t dt + \sigma \bar{S}_t dB_t.$$

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Remark

If μ and S are polynomial processes each of which with respect to its natural filtration but not jointly, then more flexibility for $\bar{S}$ is possible.

Properties

- In polynomial market weight and asset prices models the wealth process of the market portfolio can be any polynomial diffusion on $\mathbb{R}_+$ (in its natural filtration), in particular it can be an affine process.

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- It is necessarily a Black & Scholes model if S is a polynomial process in its natural filtration.
- The individual stocks can still reflect the fact that log prices of smaller stocks tend to have greater volatility than the log prices of larger ones, but allow additionally for correlation, in particular we have

$$c_{ii}^{\log S} = \frac{1}{\mu^i}\left(\frac{\sigma^2 S^i + \sum_{i\neq j}\gamma_{ij}^{\mu} S^j + \varphi\mu^i}{\bar{S}}\right), \quad c_{ij}^{\log S} = -\gamma_{ij}^{\mu} + \sigma^2 + \frac{\varphi}{\bar{S}}.$$

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- Positive excess growth rate: $\gamma_*^{\mu} = \frac{1}{2}\sum_{i=1}^{d}\frac{c_{ii}^{\mu}}{\mu^i} \geq \min_{i,j}\gamma_{ij}^{\mu}\frac{d-1}{2}$ whenever $\mu \in \mathring{\Delta}^d$ and γ is non-degenerate, which implies the existence of certain functionally generated relative arbitrage opportunities over sufficiently long time horizons

Extension with jumps

Proposition (C. (2016))

Let μ and $\bar{S}$ be independent polynomial processes (*both possibly with jumps*) *on Δ^d and $\mathbb{R}_+$ respectively. Assume that for $|\mathbf{k}| = 2$, the respective jump measures satisfy*

$$\int (\xi^{\mu})^{\mathbf{k}} K(\mu, d\xi^{\mu}) \in \mathcal{P}_{|\mathbf{k}|} \text{ and } \int (\xi^{\bar{S}})^{k} K(\bar{S}, d\xi^{\bar{S}}) \in \mathcal{P}_{k}.$$

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Remark

Note that the condition on the jump measure is anyhow satisfied for $k \geq 3$ by the polynomial property of the processes μ and $\bar{S}$.

Polynomial market weight models allowing for relative arbitrage

Theorem (C. 2016)

Let μ be a polynomial diffusion process for the market weights on Δ^d with characteristics as above such that for every i there exists some j with $\gamma_{ij} \neq 0$. Then the following assertions are equivalent:

1. *The model satisfies NUPBR, i.e*

$$\lim_{n\to\infty} \sup_{\pi\in\Pi} P[Y_T^{1,\pi} \geq n] = 0$$

   and there exist strong relative arbitrage opportunities.

2. *There exists some $i \in \{1,\ldots,d\}$ such that $b_i^\mu > 0$ on $\{\mu_i = 0\}$ and for all indices i with $b_i^\mu > 0$ on $\{\mu_i = 0\}$, we have*

$$2\beta_i^\mu + \min_{i\neq j}(2B_{ij}^\mu - \gamma_{ij}^\mu) \geq 0.$$

Optimal arbitrage - Definition

Definition

We denote by U the superhedging price of 1, that is,

$$U(T) := \inf\{q \geq 0 \,|\, \exists\, \pi \in \Pi \text{ with } Y_T^{q,\pi} \geq 1 \quad P\text{-a.s}\}$$

and we call $\frac{1}{U(T)}$ optimal arbitrage.

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Remark

In polynomial diffusion market weight models with $P[\mu_t \in \mathring{\Delta}, \forall t \in [0,T]] = 1$, this optimal arbitrage can be achieved by investing $U(T)$ and replicating the payoff $1_{\mathring{\Delta}^d}(\mu_T)$. Denoting by $g(t,\mu_t) = E_{P_0}[1_{\mathring{\Delta}^d}(\mu_T)|\mathcal{F}_t]$ the replicating Delta hedge is computed via $D_i g(t,\mu_t)$, which translate to portfolios as

$$\pi_t^i = \mu_t^i \left(\frac{D_i g(t,\mu_t)}{g(t,\mu_t)} + 1 - \sum_{j=1}^{d} \mu_t^i \frac{D_i g(t,\mu_t)}{g(t,\mu_t)} \right).$$

By approximating the payoff $1_{\mathring{\Delta}^d}(\mu_T)$ via polynomials, we obtain...

Computational aspects

Proposition (C. (2016))

Let μ be a polynomial diffusion process for the market weights on Δ^d whose parameters satisfy the conditions of the above theorem with $b_i^\mu > 0$ on $\{\mu_i = 0\}$ for all $i \in \{1, \dots, d\}$.

