Infinite dimensional Lie Theory from the point of view of Functional Analysis

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Für meine Eltern, Michael, Michaela und Hannah Zsofia Hazel.

ABSTRACT. Convenient analysis is enlarged by a powerful theory of Hille-Yosida type. More precisely asymptotic spectral properties of bounded operators on a convenient vector space are related to the existence of smooth semigroups in a necessary and sufficient way. An approximation theorem of Trotter-type is proved, too. This approximation theorem is in fact an existence theorem for smooth right evolutions of non-autonomous differential equations on convenient locally convex spaces and crucial for the following applications.

To enlighten the generically "unsolved" (even though H. Omori et al. gave interesting and concise conditions for regularity) question of the existence of product integrals on convenient Lie groups, we provide by the given approximation formula some simple criteria. On the one hand linearization is used, on the other hand remarkable families of right invariant distance functions, which exist on all up to now known Lie groups, are the ingredients: Assuming some natural global conditions regularity can be proved on convenient Lie groups. The existence of product integrals is an essential basis for Lie theory in the convenient setting, since generically differential equations cannot be solved on non-normable locally convex spaces.

The relationship between infinite dimensional Lie algebras and Lie groups, which is well understood in the regular case, is also reviewed from the point of view of local Lie groups: Namely the question under which conditions the existence of a local Lie group for a given convenient Lie algebra implies the existence of a global Lie group is treated by cohomological methods. It is shown that the considerations do not depend on the convergence of the Campbell-Baker-Hausdorff-Formula as in the original paper of W.T. van Est and Th.J. Korthagen.

Contents

Danksagung

Ein Mensch setzt sich zur Aufgabe, die Welt abzuzeichnen. Im Laufe der Jahre bevölkert er einen Raum mit Bildern von Provinzen, Königreichen, Gebirgen, Buchten, Schiffen, Inseln, Fischen, Behausungen, Werkzeugen, Gestirnen, Pferden und Personen. Kurz bevor er stirbt, entdeckt er, daß dieses geduldige Labyrinth das Bild seines Gesichts wiedergibt.

(Jorge Luis Borges)

Im wesentlichen glaube ich ja, daß ich niemals sterben werde, aber trotzdem, daß in all dem, was man so tut und schafft mehr von einem steckt als auch auf dem zweiten Blick erkennbar ist. Damit schimmern auch die Personen, die durch Unterstützung jedweder Art und durch Freundschaft in meinem Leben wichtig gewesen sind und sein werden zwischen der hochabstrakten Mathematik durch.

Meinen Eltern Erna und Josef Teichmann gebührt der erste und größte Dank, unter vielen Dingen ist für mich besonders zu erwähnen, daß Diskussion über alles mögliche, um den Sachen auf den Grund zu gehen, bei uns zu Hause eine große Rolle spielten. Im speziellen bin ich meinen Eltern dafür dankbar, diesen Stil mitbekommen zu haben für mein Leben.

Meinem Bruder, der die ersten Worte dieses Textes vor Jahren zu nachtschlafener Stunde in mein Hermes-Baby tippte, möchte ich danken, natürlich für die vielen interessanten Diskussionen, für die neuen Zugänge zum Leben, für seine fortwährende Unterstützung und die Liebe zur K-Theorie.

Meiner Tante Hilde Darani möchte ich für die schönen Gespräche über Religion und das Leben im allgemeinen danken, für die fortwährende Unterstützung und ihren Charme.

Michaela und ich sind zu Beginn unserer Zeit in Wien in das jeweilig andere Leben getreten, Monate später begannen wir dann uns auf Hannah Zsofia Hazel zu freuen. Michaela möchte ich für die ungarischen Tänze im Santo Spirito danken, natürlich die vielen Diskussionen, den Widerstandsgeist und die Liebe.

H.Z. Hazel, obwohl noch ein sehr junger Mensch, hat den größten Einfluß auf uns alle. Bei ihr sind Eigenschaften wie Lust, Freude, Widerstandskraft und Wille wunderbar ausgeprägt. An dieser Stelle möchte ich auch Sabine und Ingo Franz, Herbert Oberhuber und Sigrid Winkler für die Unterstützung unserer kleinen Familie danken.

Mit Stefan Haller hatte ich die Freude den Raum zu teilen, den wir regelmäßig in eine Spelunke mit tiefhängenden Rauchschwaden verwandelten, vielen Dank für die schönen Diskussionen. Dank gebührt auch den TeilnehmerInnen an den Wissenschaftlichen Arbeiten, insbesondere Andreas Cap, für die anregende und profunde Kritik und das Zuhören.

Mein mathematischer Weg wurde von zwei Persönlichkeiten an der Universität Graz geprägt: Peter Schöpf habe ich gleich zu Beginn meines Studiums bei einer Erläuterung logischer Probleme der Analysis 1 kennengelernt, Franz Halter-Koch etwas später bei seinen Algebra-Vorlesungen und auch als sein Sekretär. Beiden verdanke ich mathematische Bildung, viele Diskussionen und Unterstützungen. Besonders erwähnen möchte ich dabei, daß Franz Halter-Koch mich auch persönlich finanziell unterstützte, um mir die Möglichkeit zu bieten, eine andere Universität kennenzulernen und dort weiter zu studieren. Danken möchte ich außerdem Alfred Geroldinger, Ludwig Pittner, Ernst Seidel und Bernd Thaller.

Wolgang Arendt habe ich in Besançon im Studienjahr 1994/95 kennengelernt: Ihm möchte ich für die beiden Einladungen nach Ulm, wo ich meine Arbeiten präsentieren konnte, danken, auch für die schönen Diskussionen über Österreich donauaufwärts.

Mein Weg nach Wien war möglicherweise mathematisch etwas unerwartet, da meine Diplomarbeit im funktionalanalytischen Bereich angesiedelt gewesen war, aber die Offenheit Peter Michors ließ alle Zweifel schwinden. Die liberale und intellektuelle Atmosphäre um Peter Michor war für meine persönliche Entwicklung und für das Gedeihen der Dissertation wesentlich. Außerdem hat er dadurch, daß er an meine Arbeit glaubte und sie aus Mitteln des FWF (Projekt P-10037-PHY) unterstützte, nicht nur mein Leben als reiner Mathematiker ermöglicht, sondern auch mein Leben als Hausmann. Für diese liberale und seltene Haltung, nämlich bezüglich der Zeiteinteilung mir freie Hand zu lassen, möchte ich besonders danken.

Andreas Kriegl möchte ich für die vielen detaillierten Hinweise und die Redlichkeit beim Diskutieren danken, auch für die Signale der Bereitschaft dazu.

Anregungen zur Dissertation und zu meinem mathematischen Denken kamen auch von nichtmathematischer Seite, einerseits durch die Aufforderung zu erklären, was ich da so mache, andererseits durch kluge und präzise Statements nach meinen Erläuterungen. So war Martin Feldbacher vor zwei Jahren fest davon überzeugt, daß das Konzept mit den Lipschitz-Metriken aufgehen wird. Sebastian Markt legte mir nahe, einige Aussagen zur mathematischen Logik noch einmal zu überlegen. Roman Urbaner sah, daß die Erstfassung meiner dilletierenden Ausführungen zu Robert Musil doch etwas im Dunkeln liegen, wo sie vielleicht noch immer sind.

Niko Wahl und Gregor Weiss möchte ich für die Tatsache danken, mir die Bibel und das Kapital nähergebracht zu haben.

Josef Teichmann, Wien im Juli 1999

Introduction

Er war weniger wissenschaftlich als menschlich verliebt in die Wissenschaft. Er sah, daß sie in allen Fragen, wo sie sich für zuständig hält, anders denkt als gewöhnliche Menschen. Wenn man statt wissenschaftlicher Anschauungen Lebensanschauung setzen würde, statt Hypothese Versuch und statt Wahrheit Tat, so gäbe es kein Lebenswerk eines ansehnlichen Naturforschers oder Mathematikers, das an Mut und Umsturzkraft nicht die größten Taten der Geschichte weit übertreffen würde. Der Mann war noch nicht auf der Welt, der zu seinen Gläubigen hätte sagen können: Stehlt, mordet, treibt Unzucht - unsere Lehre ist so stark, daß sie aus der Jauche eurer Sünden schäumend helle Bergwässer macht; aber in der Wissenschaft kommt es alle paar Jahre vor, daß etwas was bis dahin als Fehler galt, plötzlich alle Anschauungen umkehrt oder daß ein unscheinbarer und verachteter Gedanke zum Herrscher über ein neues Gedankenreich wird, und solche Vorkommnisse sind dort nicht bloß Umstürze, sondern führen wie eine Himmelsleiter in die Höhe.

(Robert Musil, Der Mann ohne Eigenschaften)

Mathematics is not politics, but the universe of mathematics has been influencing in a subtle and silent way, but revolutionary in result, the discourses of human beings in this century. Mathematics is the most dangerous science. Here I do not think of people exploiting mathematical knowledge for their own purposes, I have the cold mathematical universe in mind, where every thinkable thought can be thought. Mathematics is the end of cynicism and this century is also characterized by the fact, that the most cynical thoughts about human beings have been realized with scientific consequence as some recent historical research on "technocrats" in the national socialist and stalinist hierarchy explains. Today's neoliberal world is much more subtle, but nevertheless governed by the structures of mathematics, which might mean liberty and prison. Unfortunately it is very difficult to find a book on history of mathematics discussing the aspect of possible sociological and psychological influences of the progress in mathematics, even marxist historicians do not although they definitively should.

The mathematical fin de siècle is in particular marked by the famous antinomies of naive set theory by Georg Cantor and Bertrand Russell, the naive concept of a set broke down. The set of all sets and the set of all sets not containing itself led to paradox situations. As a consequence David Hilbert proposed as one of his famous 23 problems at the international congress of mathematics in Paris, 1900, a program to set the fundaments of mathematics rigorously. These discoveries and their fascinating consequences changed the view towards the queen of sciences essentially, since her fundaments were in danger to be contradictory. However, the insight obtained in the following years forced the point of view, that mathematical formalism has its intrinsic limits and that there are no natural and obvious ways to choose the axioms of mathematical theories. In a way the platonic book of mathematics was closed and disappeared forever. More precisely the platonic book of mathematics was replaced by an infinity of platonic books, for number theory, for functional analysis, based on some axioms of a mathematical theory with some pages on undecideable assertions added just for fun. Some of them will maybe never be discovered, there existence is belief. The pride of mathematics was corrected to a more modest and pragmatic position.

The reason why I try point out these ideas is my constant reading of Robert Musil's "Der Mann ohne Eigenschaften" during the work on my thesis, from where I got the idea to connect the consequences of several of Robert Musil's ideas with the topics of the chapters of my thesis relating to their historical development and application. Relations are given by the way of mathematical reasoning, its universality and its arbitrary basis, on the one hand. On the other hand the progress in mathematics provided the background of the immense technical development and moral changes characterizing this century. Questions of responsibility of the scientists arise at first, more substantial questions about the future of these developments follow. Science is an adventure in mind, mystic in the best sense, so scholastics of the modern world could be mathematics, but what about the consequences? Application is the magic word in research proposals, however, this way we get personally involved. In the highly specialized universe of a working mathematician division of labour is completely realized, which is an older phenomenon with some new decisive features: Thinking and working in a defined area without questions about what other people do with your results has never been completely accepted in this century in the western world, but has some remarkable success. The spirit of mathematics is exactly given by this technocratic mentality. Richard P. Feynman resumes the question of his responsibility for the consequences of the work done in Los Alamos creating the first atomic bomb by the principle of active social irresponsibility of scientists, as John von Neumann (a professor of mine always insisted in any occasion on Johann von Neumann) did, although his situation as jew and a type of patriot is different. He solves problems he is interested in, applications - good or bad - are realized by other people, who are responsible. One can imagine the fascinated and original physicist, who proved to be a very independent character in the investigations of the Challenger catastrophe, in his universe thinking about physical problems without responsibility. Has he failed? The progress of his point of view is that he does not defend his work by some stupid patriotic or ideological arguments, the step back is that he does not think about the consequences of this "division of labour"-moral.

In my thesis convenient analysis represents the abstract playground, Hille-Yosida-Theory the dangerous, but seemingly harmless application, historically driven by the necessity to solve concrete problems. So convenient analysis is the mystics of a mathematician, motivation cames from the theory and the wish to make it as simple and as stringent as possible. Hille-Yosida-Theory has this beautiful aspects, too, but motivation is enriched by concrete problems and the wish to solve them. It can be done in a harmless way, but it is not in fact. Beauty is closely related to symmetry-groups as Hermann Weyl stated in his definition "Beauty is sligthly broken symmetry": Lie groups are a modern beauty, "superficially" investigated, but deeply understood. I insist on "superficially" since the mathematical way to think about beauty is absolutely not the cognitive one, but it sometimes helps to understand it. So the analysis of what we call beautiful cognitively leads to something which does not reproduce this coginitive beauty. The soul of our world is to analyze the phenomena in exactly this way, "die Entzauberung der Welt".

The history of infinite dimensional analysis traces back to Bernhard Riemann in the theory of manifolds and Sophus Lie concerning transformation groups. However, the development of functional analysis by Stefan Banach, Maurice Fréchet, Hans Hahn, David Hilbert, Johann von Neumann, Norbert Wiener and many others was necessary to base the ideas rigourosly. In many respects the persons, who contributed to the development of modern functional analysis and differential geometry, were involved in a passive or active way in the different political revolutions and tragedies of this century. So John von Neumann awarded the strange honor to provide the idea of Dr. Strangelove in Stanley Kubrick's "Dr. Strangelove or how I learned to love the bomb". Hans Hahn was one of the fathers of the famous circle of Vienna unifying subtle neopositivism and socialdemocratic ideas. Functional analysis is both, in several respects very close to applications in physics and technics and a chapter of the book of deep pure mathematics, since functional analysis was condensed from the analytic art of solving equations appearing in applications. With the discovery of quantum mechanics functional analysis became the playground of modern physics.

One can categorize the progress in infinite dimensional differential geometry in the following way: The first insight was the discovery of interesting questions with infinite dimensional model spaces by Bernhard Riemann and Sophus Lie. Next and independently the vector space character of many classical analytical problems was discovered and successfully discussed, Richard Courant and David Hilbert wrote the first comprehensive books on these topics. It is worth mentioning that in the early days of functional analysis rather general classes of locally convex spaces were investigated to obtain solutions of differential equations formulated on them. The polish school around Stefan Banach and Julius Schauder and the french school around Maurice Fréchet and Jacques Hadamard discovered independently the basics of modern functional analysis. The development of useful theory then concentrated on Banach and Hilbert spaces, where the theorems of Stone and Hille-Yosida were found. Banach space geometry explained how rich already this "narrow" setting is. However rather simple problems led naturally to Fréchet spaces and sophisticated methods as hard implicit function theorems were developed to solve them, done by Jürgen Moser, John Nash, Vladimir Arnold and others. From this rich background it was a small, but clever step towards a powerful calculus to base the concept of infinite dimensional manifolds, which was done by Alfred Frölicher and Andreas Kriegl. Nevertheless there are some meta-statements asserting that any theorem on general locally convex spaces is trivial in each concrete case, which is not my opinion.

Infinite-dimensional geometry is a beautiful subject unifying classical analysis and differential geometry in several ways. The Korteweg-deVrieß-equation is a geodesic equation on the Virasoro-Bott-group, Hamilton dynamics on a symplectic manifold can be viewed as a smooth one-parameter subgroup of the group of symplectomorphisms and fluid-mechanics of incompressible media has the Lie group of volume-preserving diffeomorphisms as essential scene. These few examples argue the importance of infinite-dimensional geometry and underline the necessity of subtle methods in this area. By a theorem of Hideki Omori an effective and transitive action of a Banach-Lie-group on a compact finite-dimensional manifold only is possible if the Banach-Lie-group is already finite-dimensional ([**KM97**] for more examples and details on this topic and [**Omo78**]). This striking result demonstrates that interesting infinite-dimensional geometry is always modeled on at least Fréchet spaces.

Fréchet spaces behave remarkably bad with respect to the solvability of differential equations. This is due to the fact that there is no genuine inverse function theorem, only in some tame cases. One can prove by "non-tame" methods that a so called hard inverse function theorem is valid. On the behalf of Lie groups this problem was surrounded by the invention of the concept of strong ILB-groups by Hideki Omori (see [**Omo97**]). This concept leads the Fréchet space problems as near as necessary to the Banach-space-case, such that all obvious differential equations are solvable, some inverse function theorem is valid and even some Frobenius theorem, but a weaker and simpler to prove version as the hard implicit function theorems of Nash and Moser. However, the methods involved are rather technical and neither the geometric nor the topological properties of the given Fréchet-Lie group are illuminated, because the point of view is an analytic one.

In my thesis I tried to emphasize two aspects: First one should try to enligthen the situation concerning differential equations on a given convenient space. This leads in fact to inverse function theorems and related methods, which is not pointed out here. I tried to develop a theory of Abstract Cauchy Problems on convenient vector spaces in the spirit of Hille-Yosida-Miyadera-Feller. Approximations by product integrals are considered, too, since this provides the background of numerical solutions of given equations and is useful on convenient geometries guaranteeing in a natural way the existence of approximations. Second one should try to relate the geometry of an infinite-dimensional manifold to the analytic questions. More precisely, it is necessary for the solvability of globally given differential equations to implement the geometrical and topological properties of the given manifold. In other words it could be interesting to find some geometro-analytic theory of non-linear differential equations appearing naturally on infinite-dimensional objects. In finite-dimensional theory it is possible and therefore useful to separate local and global questions.

The first chapter is devoted to convenient analysis: Smooth and analytic calculus are explained and the basics of infinite dimensional differential geometry are set. The second chapter starts with the classical concepts of Hille-Yosida theory in the strongly continuous and holomorphic case to explain how universal and limited the methods of this theory are. Then convenient Hille-Yosida-Theory is explained and an extract of the rich universe of examples on locally convex spaces is given. The main theorem of this part is the approximation theorem providing a possibility to conclude on convenient locally convex spaces the existence of time evolutions of non-autonomuous linear differential equations. The advantage of this theorem is its universality since convenient vector spaces cover the reasonable spaces of analysis on the one hand. On the other hand it is not necessary to prove an additional property of the convenient vector space under consideration, which would be very difficult in general. The third chapter is devoted to the fundamental question of my thesis: Under which conditions a convenient Lie group possesses an exponential map or even an evolution map. Several approaches, by linearization of the problem or by assuming the existence of so called Lipschitz-metrics, are discussed and provide a somehow successful approach to the problem of regularity. Here the power of the approximation theorem is widely applied. The fourth chapter concentrates on the special feature of infinite dimensional Lie group theory that local Lie groups are not necessarily enlargible to global ones, a phenomenon unknown in the finite dimensional case by Ado's theorem.

The title "Infinite dimensional Lie Theory from the point of view of Functional Analysis" is explained by the fact that Lie Theory originally starts if one is able to define a bijective mapping from the set of smooth one parameter subgroups of a topological smooth group G to a Lie algebra. The transported linear structure on the set of smooth one parameter subgroups should be expressed by Sophus Lie's famous formula

$$\lim_{n\to\infty}(\exp(\frac{t}{n}X)\exp(\frac{t}{n}Y))^n = \exp(t(X+Y)$$

which is a deep non-linear functional analytic question on the group G. The thesis is devoted to the analysis of this question.

Josef Teichmann, Vienna in July 1999

CHAPTER 1

Convenient Analysis

Aber ich glaube, daß die Menschen in einiger Zeit einesteils sehr intelligent, andernteils Mystiker sein werden. Vielleicht geschieht es, daß sich unsere Moral schon heute in zwei Bestandteile zerlegt: Ich könnte auch sagen: In Mathematik und Mystik. In praktische Melioration und unbekanntes Abenteuer.

(Robert Musil, Der Mann ohne Eigenschaften)

Convenient analysis provides the widest framework for analysis and is therefore of particular importance in today's border regions of differential geometry, for example the theory of diffeomorphism groups.

The concept of a smooth curve is obvious on locally convex spaces. The class of locally convex spaces where weakly smooth curves are exactly smooth curves is given by convenient vector spaces. A locally convex space is called convenient if and only if it is Mackey-complete, a weak concept of completeness. The final topology with respect to all smooth curves is called the c^∞-topology. If E is a convenient space, $c^\infty E$ need not be a topological vector space, since addition might be discontinuous, however up to Fréchet spaces $E = c^\infty E$. A mapping $f : U \to F$, where U is c^∞-open and F is a convenient vector space is called smooth if smooth curves are mapped to smooth curves, which is even on $\mathbb{R}^2$ obviously equivalent. The differential of a smooth mapping is simply given by its derivative along affine lines. Holomorphic calculus will be developed along these lines without surprises. For the matters of analysis and geometry we have to pay attention to the infinite dimensional features, namely, that no inverse function theorems are given in general and complementary subspaces are difficult to obtain. Furthermore the naturally given smooth topology on convenient vector spaces does not commute with products!

This first chapter is a mainly self-contained condensation of parts of ch.1, ch.2 and ch.6 of [**KM97**] with all necessary details. As a reference book for functional analytic fundaments we propose [**Jar81**].

1. Smooth Calculus

The setting of convenient vector spaces was introduced by Andreas Kriegl and Alfred Frölicher (see [**FK88**] and [**KM97**] for details and excellent references) to set up a useful calculus in infinite dimensions beyond Banach spaces. The necessity of a new foundation of calculus grew out of the analysis of infinite dimensional objects in differential geometry, e.g. diffeomorphism groups of finite dimensional manifolds or loop groups. The solution is as simple as beautiful and provides us with a useful tool to handle infinite dimensional questions. In fact we try to answer the question on which spaces it is possible to draw the conclusion

$$\text{weakly smooth (holomorphic)} \Rightarrow \text{smooth (holomorphic)}$$

In the sequel $\mathbb{N}$ denotes the natural numbers including zero, $\mathbb{R}$ the real numbers, $\mathbb{N}_+$ the positive natural numbers and $\mathbb{R}_{>0}$ the positive real numbers. By $E, F, G, ...$ we denote separated locally convex spaces.

A subset of a locally convex space is said to be bounded if and only if it is absorbed by every zero neighborhood. Recall that a locally convex space is normable if some zero neighborhood is bounded (Kolmogorow's theorem, see [**Jar81**]). The system of bounded sets is determined by the locally convex topology, but - given this system of sets - there are different locally convex topologies having the same system of bounded sets, which will be called the bornology of the space. Two locally convex topologies on a vector space shall be called compatible if the associated bornologies are the same.

We call a sequence $\{x_n\}_{n\in\mathbb{N}}$ Mackey-converging to x with quality $\{\mu_n\}_{n\in\mathbb{N}}$, where the μ_n are non-negative real numbers converging to 0, if there is a bounded set B such that $x_n - x \in \mu_n B$.

Analogously we call a sequence $\{x_n\}_{n\in\mathbb{N}}$ a Mackey-Cauchy-sequence if there is a sequence $\{t_{nm}\}_{n,m\in\mathbb{N}}$ with t_{nm} positive real numbers and $t_{nm} \to 0$ for $n, m \to \infty$ so that $x_n - x_m \in t_{nm}B$, where B is bounded. If every Mackey-Cauchy-sequence converges in E we speak of a Mackey-complete vector space. On a Mackey-complete vector space there is in general no natural locally convex topology reproducing only this concept of convergence (see [**KM97**], ch.1), but there is a finest topology in the set of locally convex topologies compatible with the system of bounded sets, called the bornological topology E_{born}. When we talk of a closed set in a convenient vector space we mean that the set is closed with respect to the topology of E_{born}. Given a bounded, absolutely convex set $B \subset E$ we can look at the localization $E_B := span(B)$ with the Minkowsky norm p_B as norm (see [**Jar81**], ch.6). Remark that convenient vector spaces are exactly those locally convex spaces, which are locally complete, i.e. for every closed, bounded and absolutely convex set B the normed space E_B is a Banach space. Mackey-completeness is consequently an apparently weak concept of completeness (see [**KM97**], ch.1).

A linear mapping $a : E \to F$ is said to be bounded if bounded sets are mapped to bounded sets. This is a concept more general than continuity and the appropriate concept with respect to differential calculus. A locally convex vector space E where every bounded linear mapping is continuous is called bornological; Fréchet spaces are bornological. By $L(E, F)$ we denote the vector space of bounded linear maps from E to F, we have a natural system of bounded sets given on this space, namely the sets of linear maps uniformly bounded on bounded sets, and a natural locally convex topology, namely the topology of uniform convergence on bounded sets; they are compatible. By E' we denote the space of bounded functionals on E, real- or complex valued, respectively. Recall that already continuous linear functionals on locally convex spaces detect bounded sets. The space of continuous linear functionals on a locally convex space is denoted by E^*. On Mackey-complete vector spaces we have a uniform boundedness principle asserting that pointwise bounded families of bounded linear maps between Mackey-complete vector spaces are uniformly bounded (see [**KM97**], ch.1). Given a pointwise bounded set of bounded linear maps in $L(F, G)$, then we can restrict the bounded linear maps to the Banach space E_B for an absolutely convex bounded and closed set B. So we obtain a pointwise bounded set of continuous linear maps in $L(E_B, F)$, where we can apply the classical uniform boundedness principle, indeed we apply directly the Baire theorem (see [**Jar81**], ch.11). Consequently the set of bounded linear maps is uniformly bounded.

Beyond Banach spaces several different differential calculi have been developed, most of them rather complicated. The main difficulty is that the composition of continuous linear mappings on locally convex spaces stops to be jointly continuous at the level of Banach spaces for any compatible topology, so a useful calculus should be working for some non-continuous mappings, too. The historic development is very well written down in [**KM97**], ch.1. In the sequel we define the main concepts of Frölicher-Kriegl-calculus to be able to apply them to our problems:

The concept of a smooth curve poses no problems in any locally convex space: A curve is called smooth if all derivatives up to arbitrary order do exist. The crucial observation is that the set of smooth curves $C^\infty(\mathbb{R}, E)$ only depends on the bornology of the locally convex space. This is due to the following mean value theorem, which guarantees that the convergence of the difference quotient to the respective derivative is Mackey, consequently smoothness only depends on the bornology.

1.1. Lemma. *Let E be a locally convex space and $c : [a, b] = I \to E$ be a continuous curve differentiable except at points in a countable subset $D \subset I$. Let A be a closed convex subset of E with $c'(t) \in A$ for $t \in I \setminus D$, then*

$$c(b) - c(a) \in (b - a)A$$

Proof. (see [**KM97**], p. 10) By the theorem of Hahn-Banach we can reduce the problem to the case $E = \mathbb{R}$. Let l be a continuous linear functional such that

$$l(c(b) - c(a)) \notin \overline{l((a - b)A)}$$

under the assumption that the assertion is not satisfied. However $l \circ c$ satisfies the hypotheses of the one-dimensional mean value theorem with convex set $\overline{l(A)}$. So we conclude by contradiction. □

By the mean value theorem we conclude that for a smooth curve c the curve

$$t \mapsto \frac{1}{t}\left(\frac{c(t) - c(0)}{t} - c'(0)\right)$$

is bounded on bounded intervals which means Mackey-convergence. Furthermore for scalarly smooth curves into to a locally convex vector space the difference quotients are Mackey-Cauchy-nets, which can simply be tested scalarly by looking at the boundedness of

$$\frac{1}{t-s}\left(\frac{c(t)-c(0)}{t}-\frac{c(s)-c(0)}{s}\right)$$

The Riemann integral is defined canonically for continuous curves, the fundamental theorem of calculus is valid and continuous functionals commute with the integral. Furthermore the Riemann sums form a Mackey-Cauchy net for smooth curves.