- *Then for every $\varepsilon > 0$ there exists a time-dependent polynomial $\mu \mapsto p(t, \mu)$ and a functionally generated portfolio defined via*

$$(\pi_t^\varepsilon)^i = \mu_t^i \left(\frac{D_i p(t, \mu_t)}{p(t, \mu_t)} + 1 - \sum_{j=1}^{d} \mu_t^i \frac{D_i p(t, \mu_t)}{p(t, \mu_t)} \right),$$

  such that $P[Y_T^{1,\pi^\varepsilon} > 1] \geq 1 - \varepsilon$.
- *Moreover, as $\varepsilon \to 0$, Y_T^{1,π^ε} converges P-a.s. to the optimal arbitrage and π^ε P-a.s. to the strategy implementing the optimal arbitrage.*

Part II

Polynomial models in SPT - Calibration results

(based on ongoing joint work with K.Gellert, M. Giuricich, A. Platts, S.Sookdeo and J.Teichmann)

Calibration of polynomial market weight and asset price model

- For the implementation of optimal arbitrages and the analysis of the performance of functionally generated portfolios only the covariance structure is of importance
  $\Rightarrow$ No drift estimation, only covariance estimation

- Model:

$$d\mu_t = \cdots dt + \sqrt{c_t^\mu} dW_t, \quad d\bar{S}_t = \cdots dt + \sigma \bar{S} dB_t,$$

  where $c_{ii}^\mu = \sum_{i \neq j} \gamma_{ij}^\mu \mu^i \mu^j$ and $c_{ij}^\mu = -\gamma_{ij}^\mu \mu^i \mu^j$ with $\gamma_{ij}^\mu = \gamma_{ji}^\mu \geq 0$ for all $i \neq j$.

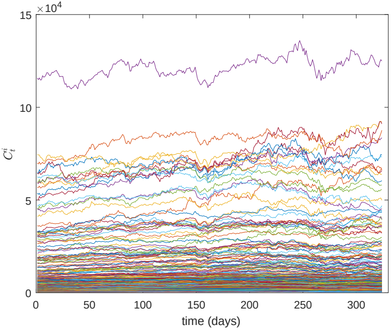
- Pathwise estimation of the integrated covariance $\langle \mu^i, \mu^j \rangle_T$ and $\langle \bar{S}, \bar{S} \rangle$ to obtain γ_{ij} and σ

Data and modeling assumptions

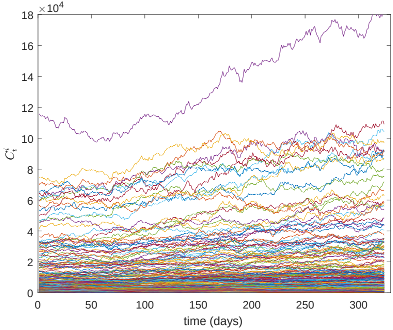
- 300 stocks representing the MSCI World Index, from August 2006 to October 2007, daily data.
- Challenge: How to estimate γ (44700 parameters) on the basis of this scarce data set?
- Assumption: Significant correlations exist only for similar magnitude market capitalizations

Comparison market and simulated trajectories

Market trajectories



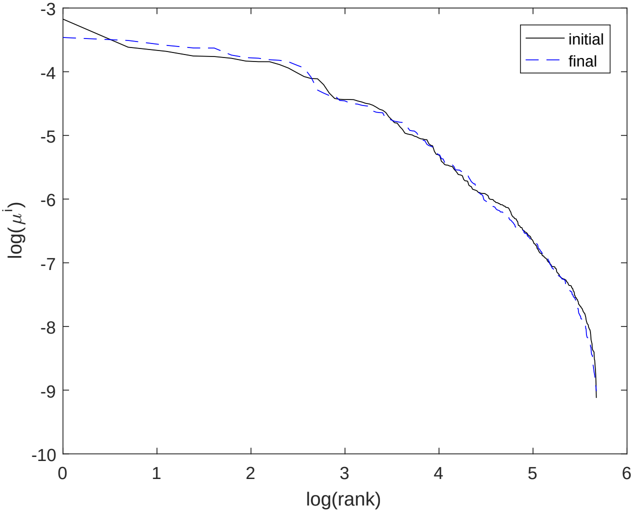
Simulated trajectories



- Stochastic volatility (coming from the influences of other stocks) and correlation structure of market and simulated data is similar despite a slightly higher volatility in the simulated data over time.

Capital distribution curves

- Preservation of the shape as well as reasonable dynamic behavior



Conclusion and Outlook

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  - Introduction of polynomial market weight and asset price models as generalization of volatility stabilized market models
  - Characterization of the existence of relative arbitrage in these models
  - Tractability properties to implement for instance (optimal) relative arbitrages
  - Successful calibration of a polynomial market weight model for 300 stocks based on relatively sparse data using a very direct method
  - Promising empirical features, e.g. capital distribution curve

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- Outlook:
  - Analyze functionally generated portfolios and the corresponding time horizons for relative arbitrage within this model class
  - Extension to measure-valued polynomial processes to treat high dimensionality and study limit behaviors

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- Outlook:
  - Analyze functionally generated portfolios and the corresponding time horizons for relative arbitrage within this model class
  - Extension to measure-valued polynomial processes to treat high dimensionality and study limit behaviors

Thank you for your attention!