1.2. Lemma (special curve lemma). *Let E be a locally convex space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence converging fast to zero, then the infinite polygon through x_n can be smoothly parametrized, i.e. there is a smooth curve with $c(\frac{1}{n}) = x_n$.*

Proof. (see [**KM97**], p. 18) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth curve with $\phi(t) = 0$ for $t \leq 0$ and $\phi(t) = 1$ for $t \geq 1$. We parametrize as follows:

$$c(t) := \begin{cases} x \text{ for } t \leq 0 \\ x_{n+1} + \phi(\frac{t-\frac{1}{n+1}}{\frac{1}{n}-\frac{1}{n+1}})(x_n - x_{n+1}) \text{ for } \frac{1}{n+1} \leq t \leq \frac{1}{n} \\ x_1 \text{ for } t \geq 1 \end{cases}$$

This parametrization has the desired properties due to fast convergence. □

The special curve lemma is useful in many proofs of convenient analysis, for example to show that a linear map between locally convex spaces is bounded if and only if it maps smooth curves to smooth curves. One direction is given by the above statements, the other direction is proved by contradiction. Assume that a linear map $f : E \to F$ carries smooth curves to smooth ones and that there is a bounded set B with $f(B)$ unbounded, then there is a sequence $\{b_n\}_{n\in\mathbb{N}}$ in B with $|l \circ f(b_n)| \geq n^{n+1}$ for $n \in \mathbb{N}$ and some $l \in F'$. $\{n^{-n}b_n\}_{n\in\mathbb{N}}$ converges fast to 0 in E and lies by the special curve lemma on the compact part of a smooth curve, but consequently $f(n^{-n}b_n)$ should be bounded.

1.3. Definition. *Let E be a locally convex space. E is said to be convenient if one of the following equivalent conditions is satisfied:*

1. *For $c \in C^\infty(\mathbb{R}, E)$ the Riemann-integral $\int_0^1 c(t)dt$ exists.*
2. *A curve $c : \mathbb{R} \to E$ is smooth if and only if $\lambda \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $\lambda \in E'$.*
3. *E is Mackey-complete.*

Similarly the concept of a *Lip*^{n}-curve depends only on the bornology of the space. $c : \mathbb{R} \to E$ is called *Lip*^{n} for $n \in \mathbb{N}$ if c is n-times differentiable and $c^{(n)}$ is locally Lipschitz on $\mathbb{R}$. The set of *Lip*^{n}-curves to E is denoted by $Lip^n(\mathbb{R}, E)$. By adapting the order of Lipschitz-differentiability we can proof similar statements for Lipschitz curves as stated for smooth curves. Obviously smooth curves are *Lip*^{n} for all orders. A *Lip*^{n}-curve $c : \mathbb{R} \to E$ "factors" over compact intervals J into a normed space of the type $E_B \hookrightarrow E$ for some absolutely convex bounded set B as a *Lip*^{n}-curve.

We fix the locally convex topology of uniform convergence of all derivatives on compact subsets of $\mathbb{R}$ on the space of smooth curves $C^\infty(\mathbb{R}, E)$ into a convenient vector space E. This is the initial topology with respect to the maps

$$C^\infty(\mathbb{R}, E) \xrightarrow{d^k} C^\infty(\mathbb{R}, E) \to l^\infty(K, E)$$

with $k \in \mathbb{N}$ and K a compact subset of the real line. $l^\infty(K, E)$ is the space of bounded maps from K to E with uniform convergence of K. Remark that this topology coincides with the usual topology on $C^\infty(\mathbb{R}, \mathbb{R}^n)$.

1.4. Lemma (general curve lemma). *Let E be a convenient vector space and $\{c_n\}$ a sequence of curves in $C^\infty(\mathbb{R}, E)$ converging fast to 0. Let $s_n \geq 0$ be a sequence of real numbers with $\sum_{n\in\mathbb{N}_+} s_n < \infty$, then there is a smooth curve $c : \mathbb{R} \to E$ and a converging sequence $\{t_n\}_{n\in\mathbb{N}_+}$ with*

$$c(t + t_n) = c_n(t) \text{ for } |t| \leq s_n$$

for $t \in \mathbb{R}$ and $n \in \mathbb{N}_+$.

Proof. (see [**KM97**], 12.2.) Let $r_n := \sum_{1\leq k<n}(\frac{2}{k^2}+2s_k)$ and $t_n := \frac{r_n+r_{n+1}}{2}$. Let $h:\mathbb{R}\to[0,1]$ be smooth with $h(t)=1$ for $t\geq 0$ and $h(t)=0$ for $t\leq -1$. We put $h_n(t):=h(n^2(s_n+t))h(n^2(s_n-t))$ and

$$c(t) := \sum_{n\in\mathbb{N}_+} h_n(t-t_n)c_n(t-t_n)$$

which exists since for any t there is at most one non-zero summand. The series of derivatives of k-th order is uniformly converging on $\mathbb{R}$, consequently c is a smooth curve. □

The final topology with respect to all smooth curves (or Lipschitz curves) is denoted by $c^\infty E$. Up to Fréchet spaces $c^\infty E = E$ topologically, but for more general convenient vector spaces $c^\infty E$ even fails to be a topological vector space. The bornological topology is the finest locally convex topology coarser than the c^∞-topology. c^∞-open subsets are the natural domains of definition of smooth mappings (see [**KM97**], ch.1) . We shall refer to this topology as smooth topology on a convenient vector space.

It is useful to deeply understand the c^∞-topology on a convenient vector space, because this is the natural topology for analytic questions on convenient vector spaces. A subset A of a locally convex vector space is c^∞-closed if the inverse image under smooth curves is closed. This is equivalent to the statement that all Mackey-converging sequences in A with fixed quality $\{\mu_n\}_{n\in\mathbb{N}}$ of positive real numbers have limits in A. The equivalence is established via the special curve lemma: Given a Mackey-converging sequence in A with quality $\{\mu_n\}_{n\in\mathbb{N}}$, then we can choose a fast converging subsequence, which lies on a compact part of a smooth curve, consequently the limit lies in A. If the limits of all Mackey-converging sequences in A with fixed quality $\{\mu_n\}_{n\in\mathbb{N}}$ lie in A, then the inverse image of a smooth curve has to be closed, because the limits $c(t)\to c(s)$ for $t\to s$ are Mackey-limits with quality $|t-s|$, so one obtains any quality.

1.5. Remark. E is convenient if and only if E is c^∞-closed in any locally convex vector space, where E is assumed to be embedded. Let E be embedded in a locally convex vector space F. If E is convenient, then a fast falling sequence in E lies on a compact part of a smooth curve in E by the special curve lemma, consequently the limit lies in E and E is c^∞-closed. E can be embedded naturally in $L(E^*_{equ},\mathbb{R})$, the space of all linear functionals on E^*, which are bounded on equicontinuous sets of continuous linear functionals, via $\delta(x)(l)=l(x)$. By the bipolar theorem we see that this is a homeomorphism and $L(E^*_{equ},\mathbb{R})$ is complete, so convenient. If E is c^∞-closed in this space any Mackey-Cauchy-sequence in E has a limit in E. $C^\infty(\mathbb{R},E)$ is convenient if and only if E is convenient. E can be embedded into $C^\infty(\mathbb{R},E)$ via constant curves as a c^∞-closed subspace, so E is convenient if $C^\infty(\mathbb{R},E)$ is convenient. Denote by $l^\infty(\mathbb{R},E)$ the projective limit of the restrictions to the compact subsets, then we obtain a convenient vector space, which can be seen directly. The mapping $c\longmapsto (c^{(n)})_{n\in\mathbb{N}}$ is an embedding of $C^\infty(\mathbb{R},E)$ to a closed subspace of $\prod_{n\in\mathbb{N}} l^\infty(\mathbb{R},E)$.

1.6. Remark. A sequence is c^∞-convergent if and only if any subsequence has a subsequence, which is Mackey-convergent. One direction is clear since Mackey-convergence implies c^∞-convergence. We show that there is a Mackey-converging subsequence of a c^∞-converging sequence: $A:=\{x_n\,|\,n\in\mathbb{N}\}$ is assumed not to contain the limit without loss of generality, so it can not be c^∞-closed, so there is a subsequence converging Mackey to some $x'\in E$, which is a fortiori equal to x. If every c^∞-converging sequence has a Mackey-converging subsequence, then every subsequence of this sequence has one. On Fréchet spaces we have consequently $c^\infty E = E$, because every converging sequence converges Mackey, so the identity $id: c^\infty E\to E$, which is continuous, has a sequentially continuous inverse, but this means continuous on a Fréchet space.

1.7. Remark. Let E be a non-normable convenient bornological vector space, then $c^\infty(E\times E')$ is not a topological vector space, but $c^\infty(E\times\mathbb{R}^n)=c^\infty E\times c^\infty\mathbb{R}^n$. The pairing $\langle .,.\rangle : E\times E'\to\mathbb{R}$ is a bounded bilinear map, so smooth and $\langle .,.\rangle : c^\infty(E\times E')\to\mathbb{R}$ is continuous. Assume that addition $+: c^\infty(E\times E')\times c^\infty(E\times E')\to c^\infty(E\times E')$ were continuous, then via the natural inclusions $c^\infty E\to c^\infty(E\times E')$, $c^\infty(E')\to c^\infty(E\times E')$ one could write the pairing on $c^\infty E\times c^\infty(E')$ as a composition of continuous maps

$$c^\infty E\times c^\infty(E')\to c^\infty(E\times E')\times c^\infty(E\times E')\xrightarrow{+} c^\infty(E\times E')\xrightarrow{\langle .,.\rangle}\mathbb{R}$$

So there exist open 0-neighborhoods $U \subset c^\infty E$, $V \subset c^\infty(E')$ with $|\langle U, V\rangle| \subset [0, \epsilon[$ for any $\epsilon > 0$. The inclusion remains if one replaces the open sets by their absolutely convex hulls, but convex c^∞-open sets are open in the bornological topology. Hence U is scalarwise bounded, since V is absorbing, so U is bounded. Consequently E has to be normable by Kolmogorow's theorem. Nevertheless for all $n \geq 1$ we have $c^\infty(E \times \mathbb{R}^n) = c^\infty E \times \mathbb{R}^n$.

1.8. Remark. There is a unique convenient vector space $\widetilde{E}$ and a bounded linear injection $i : E \to \widetilde{E}$ such that each bounded linear mapping $l : E \to F$ can be extended along i. Furthermore $i(\widetilde{E})$ is dense in the c^∞-topology (see [**KM97**], 4.29.).

1.9. Definition. *Let E, F be locally convex vector spaces and $f : U \subset E \to F$ a mapping, where $U \subset E$ is c^∞-open in E. f is said to be smooth if*

$$\forall\, c \in C^\infty(\mathbb{R}, U)\, : f \circ c \in C^\infty(\mathbb{R}, F)$$

The first derivative $df : U \times E \to F$ is simply defined by the well known formula $df(x)(v) := \frac{d}{dt} f(x + tv)|_{t=0}$ for $x \in U$, $v \in E$. Let $n \in \mathbb{N}$ be a natural number. A mapping $f : U \to F$, where $U \subset E$ is c^∞-open, is called Lip^n if

$$\forall\, c \in C^\infty(\mathbb{R}, U)\, : f \circ c \in Lip^n(\mathbb{R}, F)$$

1.10. Remark. The composition of smooth maps is smooth by definition and smooth maps detect smooth curves by composition. This vice versa relation will be axiomatized by the concept of Frölicher spaces. Given a convenient vector space E and a subset F of E' of bounded linear functionals, such that the elements of F detect the bounded sets in E. Then a curve $c : \mathbb{R} \to E$ is smooth if and only if $l \circ c$ is smooth for all $l \in F$ (see uniform $\mathcal{S}$-boundedness principle, [**KM97**], 5.22.).

1.11. Remark. The composition of Lip^k-maps is Lip^k, which can be seen by the general curve lemma. We show the Lip^0-case: Given a Lip^0-function $f : U \to \mathbb{R}$ and suppose that there is a Lip^0-curve $d : \mathbb{R} \to U$ such that $f \circ d$ is not Lip^0 around 0, then there are sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ with $u_n \neq v_n$, $|u_n| \leq \frac{1}{4^n}$ and $|v_n| \leq \frac{1}{4^n}$ such that

$$|f \circ d(u_n) - f \circ d(v_n)| \geq |u_n - v_n| 2^n n$$

By the general curve lemma we find a smooth curve c associated to $s_n = |u_n - v_n|2^n$ and $c_n(t) = d(u_n) + t\frac{d(v_n)-d(u_n)}{|(v_n-u_n)|2^n}$. Around 0 the curve maps to U and consequently we obtain

$$\frac{|f \circ c(t_n + s_n) - f \circ d(t_n)|}{s_n} = \frac{|f \circ d(u_n) - f \circ d(v_n)|}{s_n} \geq n$$

which is a contradiction. The rest is done by a sophisticated induction.

The concept of smoothness follows an idea of Boman [**Bom67**]. Up to Banach spaces this concept of smoothness coincides with all reasonable concepts, but even on $\mathbb{R}^2$ the proof of this assertion is not trivial:

1.12. Proposition. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a mapping, then the following statements are equivalent:*

1. *All iterated partial derivatives exist and are continuous.*
2. *All partial derivatives exist and are locally bounded.*
3. *For $v \in \mathbb{R}^2$ the iterated directional derivatives*

$$d_v^n f(x) := \frac{\partial^n}{\partial t^n}|_{t=0} f(x + tv)$$

exist and are locally bounded with respect to x.

4. *For all smooth curves c the composite $f \circ c$ is smooth*

Proof. For the proof see [**Bom67**] or [**KM97**], 3.4: 1. $\Rightarrow$ 2. is obvious and 2. $\Rightarrow$ 1. is given by integration, we get continuity immediately via

$$f(s, t) - f(0, 0) = s \int_0^1 \partial_1 f(\sigma s, t) d\sigma + t \int_0^1 \partial_2 f(0, \tau t) d\tau$$

1. $\Rightarrow$ 4.: by the classical chain rule.

4. $\Rightarrow$ 3.: existence of the iterated directional derivatives is clear. Assume that there is a fast converging sequence $\{x_n\}_{n\in\mathbb{N}}$ with $|d_v^p f(x_n)| \geq 2^{n^2}$. By the general curve lemma there is a smooth curve c with $c(t_n+t) = x_n + \frac{t}{2^n}v$ locally with t_n converging to 0 (by translation), then $(f\circ c)^{(p)}(t_n) = d_v^p f(x_n)\frac{1}{2^{np}}$, which yields a contradiction. Remark that this argument is valid on c^∞-open subsets of a locally convex space.

3. $\Rightarrow$ 1.: First we show the continuity of $d_v^p f$ by induction on p. For $p = 0$ we refer to 2. $\Rightarrow$ 1., for $p > 0$ we apply that $d_v^p f(.+tv) - d_v^p f(.) = t\int_0^1 d_v^{p+1} f(.+t\tau v)d\tau \to 0$ for $t \to 0$ uniformly on bounded sets. Assume that there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x with $d_v^p f(x_n) - d_v^p f(x) \geq \epsilon$, then there is $\delta > 0$ such that for $|t| < \delta$

$$d_v^p f(x_n + tv) - d_v^p f(x + tv) \geq \frac{\epsilon}{2}$$

Integration $\int_0^\delta dt$ leads to

$$d_v^{p-1} f(x_n + \delta v) - d_v^{p-1} f(x) - (d_v^{p-1} f(x + \delta v) - d_v^{p-1} f(x)) \geq \frac{\epsilon\delta}{2}$$

but by induction the left hand side converges to 0. By convolution with a Dirac sequence we can conclude: $f_\epsilon := f * \phi_\epsilon \to f$ in $C(\mathbb{R}^2,\mathbb{R})$ for any continuous function f. f_ϵ is smooth, furthermore $(d_v^p f)_\epsilon = d_v^p f_\epsilon$by the properties of convolution and the uniform convergence on bounded sets. There is a universal formula expressing iterated partial derivatives by iterated directional derivatives depending on the v's inserted. So we know that the iterated partial derivatives of f_ϵ converge to continuous functions, but then we can easily prove that f is smooth.

Remark that all these statements could be made in the case of vector-valued functions by testing scalarly. $\square$

The exponential law is a categorical statement of the type $Z^{X\times Y} = (Z^X)^Y$, where X^Y denotes the morphisms from Y to X (which should be in the category). First we provide $C^\infty(U,F)$ with the initial locally convex structure given through

$$C^\infty(U,F) \xrightarrow{c^*} C^\infty(\mathbb{R},F) \text{ for all } c \in C^\infty(\mathbb{R},U)$$

This locally convex structure can be tested by evaluation of smooth functions along smooth curves. It coincides with the usual structure on $C^\infty(\mathbb{R}^n,\mathbb{R}^m)$. In our case we shall prove $C^\infty(E\times F,G) = C^\infty(E,C^\infty(F,G))$. In order to do this it is sufficient to prove the simplest case of the exponential law:

1.13. Theorem. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary mapping, then all partial derivatives exist and are locally bounded if and only if the associated map $\check{f} : \mathbb{R} \to C^\infty(\mathbb{R},\mathbb{R})$ exists as a smooth curve.*

Proof. We show the theorem in several steps (see [**KM97**], 3.2.): By Boman's theorem we know that the existence of all partial derivatives with respective local boundedness means the existence of the associated map $\check{f} : \mathbb{R} \to C^\infty(\mathbb{R},\mathbb{R})$. We observe that $(d^q(\ \check{f}(t)))(s) = \partial_2^q f(t,s)$. To prove smoothness of $\check{f}$ we look at $c(t) = \frac{\check{f}(t+a)-\check{f}(a)}{t}$ for $t \neq 0$ and $c(0)(s) = \partial_1 f(a,s)$ and prove continuity of $c : \mathbb{R} \to C^\infty(\mathbb{R},\mathbb{R})$. This can be tested by looking at $d^q \circ c : \mathbb{R} \to C(\mathbb{R},\mathbb{R})$. By the exponential law $C(\mathbb{R}^2,\mathbb{R}) =C(\mathbb{R},C(\mathbb{R},\mathbb{R}))$ we arrive at $\widehat{d^q \circ c} : \mathbb{R}^2 \to \mathbb{R}$. If these functions are continuous, then c is continuous. However, the formulation is already the proof, since

$$\widehat{d^q \circ c}(t,s) = \int_0^1 \partial_1\partial_2^q f(t\tau,s)d\tau \text{ for all } (t,s)$$

which is continuous on $\mathbb{R}^2$. The rest is done by induction, because $d\check{f} = \check{\partial_1 f}$.

The other direction can be easily checked by writing down what has to be shown. $\square$

To be able to handle more general structures and to give a more abstract, but coherent idea of convenient calculus we introduce Frölicher spaces (smooth spaces).

1.14. Definition. *A non-empty set X, a set of curves $C_X \subset Map(\mathbb{R},X)$ and a set of mappings $F_X \subset Map(X,\mathbb{R})$ are called a Frölicher space if the following conditions are satisfied:*

1. *A map $f : X \to \mathbb{R}$ belongs to F_X if and only if $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ for $c \in C_X$.*
2. *A curve $c : X \to \mathbb{R}$ belongs to C_X if and only if $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ for $f \in F_X$*

Let X be a Frölicher space, then C_X is called the set of smooth curves, F_X the set of smooth real valued functions. Mappings between Frölicher spaces are called smooth if they map smooth curves to smooth curves. Let X, Y be Frölicher spaces then $C^\infty(X,Y)$ has a natural structure of a Frölicher space due to the following requirement:

$$C^\infty(X,Y) \overset{C(f,c)}{\to} C^\infty(\mathbb{R},\mathbb{R}) \overset{\lambda}{\to} \mathbb{R}$$

is a smooth map for $c \in C_X$, $f \in F_Y$ and $\lambda \in C^\infty(\mathbb{R},\mathbb{R})'$, where $C(f,c)(\phi) := f \circ \phi \circ c$. The Frölicher space structure generated by these smooth maps is the canonical structure on $C^\infty(X,Y)$.

1.15. Remark. Let U be c^∞-open in a convenient vector space, then the structure

$$[U, C^\infty(\mathbb{R},U), C^\infty(U,\mathbb{R})]$$

constitutes a Frölicher space. Given a convenient vector space F, the Frölicher space structure on $C^\infty(U,F)$ coincides with the structure given through the convenient structure of $C^\infty(U,F)$. We show that the smooth curves and the smooth functions are the same: A curve $d : \mathbb{R} \to C^\infty(U,F)$ is smooth with respect to the convenient structure if and only if $\lambda \circ l_* \circ c^* \circ d : \mathbb{R} \to \mathbb{R}$ is smooth for all $c \in C^\infty(\mathbb{R},U)$ and $l \in F'$, $\lambda \in C^\infty(\mathbb{R},\mathbb{R})'$. This is equivalent to the smoothness of $\lambda \circ f_* \circ c^* \circ d : \mathbb{R} \to \mathbb{R}$ for $\lambda \in C^\infty(\mathbb{R},\mathbb{R})'$, $f \in C^\infty(F,\mathbb{R})$ and $c \in C^\infty(\mathbb{R},U)$. Smooth maps are defined in the same way, consequently they coincide, since in both categories smooth functions determine smooth curves.

1.16. Proposition. *The category of Frölicher spaces is complete, cocomplete and cartesian closed.*

Proof. (see [**KM97**], 23.2.) We can look at the respective limits in the category of sets and provide them with an obvious respective Frölicher structure which is a fortiori the (co)limit in the category of Frölicher spaces.

The exponential law asserts that the natural map

$$i : C^\infty(X, C^\infty(Y,Z)) \to C^\infty(X \times Y, Z)$$

exists and is a diffeomorphism. Given $f : X \times Y \to Z$, $g : X \to C^\infty(Y,Z)$ smooth we investigate, when $i^{-1}(f) = \check{f}$, $i(g) = \widehat{g}$ is smooth:

$\check{f}$ is smooth $\iff$

$\check{f} \circ c_X$ is smooth for all $c_X \in C_X \iff$

$C(f_Z, c_Y) \circ \check{f} \circ c_X \in C^\infty(\mathbb{R},\mathbb{R})$ for all $c_X \in C_X$, $c_Y \in C_Y$, $f_Z \in F_Z \iff$

$\widehat{C(f_Z, c_Y) \circ \check{f}} \circ c_X = f_Z \circ f \circ (c_X \times c_Y) = f_Z \circ (\widehat{c_Y^* \circ \check{f}} \circ c_X) : \mathbb{R}^2 \to \mathbb{R}$ is smooth for all $c_X \in C_X$, $c_Y \in C_Y$, $f_Z \in F_Z$ by the simplest case of the exponential law.

Each smooth curve into $X \times Y$ (c_X, c_Y) is of the form $(c_X \times c_Y) \circ \Delta$ where $\Delta : \mathbb{R} \to \mathbb{R}^2$denotes the diagonal map. So $\check{f}$ is smooth if and only if f is smooth, which implies existence of i.

By this observation we already get that i is a diffeomorphism, because

$$C^\infty(\mathbb{R}, C^\infty(X, C^\infty(Y,Z)) = C^\infty(\mathbb{R} \times X \times Y, Z) = C^\infty(\mathbb{R}, C^\infty(X \times Y, Z))$$

via the mapping i. □

Convenience is not preserved by colimits, so the cocompleteness statement in the category of convenient vector spaces with smooth maps has to be replaced by a weaker one. Namely, convenience is preserved by direct limits, strict inductive limits of sequences of closed embeddings.

Sometimes it is very instructive to have good counterexamples in mind, especially in these rather subtle matters (see [**KM97**], ch.1):

1.17. Theorem. *The following statements are false:*

1. *The c^∞-closure of a subset or a linear subspace is given by the limits of all Mackey-converging sequences of the respective set in the total space.*
2. *A c^∞-dense subspace of a convenient vector space has this space as c^∞-completion.*
3. *If a space is c^∞-dense in a total space, then it is c^∞-dense in all linear spaces lying in-between.*
4. *Every bounded linear functional on a subspace can be extended to a bounded linear functional on the total space.*
5. *A linear subspace of a bornological locally convex space is bornological.*

In the following theorem we collect the relevant statements on convenient calculus, which shall be applied in the thesis (see [**KM97**], ch.1 for more detailed proofs):

1.18. Theorem. *Let E, F, G be convenient vector spaces, $U \subset E$, $V \subset F$ c^∞-open, then we obtain:*

1. *Multilinear mappings are smooth if and only if they are bounded.*
2. *If $f : U \to F$ is smooth, then $df : U \times E \to F$ and $df : U \to L(E, F)$ are smooth.*
3. *The chain rule holds.*
4. *The vector space $C^\infty(U, F)$ of smooth mappings $f : U \to F$ is again a convenient vector space with the following initial topology:*
$$C^\infty(U,F) \xrightarrow{c^*} \prod_{c \in C^\infty(\mathbb{R},U)} C^\infty(\mathbb{R},F) \xrightarrow{\lambda_*} \prod_{c \in C^\infty(\mathbb{R},U),\, \lambda \in F'} C^\infty(\mathbb{R},\mathbb{R})$$
5. *The exponential law holds, i.e.*
$$i : C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$
is a linear diffeomorphism of convenient vector spaces. Usually we write $i(f) = \hat{f}$ and $i^{-1}(f) = \check{f}$.
6. *Taylor's formula is true, if by applying cartesian closedness and obvious isomorphisms one defines the multilinear-mapping-valued higher derivatives $d^n f : U \to L^n(E, F)$ of a smooth function $f \in C^\infty(U, F)$, more precisely for $x \in U$, $y \in E$ so that $[x, x + y] = \{x + sy | 0 \leq s \leq 1\} \subset U$ we have the formula*
$$f(y) = \sum_{i=0}^{n} \frac{1}{i!} d^i f(x) y^{(i)} + \int_0^1 \frac{(1-t)^n}{n!} d^{n+1} f(x + ty)\,(y^{(n+1)}) dt$$
for all $n \in \mathbb{N}$.
7. *The smooth uniform boundedness principle is valid: A linear mapping $f : E \to C^\infty(V, G)$ is smooth (bounded) if and only if $ev_v \circ f : E \to G$ is smooth for $v \in V$, where $ev_v : C^\infty(V, G) \to G$ denotes the evaluation at the point $v \in V$.*
8. *The smooth detection principle is valid: $f : U \to L(F, G)$ is smooth if and only if $ev_x \circ f : U \to G$ is smooth for $x \in F$ (This is a reformulation of the smooth uniform boundedness principle by cartesian closedness).*
9. *Let E be a normed space and $f : U \to F$ be a Lip^0-mapping, then f is locally Lipschitz, i.e. for every point $x \in U$ there is a neighborhood $V \subset U$ with*
$$\{\frac{f(x) - f(y)}{||x - y||} | x \neq y \in V\} \text{ is bounded in } F$$

Proof. 1. Apply the exponential law $L^2(E, F) = L(E, L(E, F))$ inductively.

2. Smoothness is clear by definition, the rest is given by the exponential law. By the uniform boundedness principle for convenient vector spaces the point evaluations are sufficient.

3.-6. has already been proved, where Taylor's formula is obtained via a scalar proof.

7.-8. We need bounded linear functionals on $L(F, G)$ which detect the bornology. However, that the point evaluations detect the bornology is the assertion of the uniform boundedness principle on convenient vector spaces.

9. We assume that for a point $z \in U$ there are sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ and a linear bounded functional $l \in F'$ with $||x_n - z|| \leq \frac{1}{2^n}$, $||y_n - z|| \leq \frac{1}{2^n}$, $x_n \neq y_n$ and
$$|l(f(x_n)) - l(f(y_n))| \geq ||x_n - y_n||n$$
Now we take a Lip^0-curve with $c(t) = x_1$ for $t \leq 0$, then c runs with constant speed from x_1 to y_1 in the time interval $[0, ||x_1 - y_1||]$, then c runs from y_1 to x_2 with constant speed on the interval $[||x_1 - y_1||, ||x_1 - y_1|| + ||y_1 - x_2||]$ and so on. The construction finishes at a certain point t_∞ where we continue by z. We obtain consequently $||c(t) - c(s)|| \leq |t - s|$. Around t_∞ the Lip^0-curve lies in U, so we can look at $l \circ f \circ c$, but this yields a contradiction by construction. □

2. Analytic Calculus

The concepts of smooth convenient calculus can be easily carried over to the holomorphic world (see [**KM97**], ch.2 for details and references). In this section we assume all convenient vector spaces to be complex or equivalently that on a real convenient vector space E there is a bounded linear map, the complex structure , $J : E \to E$ with $J^2 = -id_E$, which will be referred to as complex structure. Holomorphic curves to locally convex vector spaces are a common concept, i.e. for all $|z| < 1$ the limit $\frac{c(z+w)-c(z)}{w}$ exists in E for $w \to 0$ on the unit disk. By E^* we denote the space of complex linear functionals, a complex linear functional, i.e. $l \circ J = i \cdot l$, is uniquely determined by its real part, so $E'_{\mathbb{R}} \simeq E^*$. By $\mathbb{D}$ we denote the unit disk in $\mathbb{C}$. Complex linear functionals detect boundedness, too.

Given a sequence $\{a_n\}_{n\in\mathbb{N}}$ in a convenient vector space, then by Abel's theorem, the boundedness of

$$\{r^n a_n \,|\, n \in \mathbb{N}\} \text{ for all } |r| < 1$$

in E is equivalent to strong and weak convergence of the power series $\sum_{n\in\mathbb{N}} z^n a_n$ on $\mathbb{D}$, the convergence is Mackey and uniform on compact subsets.

If a curve on the unit disk is weakly holomorphic, then for any continuous $l \in E^*$ the difference quotient $l(\frac{c(z+w)-c(z)}{w})$ extends to a holomorphic function in w on a small neighborhood of 0, so it is locally Lipschitz and consequently the difference quotient forms a Mackey-Cauchy net. If c is holomorphic, then for all continuous $l \in E^*$ $l \circ c$ is weakly holomorphic with derivative $l \circ c'$, so $\frac{1}{z}(\frac{c(z)-c(0)}{z} - c'(0))$ is complex differentiable by the previous statement. Consequently $l(\frac{1}{z}(\frac{c(z)-c(0)}{z} - c'(0)))$ is locally bounded for a bounded complex linear functional l.

The equivalence of weak and strong complex differentiability yields immediately that many theorems of classical complex analysis can be carried to the generic case: So holomorphic curves have a Taylor representation as power series, Cauchy's theorem is valid. A curve is holomorphic if and only if it is Lip^1 and the first derivative is complex linear. Furthermore if a curve c is holomorphic, then for $B \subset E$ bounded, absolutely convex and closed c factors as holomorphic curve to E_B, since it factors as Lip^1-curve and the derivative is complex linear.

Holomorphic mappings on c^∞-open subsets of a convenient vector space are those mapping holomorphic curves to holomorphic ones.

2.1. Proposition (Hartog's theorem). *Let E_1, E_2 be convenient vector spaces with U c^∞-open in $E_1 \times E_2$, then $f : U \to F$ is holomorphic if and only if it is separately holomorphic.*

Proof. (see [**KM97**], 7.9.) Since $incl_y^{-1}(E_1 \times \{y\} \cap U\ )$ is c^∞-open in E_1 $f(.,y)$ is holomorphic. Assume that f is separately holomorphic: For any holomorphic curve $(c_1, c_2) : \mathbb{D} \to U$ we consider the holomorphic mapping $c_1 \times c_2 : \mathbb{D}^2 \to E_1 \times E_2$, which is smooth. So $(c_1 \times c_2)^{-1}(U)$ is open in $\mathbb{D}^2$. Given $l \in F^*$ we know that $l \circ f \circ (c_1 \times c_2) : (c_1 \times c_2)^{-1}(U) \to \mathbb{C}$ is separately holomorphic and consequently holomorphic by the classical Hartog's theorem. Composition with the diagonal map yields the result. $\square$

Let $f : U \to F$ be a holomorphic mapping, then the derivative $df(v)(w) = \frac{d}{dz}|_{z=0} f(v + zw)$ is complex linear in w and holomorphic in both variables. Complex linearity follows from composition with a bounded complex linear functional and restriction to a two-dimensional subspace.

A multilinear mapping between convenient vector spaces is bounded if and only if it is holomorphic. Since this can be tested separately by Hartog's theorem it is sufficient to check it for complex linear functionals: One direction is already clear, assume that a complex linear functional is holomorphic: Given a sequence $\{a_n\}_{n\in\mathbb{N}}$ with $|f(a_n)| > 1$ and $\{2^n a_n |\ n \in \mathbb{N}\}$ is bounded in E. The power series $c(z) := \sum_{n\in\mathbb{N}_+}(a_n - a_{n-1})(2z)^n$ describes a holomorphic curve in E, so $f \circ c$ is holomorphic and has a power series expansion $(f \circ c)(z) = \sum_{n\in\mathbb{N}} b_n z^n$, however,

$$b_n = (f(a_n) - f(a_{n-1}))2^n$$

by linearity, so $0 = f(0) = f(c(\frac{1}{2})) = \sum_{n\in\mathbb{N}_+} f(a_n) - f(a_{n-1}) = \lim_{n\to\infty} f(a_n)$, which is a contradiction.

To finish with the basics of convenient calculus we need some properties of power series on convenient vector spaces: Let f_k be k-linear symmetric scalar valued bounded mappings on a Fréchet space E for $k \in \mathbb{N}$, then the following statements are equivalent by the Baire property (see [**KM97**], 7.14.):

1. $\sum_{k\in\mathbb{N}} f_k$ converges pointwise on an absorbing subset of E.
2. $\sum_{k\in\mathbb{N}} f_k$ converges uniformly and absolutely on some neighborhood of 0.
3. $\{f_k(x^k) \mid k \in \mathbb{N},\, x \in U\}$ is bounded on some neighborhood U of 0.
4. $\{f_k(x_1, ..., x_k) \mid k \in \mathbb{N},\, x_j \in U\}$ is bounded on some neighborhood U of 0.

If a power series $\sum_{k\in\mathbb{N}} f_k$ converges pointwise on a convenient vector space and the resulting mapping is bounded on bounded sets, then the convergence is uniform on bounded sets.

2.2. Theorem. *Let $f : U \subset E \to F$ be a mapping from a c^∞-open subset, then the following assertions are equivalent:*

1. *f is holomorphic.*
2. *For all $l \in F'$ and all absolutely convex closed and bounded subsets B the mapping $l \circ f : E_B \to \mathbb{C}$ is holomorphic.*
3. *f is holomorphic along (complex) affine lines and is bounded on bornologically compact subsets.*
4. *f is holomorphic along (complex) affine lines and is c^∞-continuous.*
5. *f is holomorphic along (complex) affine lines and at each point the first derivative is a bounded complex linear mapping.*
6. *f is given c^∞-locally by a convergent series of bounded homogeneous complex polynomials.*
7. *f is holomorphic along (complex) affine lines and in every connected component of U there is at least one point where all derivatives are bounded complex multilinear mappings.*
8. *f is smooth and the derivative is complex linear at any point.*
9. *f is Lip^1 and the derivative is complex linear at any point.*

Proof. (see [**KM97**], 7.19.) Since holomorphic curves factor over some E_B-spaces the first and second assertion are equivalent, consequently we can reduce the prove to the scalar valued case and to the case of a Banach space E, with exception of 6., which will be proved to be equivalent at the end of the proof:

1. $\Rightarrow$ 5.: The complex derivative is holomorphic in both variables, consequently bounded and complex linear in the second.

1. $\Rightarrow$ 6.: We choose a fixed point $z \in U$, along the affine lines through z the mapping f is given by a pointwise convergent power series, by classical Hartog's theorem this is true for all affine finite dimensional subspaces in U. The mapping $df : U \to E'$ is well-defined and holomorphic, since this can be tested by point evaluation, consequently we can proceed by induction to obtain that the higher derivatives are bounded complex multilinear mappings at each point, representing the function by a power series converging pointwise (see power series on Fréchet spaces).

6. $\Rightarrow$ 3.: By the above remarks uniform convergence follows immediately on a Banach space E and the resulting function is continuous.

3. $\Rightarrow$ 4.: is obvious.

4. $\Rightarrow$ 5.: By the one-dimensional Cauchy integral formula

$$df(z)(v) = \frac{1}{2\pi i}\int_{|\lambda|=1} \frac{f(z+\lambda v)}{\lambda^2} d\lambda$$

the first derivative at a point z is bounded for compact sets K for which $\{z+\lambda v \in U \mid v \in K,\, |\lambda| \le 1\}$, consequently a bounded linear functional.

6. $\Rightarrow$ 1.: Composing the power series of a curve and a power series yields a holomorphic curve on Fréchet spaces again by the Baire property.

6. $\Rightarrow$ 7.: is obvious.

7. $\Rightarrow$ 1.: By Hahn's theorem (see [**KM97**], 7.18) the limit of a sequence of almost continuous functions on a Baire space is almost continuous. The first derivative as a limit of difference quotients is almost continuous, so continuous since it is linear. Consequently the set where f is holomorphic near a point contains with a point z the star around z in U around by 5. $\Rightarrow$ 1.. The set is not empty since near the point where all derivatives are bounded multilinear the function is holomorphic. So f is holomorphic since every connected component is polygonally connected.

8. $\Rightarrow$ 9.: is obvious

9. $\Rightarrow$ 3.: f is holomorphic along (complex) affine lines and c^∞-continuous

1. $\Rightarrow$ 8.: All derivatives are again holomorphic and thus locally bounded and smooth.

For the case of a convenient space E we have to show the equivalence of the equivalent other assertions:

6. $\Rightarrow$ 1.: $f|_{U\cap E_B}$ is locally the pointwise limit of some polynomials, so holomorphic on $U \cap E_B$, consequently it is holomorphic, since holomorphic curves factor to some E_B as holomorphic curves.

1. $\Rightarrow$ 6.: Let f be holomorphic on U. The mapping $f|_{U\cap E_B}$ satisfies all equivalent assertions of the theorem for each closed absolutely convex bounded B, so f is smooth and its Taylor series converges pointwise on a c^∞-open neighborhood, since it converges on the star around z in U. $\square$

By the real chain rule and the previous theorem we obtain that the chain rule holds. The space $\mathcal{H}(U,F)$ is a closed subspace of the smooth functions, since complex linearity is preserved by limits in $C^\infty(U,F)$. We provide it with the induced convenient vector space structure:

2.3. Theorem (Cartesian closedness). *For convenient vector spaces E_1, E_2 and F, $U_i \subset E_i$ the following convenient vector spaces are isomorphic via the "unifying map"*

$$i : \mathcal{H}(U, \mathcal{H}(V,G)) \cong \mathcal{H}(U \times V, G)$$

Proof. (see [**KM97**], 7.22.) For given f the mapping $i^{-1}(f) = \check{f}$ is smooth. Its derivative is canonically associated to the first partial derivative, which is complex linear, so $\check{f}$ is holomorphic. If $\check{f}$ is holomorphic, then we conclude by cartesian closedness that f is smooth, the derivative is complex linear as composition of complex linear maps:

$$df(x,y)(u,v) = ((d\check{f})(x)v)(y) + (d \circ \check{f})(x)(y)w$$

The map i is bibounded by smooth cartesian closedness and the closedness of the holomorphic functions in $C^\infty(U,F)$. $\square$

Analytic calculus will be useful in determining the convergence of the exponential series in the second chapter.

3. Convenient Manifolds

Convenient Calculus was designed as a basis for infinite dimensional geometry (see [**KM97**] for details and references). The concept of a convenient manifold is common, only some topological questions have to be answered more precisely. As far as vector fields are concerned one already feels the difficulties of infinite dimensional differential geometry, whereas differential forms can be defined straight forward after some subtle investigations. A chart on a set M is a mapping $u : U \to u(U) \subset E_U$, where E_U is a convenient vector space and $U \subset M$, $u(U) \subset E_U$ is c^∞-open. For two charts (u_α, U_α), (u_β, U_β) the chart changing $u_{\alpha\beta} := u_\alpha \circ u_\beta^{-1} : u_{\alpha\beta}(U_{\alpha\beta}) \to u_{\alpha\beta}(U_{\alpha\beta})$, where $U_{\alpha\beta} := U_\alpha \cap U_\beta$. An atlas is a collection of charts such that the U_α form a cover of M and the chart changings are defined on c^∞-open subsets of the respective convenient spaces. A C^∞-atlas is an atlas with smooth chart changings. Two C^∞-atlases are equivalent is their union is an C^∞-atlas, a maximal C^∞-atlas is called a C^∞-structure on M (maximal is understood with respect to some carefully chosen universe of sets). A smooth (convenient) manifold is a set together with a C^∞-structure.

A smooth mapping $f : M \to N$ between smooth manifolds is defined in the canonical way, i.e. for any $x \in M$ there is a chart (V, v) with $f(x) \in V$, a chart (U, u) of M with $x \in U$ and $f(U) \subset V$, such that $v \circ f \circ u^{-1}$ is smooth. This is the case if and only if $f \circ c$ is smooth for all smooth curves $c : \mathbb{R} \to M$, where the concept of a smooth curve is easily set upon.

The final topology with respect to smooth curves or equivalently the final topology with respect to all inverses of chart mappings is the canonical topology of the smooth manifold. We assume manifolds to be smoothly Hausdorff (see the discussion in [**KM97**], p. 265), i.e. the real valued smooth functions on M separate points. In fact we shall often assume smooth regularity (see appendix 1), i.e. the smooth functions generate the topology on M, but we do not add to the general definition of a manifold. The product of smooth manifolds is defined canonically by building up the product of the atlases, however the canonically associated topology of the product of two smooth manifolds may be finer than the product topology of them, since this is true for convenient vector spaces. Smoothly regular manifolds are Frölicher spaces with the real valued smooth functions and the smooth curves as structure elements.

The concept of tangent vector is natural, but poses already several problems (see [**KM97**],28.): The kinematic tangent vectors given as equivalence classes by touching at a point of smooth curves induce bounded derivations on the algebra of germs of smooth functions at a point. Nevertheless the convenient vector space of bounded derivations of the algebra of germs at a point, the so called operational tangent vectors , cannot generically be identified with the kinematic tangent space, since there are much more. We denote the kinematic tangent space by T_aM, the operational tangent space by D_aM for $a \in M$, they are canonically isomorphic to E_α and D_0E_α for the modelling space E_α around a.

The concept of a vector bundle over smooth manifolds is well-known: Let $p : E \to M$ be a smooth mapping between smooth manifolds. A vector bundle chart is a pair (U, ψ), where U is an open subset in M and ψ is a fiber preserving diffeomorphism $\psi : E|_U := p^{-1}(U) \to U \times V$ with $pr_1 \circ \psi = p$, where V is a convenient vector space. Two vector bundle charts are called compatible if $(\psi_1 \circ \psi_2^{-1})(x, v) = (x, \psi_{12}(x)v)$ is a fiber linear isomorphism on $U_1 \cap U_2$. The mapping $\psi_{1,2} : U_1 \cap U_2 \to GL(V)$ is called the transition function between the two vector bundle charts. Atlases are defined in an obvious way. A vector bundle (E, p, M) is a smooth mapping $p : E \to M$ with an equivalence class of vector bundle atlases. p turns out to be a surjective map with a smooth right inverse, the 0-section.

A vector bundle homomorphism $\phi : (E, p, M) \to (F, q, N)$ over $\widetilde{\phi} : M \to N$ is a fiber linear smooth map with $q \circ \phi = \widetilde{\phi} \circ p$. Vector bundle homomorphisms over id are simply referred to as vector bundle homomorphisms.

There is a simple formal classification of vector bundles with standard fiber V over M. Given a cover (U_α) of M associated to an atlas, then the transition functions $\psi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(V)$ satisfy the cocycle condition

$$(C) \begin{cases} \psi_{\alpha,\beta}(x)\psi_{\beta,\gamma}(x) = \psi_{\alpha,\gamma}(x) \text{ for } x \in U_\alpha \cap U_\beta \cap U_\gamma \\ \psi_{\alpha\alpha}(x) = id \text{ for all } x \in U_\alpha \end{cases}$$

Given another atlas over the same cover with chart mappings (U_α, ϕ_α), then the functions $(\phi_\alpha \circ \psi_\alpha^{-1})(x, v) = (x, \tau_\alpha(x)v)$ for some $\tau_\alpha : U_\alpha \to GL(V)$ satisfy

$$\tau_\alpha(x)\psi_{\alpha,\beta}(x) = \phi_{\alpha,\beta}(x)\tau_\beta(x) \text{ for } x \in U_\alpha \cap U_\beta$$

We call such cocycles cohomologuous, the cohomology classes of all cocycles form a set $\check{H}^1((U_\alpha), \underline{GL(V)})$, the first Čech cohomology class of the open cover (U_α) with values in the sheaf $C^\infty(., GL(V)) = \underline{GL(V)}$. By refining the cover we can form a directed system, the direct limit is denoted by $\check{H}^1(M, \underline{GL(V)})$. This set is isomorphic to the set of all isomorphism classes of vector bundles with typical fiber V over M by standard arguments.

By the above description we can easily write down several basic constructions with vector bundles: Given a covariant functor $\mathcal{F}$ from the category of convenient spaces with bounded linear maps, such that $L(V, W) \to L(\mathcal{F}(V), \mathcal{F}(W))$ is smooth. We will refer to such a functor as smooth functor. For a cocycle (U_α, ϕ_α) of the vector bundle (E, p, M) with typical fiber V we define via $\mathcal{F}(\phi_{\alpha,\beta})(x) = \mathcal{F}(\phi_{\alpha,\beta}(x))$ a new cocycle. The cocycle condition and cohomology are preserved by covariance and the functor properties, consequently there is a smooth vector bundle $\mathcal{F}((E, p, M))$ over M with typical fiber $\mathcal{F}(V)$. The same construction is valid for contravariant functors via $\mathcal{F}(\phi_{\alpha,\beta})(x) = \mathcal{F}(\phi_{\alpha,\beta}^{-1}(x))$, we take as cocycle $\mathcal{F}(\phi_{\alpha,\beta}^{-1})$, such that the cocycle condition is satisfied. So many known functors of linear algebra can be extended to the curved vector bundle case.

The pullback (f^*E, f^*p, M) of a vector bundle (E, p, N) along $f : M \to N$ is defined in an analogous way by pulling back the cocycles to a cocycle over the pulled back cover.

The definition of the kinematic and operational tangent bundle is now straight forward. Given a smooth manifold with atlas $(M \supset U_\alpha \overset{u_\alpha}{\to} E_\alpha)$, then we consider the equivalence relation $\partial_\alpha \sim \partial_\beta$ if and only if $D(u_{\alpha\beta})\partial_\beta := \partial_\beta(u_{\alpha\beta})^* = \partial_\alpha$ for derivations $\partial_\alpha \in D(u_\alpha(U_\alpha))$ and $\partial_\beta \in D(u_\beta(U_\beta))$ on the disjoint union

$$\sqcup_\alpha D(u_\alpha(U_\alpha))$$

with $D(V) = V \times D_0E$, where $V \subset E$ is c^∞-open. The quotient set is called the operational tangent bundle DM with the obvious footpoint projection $\pi_M : DM \to M$. We define $DU_\alpha := \pi_M^{-1}(U_\alpha)$ and $Du_\alpha : DU_\alpha \to D(u_\alpha(U_\alpha))$ via $Du_\alpha(\partial_\alpha) = \partial_\alpha$, consequently $Du_\alpha(\partial_\beta) = D(u_{\alpha\beta})\partial_\beta$. So the charts are given by (DU_α, Du_α) and they form a smooth atlas since the chart changings are given by

$$Du_\beta \circ Du_\alpha^{-1} = D(u_{\alpha\beta}) : D(u_\alpha(U_\alpha)) \to D(u_\beta(U_\beta))$$

by $D(u_{\alpha\beta})(x,\partial) = (u_{\alpha\beta}(x), D(u_{\alpha\beta})\partial)$. So DM becomes a smooth manifold since DM is automatically smoothly Hausdorff if M is. The kinematic tangent bundle is constructed in the finite dimensional manner without any problems.

The tangent mappings are given in the following way: For $f : M \to N$ a smooth mapping $Df : DM \to DN$ is defined via $D_x f(\partial_x)(h) = \partial_x(h \circ f)$ for h a smooth germ around $f(x)$ for $x \in M$. Df is a smooth mapping and restricts to the kinematic tangent space, where we obtain a smooth mapping $Tf : TM \to TN$. This tangent map can be given classically via mapping curves. All known functorial properties are preserved in this general setting.

$C^\infty(M,F)$ is a convenient vector space with the obvious convenient structure given by the set of smooth curves, the spaces of sections to vector bundles carry convenient structures, too, by pointwise given addition and scalar multiplication, which can be extended to smooth mappings on the whole bundle. We shall not review these structure since we only need the simple fact. We conclude the section by a collection of definitions and results on kinematic vector fields, differential forms and the de Rham-complex: Vector fields are defined as derivations of the sheaf of smooth functions on open sets of a smooth manifold M. If M is smoothly regular, then the vector fields coincide with the derivations of the algebra of smooth functions on M by standard arguments. The vector fields are the smooth sections in the operational tangent bundle DM. The situation becomes again more complicated since the commutator of smooth vector fields, the Lie bracket, is well-defined, but there are some structural problems with the preservation of the degree of the vector field (see [**KM97**], ch.6, section 28). However it is true, that the commutator of two kinematic vector fields is a kinematic one.

Two vector fields $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ are called f-related for a smooth map $f : M \to N$ if $Df \circ X = Y \circ f$. We obtain a bounded Lie algebra homomorphism $f^* : \mathfrak{X}(N) \to \mathfrak{X}(M)$ if f is a local diffeomorphism $f : M \to N$ (i.e. $T_x f$ is invertible on each fiber for $x \in M$). In terms of smooth sections $Y \in C^\infty(N \leftarrow DN)$ the pull back reads as follows: $(f^*Y)(x) = (T_x f)^{-1}(Y_{f(x)})$.

By definition only kinematic vector fields can have integral curves, i.e. a curve $c : J \to M$ with $\frac{d}{dt}c(t) = X(c(t))$. A flow of a kinematic vector field X is a smooth map $Fl^X : U \subset M \times \mathbb{R} \to M$ with the following properties:

-$U \cap (\{x\} \times \mathbb{R})$ is a connected open interval for $x \in M$.

-If $Fl^X_s(x)$ exists, then $Fl^X_{t+s}(x)$ exists if and only if $Fl^X_t(Fl^X_s(x))$ exists, and they are equal.

-$Fl^X_0(x) = x$ for $x \in M$.

-$\frac{d}{dt}Fl^X_t(x) = X(Fl^X_t(x))$

Each kinematic vector field X possessing a flow has a maximal flow Fl^X producing integral curves, furthermore X is Fl^X_t-related for any t. The Lie derivative along a vector field can be defined in terms of flows, but we need for later purposes a slightly more general concept: Let $\phi : U \subset M \times \mathbb{R} \to M$ such that $(t,x) \mapsto (t,\phi(t,x))$ is a diffeomorphism of two open subsets U and V in $M \times \mathbb{R}$. Furthermore $\phi_0(x) = x$ and $\frac{d}{dt}|_{t=0}\phi_t(x) = X(x)$. Then we can define $L_X f = \frac{d}{dt}|_{t=0}\phi_t^* f$ and $L_X Y = \frac{d}{dt}|_{t=0}\phi_t^* Y$ for $f \in C^\infty(M,\mathbb{R})$ and a vector field $X \in C^\infty(M \leftarrow DM)$. We obtain $L_X f = Df \circ X$ and $L_X Y = [X,Y]$. This important alternative definition of the Lie bracket will be useful to define a Lie algebra structure on the right (or left) invariant vector fields of a Lie group.

Let $X, Y \in \mathfrak{X}(M)$ be two vector fields admitting local flows, which are f-related by a smooth map $f : M \to M$, then $f \circ Fl^X_t = Fl^Y_t \circ f$ whenever both sides are defined. This is the basis of the theory of first integrals of differential equations.

For differential forms there are several approaches, which seem to be exchangeable from the finite dimensional point of view (see [**KM97**], 33.). However the infinite dimensional setting forces us to a definite choice of the definition if we want to make the following operations work on differential forms: The Lie derivative along a vector field X, the pull back along a smooth map and the differential d . We define differential forms as smooth sections in the vector bundle of alternating bounded multilinear mappings from the tangent bundle to the trivial bundle $M\times \mathbb{R}$.

The wedge product of two differential forms is given by the wedge product of two forms, which is a classical concept, the antisymmetrization of the tensor product of two forms. We obtain a graded commutative convenient algebra. The insertion operator is well-defined, too.

The pull-back along a function $f : M \to N$ is given in classical terms, the differential d, too: For $\omega \in C^\infty(M \leftarrow L^k_{alt}(TM, M \times \mathbb{R}))$ we have

$$(d\omega)(x)(X_0, ..., X_k) = \sum_{i=0}^{k}(-1)^i X_i(\omega \circ (X_0, ..., \widehat{X_i}, ..., X_k)) +$$
$$\sum_{i<j}(-1)^{i+j}\omega \circ ([X_i, X_j], X_0, ..., \widehat{X_i}, ..., \widehat{X_j}, ..., X_k)$$

The Lie derivative along a kinematic vector field is well-defined via

$$(L_X\omega)(X_1, ..., X_k) = X(\omega(X_1, ..., X_k)) - \sum_{i=1}^{k}\omega(X_0, ..., [X, X_i], ..., X_k))$$

for kinematic vector fields $X, X_0, ..., X_k$. The Lie derivative is again given by

$$\frac{d}{dt}|_{t=0}\phi_t^*\omega = L_X\omega$$

where we apply the above notation. Differential forms of order k are denoted by $\Omega^k(M)$ and form a convenient vector space. We can collect the crucial results (see [**KM97**], 33.18.):

3.1. Theorem. *Let M be a smooth manifold, then on the graded commutative algebra of differential forms the following assertions are valid:*

1. i_X, d *are derivations of degree* $-1, +1$
2. L_X *is a derivation of degree* 0.
3. $[L_X, d] = 0$
4. $[i_X, d] = L_X$
5. $[L_X, L_Y] = L_{[X,Y]}$
6. $[L_X, i_Y] = i_{[X,Y]}$
7. $[i_X, i_Y] = i_{[X,Y]}$

where $[.,.]$ *denotes the graded commutator for graded derivations.*

Naturally the Poincaré Lemma is valid for star-shaped domains on a smooth manifold and the Mayer-Vietoris sequence is exact, so all ingredients of de Rham's cohomology theory have been developed in this fairly general setting.

Smooth regularity (see appendix 1) asserts that the smooth topology on N is initial with respect to the smooth functions in $C^\infty(N, \mathbb{R})$, which is not apriori clear, since there need not be enough global smooth functions. Smooth regularity is indeed a reasonable assumption for smooth manifolds, since otherwise it is impossible to make the rare possible conclusions from local to global in infinite dimensions. If M is smoothly regular, then each germ at a point has a global representative (see [**KM97**], 27.21).

3.2. Lemma. *Given a convenient manifold N. Let $\{c_n\}_{n\geq 0} \subset C^\infty(M, N)$ be a sequence of smooth mappings from a finite dimensional compact manifold M to N, such that for all $m \in M$ the sequence $\{c_n(m)\}_{n\geq 0}$ lies in a sequentially compact set with respect to the topology $c^\infty N$. Let furthermore $c_n^* : C^\infty(N, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ be a Mackey-Cauchy sequence:*

1. *Since N is smoothly Hausdorff by the definition of a smooth manifold, there is a smooth curve $c \in C^\infty(M, N)$ with $c_n(m) \overset{n\to\infty}{\to} c(m)$ for all $m \in M$.*
2. *If N is additionally smoothly regular, then for any $m \in M$ there is a chart (u, U) around $c(m)$ such that almost all c_n lie locally around m (at some fixed open neighborhood V of m) in U and all derivatives of $u \circ c_n$ converge Mackey uniformly on V to the derivatives of $u \circ c$.*

Proof. For any point m there exists at least one adherence point of $\{c_n(m)\}_{n\geq 0}$. By $c_n^* : C^\infty(N, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ Mackey-Cauchy convergent to some bounded linear map A the adherence point has to be unique since smooth functions are continuous with respect to $c^\infty N$ and they separate points by definition of a smooth manifold, we denote the unique adherence point by $c(m)$. Consequently there is a mapping $c : M \to N$ which is the pointwise limit of $\{c_n\}_{n\geq 0}$. The limit of $\{f \circ c_n\}_{n\geq 0}$ is a smooth functions and by continuity equal to $f \circ c$ for all $f \in C^\infty(N, \mathbb{R})$, so c is smooth by the defintion of smoothness. This proves the first assertion.

For the second assertion we need a non-negative bump function f with respect to a chart (u, U) around $c(m)$ taking the value 1 at a small neighborhood of $c(m)$. By uniform convergence of $f \circ c_n$ to $f \circ c$ on a small closed neighborhood V of m we see that on V almost all c_n lie in U. By multiplication of $f \circ u^{-1}$ with any linear functional l on the model space we get gobal functions on N representing locally around $c(m)$ each linear functional. Consequently we obtain that all derivatives of $u \circ c_n$ converge at the point m Mackey to the respective derivative of $u \circ c$ by $l \circ u \circ c_n \to l \circ u \circ c$ Mackey in all derivatives and a quality independent of l. $\square$

CHAPTER 2

Convenient Hille-Yosida-Theory

Man hat eine zweite Heimat in der alles, was man tut, unschuldig ist.
(Robert Musil, Der Mann ohne Eigenschaften)

Much work has been done to make a theory on strongly continuous semigroups on locally convex spaces, however, some main ingredients for useful theories on locally convex spaces have been neglected: First the appropriate setting of calculus should be fixed. This is not really easy due to the surprising fact that some natural mappings on locally convex spaces behave like smooth mappings, but are not continuous. Second the appropriate class of semigroups should be specified: Banach spaces are like a simple piano with one single key, the norm, one can play with. Locally convex spaces are like an organon with infinitely many keys, which can be used to encode the properties of a problem, but additionally a more complicated instruments allows higher symmetry of the pieces played with. Third the appropriate approximation procedures should be specified to be able to produce complex solutions from simple ones.

1. Classical concepts and modern points of view

In the fifties the fundaments of solvability of initial value problems were laid down by connecting knowledge of spectral properties with the solvability of Abstract Cauchy Problems. The old method of solving ordinary differential equations by Laplace transforms awarded new merits. These fundaments were enriched by additional assumptions on the considered Banach spaces as for example lattice structures or C^*-algebra-structures. In this section X, Y denote Banach spaces, T, S C_0-semigroups of continuous linear operators. A semigroup homomorphism is understood to map the identity to the identity. As reference books for strongly continuous semigroups we propose [**EN99**], [**Kan95**] and [**Nag86**].

1.1. Definition (Abstract Cauchy Problem). *Let $(A, D(A))$ be a closed operator on a Banach space X, the Abstract Cauchy Problem (ACP) associated to A with initial value $f \in D(A)$ is the solution of*

$$u \in C^1(\mathbb{R}_{\geq 0}, X)$$
$$u(0) = f \text{ and } u(t) \in D(A) \text{ for } t \geq 0$$
$$\frac{d}{dt}u(t) = Au(t)$$

1.2. Definition (C_0-semigroup). *A strongly continuous semigroup of linear operators on a Banach space X is a semigroup homomorphism $T : \mathbb{R}_{\geq 0} \to L(X)$ with*

$$\lim_{t \downarrow 0} T_t x = x$$

These semigroups are often referred to as C_0-semigroups.

1.3. Remark. T is a strongly continuous semigroup if and only if $T : \mathbb{R}_{\geq 0} \to L(X)_s$ is continuous, where the bounded linear operators carry the strong topology. By Banach-Steinhaus (uniform boundedness principle) there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$||T_t|| \leq M \exp(\omega t) \text{ for } t \geq 0$$

The smallest possible value of ω is called the growth bound $\omega(T)$ and might be $-\infty$, the formula

$$\omega(T) = \lim_{t \downarrow 0} \frac{\ln ||T_t||}{t}$$

is valid. A semigroup T is called bounded if $||T_t|| \leq M$ for $t \geq 0$, T is called contraction semigroup if $M = 1$ is a possible choice (see [**Nag86**] for details).

1.4. Definition. *Let T be a strongly continuous semigroup, then the infinitesimal generator $(A, D(A))$ is defined in the following way:*

$$D(A) = \{x \in X \mid \lim_{t\downarrow 0} \frac{T_t x - x}{t} \text{ exists } \}$$
$$Ax = \lim_{t\downarrow 0} \frac{T_t x - x}{t} \text{ for } x \in D(A)$$

1.5. Remark. Generically the operator $(A, D(A))$ is unbounded on the Banach space X. The subtle relation between the infinitesimal generator and the global object, the strongly continuous semigroup, is the subject of Hille-Yosida-Theory.

1.6. Theorem. *Let $(A, D(A))$ be the generator of a strongly continuous semigroup, then the following assertions are valid:*

1. *If $x \in D(A)$, then $T_t x \in D(A)$ for $t \geq 0$*
2. *The map $t \mapsto T_t x$ is differentiable if and only if $x \in D(A)$:*
$$\frac{d}{dt} T_t x = AT_t x = T_t Ax \text{ for } x \in D(A)$$
3. *For $x \in X$ and $\phi \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ the integral $\int_0^t \phi(s)T_s x ds$ lies in $D(A)$ for any $t \geq 0$ and*
$$A \int_0^t \phi(s)T_s x ds = \phi(t)T_t x - \phi(0)x - \int_0^t \phi'(s)T_s x ds$$
4. *The domain $D(A)$ is dense in X and $(A, D(A))$ is a closed operator. Furthermore $(D(A^n), ||.||+ ||A.||+ ... +||A^n.||)$ are Banach spaces with $D(A^n)$ dense in X. The intersection $\cap_{n\geq 1} D(A^n)$ is dense in X and equipped with the initial topology via $A^n : \cap_{n\geq 1} D(A^n) \to X$, $n \in \mathbb{N}$ a Fréchet space.*
5. *There is only one semigroup with infinitesimal generator A.*

Proof. The first and second statement are clear by definition: The integral exists and for $h < t$ we obtain

$$\frac{T_h - id}{h} \int_0^t \phi(s)T_s x ds = \frac{1}{h}(\int_h^{t+h} \phi(s-h)T_s x ds - \int_0^t \phi(s)T_s x ds) =$$
$$= -\int_h^{t+h} \frac{\phi(s) - \phi(s-h)}{h} T_s x ds + \frac{1}{h}(\int_h^{t+h} \phi(s)T_s x ds - \int_0^t \phi(s)T_s x ds) =$$
$$= -\int_h^{t+h} \frac{\phi(s) - \phi(s-h)}{h} T_s x ds + \frac{1}{h}\int_t^{t+h} \phi(s)T_s x ds - \frac{1}{h}\int_0^h \phi(s)T_s x ds) \overset{h\to 0}{\to}$$
$$= -\int_0^t \phi'(s)T_s x ds + \phi(t)T_t x - \phi(0)x$$

which proves the third assertion. Taking a Dirac sequence $\{\phi_\epsilon\}_{\epsilon>0}$ right from zero with support in $\mathbb{R}_{>0}$ we see that $\int_0^1 \phi_\epsilon(s)T_s x ds \to x$ for $x \in X$ and

$$A \int_0^1 \phi_\epsilon(s)T_s x ds = \int_0^1 \phi_\epsilon'(s)T_s x ds$$

Consequently all domains $D(A^n)$ and their intersection are dense in X. Closedness of A will be proved directly. $T_t x - x = \int_0^t T_t Ax dt$ by 2. and if $x_m \to x$ and $Ax_m \to y$, then by continuity $T_t x - x = \int_0^t T_t y dt$, so $x \in D(A)$ and $y = Ax$. So $(D(A), ||.|| + ||A.||)$ is a Banach space. Assume that $(D(A^n), ||.|| + ||A.|| + ... + ||A^n.||)$ is a Banach space as inductive hypothesis. By 2. $D(A^n)$ is T_t-invariant, we can restrict the semigroup to $D(A^n)$ and obtain a semigroup of bounded linear operators, which is strongly continuous in the complete topology. The infinitesimal generator is $A|_{D(A^{n+1})}$, which is closed by induction. So $(D(A^{n+1}), ||.|| + ||A.|| + ... + ||A^{n+1}.||)$ is a Banach space. This establishes by the way the completeness of the intersection as a Fréchet space.

Assume that there is another strongly continuous semigroup S with infinitesimal generator A, then

$$T_t x - S_t x = \int_0^t \frac{d}{ds}(S_{t-s}T_s x)ds = \int_0^t (S_{t-s}T_s)(Ax - Ax)ds = 0$$

for $x \in D(A)$, consequently by continuity $S = T$. $\square$

1.7. Remark. (see [**Nag86**] for detailed comments) The abstract Cauchy Problem associated to a closed operator A is uniquely solvable for every $f \in D(A)$ if and only if $(A_1, D(A^2))$ is the infinitesimal generator of a strongly continuous semigroup T^1 on the Banach space $E_1 := (D(A), ||.|| + ||A.||)$. $A_1 f = Af$ for $f \in D(A^2)$. Assume unique solvability on $D(A)$ of the abstract Cauchy problem associated to A. Then we can associate a strongly continuous semigroup T^1 of linear operators. We have to prove that T^1_t is continuous and that the infinitesimal generator is $(A_1, D(A^2))$: We denote the unique C^1-solution of (ACP) with initial value f by $u(.,f)$. $T^1_t f = u(t,f)$ for $f \in D(A)$ and $t \geq 0$. We investigate $\eta : E_1 \to C([0,t], E_1)$, $f \mapsto u(.,f)$: Let $f_n \to f$ and $\eta(f_n) \to v$ be sequences, then $u(s, f_n) = f_n + \int_0^s Au(r, f_n)dr \to v(s) = f + \int_0^s Avdr$, then v is continuously differentiable on $[0,t]$. Continuing by T^1 gives the desired $v(s) = u(s,f)$ by uniqueness on $[0,t]$. So η has closed graph and is consequently continuous, evaluation at $s \in [0,t]$ yields that T^1 is a strongly continuous semigroup of bounded linear operators. Denote the infinitesimal generator of $T^1$by B. First we show that $T^1_t Af = AT^1_t f$ for $f \in D(A^2)$. $v(t) := f + \int_0^t u(s, Af)ds$, then $\frac{d}{dt}v(t) = u(t, Af) = Af + \int_0^t Au(r, Af)dr = Av(t)$, so $v(t) = u(t,f)$ and $Au(t,f) = \frac{d}{dt}v(t) = u(t, Af)$. The rest is given by stating the definitions.

Assume that there is a strongly continuous semigroup T^1 with infinitesimal generator A_1, then unique solvability follows from the definitions directly.

The existence of a strongly continuous semigroup T with infinitesimal generator A, a closed operator, is equivalent to unique solvability of (ACP) on $D(A)$ and $\rho(A) \neq \emptyset$, because $(\lambda - A)^{-1} : E \to E_1$ is an equivalence between the two semigroups if they exist in the sense of represenation theory.

The relation between spectral properties of the infinitesimal generator and C_0-semigroups is clarified by the Laplace transform. The existence of the Laplace transform leads to several strong properties of the resolvent and these strong asymptotic properties are seen to be sufficient for the existence. For $\phi_\lambda(t) = \exp(-\lambda t)$ we see that for $\lambda > \omega$ the limit $t \to \infty$ exists, consequently we arrive at the formula by 1.6.2.:

$$(\lambda - A)\int_0^\infty \exp(-\lambda t)T_t x dt = x$$

for all $x \in X$ and $\lambda > \omega$. This formula is valid for complex Banach spaces, too. In this case we can assert the half-plane right from ω lies in the resolvent set, we will emphasize further properties of the resolvent set in the case of analytic semigroups.

Consequently the Laplace transform for $\lambda > \omega$ is the resolvent $R(\lambda, A) := (\lambda - A)^{-1}$ of the infinitesimal generator. If the resolvent set $\rho(A)$ of a closed operator A on a complex Banach space is not empty, then it is open and the resolvent is analytic and satisfies the resolvent equation

$$R(\lambda, A)R(\mu, A)(\mu - \lambda) = R(\lambda, A) - R(\mu, A)$$

for $\lambda, \mu \in \rho(A)$. Immediately this functional equation leads to

$$R(\lambda, A)^{(n)} = (-1)^n n! R(\lambda, A)^{n+1}$$

On the other hand by n-fold differentiation of the Laplace transform one obtains

$$\int_0^\infty (-1)^n t^n \exp(-\lambda t)T_t x dt$$

The identity leads to the well-known condition:

$$R(\lambda, A)^{n+1}x = \int_0^\infty \frac{t^n}{n!}\exp(-\lambda t)T_t x dt$$

for $\Re\lambda > \omega$ and $x \in X$, $n \in \mathbb{N}$. The estimate of the right hand side via the exponential growth constants of T leads to the Hille-Yosida-condition by partial integration:

$$||R(\lambda, A)^n|| \leq \frac{M}{(\Re\lambda - \omega)^n}$$

for $\Re\lambda > \omega$ and $n \in \mathbb{N}$. Surprisingly this simple asymptotic condition on the resolvent is already sufficient for the existence of a strongly continuous semigroup with infinitesimal generator A, which is the contents of the Feller-Hille-Miyadera-Philipps-Yosida-Theorem:

1.8. Theorem (Hille-Yosida-Theorem). *Let X be a Banach space and $(A, D(A))$ a densely defined closed operator, then the following assertions are equivalent:*

1. *A is the infinitesimal generator of a strongly continuous semigroup.*
2. *There are constants $M \geq 1$, $\omega > 0$ such that $]\omega, \infty[\subset \rho(A)$ and*

$$||R(\lambda, A)^n|| \leq \frac{M}{(\lambda - \omega)^n}$$

for $n \in \mathbb{N}$ and $\lambda > \omega$.

Proof. (see [**Kan95**], 1.17.) Necessity was already shown by the previous remarks, so we assume 2.: We define $A_\lambda := \lambda A R(\lambda, A) = \lambda(\lambda R(\lambda, A) - id)$, so a continuous operator. For $x \in D(A)$

$$||AR(\lambda, A)x|| = ||R(\lambda, A)Ax|| \leq \frac{M}{\lambda - \omega}||Ax|| \overset{\lambda \to \infty}{\to} 0$$

$$||AR(\lambda, A)|| = ||\lambda R(\lambda, A) - id|| \leq \frac{\lambda M}{\lambda - \omega} + 1$$

and by denseness of $D(A)$ we can conclude that $AR(\lambda, A)x \to 0$ for $x \in X$, so $\lambda R(\lambda, A)x \to x$ for $x \in X$. This means finally $A_\lambda x = \lambda R(\lambda, A)Ax \to Ax$ for $x \in D(A)$. Since A_λ is bounded the exponential exists and satisfies the following estimate on $]2\omega, \infty[$

$$\begin{aligned}
||\exp(tA_\lambda)|| &\leq \exp(-\lambda t) \sum_{n \in \mathbb{N}} \frac{t^n \lambda^{2n}}{n!} ||R(\lambda, A)^n|| \\
&\leq \exp(-\lambda t) \sum_{n \in \mathbb{N}} \frac{t^n \lambda^{2n}}{n!} \frac{M}{(\lambda - \omega)^n} \\
&\leq M \exp(t \frac{\omega \lambda}{\lambda - \omega}) \leq M \exp(2\omega t)
\end{aligned}$$

for $t \geq 0$. Consequently we can hope for convergence if λ tends to infinity:

$$\begin{aligned}
||\exp(tA_\lambda)x - \exp(tA_\mu)x|| &\leq || \int_0^t \frac{d}{ds}(\exp((t-s)A_\lambda)\exp(sA_\mu))x ds|| \\
&\leq || \int_0^t \exp((t-s)A_\lambda)\exp(sA_\mu)(A_\mu x - A_\lambda x) ds|| \\
&\leq M^2 \exp(4\omega t) t ||A_\mu x - A_\lambda x|| \to 0
\end{aligned}$$

as λ, μ tend to infinity uniformly on bounded intervals for $x \in D(A)$. By the exponential boundedness $\exp(tA_\lambda)x$ converges uniformly on bounded intervals to $T_t x$ for all $x \in X$, where the limit $T_t x$ is a continuous curve and T_t is a bounded linear operator. The semigroup property is preserved by the limit, too. We have to show, that the generator A' of the strongly continuous semigroup T is A. The formula $\exp(A_\lambda t)x - x = \int_0^t \exp(A_\lambda s) A_\lambda x ds$ tends to

$$T_t x - x = \int_0^t T_s A x ds$$

as $\lambda \to \infty$ for $x \in D(A)$. Consequently $x \in D(A')$ and $A'x = Ax$. By the asymptotic properties of $\exp(A_\lambda t)$ we know by application of the Laplace transform that $\lambda - A' = \lambda - A$ are both one-to-one and onto X for $\lambda > 2\omega$, so $A = A'$. □

1.9. Corollary. *$(A, D(A))$ is the generator of a contraction semigroup if and only if $]0,\infty[\subset \rho(A)$ and*

$$||R(\lambda, A)|| \leq \frac{1}{\lambda}$$

for $\lambda > 0$.

To calculate semigroups associated to perturbations approximation formulas are of great importance in the theory of strongly continuous semigroups. In many respects the first appearance of a common perturbation formula is Sophus Lie's

$$\lim_{n\to\infty}(\exp(\frac{t}{n}A)\exp(\frac{t}{n}B))^n = \exp(t(A+B))$$

which was enlarged in several directions in this century.

1.10. Theorem (Chernoff's approximation Theorem). *Let X be a Banach space, $c : \mathbb{R}_{\geq 0} \to L(X)$ curve of uniformly power-bounded operators, i.e. there is $s_0 > 0$ and $M \geq 1$ such that $||c(t)^n|| \leq M$ for $t \in [0, s_0]$ and $n \in \mathbb{N}$. If there is a dense subset $D \subset X$ such that*

$$\lim_{t\downarrow 0}\frac{c(t)x - x}{t} = Ax \text{ for } x \in D$$

and there is $\lambda_0 > 0$ with $(\lambda_0 - A)D$ is dense in X, then $\overline{A}$ is the infinitesimal generator of a strongly continuous semigroup T and

$$s\text{-}lim_{n\to\infty} c(\frac{t}{n})^n = T_t$$

uniformly on compact subsets of $\mathbb{R}_{\geq 0}$.

Proof. (see [**EN99**], ch.3, 5.2.) The essential estimate is a simple consideration on power-bounded operators. Let $S \in L(X)$ be power-bounded via $||S^n|| \leq M$ with given $M \geq 1$, then

$$||\exp(n(S - id))x - S^n x|| \leq \sqrt{n}M||Sx - x||$$

for $n \in \mathbb{N}$ and $x \in X$. Let $n \geq 1$ be fixed, then

$$\exp(n(S - id)) - S^n = \exp(-n)\sum_{k=0}^{\infty}\frac{n^k}{k!}(S^k - S^n)$$

but $||S^k x - S^n x|| \leq |n-k|M||Sx - x||$ by telescoping. So we can estimate

$$\begin{aligned}
||\exp(n(S - id))x - S^n x|| &\leq \exp(-n)\sum_{k=0}^{\infty}(\frac{n^k}{k!})^{\frac{1}{2}}(\frac{n^k}{k!})^{\frac{1}{2}}M|n-k|\cdot||Sx - x|| \\
&\leq M\exp(-n)(\sum_{k=0}^{\infty}\frac{n^k}{k!})^{\frac{1}{2}}(\sum_{k=0}^{\infty}\frac{n^k}{k!}|n-k|^2)^{\frac{1}{2}}\cdot||Sx - x|| \\
&\leq M\exp(-n)\exp(\frac{n}{2})\sqrt{n}\exp(\frac{n}{2})\cdot||Sx - x|| \\
&\leq \sqrt{n}M||Sx - x||
\end{aligned}$$

where we applied the Cauchy-Schwarz inequality and the identity $\sum_{k=0}^{\infty}\frac{n^k}{k!}|n-k|^2 = n\exp(n)$.

Now we define $A_s x := \frac{c(s)x - x}{s}$ for $s_0 \geq s > 0$ and $x \in D$. $||\exp(tA_s)|| \leq M$ for $t \geq 0$ by the boundedness condition on c. As in the proof of the Hille-Yosida-Theorem we conclude that there is a semigroup T and s-$\lim_{n\to\infty}\exp(A_{\frac{t}{n}}t) = T_t$ uniformly on compact subsets of $\mathbb{R}_{\geq 0}$ with infinitesimal generator A', furthermore we know that A' is a closed extension of A. There is $\lambda_0 > 0$ with $(\lambda_0 - A)D$ dense, but $(\lambda_0 - A')$ is invertible as continuous operator, consequently the closure of A is A'. Anyway,

$$\begin{aligned}
||\exp(A_{\frac{t}{n}}t)x - c(\frac{t}{n})^n x|| &= ||\exp(n(c(\frac{t}{n}) - id))x - c(\frac{t}{n})^n x|| \leq \\
&\leq \sqrt{n}M||c(\frac{t}{n})x - x|| \leq \frac{t}{\sqrt{n}}||A_{\frac{t}{n}}x|| \to 0
\end{aligned}$$

for $x \in D$ and $n \to \infty$. By the boundedness properties we conclude strong uniform convergence on compact intervals in $\mathbb{R}_{\geq 0}$. $\square$

The investigation of different classes of C_0-semigroups leads immediately to the rich and applicable class of analytic semigroups. All Banach spaces are assumed to be complex in the sequel:

1.11. Remark. On complex Banach spaces the following assertions are equivalent for a closed operator $(A, D(A))$ due to the Laplace integral formula in the complex case:

1. A is the infinitesimal generator of a strongly continuous semigroup.
2. There are constants $M \geq 1$, $\omega > 0$ such that $]\omega, \infty[\subset \rho(A)$ and
$$||R(\lambda, A)^n|| \leq \frac{M}{(\lambda - \omega)^n}$$
for $n \in \mathbb{N}$ and $\lambda > \omega$.
3. There are constants $M \geq 1$, $\omega > 0$ such that $\{\lambda \in \mathbb{C} \mid \Re\lambda > \omega\} \subset \rho(A)$ and
$$||R(\lambda, A)^n|| \leq \frac{M}{(\Re\lambda - \omega)^n}$$
for $n \in \mathbb{N}$ and $\lambda > \omega$.

1.12. Definition. *A closed linear operator $(A, D(A))$ with dense domain $D(A)$ in a Banach space X is called sectorial of angle δ if there exists $0 < \delta < \frac{\pi}{2}$ such that the sector*
$$\Sigma_{\frac{\pi}{2}+\delta} = \{\lambda \in \mathbb{C} \mid |\arg(z)| < \frac{\pi}{2} + \delta\} \setminus \{0\}$$
is contained in the resolvent set $\rho(A)$ and for each $\epsilon \in]0, \delta[$ there exists M_ϵ such that
$$\|R(\lambda, A)\| \leq \frac{M_\epsilon}{|\lambda|} \text{ for all } \lambda \in \Sigma_{\frac{\pi}{2}+\delta}$$

1.13. Definition (Analytic Semigroups). *Let $0 < \delta \leq \frac{\pi}{2}$ be given. A family of bounded linear operators $\{T(z)\}_{z\in\Sigma_\delta\cup\{0\}}$ is called analytic semigroup if it is a semigroup homomorphism on $\Sigma_\delta \cup \{0\}$, analytic on Σ_δ and if it is strongly continuous on $\Sigma_{\delta'} \cup \{0\}$ for $0 < \delta' < \delta$. If additionally the analytic semigroup is uniformly bounded on $\Sigma_\delta \cup \{0\}$, then we call it bounded analytic semigroup.*

1.14. Remark. Let T be an analytic semigroup on Σ_δ, then there are constants M_ϵ and ω_ϵ such that
$$\|T(z)\| \leq M_\epsilon \exp(\omega_\epsilon |z|) \text{ for all } z \in \Sigma_\epsilon$$
with $\epsilon \in]0, \delta[$. The proof is an application of the Banach-Steinhaus-theorem on Banach spaces (uniform boundedness principle). Therefore we can create bounded analytic semigroups on the given sector Σ_ϵ with infinitesimal generator $A - \omega_\epsilon$, consequently it is sufficient to analyze the theory of bounded analytic semigroups (see [**EN99**] for detailed comments and examples).

For sectorial operators and appropriate paths γ the exponential formula can be given by a Cauchy integral formula.

1.15. Definition. *Let $(A, D(A))$ be a sectorial operator of angle δ, define $T_0 = id$ and continuous operators $T(z)$ for $z \in \Sigma_\delta$ by*
$$T(z) := \frac{1}{2\pi i} \int_\gamma \exp(\mu z) R(\mu, A) d\mu$$
where γ is any piecewise smooth curve in $\Sigma_{\frac{\pi}{2}+\delta}$ going from $+\infty \exp(-i(\frac{\pi}{2} + \delta'))$ to $+\infty \exp(i(\frac{\pi}{2} + \delta'))$ for some $\delta' \in]|\arg(z)|, \delta[$.

We obtain the following properties of the operators $T(z)$:

1.16. Theorem. *Let $(A, D(A))$ be a sectorial operator of angle δ. Then the bounded linear operators satisfy the following assertions:*

1. *$\|T(z)\|$ is uniformly bounded for $z \in \Sigma_{\delta'}$ if $0 < \delta' < \delta$.*
2. *T is analytic on Σ_δ.*
3. *T is a semigroup homomorphism from the semigroupunder addition $\Sigma_\delta \cup \{0\}$ to the semigroup of bounded linear operators on X.*
4. *The map $z \mapsto T(z)$ is strongly continuous in $\Sigma_{\delta'} \cup \{0\}$ for $0 < \delta' < \delta$.*

Proof. (see [**EN99**], ch.2, 4.3.) First we show that the integral is well-defined and produces a uniformly bounded family of continuous linear operators. We verify that the integral converges uniformly for $z \in \Sigma_{\delta'}$ if $0 < \delta' < \delta$. We choose a special path $\gamma_r = \gamma_{1,r} + \gamma_{2,r} + \gamma_{3,r}$, coming in from ∞ with angle $-(\frac{\pi}{2} + \delta - \epsilon)$ until the radius $r = \frac{1}{|z|}$ is reached, then moving along the circle with radius r to the angle $(\frac{\pi}{2} + \delta - \epsilon)$ and passing along the radius to ∞, where $\epsilon = \frac{\delta - \delta'}{2}$.

We collect the following estimates

$$\frac{1}{|\mu z|}\Re(\mu z) = \cos(\arg \mu + \arg z) \leq \cos(\frac{\pi}{2} + \epsilon) = -\sin \epsilon \text{ for } \mu \in \gamma_{3,r} \cup \gamma_{1,r},\ z \in \Sigma_{\delta'}$$
$$|\exp(\mu z)| \leq \exp(-|\mu z| \sin \epsilon) \text{ for } \mu \in \gamma_{3,r} \cup \gamma_{1,r},\ z \in \Sigma_{\delta'}$$

and put them together in the integral

$$\begin{aligned}||\int_{\gamma_r} \exp(\mu z) R(\mu, A) d\mu|| &\leq 2M_\epsilon \int_{\frac{1}{|z|}}^{\infty} \frac{1}{\rho} \exp(-\rho |z| \sin \epsilon) d\rho + 2\pi e M_\epsilon \leq \\ &\leq 2M_\epsilon \int_{1}^{\infty} \frac{1}{\rho} \exp(-\rho \sin \epsilon) d\rho + 2\pi e M_\epsilon\end{aligned}$$

for $z \in \Sigma_{\delta'}$. Consequently the integral converges uniformly on compact subsets for $z \in \Sigma_{\delta'}$ and is uniformly bounded on $\Sigma_{\delta'}$. Furthermore by the Cauchy theorem of complex analysis the contour can be chosen "arbitrarily" and the resulting mapping T is holomorphic on Σ_δ.

The semigroup property follows from the resolvent equation: We choose some constant $c > 0$ such that $\gamma_1 \cap (\gamma_1 + c) = \emptyset$, then for $z_1, z_2 \in \Sigma_{\delta'}$ we obtain via $(\gamma = \gamma_1)$

$$\begin{aligned}T(z_1)T(z_2) &= \frac{1}{(2\pi i)^2} \int_\gamma \int_{\gamma + c} \exp(\mu z_1) R(\mu, A) \exp(\lambda z_2) R(\lambda, A) d\mu d\lambda \\ &= \frac{1}{(2\pi i)} \int_\gamma \exp(\mu z_1) R(\mu, A) (\frac{1}{(2\pi i)} \int_{\gamma + c} \frac{\exp(\lambda z_2)}{\lambda - \mu} d\lambda) d\mu - \\ &- \frac{1}{(2\pi i)} \int_\gamma \exp(\lambda z_2) R(\lambda, A) (\frac{1}{(2\pi i)} \int_{\gamma + c} \frac{\exp(\mu z_1)}{\lambda - \mu} d\mu) d\lambda \\ &= \frac{1}{(2\pi i)} \int_\gamma \exp(\mu(z_1 + z_2)) R(\mu, A) du\end{aligned}$$

which is the desired relation. Strong continuity is proved by the following simple observation:

$$\begin{aligned}T(z)x - x &= \frac{1}{2\pi i} \int_\gamma \exp(\mu z)(R(\mu, A)x - \frac{1}{\mu} x) d\mu \\ &= \frac{1}{2\pi i} \int_\gamma \frac{\exp(\mu z)}{\mu} R(\mu, A) A x d\mu \to \frac{1}{2\pi i} \int_\gamma \frac{1}{\mu} R(\mu, A) A x d\mu = 0\end{aligned}$$

as $z \to 0$ in $\Sigma_{\delta'}$ by Lebeque's dominated convergence theorem. □

The generator of the semigroup restricted to $\mathbb{R}_{\geq 0}$ T defined by the Cauchy formula is the sectorial operator $(A, D(A))$. We calculate the Laplace transform of T for $\lambda > 2$

$$\begin{aligned}\int_0^{t_0} \exp(-\lambda t) T(t) x dt &= \frac{1}{2\pi i} \int_\gamma \frac{\exp(t_0(\mu - \lambda)) - 1}{\mu - \lambda} R(\mu, A) x d\mu \\ &= R(\lambda, A)x + \frac{1}{2\pi i} \int_\gamma \frac{\exp(t_0(\mu - \lambda))}{\mu - \lambda} R(\mu, A) x d\mu\end{aligned}$$

by Fubini's theorem, but the last term tends to zero as $t_0 \to \infty$, consequently the resolvents coincide. We shall collect some already rather complicated results on analytic bounded semigroups:

1.17. Theorem. *Let $(A, D(A))$ be a closed operator on a Banach space X, then the following assertions are equivalent:*

1. *A generates a bounded analytic semigroup on some $\Sigma_\delta \cup \{0\}$.*
2. *There exists $\theta \in]0, \frac{\pi}{2}[$ such that $\exp(\pm i\theta)A$ generate bounded strongly continuous semigroups.*

3. *A generates a bounded strongly continuous semigroup T such that $T(t)(X) \subset D(A)$ for $t > 0$ and*

$$M := \sup_{t>0} ||tAT(t)|| < \infty$$

4. *A generates a bounded strongly continuous semigroup T and there exists a constant $C > 0$ such that for $r > 0$ and $s \neq 0$*

$$||R(r + is, A)|| \leq \frac{C}{|s|}$$

5. *A is a sectorial operator.*

Proof. (see [**EN99**], ch. 2, 4.6.) We shall prove 1. $\Rightarrow$ 2. $\Rightarrow$ 4. $\Rightarrow$ 5. $\Rightarrow$ 3. $\Rightarrow$ 1.:

1. $\Rightarrow$ 2.: For $\theta \in]0, \delta[$ we define $T^\theta_t = T_{\exp(i\theta)t}$ which is a strongly continuous semigroup of continuous linear operators on X. By Laplace transform we obtain $R(1, A) = \exp(i\theta)R(\exp(i\theta), A_\theta)$ where A_θ denotes the infinitesimal generator of T^θ. Consequently $\exp(i\theta)A = A_\theta$. Substituting θ by $-\theta$ yields the assertion.

2. $\Rightarrow$ 4.: $\exp(-i\theta) = a - ib$ for $a, b > 0$. Applying the Hille-Yosida-Theorem to the semigroup $T^{-\theta}$ we obtain

$$\begin{aligned} ||R(r + is, A)|| &= || \exp(-i\theta)R(\exp(-i\theta)(r + is), \exp(-i\theta)A)|| \\ &\leq \frac{C_1}{ar + bs} \leq \frac{C}{s} \text{ for } r, s > 0 \end{aligned}$$

Analogously for $s < 0$ with the generator $\exp(i\theta)A$.

4. $\Rightarrow$ 5.: The formula

$$||R(\lambda, A)|| \geq \frac{1}{dist(\lambda, \sigma(A))} \text{ for } \lambda \in \rho(A)$$

which is valid for any closed operator with non-empty resolvent set by investigating the Taylor series of the analytic function R

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\lambda - \mu)^n (-1)^n R(\mu, A)^{n+1}$$

Hence we obtain that $i\mathbb{R} \setminus \{0\} \subset \rho(A)$. Since $\Sigma_{\frac{\pi}{2}} \subset \rho(A)$ by the property that A generates a bounded strongly continuous semigroup. Furthermore $||R(\lambda, A)|| \leq \frac{M}{|\lambda|}$ for $\Re\lambda \geq 0$ and $\lambda \neq 0$ by combining the given estimate and the Hille-Yosida estimate. So we can apply the Taylor series of the analytic function R to extend it to the sector $\Sigma_{\frac{\pi}{2}+\delta-\epsilon}$ for some constants with the necessary estimates.

5. $\Rightarrow$ 3.: The sectorial operator A generates a bounded analytic semigroup and consequently $T_tX \subset D(A)$ for $t > 0$. The operator AT_t is continuous by the closed graph theorem, by the resolvent equation and Cauchy theorem we can estimate along $\gamma_{\frac{1}{t}}$:

$$\begin{aligned} ||AT_t|| &= ||A \int_{\gamma_{\frac{1}{t}}} \exp(\mu t)R(\mu, A)d\mu|| = \\ &= || \int_{\gamma_{\frac{1}{t}}} \exp(\mu t)(\mu R(\mu, A) - id)d\mu|| = \\ &= || \int_{\gamma_{\frac{1}{t}}} \exp(\mu t)\mu R(\mu, A)d\mu|| \leq \\ &\leq \frac{M}{t} \end{aligned}$$

for $t > 0$.

3. $\Rightarrow$ 1.: T is smooth on $\mathbb{R}_{>0}$ by assumption, furthermore $(AT_{\frac{t}{n}})^n = A^nT_t = T^{(n)}_t$ for $n \geq 0$ and $t > 0$, hence $\frac{1}{n!}||T^{(n)}_t|| \leq (\frac{n}{t})^n M^n \leq (\frac{eM}{t})^n$ by Stirling's formula. Consequently we can estimate the

remainder in Taylor's formula

$$T_{t+h}x = \sum_{k=0}^{n} \frac{h^k}{k!} T_t^{(k)} x + \frac{1}{n!} \int_t^{t+h} (t+h-s)^n T_s^{(n+1)} x ds \text{ for } |h| < t$$

and extend the semigroup analytically to a sector Σ_δ, all the desired properties follow easily. □

1.18. Remark. If A generates a bounded strongly continuous group T, then A^2 generates a bounded analytic semigroup S of angle $\frac{\pi}{2}$. By the Hille-Yosida-Theorem there is a constant M such that

$$||R(\lambda, A)^n|| \leq \frac{M}{|\lambda|^n}$$

for $\lambda \neq 0$. $R(-\lambda, A)R(\lambda, A) = -(\lambda^2 - A)^{-1}$ for $\lambda \neq 0$, consequently $]0, \infty[\subset \rho(A^2)$. Since A generates a bounded group there is a constant $N \geq 1$ such that

$$||R(\lambda, A^2)|| \leq \frac{N^2}{(\Re\sqrt{\lambda})^2} \leq \frac{1}{|\lambda|}(\frac{|\lambda| N^2}{(\Re\sqrt{\lambda})^2}) \leq \frac{M_\delta}{|\lambda|}$$

for $\lambda \in \Sigma_{\frac{\pi}{2}-\delta}$ with $M_\delta = \frac{N^2}{\cos^2(\frac{\pi-\delta}{2})}$. So A^2 generates a bounded analytic semigroup. Take $L^p(\mathbb{R}^n, X)$ for a Banach space X and $1 \leq p < \infty$. The strongly continuous translation groups in direction of the canonical basis T^i commute and are contraction, so their generators $\frac{\partial}{\partial x^i}$ commute and $\frac{\partial^2}{\partial x^{i2}}$generate bounded analytic semigroups S^i in each direction: $S = S^1 \cdot ... \cdot S^n$ is an analytic semigroup with infinitesimal generator the closure of the Laplace operator in the respective space Δ_p. So the solution of the heat equation is analytic of angle $\frac{\pi}{2}$. Nevertheless there is no way to continue the semigroup strongly continuously to the left, because there we find the spectrum of the Laplace operator $\sigma(\Delta_p) = [-\infty, 0]$, but there exists an idea how the semigroup should behave on the left hand side. This discussion shall be continued after the second section.

2. Smooth semigroups

A convenient algebra is a convenient vector space with smooth associative multiplication. Furthermore we always assume it to be unital. Smooth Semigroups are smooth homomorphisms from $\mathbb{R}_{\geq 0}$ to a convenient algebra A. In this chapter we develop a theory of asymptotic resolvents by which one can provide a necessary and sufficient criterion wether a smooth semigroup exists given the infinitesimal generator. Furthermore we argue why it is convenient to consider this theory in the case of C_0-semigroups on a convenient topological vector space applying some smooth vector arguments.

For the purpose of estimates we need some Landau-like terminology in convenient vector spaces. We shall only apply the symbol O:

2.1. Definition (Landau symbol O). *Let E be a convenient vector space, $c : D \to E$ for $D \subset \mathbb{R}^n$ some non-empty subset, an arbitrary mapping. Let $d : D \to \mathbb{R}$ be some non-negative function, then we say that c has growth d on D if there is a closed absolutely convex and bounded subset $B \subset E$ so that*

$$c(x) \in d(x)B \text{ for all } x \in D \quad .$$

We write $c = O(d)$ on D or $c(x) = O(d(x))$, applying Landau's symbol O.

The next lemma provides us with the basic rules of Landau's symbol, which will be applied to prove the version of the Hille-Yosida-Theorem comfortably:

2.2. Lemma. *Let E be a convenient vector space, $D \subset \mathbb{R}^n$ some non-empty subset, $c_1, c_2 : D \to E$ some arbitrary mappings, $d_1, d_2 : D \to \mathbb{R}$ some non-negative functions:*

1. *If $c_i = O(d_i)$ for $i = 1, 2$, then $c_1 + \mu c_2 = O(d_1 + |\mu| d_2)$ for all $\mu \in \mathbb{K}$.*
2. *If $c_1 = O(d_1)$ and $c_2 = O(d_2)$ and $d_1 \leq d_2$, then $c_1 = O(d_2)$.*
3. *Let $c : D \to E$ be some function, $d : D \to \mathbb{R}$ some non-negative function, $\phi : E \to F$ a bounded linear map. If $c = O(d)$, then $\phi \circ c = O(d)$.*

4. *Let $\{c_\lambda\}_{\lambda\in[a,b]}$ be a pointwise Riemann-integrable family of mappings from D to E and $\{d_\lambda\}_{\lambda\in[a,b]}$ a pointwise Riemann-integrable family of positive mappings from D to $\mathbb{R}$. If $c = O(d)$ on $D \times [a,b]$, then*

$$\int_a^b c_\lambda d\lambda = O(\int_a^b d_\lambda d\lambda)$$

Remark that the last property is sufficient for pointwise Mackey-convergence to $\int_a^\infty c_\lambda d\lambda$ if the hypotheses of 4. are satisfied and $\int_a^\infty d_\lambda d\lambda$ exists pointwise.

Proof. Since the absolutely convex hull of closed, bounded subsets of a convenient vector space is closed, bounded and absolutely convex the first two properties are true, the third one is the generalization. The fourth property is due to the convergence of Riemannian sums and the first two properties. By the fourth property we obtain pointwise Mackey-convergence on D to a limit in E by the convergence of $\int_a^\infty d_\lambda d\lambda$. $\square$

2.3. Definition. *Let A be a convenient algebra and $T : \mathbb{R}_{\geq 0} \to A$ a smooth semigroup homomorphism referred to as smooth semigroup, then*

$$a := \lim_{h\downarrow 0} \frac{T_h - e}{h}$$

is called the infinitesimal generator of the smooth semigroup T. Given $b > 0$ the family $\{R(\lambda)\}_{\lambda>0}$ with

$$R(\lambda) := \int_0^b \exp(-\lambda t) T_t dt$$

is a called a standard asymptotic resolvent family of a. We omit the dependence on b.

2.4. Proposition. *Let A be a convenient algebra, T a smooth semigroup, then the following formulas are valid:*

1. *Let a be the infinitesimal generator of T, then $\frac{d}{dt}T_t = aT_t = T_t a$ for all $t \in \mathbb{R}_{\geq 0}$.*
2. *The semigroup is uniquely determined by a.*
3. *For all $b \in \mathbb{R}_{>0}$ the following integral exists in A:*

$$R(\lambda) = \int_0^b \exp(-\lambda t) T_t dt \text{ for all } \lambda \in \mathbb{R}$$

4. *For all $\lambda, \mu \in \mathbb{R}$, $b \in \mathbb{R}_{>0}$ we obtain:*

$$(\lambda - a)R(\lambda) = id - \exp(-\lambda b)T_b$$
$$R(\lambda)R(\mu) = R(\mu)R(\lambda)$$
$$R(\lambda)a = aR(\lambda)$$

5. *$R : \mathbb{R}_{>0} \to A$ is real analytic and the set*

$$\{\frac{\lambda^{n+1}}{n!} R^{(n)}(\lambda) \mid \lambda > 0 \text{ and } n \in \mathbb{N}\}$$

is bounded in A.

Proof. The first assertion follows from boundedness of the multiplication:

$$T_t a = T_t \lim_{h\downarrow 0} \frac{T_h - id}{h} = \lim_{h\downarrow 0} \frac{T_{t+h} - T_h}{h} = \frac{d}{dt} T_t = \lim_{h\downarrow 0} \frac{T_h - id}{h} T_t = T_t a$$

Suppose that there is another semigroup associated to a, more precisely, let S, T be smooth semigroups in A with

$$a = \lim_{h\downarrow 0} \frac{T_h - id}{h} = \lim_{h\downarrow 0} \frac{S_h - id}{h}$$

then the curve $c(r) = T(t-r)S(r)$ on $[0,t]$ for $t > 0$ arbitrary is smooth and $c'(r) = -ac(r) + c(r)a = 0$, consequently $T_t = c(0) = c(t) = S_t$.

The existence of the integral and the commutation relations are clear, the only assertion to prove is the asymptotic condition:

$$
\begin{aligned}
(\lambda - a)R(\lambda) &= \int_0^b \exp(-\lambda t)(\lambda - a)T_t dt = \\
&= \int_0^b -\frac{d}{dt}(\exp(-\lambda t)T_t)dt = id - \exp(-\lambda b)T_b
\end{aligned}
$$

Differentiation under the integral yields

$$\frac{\lambda^{n+1}}{n!}R^{(n)}(\lambda) = (-1)^n \lambda^{n+1} \int_0^b \exp(-\lambda t)\frac{t^n}{n!}T_t dt$$

$T = O(1)$ on any bounded interval in $\mathbb{R}_{\geq 0}$, so

$$\frac{\lambda^{n+1}}{n!}R^{(n)}(\lambda) = O\left(\lambda^{n+1}\int_0^b \exp(-\lambda t)\frac{t^n}{n!}dt\right) = O(1)$$

for all $\lambda > 0$ and $n \in \mathbb{N}$. So the estimate and real analyticity are proved, since the remainder of the Taylor series converges to zero. □

2.5. Definition. *Let $a \in A$ be a given element of the convenient algebra A, a smooth map $R : \mathbb{R}_{>\omega} \to A$ is called asymptotic resolvent for $a \in A$ if*

1. *$aR(\lambda) = R(\lambda)a$ and $R(\lambda)R(\mu) = R(\mu)R(\lambda)$ for $\lambda, \mu > \omega$.*
2. *$(\lambda - a)R(\lambda) = e + S(\lambda)$ with $S : \mathbb{R}_{>\omega} \to A$ smooth and there is are constants $b > 0$ so that the set*

$$\{\frac{\exp(b\lambda)}{b^k}S^{(k)} \mid \lambda > \omega \text{ and } k \in \mathbb{N}\}$$

is bounded in A.

2.6. Remark. By Proposition 2.4 the standard asymptotic resolvent family is an asymptotic resolvent. The estimate of Definition 2.5 can be generalized to the assertion that

$$\{\frac{\exp(b\lambda)}{c^k}S^{(k)} \mid \lambda > \omega \text{ and } k \in \mathbb{N}\}$$

is bounded in A with some $c \geq b > 0$. The following theorems stay valid, but the calculations get more complicated. We restrict ourselves to Definition 2.5. The case $S = 0$ is equivalent to the choice $b = \infty$, which is not always possible, because there are semigroups with rapid growth and infinitesimal generators without classical spectral theory, respectively. For the standard asymptotic resolvent $S(\lambda) = -\exp(\lambda b)T_b$.

The following theorem is the generalization of the Hille-Yosida-Theorem to the convenient case, it is useful in the analysis of Abstract Cauchy Problems on locally convex vector spaces (see [**Ouc73**] for the idea of the proof).

2.7. Theorem (convenient Hille-Yosida-Theorem). *Let A be a convenient algebra and $a \in A$ an element, then a is the infinitesimal generator of a smooth semigroup T in A if and only if there is an asymptotic resolvent R for the operator $a \in A$ with*

$$\{\frac{\lambda^{n+1}}{n!}R^{(n)}(\lambda) \mid \lambda > \omega \text{ and } n \in \mathbb{N}\}$$

a bounded set in A (Hille-Yosida-condition).

Proof. If a is the infinitesimal generator of a smooth semigroup in A, then there is by prop. 2.4. an asymptotic resolvent so that the above conditions are satisfied.

Let R be an asymptotic resolvent defined on $\mathbb{R}_{>\omega}$ satisfying the hypotheses, then $\lambda R(\lambda) = e + aR(\lambda) + O(\exp(-b\lambda))$ by 2.5.2, consequently $\lim_{\lambda\to\infty} \lambda R(\lambda) = e$ is a Mackey-limit. $a_\lambda := -\lambda + \lambda^2 R(\lambda) = \lambda(-e+\lambda R(\lambda)) = \lambda R(\lambda)a + O(\lambda \exp(-b\lambda)) \to a$ as Mackey-limit for $\lambda \to \infty$. Differentiating the equation $(\lambda - a)R(\lambda) = e + S(\lambda)$ $(k+1)$-times we obtain $(\lambda - a)R^{(k+1)}(\lambda) + (k+1)R^{(k)}(\lambda) = S^{(k+1)}(\lambda)$ for $k \in \mathbb{N}$. Multiplication with $R(\lambda)$ yields

$$R^{(k+1)}(\lambda) + (k+1)R(\lambda)R^{(k)}(\lambda) = R(\lambda)S^{(k+1)}(\lambda) - S(\lambda)R^{(k+1)}(\lambda)$$

for $k \in \mathbb{N}$. Putting together the hypotheses $R(\lambda)S^{(k+1)}(\lambda) = O(\frac{\exp(-\lambda b)b^{k+1}}{\lambda})$ and $S(\lambda)R^{(k+1)}(\lambda) = O(\exp(-\lambda b)\frac{(k+1)!}{\lambda^{k+2}})$ we arrive at

$$R^{(k+1)}(\lambda) + (k+1)R(\lambda)R^{(k)}(\lambda) = O(\exp(-\lambda b)\frac{(k+1)! + (b\lambda)^{k+1}}{\lambda^{k+2}})$$

for $k \in \mathbb{N}$ and $\lambda > \omega$. Now we try to define out of these data a smooth semigroup T. Let $t \in [0, \frac{b}{4}]$, then

$$T_t(\lambda) := \exp(-\lambda t)\left(e + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\lambda^2 t)^{k+1}}{(k+1)!} R^{(k)}(\lambda)\right)$$

for $\lambda > \omega$. Looking at the growth for $\lambda > \omega$ and $t \in [0, \frac{b}{4}]$ we obtain

$$T_t(\lambda) = O(\exp(-\lambda t)(1 + \sum_{k=0}^{\infty} \frac{(\lambda t)^{k+1}}{(k+1)!})) = O(1)$$

for $k \in \mathbb{N}$ and $\lambda > \omega$ by the Hille-Yosida-condition, which implies the existence of $T_t(\lambda)$ uniformly on compact intervals in λ and t as a Mackey-limit by the Cauchy condition on the convergence of infinite series. By inserting the Hille-Yosida condition in the termwise derived series we obtain the uniform convergence by the Cauchy condition on compact intervals in λ and t, which leads to smoothness of $T_t(\lambda)$ in t, even at the boundary points $t = 0$ and $t = \frac{b}{4}$: We obtain

$$\begin{aligned}
\frac{d}{dt}T_t(\lambda) =& -\lambda T_t(\lambda) + \lambda^2 \exp(-\lambda t)(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{(\lambda^2 t)^k}{k!} R^{(k)}(\lambda))\\
=& -\lambda T_t(\lambda) + \lambda^2 \exp(-\lambda t)(R(\lambda) + \sum_{k=0}^{\infty} \frac{(-\lambda^2 t)^{k+1}}{(k+1)!(k+1)!} R^{(k+1)}(\lambda))\\
=& -\lambda T_t(\lambda) + \lambda^2 \exp(-\lambda t)\left(R(\lambda) + \sum_{k=0}^{\infty} (-1)^k \frac{(\lambda^2 t)^{k+1}}{(k)!(k+1)!} R(\lambda)R^{(k)}(\lambda)\right)\\
&+ \lambda^2 \exp(-\lambda t)\sum_{k=0}^{\infty} \frac{(-\lambda^2 t)^{k+1}}{(k+1)!(k+1)!} S_k(\lambda)
\end{aligned}$$

with $S_k(\lambda) = R^{(k+1)}(\lambda) + (k+1)R(\lambda)R^{(k)}(\lambda)$. The last sum on the right hand side is of order

$$\lambda^2 \exp(-\lambda t)\sum_{k=0}^{\infty} \frac{(\lambda^2 t)^{k+1}}{(k+1)!(k+1)!} \exp(-\lambda b)\frac{(k+1)! + (b\lambda)^{k+1}}{\lambda^{k+2}}$$

This term can be estimated by

$$\begin{aligned}
&=\lambda \exp(-\lambda(t+b))\left(\sum_{k=0}^{\infty}(\frac{(\lambda^2 tb)^{k+1}}{(k+1)!(k+1)!} + \sum_{k=0}^{\infty} \frac{(\lambda t)^{k+1}}{(k+1)!}\right)\\
&\leq\lambda\left(\exp(-\lambda(\sqrt{t} - \sqrt{b})^2) + \exp(-\lambda b)\right)\\
&\leq 2\lambda \exp(-\lambda\frac{b}{4})
\end{aligned}$$

for $t \in [0, \frac{b}{4}]$, since $(\sqrt{t} - \sqrt{b})^2$ attains the minimum $\frac{b}{4}$. The middle term equals

$$\lambda^2 \exp(-\lambda t)\left(R(\lambda) + \sum_{k=0}^{\infty} (-1)^k \frac{(\lambda^2 t)^{k+1}}{(k)!(k+1)!} R(\lambda)R^{(k)}(\lambda)\right) = R(\lambda)\lambda^2 T_t(\lambda)$$

by definition. Consequently we arrive at the equation by $a_\lambda = -\lambda + \lambda^2 R(\lambda)$:

$$\frac{d}{dt}T_t(\lambda) = a_\lambda T_t(\lambda) + O(\lambda \exp(-\lambda\frac{b}{4}))$$

for $t \in [0, \frac{b}{4}]$ and $\lambda > \omega$. Finally we can calculate the difference

$$T_t(\lambda) - T_t(\mu) = \int_0^t \frac{d}{ds}\left(T_s(\lambda)T_{t-s}(\mu)\right) ds$$

because $T_0(\lambda) = e$ and so by the commutation relations

$$T_t(\lambda) - T_t(\mu) = \int_0^t T_s(\lambda)T_{t-s}(\mu)(a_\lambda - a_\mu) ds + O(\lambda \exp(-\lambda\frac{b}{4})) + O(\mu \exp(-\mu\frac{b}{4}))$$

we are lead to uniform Mackey-convergence on $[0, \frac{b}{4}]$ of $T_t(\lambda)$ as $\lambda \to \infty$. We denote the limit by T_t. Due to uniform convergence on $[0, \frac{b}{4}]$ and the Mackey-property of the limits we obtain $T_t(\lambda)a_\lambda \to T_t a$, consequently the first derivatives of $T_t(\lambda)$ converge uniformly in t to $aT_t = T_t a$, which guarantees Lipschitz-differentiability of order Lip^1 of T_t with derivative aT_t. Since multiplication with a is a bounded operation we see that the first derivative is Lip^1, too. Consequently T_t is smooth on $[0, \frac{b}{4}]$. Given $t, s \in [0, \frac{b}{4}]$ with $t + s \in [0, \frac{b}{4}]$, then

$$T_{t+s} - T_t T_s = \int_0^t \frac{d}{du}\left(T_{t-u}T_{s+u}\right) du = \int_0^t T_{t-u}T_{s+u}(a - a) du = 0 \quad .$$

So T is a smooth semigroup in A with generator a, which is the desired assertion. □

2.8. Remark. If A is a convenient algebra, then also $l^\infty(\mathbb{R}, A)$ (mappings bounded on compact sets), $C^\infty(\mathbb{R}, A)$. In all cases the application of the theorem yields some results on locally bounded (smooth) families of smooth semigroups. However, the relations on the infinitesimal generators in the smooth case cannot be written down without involving derivatives in the parameter.

The following theorem provides a reproduction formula, given an asymptotic resolvent, we can calculate the smooth semigroup. The formula is apparently complicated, but all the known reproduction formulas from classical theory follow (see [**Ouc73**] for the idea of the proof):

2.9. Theorem (Reproduction formula). *Let $a \in A$ be an element, $R : \mathbb{R}_{>\omega} \to A$ an asymptotic resolvent of a satisfying the Hille-Yosida condition and T the associated smooth semigroup by 2.7, then*

$$\lim_{n\to\infty} \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{n}{t}\right)^n R^{(n-1)}\left(\frac{n}{t}\right) = T_t$$

uniformly on compact intervals in $[0, b[$ as Mackey-limit, where at $t = 0$ the term is given by e.

Proof. First we show that the term can be continued by e at $t = 0$, therefore we apply the formula

$$R^{(k+1)}(\lambda) + (k+1)R(\lambda)R^{(k)}(\lambda) = O\left(\exp(-\lambda b)\frac{(k+1)! + (b\lambda)^{k+1}}{\lambda^{k+2}}\right)$$

for $k \in \mathbb{N}$ and $\lambda > \omega$ from the proof of Theorem 2.7. We prove that

$$\lim_{\lambda\to\infty} \frac{(-1)^n \lambda^{n+1}}{n!} R^{(n)}(\lambda) = e$$

is a Mackey-limit. For $n = 0$ the assertion was proved at the beginning of the previous demonstration, we assume by induction, that it is valid for some $n \geq 0$. With the formula of the proof of theorem 2.7. we obtain by applying the commutation relations

$$\begin{aligned}\frac{(-1)^{n+1}\lambda^{n+2}}{(n+1)!}R^{(n+1)}(\lambda) =& \frac{(-1)^n \lambda^{n+1}}{n!} R^{(n)}(\lambda)R(\lambda)\lambda \\ &+ O(\exp(-\lambda b)\frac{(n+1)! + (b\lambda)^{n+1}}{(n+1)!})\end{aligned}$$

and we can insert the hypotheses of induction. Letting λ tend to infinity we arrive at the limit result by induction (we need the case $n = 0$ and the induction hypothesis), the Mackey-property is also proved.

$$S_n(t) := \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{n}{t}\right)^n R^{(n-1)}\left(\frac{n}{t}\right)$$

is defined for $t \in [0,b[$ and $n > b\omega$. Remark that

$$S_n(t) = O(1)$$

for $n > b\omega$ and $t \in [0,b[$ by the Hille-Yosida condition. Let $0 < t < b$ be given, then

$$\begin{aligned}\frac{d}{dt}S_n(t) = &-\frac{n}{t}S_n(t) + \frac{(-1)^n}{(n-1)!}\left(\frac{n}{t}\right)^n R^{(n)}\left(\frac{n}{t}\right)\frac{n}{t^2} = \\ = &-\frac{n}{t}S_n(t) + \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{n}{t}\right)^{n+2} R^{(n-1)}\left(\frac{n}{t}\right)R\left(\frac{n}{t}\right) + \frac{1}{n!}\left(\frac{n}{t}\right)^{n+2}G_n(t) = \\ = &a_{\frac{n}{t}}S_n(t) + \frac{1}{n!}\left(\frac{n}{t}\right)^{n+2}G_n(t)\end{aligned}$$

by the above formula with $G_n(t) := O(\exp(-\frac{n}{t}b)(\frac{t}{n}b^n + (\frac{t}{n})^{n+1}n!))$:

$$\begin{aligned}\frac{1}{n!}\left(\frac{n}{t}\right)^{n+2}\exp\left(-\frac{n}{t}b\right)\left(\frac{t}{n}b^n + \left(\frac{t}{n}\right)^{n+1}n!\right) &= \exp\left(-\frac{n}{t}b\right)\left(\frac{1}{n!}\left(\frac{n}{t}\right)^{n+1}b^n + \frac{n}{t}\right) \leq \\ &\leq K_1\exp\left(-\frac{n}{t}b\right)\left(\frac{1}{n!}\left(\frac{nb}{t}\right)^{n+1} + \frac{n}{t}\right)\end{aligned}$$

with a constant $K_1 \geq 1$. Now we apply Stirling' s formula $n! \sim n^n \exp(-n)\sqrt{2\pi n}$ (see [**Kno51**], ch. 14, for remarkable details), consequently

$$\begin{aligned}\leq &K_2\exp\left(-\frac{n}{t}b\right)\left(\exp(n)\sqrt{n}\left(\frac{b}{t}\right)^{n+1} + \frac{n}{t}\right) \\ = &K_2\left(\exp\left(n\left(1-\frac{b}{t}\right)\right)\sqrt{n}\left(\frac{b}{t}\right)^{n+1} + \exp\left(-\frac{n}{t}b\right)\frac{n}{t}\right)\end{aligned}$$

The function $f(x) = x^m \exp(-nx)$ is decreasing on the interval $[\frac{m}{n}, \infty[$. Given $0 < t_0 < b$, $t \in [0, t_0]$, $n \in \mathbb{N}$ with $\beta := \frac{b}{t_0} \geq 1 + \frac{1}{n}$, then

$$\exp\left(-n\frac{b}{t}\right)\left(\frac{b}{t}\right)^{n+1} \leq \beta^{n+1}\exp(-n\beta)$$

$$\exp\left(-n\frac{b}{t}\right)\left(\frac{b}{t}\right) \leq \beta\exp(-n\beta)$$

Inserted in our formula we arrive at

$$\leq K_3(\sqrt{n}\beta^{n+1}\exp(n(1-\beta)) + n\beta\exp(-n\beta))$$

However, $\beta\exp(1-\beta) < 1$ for $\beta > 1$, so the term in question tends to zero as $n \to \infty$ uniformly in t on compact intervals in $[0,b[$. The following formula prepares the result:

$$\begin{aligned}S_n(t) - T_t = &\int_0^t \frac{d}{ds}(S_n(s)T_{t-s})\,ds = \\ = &\int_0^t S_n(s)T_{t-s}(a_{\frac{n}{s}} - a)ds + \int_0^t \frac{1}{n!}\left(\frac{n}{t}\right)^{n+2}G_n(s)ds\end{aligned}$$

for $n > b\omega$. Given $t \in [0, t_0]$ we obtain by boundedness of S_n and the above convergence of the perturbation $\frac{1}{n!}(\frac{n}{t})^{n+2}G_n(s)$ the result and the Mackey-property. □

3. ACP and Zoology of locally convex Spaces

Semigroups will be denoted by $S, T, \ldots$, their infinitesimal generators by a, b. We use the conventions of semigroup theory: $T_t = T(t)$. The interest in semigroup theory stems from properties of the solutions of Abstract Cauchy Problems on convenient vector spaces. Let E be a convenient vector space, $a \in L(E)$ a bounded operator, then, given $x \in E$, a solution of $ACP(a)$ (*Abstract Cauchy Problem associated to* a) is a curve $x : \mathbb{R}_{\geq 0} \to E$ satisfying

$$x \in Lip^1(\mathbb{R}_{\geq 0}, E) \text{ and } x(0) = x$$

$$\frac{d}{dt}x(t) = ax(t) \text{ for all } t \in \mathbb{R}_{\geq 0}$$

If $ACP(a)$ has a unique solution for every $x \in E$, one can form a semigroup T of linear mappings on E, we call such an Abstract Cauchy Problem well-posed. If the space E is webbed in a locally convex topology compatible with the bornology on E, then the semigroup is a smooth semigroup of bounded linear mappings on E:

3.1. Proposition. *Let E be a webbed locally convex vector space, such that E_{born} is Baire, $a \in L(E)$ a bounded, linear operator, then the following assertions are equivalent:*

1. *For any $x \in E$ the Abstract Cauchy Problem $ACP(a)$ has a unique solution with initial value x.*
2. *The mapping $T : \mathbb{R}_{\geq 0} \to L(E)$, $t \mapsto (x \mapsto x(t))$, where $x(t)$ denotes the value of the unique solution of $ACP(a)$ with initial value x at time t, is well-defined, smooth and*
$$\frac{d}{dt}T_t = aT_t$$
for $t \in \mathbb{R}_{\geq 0}$.

Proof. The step from the second to the first assertion is valid in general for any locally convex space E. The other direction is a little bit more complicated:

We denote by $T_.x$ the unique solution with initial value $x \in E$. Remark that by definition this solution is smooth, so $T_.x \in C^\infty([0,\infty[,E)$. Furthermore by uniqueness the family $\{T_t\}_{t\in\mathbb{R}_{\geq 0}}$ is a semigroup of linear operators on E. We define $\eta : E_{born} \to C^\infty([0,\infty[,E_{born})$ by $\eta(x) = T_.x$, which is a linear mapping. We show that it has closed graph. Let $\{x_i\}_{i\in I}$ be a converging net with limit $x \in E_{born}$ and $\eta(x_i) \to y$ with $y \in C^\infty([0,\infty[,E_{born})$, then
$$\eta(x_i)(s) = x_i + \int_0^s a\eta(x_i)(t)dt$$
for $s \geq 0$. Passing to the limit we obtain
$$y(s) = x + \int_0^s ay(t)dt$$
for $s \geq 0$. So y is a solution of $ACP(a)$ with initial value x, consequently $y = \eta(x)$ and η has a closed graph. If E is webbed, E_{born} is webbed and Baire by assumption, consequently η continuous. $ev_t \circ \eta = T_t$ is continuous on E_{born}, so T is a smooth semigroup of bounded linear operators. ☐

Proposition 3.1 justifies the introduction of the notion of a smooth semigroup. A smooth semigroup on E is a smooth semigroup in $L(E)$. On webbed spaces with E_{born} Baire it is sufficient to have a theory of smooth semigroup to be able to handle well posed Abstract Cauchy Problems.

In the category of sequentially complete locally convex spaces one can associate to each strongly continuous semigroup of continuous operators a smooth semigroup of bounded operators on the sequentially complete vector space of smooth vectors, consequently we only treat smooth semigroups, as the concept of strongly continuous semigroups only makes sense for calculation on sequentially complete vector spaces.

3.2. Definition. *Let E be a convenient locally convex vector space, T a semigroup of continuous linear operators. T is called C_0-semigroup if $\lim_{t\downarrow 0} T_tx = x$ for $x \in E$.*

The theory of C_0-semigroups on convenient vector spaces can be developed analogously to the theory of smooth semigroups, if enough smooth vectors exist. Remark that this is the case if the space is sequentially complete. Anyway we do not need local equicontinuity as assumed in most of the articles on the subject.

3.3. Proposition. *Let E be a convenient locally convex vector space, T a C_0-semigroup of continuous linear operators on E, such that the "smooth vectors" $S(\phi,t)x = \int_0^t \phi(s)T_sxds$ exist for $t \geq 0$ and $\phi \in C_c^\infty(\mathbb{R}_{>0})$, then the linear subspace*
$$E^\infty := \{x \in E \mid t \mapsto T_tx \text{ is smooth } \}$$
of smooth vectors is dense in E. Let a denote the infinitesimal generator of T. The initial locally convex topology $(a|_{E^\infty})^n : E^\infty \to E$ for $n \in \mathbb{N}$ is convenient and the restriction $T|_{E^\infty}$ is a smooth semigroup with infinitesimal generator $a|_{E^\infty}$.

Proof. The proof is done by standard semigroup theory:

$$\mathcal{D}(a) := \{x \in E \mid \lim_{t \downarrow 0} \frac{1}{t}(T_t x - x) \text{ exists } \}$$
$$ax := \lim_{t \downarrow 0} \frac{1}{t}(T_t x - x) \text{ for } x \in \mathcal{D}(a)$$

By continuity we obtain $aT_t x = T_t ax$ for $x \in \mathcal{D}(a)$ and that a is a closed operator, furthermore the smooth vectors commute with a. Another easy calculation gives

$$a \int_0^t \phi(s) T_s x ds = - \int_0^t \phi'(s) T_s x ds$$

for ϕ smooth with support in $\mathbb{R}_{>0}$ (for the arguments see the proof of theorem 1.6). Consequently the notion "smooth vector" is justified as the image under a again lies in $\mathcal{D}(a)$. However these vectors lie dense in E as we can choose a Dirac sequence right from zero and hence $\mathcal{D}(a)$ is dense in E. Providing $\mathcal{D}(a)$ with the graph norm yields a new convenient locally convex space, where we can apply the same procedure: The semigroup $T^{(1)}$ steming from T via restriction is a strongly continuous semigroups of continuous linear opearators and smooth vectors exist in $\mathcal{D}(a)$. E^∞ is given as the intersection of all these spaces and equivalently as the domain of definition of all a^k for $k \in \mathbb{N}$. The above described topology is a convenient locally convex topology as it lies in the domain of definition of all a^k, $k \in \mathbb{N}$. Again by the smooth vectors we conclude that E^∞ is dense. □

Remark that this proof can be generalized to strongly continuous group homomorphisms of finite dimensional Lie groups to continuous linear operators on locally convex spaces (see [**KM97**], compare to 49.4.) assuming the existence of smooth vectors.

The rest of this section is devoted to the analysis of examples. Therefore we need a principle referred to as Holmgren's principle (see [**LS93**], p. 133-158 for details) concerned with the dual ACP on the space E' of bounded functionals on E. The pairing will be denoted by $\langle ., . \rangle$:

3.4. Proposition. *Let E be a convenient vector space, $a \in L(E)$ a linear operator:*

1. *If $ACP(a)$ is uniquely solvable for every initial value on E and $ACP(a')$ is uniquely solvable for every initial value on E', then the solutions determine smooth semigroups of bounded operators on E and E', respectively. They are dual to each other.*
2. *Let $x : \mathbb{R}_{\geq 0} \to E$ be a (nontrivial) solution of $ACP(a)$ with initial value $x(0) = 0$, then for every solution $y : \mathbb{R}_{\geq 0} \to E'$ of $ACP(a')$ we have:*
$$\forall s, t \in \mathbb{R}_{\geq 0}, n \in \mathbb{N} : \langle x^{(n)}(s), y(t) \rangle = 0$$
3. *Let $y : \mathbb{R}_{\geq 0} \to E'$ be a (nontrivial) solution of $ACP(a')$ with initial value $y(0) = 0$, then for every solution $x : \mathbb{R}_{\geq 0} \to E$ of $ACP(a)$ we have:*
$$\forall s, t \in \mathbb{R}_{\geq 0}, n \in \mathbb{N} : \langle x(s), y^{(n)}(t) \rangle = 0$$

In other words the non-uniqueness of the ACP associated to a or a', respectively, determines forbidden zones for the dual problem, that means subspaces where solutions of the dual problems cannot pass by.

Proof. The first assertion follows from the observation that the semigroups of (possibly unbounded) operators T on E and S on E' are dual to each other, consequently bounded. For the proof we look at the following curve. Let $t > 0$, $x \in E$, $y \in E'$ be fixed, then

$$c(s) := \langle T_{t-s} x, S_s y \rangle \text{ for } s \in [0, t]$$

is a smooth curve with derivative zero, because of the boundedness of the pairing, so $c(0) = \langle T_t x, y \rangle = c(t) = \langle x, S_t y \rangle$. A mapping is bounded if the composition with all bounded functionals is bounded, consequently the given semigroups are semigroups of bounded linear maps.

For the last two assertions we have to examine a classical object, the shift semigroup on $C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ given by

$$(S_s f)(t) = f(t+s) \text{ for all } t, s \geq 0 \quad .$$

This is a smooth semigroup for the convenient vector space $C^\infty(\mathbb{R}_{>0},\mathbb{R})$ associated to $\frac{\partial}{\partial t}$, so the solutions of the associated Abstract Cauchy Problem are unique. Taking the data from point 2. we can define $f(t,s) := \langle x(s), y(t)\rangle$:

$$\frac{\partial}{\partial s}f(t,s) = \langle ax(s), y(t)\rangle = \langle x(s), a'y(t)\rangle = \frac{\partial}{\partial t}f(t,s)$$

for $s,t \geq 0$. Using cartesian closedness we obtain that $f_s := f(.,s)$ for $s \geq 0$ is a solution of the abstract Cauchy problem given above. This means that

$$f(t,s) = (S_s f_0)(t) = f_0(t+s) = 0$$

because $x(0) = 0$. Derivations in s-direction give the desired assertion. Taking the data of 3. we can proceed in the same manner by interchanging the roles. $\square$

As a corollary we obtain the following relaxation of the hypotheses in 3.4.

3.5. Corollary. *Let E be a convenient vector space, $a \in L(E)$ a linear operator. If $ACP(a)$ is solvable for every initial value on E and $ACP(a')$ is solvable for every initial value on E', then the solutions determine smooth semigroups of bounded operators on E and E', respectively.*

PROOF. By Proposition 3.4 we obtain that the solutions have to be unique, because they exist for all initial values. Therefore we can apply 1 of 3.4. $\square$

The question wether the exponential of a bounded operator exists as a series is solved by the following useful proposition:

3.6. Proposition. *Let E be a complex convenient vector space, $a \in L(E)$ a linear bounded operator with $ACP(a)$ and $ACP(ia)$ uniquely solvable by a smooth group of bounded operators, respectively, then the exponential series converges to a holomorphic mapping* $\exp(ta)$ *uniformly on compact sets in $\mathbb{C}$.*

PROOF. (see [**LS93**], p. 146 for the idea of the proof) Let T be the semigroup associated to a and S be the semigroup associated to ia, then we investigate the mapping $(s,t) \longmapsto T_s S_t$, which satisfies the differential equations

$$\frac{\partial}{\partial s}f(s,t) = i\frac{\partial}{\partial t}f(s,t)$$

Consequently the Cauchy-Riemann equations for the given mapping are satisfied. Consequently the mapping in question is weakly holomorphic and by convenience holomorphic. However this already guarantees the convergence of the exponential series. $\square$

The following examples explain the problems which can occur within the analysis of ACP on even well-examined convenient vector spaces. Non-uniqueness and Non-solvability are generic phenomena in this setting, but sometimes only non-normable spaces provide a good basis to analyze a problem.

3.7. Example (rapid growth). Let E be the space of entire functions on the complex plane $\mathcal{H}(\mathbb{C})$ and define a to be the multiplication operator by the function id, then the $(ACP)(a)$

$$\frac{\partial}{\partial t}f(t,z) = zf(t,z)$$

is solvable. Nevertheless the solution $\exp(zt)f(z)$ grows faster in t than any exponential in the natural topology of uniform convergence on compact sets on $\mathcal{H}(\mathbb{C})$ for any non-zero initial value.

3.8. Example (non-solvability). Let E be the Fréchet space of rapidly decreasing sequences with values in $\mathbb{R}$. We shall analyze the following ACP associated to the operator $a : s \to s$ given by

$$a(x_1, x_2, x_3, ...) = (0, 1^2 x_1, 2^2 x_2, 3^2 x_3, ...)$$

for a sequence $\mathbf{x} = (x_1, x_2, ...) \in s$. The dual space of s is a inductive limit of Banach spaces, namely the space of slowly increasing sequences, more precisely: Let $d \in \mathbb{N}_+$ be fixed, then the Banach space E_d is defined in the following way:

$$E_d := \{\mathbf{y} \in \mathbb{R}^\infty | \, p_d(\mathbf{y}) < \infty\} \quad ,$$

where $p_d(\mathbf{y}) := \sup_{k\in\mathbb{N}_+} |\frac{y_k}{k^d}|$ is a norm. $s' = \operatorname{inj\,lim}_{d\to\infty} E_d$. Applying the pairing we can calculate the adjoint of a:

$$a'(y_1, y_2, y_3, ...) = (1^2 y_2, 2^2 y_3, 3^2 y_4, ...)$$

for a sequence $\mathbf{y} \in s'$. Now we can try to solve the ACP associated to a and a'. In both cases we can solve the equations componentwise, supposing that solutions exist, we start with the adjoint problem: Let $\mathbf{y} : \mathbb{R}_{\geq 0} \to s'$ be a smooth solution of the differential equation, then

$$\frac{d}{dt} y_k(t) = k^2 y_{k+1}(t)$$

for all $k \in \mathbb{N}_+$. So y_1 determines all other components by derivation. Given $f \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$, then

$$y_k := \frac{1}{(k-1)!^2} f^{(k)}$$

for $k \in \mathbb{N}_+$ is a solution of the componentwise problem. We obtain a solution for any initial value in s' by a direct argument, but we do not obtain unicity. Some functions infinitely flat at zero provide us with enough nontrivial solutions to the initial value zero in s'. To prove these statements one has to investigate carefully the derivatives of functions with compact support on $\mathbb{R}$. The following theorem is a special case of Theorem 1.4.2. in [**Hör83**]:

Let $I \subset R$ be an open interval and $K \subset I$ a compact subset, $d := \inf\{|x-y|\,|x \in K,\ y \in \complement I\}$. Let $\{d_j\}_{j\in\mathbb{N}_+}$ be a sequence of positive real numbers so that $\sum_{j=1}^\infty d_j < d$, then there exists a function $\phi \in C_0^\infty(I, R)$, $0 \leq \phi \leq 1$ and ϕ equal to 1 in a neighborhood of K, such that

$$|\phi^{(k)}(x)| \leq C^k \frac{1}{d_1 d_2 \dots d_k}$$

for $k \in \mathbb{N}_+$ and $x \in I$. $C = 2$ is one possible choice, in general it depends on the dimension of the space.

We fix the interval $I =]-3, 3[$, $K = \{0\}$ and choose $d_j = (\frac{1}{j})^{\frac{3}{2}}$. Then we get by the lemma a function ϕ satisfying the above estimates. If we define $c : \mathbb{R} \to s'$ via

$$c_k^0(t) := \frac{1}{(k-1)!^2} \phi^{(k)}(t) \text{ for } k \in \mathbb{N}_+$$

then we obtain a smooth curve factoring for example over $E_2 \hookrightarrow s'$, one can take the norming linear functional $l(\mathbf{x}) = \sum_{i=0}^\infty \frac{x_k}{k^2}$ for $\mathbf{x} \in E_2$ to prove the assertion. This curve has compact support and is consequently a nontrivial solution with initial value zero. Given $l \in \mathbb{N}$, we apply the indefinite integral operator $\int_0^t .dt$ l-times to ϕ, the result is denoted by ϕ_l, $\phi_0 = \phi$. ϕ_l verifies the following translated estimates:

$$|{\phi_l}^{(k)}(x)| \leq C^{k-l} \frac{1}{d_1 d_2 \dots d_{k-l}}$$

for $x \in \mathbb{R}$ and $k \geq l$. So we can define curves $c^l : \mathbb{R} \to s'$ factoring over $E_2 \hookrightarrow s'$ by the same procedure. We can show directly that each polynomial on $\mathbb{R}$ can be used to produce a solution. Now we continue inductively: c^1 is a curve with compact support having initial value e_1. Translating this solution in $\mathbb{R}$ we obtain a curve d_1 having initial value zero and attaining e_1 in some distance from $t = 0$. c^2 has initial value e_2, but does no longer have compact support, however far away from zero c_1^2 is given by a polynomial of first degree. We translate the curve and subtract the solution formed by the polynomial, so we obtain a curve d_2 having initial value zero and attaining somewhere the value $ce_1 + e_2$, $c \in \mathbb{R}$. This procedure can be continued and we get a sequence of nontrivial solutions $\{d_n\}_{n\in\mathbb{N}_+}$ with initial value zero. For each $n \in \mathbb{N}_+$ there is a $t > 0$ so that $d_n(t) = \sum_{i=1}^{n-1} c_i e_i + e_n$ with $c_i \in \mathbb{R}$. With this sequence we can exclude all possible nontrivial initial values for $ACP(a)$ with respect to the existence of a solution, by a refinement of Proposition 3.4 we can assert that not even on an interval, there is a solution of the problem.

As one example we try to solve $ACP(a)$ for initial value e_1. Suppose a smooth solution $\mathbf{x}$ exists, so we can look at it componentwise:

$$\frac{d}{dt} x_k(t) = (k-1)^2 x_{k-1}(t)$$

for $k \geq 2$. $\frac{d}{dt}x_1(t) = 0$. Solving this recursive problem yields $x_k(t) = (k-1)!t^{k-1}$ for $k \in \mathbb{N}_+$ and $t \in \mathbb{R}_{>0}$, which is not in s for $t > 0$.

3.9. Example (nearly non-solvability). We analyze the heat equation on $C^\infty(\mathbb{T}^n, \mathbb{C})$ for negative time direction: Remark that the Laplacian generates an analytic semigroup of angle $\frac{\pi}{2}$ on $L^p(\mathbb{T}^n, \mathbb{C})$ for $1 \leq p < \infty$ by the looking at the bounded strongly continuous translation groups. The smooth functions on the n-torus are the smooth vectors of the Laplacian with respect to $L^p(\mathbb{T}^n, \mathbb{C})$, which can be calculated by Sobolev space methods or by looking at Fourier series on $L^2(\mathbb{T}^n, \mathbb{C})$, where we see that the smooth vectors are exactly those falling faster than any "polynomial" to zero: Transforming the smooth vectors by Fourier development we obtain on $s(\mathbb{Z}^n, \mathbb{C})$ the operator $\Delta((x_\alpha)_{\alpha \in \mathbb{Z}^n}) = (-||\alpha||_2^2 x_\alpha)_{\alpha \in \mathbb{Z}^n}$, which is bounded. The resolvent is given through $R(\lambda, \Delta) = (\lambda + ||\alpha||_2^2)^{-1}$. The smooth semigroup $\exp(\Delta t)$ is well behaved and can be easily inverted on the dense subspace of finite sequences. We obtain a pointwise smooth semigroup of unbounded linear operators on the finite sequences in $s(\mathbb{Z}^n, \mathbb{C})$, the trajectories are unbounded for a non-zero starting vector. Nevertheless on the sequence space of "very, very fast" falling sequences ($x_\alpha = O(\exp(-n||\alpha||_2^2)$ for all $n \geq 1$ on $\mathbb{Z}^n$) the heat equation can be solved by a smooth group in real time direction, but also in the imaginary time direction, so the exponential exists in the given sequence space.

Nevertheless there are vector spaces, where all Abstract Cauchy Problems are solvable, but not uniquely. The following theorem treats some infinite products of real or complex lines, which have this property, even more. Let $a \in L(E)$ be some bounded linear operator, $f \in C^\infty(\mathbb{R}_{\geq 0}, E)$ some function, then a Lip^1-curve $x : \mathbb{R}_{\geq 0} \to E$ is the solution of the *inhomogeneous Abstract Cauchy Problem* $ACP(a, f)$ with initial value x if $x(0) = x$ and $\frac{d}{dt}x(t) = ax(t) + f(t)$. Remark that such a solution has to be smooth.

3.10. Proposition. *Let B be a non-empty set, $a \in L(\mathbb{K}^B)$ a bounded linear operator, $\mathbf{f} \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{K}^B)$, then there is a solution of $ACP(a, \mathbf{f})$ for any initial value $\mathbf{x} \in \mathbb{K}^B$.*

Proof. (see [**Shk92**] for the idea of the proof) The proof is based on an inductive procedure involving the adjoint map $a' \in L(\mathbb{K}^{(B)})$. First we take $B = \mathbb{N}$. We construct an algebraic basis of $\mathbb{K}^{(\mathbb{N})}$:

$$V_0 := \langle \{ {a'}^k \mathbf{e}_0 \mid k \in \mathbb{N} \} \rangle_{\mathbb{K}^{(\mathbb{N})}}$$

There are two possibilities, either V_0 is finite dimensional, then there is $n \in \mathbb{N}_+$ minimal with ${a'}^n \mathbf{e}_0 \in \langle \{ {a'}^k \mathbf{e}_0 \mid 0 \leq k \leq n-1 \} \rangle$, or V_0 is infinite dimensional, then the family $\{ {a'}^n \mathbf{e}_0 \}_{n \in \mathbb{N}}$ is linearly independent. Remark that there is a canonical basis in V_0, namely $\mathbf{t}_i^0 = {a'}^i \mathbf{e}_0$. We obtain $a' \mathbf{t}_i^0 = \mathbf{t}_{i+1}^0$ if $i < \dim V_0$ and $a' \mathbf{t}_{\dim V_0 - 1}^0 \in \langle \{ \mathbf{t}_i^0 \mid 0 \leq i \leq \dim V_0 - 1 \} \rangle$ if $\dim V_0 < \infty$. This is the first step of induction. Assuming that we have constructed $V_0 \subset V_1 \subset \cdots \subset V_{m-1}$ for $m \geq 1$ with $\{ \mathbf{e}_0, \ldots, \mathbf{e}_{m-1} \} \in V_{m-1}$ and the following properties: $a' V_i \subset V_i$ for $0 \leq i \leq m-1$, $a' \mathbf{t}_i^k = \mathbf{t}_{i+1}^k$ if $i < \dim V_k / V_{k-1}$ and $a' \mathbf{t}_{\dim V_k / V_{k-1} - 1}^k \in \langle \{ \mathbf{t}_i^k \mid 0 \leq i \leq \dim V_k / V_{k-1} - 1 \} \rangle$ if $\dim V_k / V_{k-1} < \infty$ for $1 \leq k < m$. We define $r := \min \{ n \in \mathbb{N} \mid \mathbf{e}_n \notin V_{m-1} \}$ and $\tilde{V_m} := \langle \{ {a'}^k \mathbf{e}_r \mid k \in \mathbb{N} \} \rangle$. There are two cases to be worked out:

If ${a'}^k \mathbf{e}_r \notin \langle V_{m-1} \cup \{ {a'}^i \mathbf{e}_r \mid 0 \leq i < k \} \rangle$ for $k \in \mathbb{N}$, then $V_m := \langle V_{m-1} \cup \tilde{V_m} \rangle$. If the condition is not satisfied, we take the smallest $k \in \mathbb{N}$, where it is broken and define $V_m := \langle V_{m-1} \cup \{ {a'}^j \mathbf{e}_r \mid 0 \leq j < k \} \rangle$. In both cases there is an appropriate basis satisfying the above properties. By induction we obtain that there is an algebraic basis $\mathbf{t}_i^k$ of $\mathbb{K}^{(\mathbb{N})}$, which we can order to a matrix

$$B := \begin{pmatrix} {\mathbf{t}_0^0}^\top \\ {\mathbf{t}_1^0}^\top \\ \vdots \end{pmatrix}$$

representing an element of $L(\mathbb{K}^{\mathbb{N}})$. By construction of the algebraic basis we obtain that B has a left inverse C, $CB = id$, $C \in L(\mathbb{K}^{\mathbb{N}})$ and that $Ba = JB$ with a matrix J of the following type

$$J = \left(\begin{array}{ccc} \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ * & * & * & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} & \cdots \\ \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ * & * & * & * \end{pmatrix} & \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ * & * & * & 0 \end{pmatrix} & \cdots \\ \vdots & \vdots & \ddots \end{array} \right)$$

which explains the interest in the procedure. Remark that in the case that the width of a step is infinite the last line of unknown entries vanishes, because there is no last line. Now we are able to transform the inhomogeneous Abstract Cauchy Problem reasonably to a solvable problem: $\frac{d}{dt}\mathbf{x}(t) = a\mathbf{x}(t) + \mathbf{f}(t)$ is the regarded equation. Assuming that there is a solution $\mathbf{x}$, then $B\mathbf{x} =: \mathbf{y}$ satisfies the equation $\frac{d}{dt}\mathbf{y}(t) = J\mathbf{y}(t) + B\mathbf{f}(t)$ and if this equation has a solution $\mathbf{y}$, then $C\mathbf{y}$ satisfies the original equation. The transformed equation can easily be solved by finite dimensional methods for all times.

The general step is made by transfinite induction by well-ordering the given set B. The procedure, however, remains the same as in the denumerable case: B can be assumed to be the set of all ordinals not exceeding some infinite ordinal number ρ. Given $f \in C^\infty(\mathbb{R}, \mathbb{R}^B)$ we write the system to solve $x'_\alpha = \sum_{\beta=1}^{\rho} a_{\alpha\beta}x_\beta + f_\alpha$, where the initial value is given by x_α. $(a_{\alpha.}) \in \mathbb{R}^{(B)}$ and the set $N_\alpha := \{\beta \in B \,|\, a_{\alpha\beta} \neq 0\}$ is finite. By transfinite induction we construct sets of smooth function $X_\alpha = (x_\alpha)_{\alpha \in M_\alpha}$ for $\alpha \in B$ such that

1. If $\beta \leq \alpha$, then $\beta \in M_\alpha$ and $M_\beta \subset M_\alpha$.
2. If $\beta \in M_\alpha$, then $N_\beta \subset M_\alpha$ and

$$x'_\beta = \sum_{\gamma \in N_\beta} a_{\beta\gamma}x_\gamma + f_\beta,\; x_\beta(0) = x_\beta$$

As induction base we apply the following construction: $M_1^0 = \{1\}$, $M_1^n = M_1^{n-1} \cup (\cup_{\alpha \in M_1^{n-1}} N_\alpha)$ for $n \in \mathbb{N}$, then $M_1 := \cup_{k\in\mathbb{N}} M_1^k$, which is a countable set, furthermore we can consider the restriction of a to $\mathbb{R}^{M_1}$ due to the construction. So we find a solution set of smooth functions verifying 1. and 2. by the countable case. The induction case splits into two subcases: Let $1 < \gamma \leq \rho$ be a given ordinal number, such that for all $\beta < \gamma$ the sets M_β and X_γ are constructed. If $\gamma \in \cup_{\beta<\gamma} M_\beta$, then we define $M_\gamma := \cup_{\beta<\gamma} M_\beta$ and $X_\gamma := (x_\alpha)_{\alpha \in M_\gamma}$ and all the properties are satisfied. If $\gamma \notin \cup_{\beta<\gamma} M_\beta =: K_\gamma$, then we construct via $M_\gamma^0 := K_\gamma \cup \{\gamma\}$ and $M_\gamma^n = M_\gamma^{n-1} \cup (\cup_{\alpha \in M_\gamma^{n-1}} N_\alpha)$ a sequence of sets, whose union is denoted by M_γ. Remark that $M_\gamma \setminus K_\gamma$ is countable, we look at the Cauchy problem at $\mathbb{R}^{M_\gamma \setminus K_\gamma}$ by inserting the given solutions over K_γ and solve it by the solution theorem in the countable case, so we arrive by transfinite induction at a solution set X_ρ. □

This is one extreme case of solvability on a special type of locally convex spaces. Remark that this restricts solvability on the dual space, because the solutions maybe non-unique. Another extreme case is given by so called LN-spaces (see [**LS93**], p. 148-155 for the proof in the case $X = \mathbb{R}$): This indicates a class of nuclear Fréchet spaces E, where for all $a \in L(E)$ the exponential $\exp(at)$ exists for all times t, which is a surprising fact. We summarize first some properties of sequence spaces to derive finally the results: An infinite matrix $(a_{pn})_{p,n\geq 1}$ with the properties $a_{p+1n} \geq a_{pn} \geq 1$, $a_{pn} \leq a_{pn+1}$ and $\sum_{n\geq 1} \frac{a_{pn}}{a_{p+1n}} < \infty$ is the starting point of our considerations. The sequence space $L(a_{pn}, X)$ is the linear space of sequences of Banach space vectors (x_n) with $||(x_n)||_p = \sup_{n\geq 1} a_{pn}||x_n|| < \infty$. With these norms we obtain a nuclear complex Fréchet space. The norms $||(x_n)||_{(p)} = \sum_{n\geq 1} a_{pn}||x_n||$ are equivalent via $||(x_n)||_p \leq ||(x_n)||_{(p)}$ and $||(x_n)||_{(p)} = \sum_{n\geq 1} a_{pn}||x_n|| = \sum_{n\geq 1} \frac{a_{pn}}{a_{p+1n}}||(x_n)||_{p+1}$ for $p \geq 1$. Providing $L(a_{pn})$ with the $||.||_p$-norms we obtain for an infinite matrix $T = (t_{ij})$ of linear continuous operators on X defining an operator in $L(L(a_{pn}, X))$ to be continuous if and only if for $p \geq 1$ there is $q \geq 1$ such that $\sup_{i\geq 1} \sum_{j\geq 1} ||t_{ij}|| \frac{b_{pi}}{a_{pj}} =: ||T||_q^p < \infty$. Providing $L(a_{pn})$ with the

$||.||_{(p)}$-norms we obtain analogously for an infinite matrix of linear continuous operators $T = (t_{ij})$ to be continuous if and only if for $p \geq 1$ there is $q \geq 1$ such that $\sup_{j\geq 1} \sum_{i\geq 1} |t_{ij}| \frac{b_{pi}}{a_{pj}} =: ||T||^{(p)}_{(q)} < \infty$. The two conditions are equivalent by the equivalence of the Fréchet space topologies. The following simple assertions can be proved immediately (see [**LS93**]):

1. Let T^1 and T^2 be matrices with T^2 continuous and $||t^1_{ij}|| \leq ||t^2_{ij}||$, then T^1 is continuous. The deeper fact behind is that we work in a type of Fréchet lattice if X is a Banach lattice.
2. A sequence of matrices T^n converges to T if and only if for $p \geq 1$ there is $q \geq 1$ such that $||T^n - T||^p_q \to 0$ (or equivalently $||T^n - T||^{(p)}_{(q)} \to 0$).
3. A family of matrices (T^α) is equicontinuous if and only if the supremum-matrix $(\sup_\alpha ||t^\alpha_{ij}||)$ is continuous on $L(a_{pn}) := L(a_{pn}, \mathbb{R})$.
4. The linear map $diag : L(L(a_{pn})) \to L(L(a_{pn}))$ mapping a matrix to the matrix of its diagonal entries is continuous.

3.11. Lemma. *If the matrix of a sequence space $L(a_{pn}, X)$ has the property*

$$\forall p_1, p_2 \geq 1 \exists q \geq 1 : \sup_{n\geq 1} \frac{a_{p_1 n} a_{p_2 n}}{a_{qn}} < \infty \tag{(C)}$$

then the following assertions are equivalent:

1. *For each $p \geq 1$ there is $q \geq 1$ such that $\sup_{n\geq 1} \frac{\exp(a_{pn})}{a_{qn}} < \infty$.*
2. *For all $T : L(a_{pn}, X) \to L(a_{pn}, X)$ continuous and linear the linear continuous operator $diag(T)$ has the property that its exponential series converges.*

Proof. (see [**LS93**], p. 148-155) The equivalence is easily established: Assume that 1. holds, given a continuous diagonal matrix (λ_i) with λ_i linear continuous operators, then there is $p \geq 1$ with $\sup_{n\geq 1} ||\lambda_n|| \frac{a_{1n}}{a_{pn}} < \infty$, so $||\lambda_n|| \leq M a_{pn}$ for a positive constant $M > 0$ and $n \geq 1$. By 1. there is a constant $C > 0$ and $p_1 \geq 1$ such that $\exp(a_{pn}) \leq C a_{p_1 n}$, by condition (C) for all $p_2 \geq 1$ there is $q \geq 1$ such that

$$\forall p_1, p_2 \geq 1 \exists q \geq 1 : \sup_{n\geq 1} \frac{a_{p_1 n} a_{p_2 n}}{a_{qn}} < \infty$$

consequently for $|t| \leq \frac{1}{M}$

$$\sup_n \exp(t\lambda_n) \frac{a_{p_2 n}}{a_{qn}} \leq \exp(a_{pn}) \frac{a_{p_2 n}}{a_{qn}} < \infty$$

which means continuity of the diagonal operator with entries $\exp(t\lambda_n)$ for $|t| \leq \frac{1}{M}$. By looking at the norming linear functionals $l_p((x_n)) := \sum_{n\geq 1} a_{pn} x_n$ we can prove that the application of the diagonal matrix on an element is smooth, so the diagonal operator is smooth by the smooth detection principle. By complexification of the space the same conclusion is valid for $(i\lambda_n)$, so the exponential series converges.

Assume that 2. holds, then for any $p \geq 1$ the diagonal operator with entries a_{pn} is continuous by (C) and the diagonal operator with entries $\exp(a_{pn})$ is continuous, hence for all $p \geq 1$ there is some $q \geq 1$ such that $\sup_n \exp(a_{pn}) \frac{a_{1n}}{a_{qn}} < \infty$, which proves the desired assertion. □

3.12. Definition. *A sequence space with values in a Banach space X is called of class LSN if the following four conditions are satisfied:*

1. *For all $1 \leq p < q$ and for all $n \geq 1$: $\frac{a_{pn+1}}{a_{qn+1}} \leq \frac{a_{pn}}{a_{qn}}$.*
2. *There is $r \geq 1$ such that for any $q \geq 1$ there is $p \geq 1$ with $\sup_{n\geq 1} \frac{a_{qn+1}}{a_{pn+1}} \frac{a_{pn}}{a_{rn}} < \infty$.*
3. *For all $p \geq 1$ there is $q \geq 1$ such that for all $r \geq 1$: $\sup_{n\geq 1} \frac{a_{pn+1}}{a_{qn+1}} \frac{a_{rn}}{a_{pn}} < \infty$.*
4. *For all $p \geq 1$ there is $q \geq 1$ such that $\sup_{n\geq 1} \frac{\exp(a_{pn})}{a_{qn}} < \infty$.*

3.13. Proposition. *If the sequence space $L(a_{pn}, X)$ belongs to the class LSN, then condition (C) is satisfied and for any continuous linear map with zero diagonal $||T||^p_p < \infty$ for p large enough.*

Proof. The rather technical proof can be found in [**Lob79**]. □

Now we can prove the main theorem of this constructive theory:

3.14. Theorem. *Let E be a sequence space of class LSN, then any inhomogeneous non-autonomous Cauchy problem with smooth entries*

$$x'(t) = A(t)x(t) + f(t)$$

is uniquely solvable for all times by a smooth curve x for any initial value x_0 at time t_0.

Proof. (see [**LS93**], p. 148-155) First we observe that unique solvability of the homogenous non-autonomous Cauchy problem implies the inhomogeneous assertion, since the smooth evolution family $R(t, t_0)x_0$ (unique solution with initial value x_0 at time t_0) produces via

$$\phi(t) := R(t, t_0)x_0 + \int_{t_0}^{t} R(t, s)f(s)ds$$

a solution of the inhomogeneous problem:

$$\phi'(t) = A(t)R(t, t_0)x_0 + R(t, t)f(t) + \int_{t_0}^{t} A(t)R(t, s)f(s)ds = A(t)\phi(t) + f(t)$$

Uniqueness follows directly. Consequently the non-autonomous case has to be solved: Given a smooth curve of operators $A(t)$, then we can form $A_1(t) = diag(A(t))$ and $A_2(t) = A - A_1(t)$, both smooth curves. First we solve the equation for A_1on an open finite interval $]a, b[$. The supremum of A_1 over the interval will be denoted by $(\lambda_n) = B := (\sup_{a \leq t \leq b} ||A_{nn}(t)||)$ and is a continuous operator since A_1 is an equicontinuous family on the interval $]a, \bar{b}[$. Hence the diagonal matrix with entries

$$\exp(\int_{t_0}^{t} A_{nn}(s)ds)$$

is continuous since all entries are bounded by $\exp((b-a)\lambda_n)$, which constitutes a continuous diagonal matrix by the LSN-properties. Evaluating at one point x_0 and combining with the linear functional l_p yields smoothness of the constructed evolution operator $R(t, t_0)$. We can immediately extend this operator to the whole real line. The equation $y'(t) = H(t)y(t)$ with $H(t) = R(t, t_0)A_2(t)R(t_0, t)$ can be solved by the observation that $H(t)$ has zero diagonal and therefore - by the LSN-conditions - for large p $||H(t)||_p^p < \infty$. Consequently we can pass to the Banach space $\overline{(E, ||.||_p)}$, where we obtain a Lipschitz curve for large p, so the equation is classically solvable there and we get back smooth unique solutions for any initial value at time t_0. The solutions of the equations $y'(t) = H(t)y(t)$ for t_0 and the equation $x'(t) = A(t)x(t)$ for initial value x_0 at t_0, respectively, are mapped to each other via $x(t) = R(t_0, t)y(t)$, which follows from an easy calculation. So the desired existence and uniqueness assertions are proved. ☐

3.15. Remark. Defining $a_{pn} = b_{pn!}$ with $b_{p1} = p$ and $b_{pn+1} = p^{b_{pn}}$ for $n \geq 1$ provides an example of a LSN-space (see [**LS93**], p.150).

3.16. Remark. On LSN-spaces even non-linear equations $x'(t) = f(x(t))$ for smooth functions on an open subset admit a local flow on the open subset (see [**LS93**], pp. 158-163). So manifolds modelled on LSN-spaces admit local flows for any vector field.

3.17. Remark. Many classical differential operators on compact manifolds can be transformed to continuous operators on sequence spaces. LSN-spaces provide an example of a sort of common domain of definition, where all Cauchy problems are solvable for all times.

4. The Trotter approximation theorem for product integrals

In the sequel we apply the Landau-like symbols to shorten the proof as in section 2: Given a mapping c from a set M to a convenient vector space E we write $c = O(d)$ if there is a mapping $d : M \to \mathbb{R}_{\geq 0}$ and a bounded absolutely convex set $B \subset E$ such that $c(m) \in d(m)B$ for all $m \in M$. Remember that a sequence $\{x_n\}_{n\in\mathbb{N}}$ is Mackey-convergent if there is a sequence of positive real numbers $\{\mu_n\}_{n\in\mathbb{N}}$ with $\mu_n \downarrow 0$ such that $x_n = O(\mu_n)$, see section 2.1.

Given a smooth curve $X : \mathbb{R} \to A$ we try to solve the following ordinary differential equation

$$\text{(R)} \qquad \frac{d}{dt}x(t) = X(t)x(t)$$

with initial value $x(s) = x_s$ at the point s for $t \geq s$. If there is a smooth family of solutions c_s for initial value e at any point s, then there is a smooth family of solutions for all initial values x at any point s given through the curves $t \mapsto c_s(t)x$ for $t \geq s$. If there is a smooth family of smooth solutions c_s for initial value e at any point s with the propagation condition

$$c_s(t)c_r(s) = c_r(t) \text{ for } t \geq s \geq r$$

then the solutions of the equation are unique for all initial values at any point in time. From the defining property of the solution family $c_s(t)$ we obtain

$$0 = \frac{d}{ds}c_s(t)c_r(s) = (\frac{d}{ds}c_s(t))c_r(s) + c_s(t)X(s)c_r(s)$$

which yields evaluated at $s = r$

$$\frac{d}{ds}c_s(t) = -c_s(t)X(s)$$

for $s \leq t$. So $s \mapsto c_s(t)$ is a smooth family of solutions for initial value e at any point in time of a ordinary differential equation of the type

$$\text{(L)} \qquad \frac{d}{ds}y(s) = y(s)Y(s)$$

but in the negative time direction. Consequently we obtain by looking at another smooth solution $\widetilde{c_r}(t)$ from r to t

$$\frac{d}{ds}c_s(t)\widetilde{c_r}(s) = (\frac{d}{ds}c_s(t))\widetilde{c_r}(s) + c_s(t)X(s)\widetilde{c_r}(s) = 0$$

which allows the desired conclusion of unicity. If a smooth solution family satisfying the propagation condition exists for (R) we call it the right evolution of the curve X. If a smooth solution family satisfying the propagation condition exists for (L), we call it the left evolution of the Y.

With the concept of product integrals (see [**Mil83**], [**Omo97**] on Lie groups) we try to approximate right evolutions of a given curve X and obtain in fact an existence theorem.

4.1. Definition (Product integral). *Let A be a convenient algebra. Given a smooth curve X: $\mathbb{R} \to A$ and a smooth mapping $h : \mathbb{R}^2 \to A$ with $h(s,0) = e$ and $\frac{\partial}{\partial t}|_{t=0}h(s,t) = X(s)$, then we define the following finite products of smooth curves*

$$p_n(s,t,h) := \prod_{i=0}^{n-1} h(s + \frac{(n-i)(t-s)}{n}, \frac{t-s}{n})$$

for $a, s, t \in \mathbb{R}$. If p_n converges in all derivatives to a smooth curve $c : \mathbb{R} \to A$, then c is called the product integral of X or h and we write $c(s,t) = \prod_s^t \exp(X(s)ds)$ or $c(s,t) =: \prod_s^t h(s,ds)$. The case $h(s,t) = c(t)$ with $p_n(s,t,h) = c(\frac{t-s}{n})^n$ is referred to as simple product integral .

4.2. Theorem (Approximation theorem). *Let A be convenient algebra. Given $X : \mathbb{R} \to A$ and a smooth mapping $h : \mathbb{R} \times \mathbb{R}_{\geq 0} \to A$ with $h(r,0) = e$ and $\frac{\partial}{\partial t}|_{t=0}h(r,t) = X(r)$. Suppose that for every fixed $s_0 \in \mathbb{R}$, there is $t_0 > s_0$ such that $p_n(s,t,h) = O(1)$ on $\mathbb{N} \times \{(s,t) \in [s_0,t_0]^2 \mid s \leq t\}$. Then the product integral $\prod_s^t h(r,dr)$ exists and the convergence is Mackey in all derivatives on compact (s,t)-sets for $s \leq t$. Furthermore the product integral is the right evolution of X.*

4.3. Remark. The hypothesis on the product integrals will be referred to as *boundedness condition* .

PROOF. Literally the condition on the approximations is the following: There is an absolutely convex bounded and closed set B such that for $s_0 \leq s \leq t \leq t_0$ and $n \in \mathbb{N}$

$$p_n(s,t,h) \in B$$

We have to derive some more subtle boundedness conditions: Therefore we apply Taylor development several times to get the necessary order estimates. First we show that the first derivative of the product integral is bounded, too.

$$\frac{d}{d\delta}\prod_{i=0}^{n-1} h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n}) \in$$
$$\sum_{j=0}^{n-1}\prod_{i=0}^{j-1} h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})\cdot C\frac{1}{n}\cdot\prod_{i=j+1}^{n-1} h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})$$

for $\delta\in[0,t_0-s_0]$ with a closed absolutely convex bounded set C, such that

$$\begin{aligned}\frac{d}{d\delta}h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n}) &= \partial_1 h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})\frac{(n-i)}{n}+\partial_2 h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})\frac{1}{n} =\\ &= \frac{(n-i)\delta}{n^2}\int_0^1 \partial_2\partial_1 h(s+\frac{(n-i)\delta}{n},r\frac{\delta}{n})dr\\ &\quad +\partial_2 h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})\frac{1}{n}\in C\frac{1}{n}\end{aligned}$$

for $\delta\in[0,t_0-s_0]$ and $n\in\mathbb{N}$. Remark that $\partial_1^k h(s,0)=0$ for $s\in\mathbb{R}$, $k\geq 1$. The other factors in the above sum are of type p_m with adjusted lower and upper bound and step width,

$$\prod_{i=0}^{j-1} h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n}) = p_j(s+\frac{(n-j)\delta}{n},s+\delta,h)$$
$$\prod_{i=j+1}^{n-1} h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n}) = p_{n-j-1}(s,s+\frac{(n-j-1)\delta}{n},h)$$

so bounded by assumption on the respective intervals in $[s_0,t_0]$. This allows to conclude boundedness of the first derivative with respect to t. Repeating this procedure we obtain by induction for $k=1,2$ that

$$\frac{d^k}{d\delta^k}\prod_{i=0}^{n-1} h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n}) = O(1)$$

for $\delta\in[0,t_0-s_0]$ and $n\in\mathbb{N}$, since

$$\begin{aligned}\frac{d^2}{d\delta^2}h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n}) &= \partial_1^2 h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})\frac{(n-i)^2}{n^2}+2\partial_1\partial_2 h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})\frac{1}{n}\frac{(n-i)}{n}+\\ &\quad +\partial_2^2 h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})\frac{1}{n^2} =\\ &= \frac{(n-i)^2\delta}{n^3}\int_0^1 \partial_2\partial_1^2 h(s+\frac{(n-i)\delta}{n},r\frac{\delta}{n})dr+2\partial_1\partial_2 h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})\frac{1}{n}\frac{(n-i)}{n}+\\ &\quad +\partial_2^2 h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})\frac{1}{n^2}\in D\frac{1}{n}\end{aligned}$$

and

$$\frac{d^2}{d\delta^2}\prod_{i=0}^{n-1} h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n}) \in$$
$$\sum_{0\leq j<k\leq n-1}\prod_{i=0}^{j-1} h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})\cdot C\frac{1}{n}\cdot\prod_{i=j+1}^{k-1} h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})\cdot$$
$$C\frac{1}{n}\cdot\prod_{i=k+1}^{n-1} h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})+$$
$$+\sum_{j=0}^{n-1}\prod_{i=0}^{j-1} h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})\cdot D\frac{1}{n}\cdot\prod_{i=j+1}^{n-1} h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})$$

where D is a bounded closed absolutely convex subset of A. This means in first consequence that

$$\text{(E)} \qquad h(s+\delta,\delta) - p_m(s,s+\delta,h) = O(\delta^2)$$

by Taylor's formula up to second order uniformly in m, since $h(s,0)=e$ and $\frac{d}{d\delta}|_{\delta=0} h(s+\delta,\delta) = X(s)$ and $\frac{d}{d\delta}|_{\delta=0} \prod_{i=0}^{n-1} h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n}) = X(s)$.

Now we calculate

$$p_n(s,t,h) - p_{nm}(s,t,h) = \sum_{j=0}^{n-1}\left(\prod_{i=0}^{j-1} h(s+\frac{(n-i)(t-s)}{n},\frac{t-s}{n})\cdot\right.$$

$$\left(h(s+\frac{(n-j)(t-s)}{n},\frac{t-s}{n}) - \prod_{i=jm}^{(j+1)m-1} h(s+\frac{(nm-i)(t-s)}{nm},\frac{t-s}{nm})\right)\cdot$$

$$\left.\prod_{i=(j+1)m}^{nm-1} h(s+\frac{(nm-i)(t-s)}{nm},\frac{t-s}{nm})\right)$$

in the spirit of the following formula

$$\text{(S)} \qquad a_1\cdot\ldots\cdot a_n - b_1\cdot\ldots\cdot b_n = \sum_{j=1}^{n} a_1\cdot\ldots\cdot a_{j-1}(a_j-b_j)b_{j+1}\cdot\ldots\cdot b_n$$

which is true in any associative algebra. For the middle factor of the above series we observe that

$$h(s+\frac{(n-j)(t-s)}{n},\frac{t-s}{n}) = h(s+\frac{(n-j-1)(t-s)}{n}+\frac{t-s}{n},\frac{t-s}{n})$$

The other term is the m-th approximation for a product integral with lower border $s+\frac{(n-j-1)(t-s)}{n}$ and upper border $s+\frac{(n-j)(t-s)}{n}$:

$$\prod_{i=jm}^{(j+1)m-1} h(s+\frac{(nm-i)(t-s)}{nm},\frac{t-s}{nm}) = p_m(s+\frac{(n-j-1)(t-s)}{n},s+\frac{(n-j)(t-s)}{n},h)$$

Via the estimate (E) with $\delta=\frac{t-s}{n}$ we arrive at

$$p_n(s,t,h) - p_{nm}(s,t,h) = O(\frac{(t-s)^2}{n})$$

on $\{m|m\in\mathbb{N}\}\times\{(s,t)\in[s_0,t_0]^2\,|\,s\le t\}$, which provides the Mackey-Cauchy property for the above sequence of mappings. Convergence in all derivatives follows by redoing the above program: Calculating the derivative of order k needs the binomial formula

$$\begin{aligned}
\frac{d^k}{d\delta^k}h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n}) &= \sum_{j=0}^{k}\binom{k}{j}\partial_1^j\partial_2^{k-j}h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n})(\frac{(n-i)}{n})^j(\frac{1}{n})^{k-j} = \\
&= X^{(k)}(s+\frac{(n-i)\delta}{n})\frac{\delta}{n}(\frac{(n-i)}{n})^k + \\
&+ kX^{(k-1)}(s+\frac{(n-i)\delta}{n})(\frac{(n-i)}{n})^{k-1}\frac{1}{n} + O(\frac{1}{n^2}) = \\
&= X^{(k)}(s+\frac{(n-i-1)\delta}{n})\frac{\delta}{n}(\frac{(n-i)}{n})^k + \\
&+ kX^{(k-1)}(s+\frac{(n-i-1)\delta}{n})(\frac{(n-i)}{n})^{k-1}\frac{1}{n} + O(\frac{1}{n^2})
\end{aligned}$$

for $k\ge 1$, where we used Taylor development for the last line. The terms of order $\frac{1}{n}$ cancel away in our summation procedure (S), since this formula holds for all smooth h with $h(s,0)=e$ and $\partial_2 h(s,0)=X(s)$, hence also for $p_m(s,s+t,h)$ (remark that $p_n(s,s+t,p_m(s,s+t,h)) = p_{nm}(s,s+t,h)$) and we deal with differences of the type

$$\frac{d^k}{d\delta^k}h(s+\frac{(n-i)\delta}{n},\frac{\delta}{n}) - \frac{d^k}{d\delta^k}p_m(s+\frac{(n-i-1)\delta}{n},s+\frac{(n-i)\delta}{n},h) = O(\frac{1}{n^2})$$

Hence this is sufficient for convergence

$$\frac{\partial^k}{\partial t^k}(p_n(s,t,h) - p_{nm}(s,t,h)) = O(\frac{1}{n})$$

for $k \geq 1$. The limit will be denoted by $c(s,t)$ on the interval $[s_0, t_0]$. Differentiating these equations in s we obtain the same estimates, which are sufficient for convergence, too. "Sufficient for convergence" will be explained symbolically: Differentiating k-times yields with the above summation procedure (S) n^{k+1} terms to sum up (see the formula for the second derivative above). There are n terms where order of differentiation k appears, $O(n^2)$ terms where two orders smaller than k appear, but with sum k, $O(n^3)$ terms where three orders smaller than k with sum k appear,... Applying our summation procedure (S) to the n terms where order k of differentiation appears we get n^2 terms: n terms involve k-th derivative, so the difference is of order $\frac{1}{n^2}$, the other $n^2 - n$ terms involve oridnary factors, so the difference is of order $\frac{1}{n^2}$, but the there is some outer factor $\frac{1}{n}$. So we get inductively our order estimate:

$$O((n^2-n)\frac{1}{n^2}\frac{1}{n} + n\frac{1}{n^2} + (n^3 - 2n^2)\frac{1}{n^2}\frac{1}{n^2} + 2n^2\frac{1}{n}\frac{1}{n^2} +$$
$$+(n^4 - 3n^3)\frac{1}{n^2}\frac{1}{n^3} + 3n^3\frac{1}{n^2}\frac{1}{n^2} + ... + (n^{k+1} - kn^k)\frac{1}{n^2}\frac{1}{n^k} + kn^k(\frac{1}{n^2})^k) = O(\frac{1}{n})$$

Consequently $c(s,t)$ is smooth for $s \leq t$ in $[s_0, t_0]$ and the convergence takes place as Mackey-convergence in $C^\infty(\{(s,t) \in [s_0,t_0]^2 \,|\, s \leq t\}, A)$ with quality $\frac{1}{n}$ in each derivative.The propagation condition follows with standard arguments on continuity with respect to the smooth topology:

$$c(r,t) = c(s,t)c(r,s)$$

for $(t-r)q + r = s$ with $q \in \mathbb{Q}$, $0 < q < 1$ by construction and everywhere by continuity.

We calculate $\frac{\partial}{\partial t}c(s,s) = X(s)$ via uniform convergence of the derivative, then differentiation of the propagation condition yields the result

$$\frac{\partial}{\partial t}c(r,t) = X(t)c(r,t)$$

for $t = s$ and $r \leq s \leq t \in [s_0, t_0]$, so $c(r,t)$ is smooth in t. Looking at the situation of an arbitrary interval we can multiply existing product integrals to get an arbitrary one: Given $s < t$ we can cover this compact interval by intervals of length $\frac{t-s}{k}$ for k large enough, such that on the cover-intervals our estimates are valid.

$$p_{mk}(s,t,h) = p_m(s, s + \frac{t-s}{k}, h) \cdot ... \cdot p_m(t - \frac{t-s}{k}, t, h)$$

and $p_{mk}(s,t,h) - p_{mk+r}(s,t,h) = O(\frac{t-s}{mk})$ for $0 \leq r < k$, so we get the desired boundedness condition on the interval $[s,t]$. $\square$

The next corollary asserts smooth dependence on the smooth curve X, which will be useful in the sequel:

4.4. Corollary. *Let A be convenient algebra. Given a smooth curve $X : \mathbb{R}^2 \to A$ and a smooth mapping $h : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to A$ with $h(r_1, r_2, 0) = e$ and $\frac{\partial}{\partial t}h(r_1,r_2,0) = X(r_1,r_2)$. Suppose that for every fixed $s_0 \in \mathbb{R}$ and a compact r_1-interval, there is $t_0 > s_0$ such that $p_n(s,t,h)(r_1) = O(1)$ on $\mathbb{N} \times \{(s,t) \in [s_0,t_0]^2 \,|\, s \leq t\}$ and the compact r_1-intervals. Then the product integral $\prod_s^t h(r_1, r_2, dr_2)$ exists as smooth mapping on $\mathbb{R} \times \{(s,t) \,|\, s \leq t\}$ and the convergence is Mackey in all derivatives on compact (r_1, s, t)-sets for $s \leq t$.*

Proof. By inheritance we obtain that $C^\infty(\mathbb{R}, A)$ is a convenient algebra and the above condition tells in fact that the product integrals lie in a bounded set in this algebra on compact (s,t)-sets for $s \leq t$ and $n \in \mathbb{N}$. Consequently we arrive at the desired result. The boundedness in $C^\infty(\mathbb{R}, A)$ follows from direct calculation since $\partial_1^k h(r_1, r_2, t) = O(t)$ on compact (r_1, r_2)-sets. $\square$

5. Analytic Semigroups

As in classical theory analytic semigroups provide an important setting for the analysis of distinguished properties of solutions of abstract Cauchy problems. Our setting is convenient for holomorphic semigroups, too. A holomorphic semigroup S is a holomorphic mapping from $\Sigma_\delta \subset \mathbb{C}$, where Σ_δ denotes the open cone with vertex 0 and angle $\delta \in [0, \pi]$, such that $S(\lambda_1 + \lambda_2) = S(\lambda_1)S(\lambda_2)$ for all $\lambda_1, \lambda_2 \in \Sigma_\delta \cup \{0\}$. Furthermore $S(\exp(i\theta)t)$ is smooth for $t \geq 0$ and $\theta \in]-\delta, \delta[$. The infinitesimal generator is the derivative at 0 of $S(.)$ on the real axis. For the rest of the section we assume the convenient spaces to be complex.

5.1. Proposition (Structure theorem for analytic semigroups). *Let A be a complex convenient algebra, $a \in A$. Then a is the generator of a holomorphic semigroup S on Σ_δ for some $\delta \in]0, \frac{\pi}{2}[$ if and only if $\exp(\pm i\theta)a$ is the infinitesimal generator of a smooth semigroup for $\theta \in]0, \frac{\pi}{2}[$.*

Proof. One direction is obvious by the uniform boundedness principle. Assume that $\exp(\pm i\theta)a$ is the infinitesimal generator of a smooth semigroup for $\theta \in]0, \frac{\pi}{2}[$: We denote the associated smooth semigroups by $T^{\pm\theta}$, then the smooth semigroup $T^{\theta}_{kt}T^{-\theta}_{lt}$ has infinitesimal generator $(k\exp(i\theta) + l\exp(-i\theta))a$ for $k, l > 0$. Consequently for any $\theta' \in]-\theta, \theta[$ there is a smooth semigroup $T^{\theta'}$ with generator $\exp(i\theta')a$ and T is smooth for $t > 0$ and $\theta' \in]-\theta, \theta[$, so $T : \Sigma_\theta \to A$ is smooth and $T(\exp(i\theta')t)$ is a smooth semigroup. Differentiating $T(a + ib)$ along the axes inside Σ_θ yields that T is holomorphic, since $i\frac{\partial}{\partial a}T(a + ib) = \frac{\partial}{\partial b}T(a + ib)$. □

5.2. Corollary. *Let A be a complex convenient algebra. A semigroup T with angle $\delta > \frac{\pi}{2}$ can be extended to a holomorphic group $T : \mathbb{C} \to A$ and therefore the exponential series converges.*

Proof. By the above methods there are smooth groups with generator a and ia. Consequently the exponential series converges by proposition 3.6.. □

The Hille-Yosida-Miyadera-Theorem for smooth semigroups can be reformulated in the complex case, which is the basis of the holomorphic version of the Hille-Yosida-Theorem.

5.3. Definition. *Let $a \in A$ be a given element of the complex convenient algebra A and $\delta \in]0, \pi[$. A smooth map $R : \Sigma_\delta \to A$ is called asymptotic resolvent for $a \in A$ if*

1. *$aR(\lambda) = R(\lambda)a$ and $R(\lambda)R(\mu) = R(\mu)R(\lambda)$ for $\lambda, \mu \in \Sigma_\delta$.*
2. *$(\lambda - a)R(\lambda) = e + S(\lambda)$ with $S : \Sigma_\delta \to A$ smooth and there is are constants $b > 0$ so that the set*
$$\{\frac{\exp(b\Re\lambda)}{b^k}S^{(k)}(\lambda) \mid \lambda \in \Sigma_\delta + \omega \text{ and } k \in \mathbb{N}\}$$
is bounded in A.

5.4. Theorem (Hille-Yosida-Theorem). *Let A be a complex convenient algebra and $a \in A$ an element, then a is the infinitesimal generator of a smooth semigroup T in A if and only if there is an asymptotic resolvent $R : \Sigma_{\frac{\pi}{2}} + \omega \to A$ for a with*
$$\{\frac{(\Re\lambda)^{n+1}}{n!}R^{(n)}(\lambda) \mid \Re\lambda > \omega \text{ and } n \in \mathbb{N}\}$$
a bounded set in A (Hille-Yosida-condition).

By the above proposition we can assert the following complex version of the Hille-Yosida-Theorem, anyway we could prove it directly by the methods of section 2:

5.5. Theorem (complex Hille-Yosida-Theorem). *Let A be a complex convenient algebra and $a \in A$ an element, then a is the infinitesimal generator of a holomorphic semigroup T in A of angle δ if and only if there is an asymptotic resolvents $R : \Sigma_\delta \to A$ for a with*
$$\{\frac{|\lambda|^{n+1}}{n!}R^{(n)}(\lambda) \mid \lambda \in \Sigma_{\delta-\epsilon} \text{ and } n \in \mathbb{N}\}$$
bounded in A for a given $\epsilon \in]0, \delta[$, where the perturbation term satisfies that
$$\{\frac{\exp(b|\lambda|)}{b^k}S^{(k)}(\lambda) \mid \lambda \in \Sigma_{\delta-\epsilon} \text{ and } k \in \mathbb{N}\}$$
is bounded in A with a constant $b > 0$.

Proof. Given a holomorphic semigroup of angle δ we obtain smooth semigroups T^θ with generator $\exp(i\theta)a$ for all $\theta \in]-\delta,\delta[$. Given $\epsilon \in]0,\delta[$ we take the standard asymptotic resolvents associated to $T^{\pm\theta}$ for $|\theta| \leq \delta-\epsilon$ and denote them by $R^{\pm\theta}:\mathbb{R}_{>0}\to A$. We define $R(|\lambda|\exp(i\theta)) = R^{-\theta}(|\lambda|)\exp(-i\theta)$. The asserted estimates are satisfied, since $S(|\lambda|\exp(i\theta)) = \exp(-|\lambda|b)T_{\exp(-i\theta)b}$.

A asymptotic resolvent satisfying the given estimates produces smooth semigroups with generators $\exp(i\theta)a$ for all $\theta \in]-\delta,\delta[$, since an asymptotic resolvent satisfying the Hille-Yosida condition for $\exp(i\theta)a$ is given through

$$R^\theta(\lambda) = \exp(-i\theta)R(\exp(-i\theta)\lambda)$$

for $\lambda > 0$. So there is an analytic semigroup with generator a of angle δ. $\square$

5.6. Remark. As in the Banach space case there can be given several different characterizations of analytic semigroups of angle δ by similar methods. Especially there are formulations for exponentially bounded holomorphic semigroups, where the resolvent is defined on a cone $\Sigma_{\delta+\frac{\pi}{2}} \setminus \{\lambda \in \mathbb{C} \mid |\lambda| < \omega\}$.

5.7. Example (bounded analytic semigroups). The case of bounded analytic semigroups with angle can be treated by classical methods as in section 1: A holomorphic semigroup of angle δ is called bounded if for any fixed $\epsilon \in]0,\delta[$ $T_z = O(1)$ on $\Sigma_{\delta-\epsilon}$. Then the resolvent exists at least on $\Sigma_{\delta+\frac{\pi}{2}}$ and satisfies the sectorial estimates

$$\{|\lambda|R(\lambda,a) \mid \lambda \in \Sigma_{\delta+\frac{\pi}{2}-\epsilon}\}$$

is bounded for any $\epsilon \in]0,\delta[$. Given an operator $a \in A$ satisfying this estimate, then the semigroup can be defined by the following formula: For $z \in \Sigma_{\delta+\frac{\pi}{2}}$

$$T(z) := \frac{1}{2\pi i}\int_\gamma \exp(\mu z)R(\mu,A)d\mu$$

where γ is any piecewise smooth curve in $\Sigma_{\frac{\pi}{2}+\delta}$ going from $+\infty\exp(-i(\frac{\pi}{2}+\delta'))$ to $+\infty\exp(i(\frac{\pi}{2}+\delta'))$ for some $\delta' \in]|\arg(z)|,\delta[$.

6. Refinements, Applications and Examples

We analyze special unital convenient algebras and the case of simple product integrals by several well-known examples

1. The boundedness condition is always satisfied up to the level of unital locally m-convex convenient algebras A (the only completeness assumption is Mackey-completeness). Let $c:\mathbb{R}_{\geq 0}\to A$ be a smooth curve passing through the identity at zero. Let $p: A\to\mathbb{R}$ be a continuous seminorm satisfying $p(ab)\leq p(a)p(b)$ and $p(e)=1$. A set of seminorms of this type can be chosen on any unital locally m-convex convenient algebra A. Then we obtain for a given $s\in\mathbb{R}_{>0}$
$$p(c(\frac{t}{n})^n) \leq p(c(\frac{t}{n}))^n \leq (1+\frac{Kt}{n})^n \leq \exp(Kt)$$
for $t\in[0,s]$. The constant K depends on c and s, indeed we obtain
$$K = \sup_{t\in[0,s]} p(c'(t))$$
In this case we obtain a smooth one-parameter group in each direction.
2. It is easy to construct examples, where the boundedness condition is not satisfied: Take $A = L(s)$ the unital convenient algebra of bounded (which is equivalent to "continuous" on Fréchet spaces) operators on the space of rapidly decreasing sequences s. We take for $a: s\to s$ the following bounded operator $a(x_1,x_2,x_3,...) = (0,1^2x_1,2^2x_2,3^2x_3,...)$, then the Abstract Cauchy Problem associated to a has no nontrivial solutions. Consequently no semigroup with generator a exists. Anyway a can be decomposed into two nilpotent operators of order 2:
$$a_1(x_1,x_2,x_3,...) = (0,1^2x_1,0,3^2x_3,0,...)$$
$$a_2(x_1,x_2,x_3,...) = (0,0,2^2x_2,0,4^2x_4...)$$

$a = a_1 + a_2$, ${a_1}^2 = 0$ and ${a_2}^2 = 0$. We define $c(t) = \exp(a_1 t)\exp(a_2 t)$ for $t \in \mathbb{R}$. For this smooth curve the boundedness condition cannot be satisfied, otherwise a smooth semigroup with generator a would exist, which is a contradiction (see example 3.8.).

Approximations of smooth semigroups are provided by Trotter formulas, which are proved to be a type of existence theorem within the theory, since one does not need the existence of the solution to make a Trotter approximation converge:

6.1. Corollary. *Let $c : \mathbb{R} \times \mathbb{R}_{\geq 0} \to A$ a smooth curve into a convenient algebra A with $c(s, 0) = e$ and for any compact s-interval there is $r > 0$ such that*

$$\{c(s, \frac{t}{n})^n \,|\, 0 \leq t \leq r,\, n \in \mathbb{N}\} \text{ is bounded in } A,$$

then the limit $\lim_{n\to\infty} c(s, \frac{t}{n})^n$ exists uniformly on compact intervals in $\mathbb{R}^2$ in all derivatives. Furthermore the resulting family $T_t(s)$ is a smooth family of smooth semigroups with infinitesimal generator $\frac{\partial}{\partial t} c(s, 0)$ for $s \in \mathbb{R}$.

6.2. Corollary. *Let E be a convenient vector space and $c : \mathbb{R}_{\geq 0} \to L(E)$ a smooth curve passing through the identity at zero. There is $s > 0$ so that for every $x \in E$ the set*

$$\{c(\frac{t}{n})^n x | 0 \leq t \leq s\}$$

is bounded in E, then the boundedness condition is satisfied for c in $L(E)$.

The main theorem of the previous part is in fact an existence theorem. The question, what is implied by the existence of a semigroup was not treated, more precisely: Let E be a convenient vector space, T a semigroup of bounded linear operators on E with infinitesimal generator $a \in L(E)$. If $c : \mathbb{R}_{\geq 0} \to E$ is a smooth curve, so that $c(0) = id$ and $c'(0) = a$ are satisfied, does the sequence $\{c(\frac{t}{n})^n\}_{n\in\mathbb{N}}$ converge in some sense to the semigroup T ? The question seems to be difficult. First we prove some general result in the direction, then we try to work out an additional assumption which guarantees the positive answer to the raised question.

6.3. Proposition. *Let E be a convenient vector space, T a semigroup of bounded linear operators on E with infinitesimal generator $a \in L(E)$ and $c : \mathbb{R}_{\geq 0} \to E$ a smooth curve, so that $c(0) = id$, $c'(0) = a$. Given $s_0 > 0$, for every $x \in E$ there exists $k \in \mathbb{N}$ and $s \geq s_0$, so that the set*

$$\{\frac{1}{n^k} c(\frac{t}{n})^n x | n \in \mathbb{N},\, 0 \leq t \leq s\}$$

is bounded in E. Then the boundedness condition is satisfied for the given curve c, consequently the sequence $\{c(\frac{t}{n})^n\}_{n\in\mathbb{N}}$ converges to a smooth semigroup T_t uniformly on compact subsets of $[0, \infty[$ in all derivatives.

Proof. We apply the same methods as in the proof of the main approximation theorem, but we use the formulas pointwise to subtilize the results. Let $x \in E$ be given, then we obtain by the above

$$T_t x - c(\frac{t}{n})^n x = \sum_{i=1}^{n} T_{\frac{t(i-1)}{n}} (T_{\frac{t}{n}} - c(\frac{t}{n})) c(\frac{t}{n})^{n-i} x$$

for $n \in \mathbb{N}$ and $t \in \mathbb{R}_{>0}$. The middle term is estimated in the usual way

$$T_{\frac{t}{n}} - c(\frac{t}{n}) \in \frac{t^2}{n^2} C \text{ for all } k \in \mathbb{N}, t \in [0, s] \quad .$$

for a given $s > 0$. By hypothesis there is a bounded set B and positive number $k := k(x) \in \mathbb{N}$ so that on $[0, s] := [0, s(x)]$ with $s(x) \geq s_0$

$$c(\frac{t}{n})^n x \in n^k B \quad .$$

Inserting all estimates we obtain

$$T_t x - c(\frac{t}{n})^n x \in \frac{t^2}{n^2} \sum_{i=1}^{n} T_{\frac{t(i-1)}{n}} C (n-i)^k B \quad ,$$

which means that the assumed estimate can be improved on $[0,s]$. We arrive finally at

$$c(\frac{t}{n})^n x \in n^{k-1}B'$$

on the interval $[0,s]$. Repeating this procedure k times we arrive at the result that for any $x \in E$ there is $s \geq s_0$ so that

$$\{c(\frac{t}{n})^n x | n \in \mathbb{N},\, 0 \leq t \leq s\}$$

is bounded in E. □

By the same methods we can prove a version of this proposition on convenient algebras:

6.4. Proposition. *Let A be a convenient algebra, T a smooth semigroup with infinitesimal generator a and $c : \mathbb{R}_{\geq 0} \to A$ a smooth curve, so that $c(0) = e$ and $c'(0) = a$. If there exists $s > 0$ and $k \in \mathbb{N}$ so that*

$$\{\frac{1}{n^k} c(\frac{t}{n})^n | n \in \mathbb{N},\, 0 \leq t \leq s\}$$

is bounded in A, then the boundedness condition is satisfied for the given curve c.

These two propositions provide us with sufficient conditions on a curve to guarantee the convergence to the given semigroup, nevertheless this property need not be true in general.

The classical Hille-Yosida-Theorem is another corollary of the approximation theorem:

6.5. Corollary. *Let X be a Banach space, $A : \mathcal{D}(A) \to X$ an operator such that there are constants $\omega_0 > 0$ and $M \geq 1$ with $[\omega_0, \infty[\subset \rho(A)$ and*

$$||(\frac{1}{\lambda - A})^n|| \leq \frac{M}{(\lambda - \omega_0)^n}$$

for $\lambda > \omega_0$. Then A is the generator of a C_0-semigroup T on X and

$$s\text{-}\lim_{n\to\infty} (id - A\frac{t}{n})^{-n} = T_t$$

uniformly on compact intervals in $\mathbb{R}_{\geq 0}$.

Proof. We shall work in the Fréchet space $F = \mathcal{D}(A^\infty)$. There the resolvent $R(\lambda, A)$ can be restricted to a well defined smooth mapping for $\lambda > \omega_0$ satisfying the following estimate

$$(R(\lambda, A)|_F)^n = O(\frac{1}{(\lambda - \omega_0)^n})$$

The curve $c(t) := \frac{1}{t} R(\frac{1}{t}, A)$ is smooth and the boundedness condition is satisfied for $\frac{1}{t} > \omega_0$. Consequently there is a smooth semigroup T with

$$\lim_{n\to\infty} c(\frac{t}{n})^n = T_t$$

Mackey on compact intervals in $\mathbb{R}_{\geq 0}$. By uniformity we can extend to the Banach space, where we obtain strong convergence. □

6.6. Example. In [**Wen85a**], [**Wen85b**] perturbation and approximation theorems are considered in the spirit of strongly continuous theory. In the convenient setting they are simple corollaries of the given theorems.

By the same methods one can prove another type of approximation theorem like Trotter-Kato (see [**EN99**] for a discussion in the Banach space case) asserting that a sequence of smooth semigroups bounded on a compact interval containing zero, where the infinitesimal generators form a Mackey-Cauchy sequence, converges in $C^\infty(\mathbb{R}_{\geq 0}, A)$ to a semigroup with infinitesimal generator the limit of the sequence of infinitesimal generators. This theorem has some interesting applications:

6.7. Proposition (Convergence theorem). *Let A be a convenient algebra, $\{T_n\}_{n\in\mathbb{N}}$ a sequence of smooth semigroups with infinitesimal generators $\{a_n\}_{n\in\mathbb{N}}$. If $\{a_n\}_{n\in\mathbb{N}}$ is a Mackey-Cauchy sequence and $\{T_n(t) | 0 \leq t \leq s\}$ is bounded in A (which is equivalent to boundedness in $C^\infty(\mathbb{R}_{\geq 0}, A)$), then there is a semigroup T with infinitesimal generator $a := \lim_{n\to\infty} a_n$ and*

$$\lim_{n\to\infty} T_n = T$$

in $C^\infty(\mathbb{R}_{\geq 0}, A)$.

Proof. We show that $\{T_n\}_{n\in\mathbb{N}}$ is a Mackey-Cauchy sequence in $C^\infty(\mathbb{R}_{\geq 0}, A)$. To do this we show that all derivatives converge uniformly on compact subsets of $\mathbb{R}_{\geq 0}$ in A. Let $I \subset \mathbb{R}_{>0}$ be compact, then we obtain

$$\begin{aligned} T_n^{(k)}(t) - T_m^{(k)}(t) =& a_n^k T_n(t) - a_m^k T_m(t) = (a_n^k - a_m^k) T_n(t) + \\ &+ a_m^k (T_n(t) - T_m(t)) \in (a_n^k - a_m^k) B + tC(a_n - a_m) D \end{aligned}$$

for $k, n, m \in \mathbb{N}$, $t \in I$, where B, C, D denote appropriately chosen absolutely convex, closed bounded sets, depending on k, t, but not on m, n. By the Mackey-Cauchy-property we obtain that

$$a_n^k - a_m^k = \sum_{i=0}^{k} a_n^{i-1}(a_n - a_m) a_m^{k-i} \in t_{nm} D'$$

for a bounded, absolutely convex and closed subset of A and the given double sequence $\{t_{nm}\}_{n,m\in\mathbb{N}}$ measuring the Mackey-convergence of $\{a_n\}_{n\in\mathbb{N}}$. Putting all together we obtain the desired result: The given sequence of smooth semigroups is a Mackey-Cauchy sequence, consequently there is a smooth curve in the convenient space $C^\infty(\mathbb{R}_{\geq 0}, A)$ being the limit. A fortiori this is a smooth semigroup by boundedness of the multiplication. □

6.8. Example (infinite dimensional heat equations). (see [**ADEM97**] for a discussion) In the theory of infinite dimensional heat equations some recent advances have been made by a Trotter-Kato-type formula:
Let A be a unital convenient algebra, $\{T_n\}_{n\in\mathbb{N}}$ be a commuting sequence of smooth semigroups with infinitesimal generators $\{a_n\}_{n\in\mathbb{N}}$, such that $S_n(t) = \prod_{i=0}^n T_i(t)$ for $t \in R_+$ satisfies the boundedness-hypotheses of the convergence theorem proposition 6.7. and $b_n = \sum_{i=0}^n a_i$ for $n \in N$ is a Mackey-Cauchy sequence, then the infinite product

$$\prod_{i=0}^{\infty} T_i := \lim_{n\to\infty} S_n$$

of the sequence of semigroups exists in $C^\infty(\mathbb{R}_{\geq 0}, A)$ and is a smooth semigroup with infinitesimal generator $\sum_{i=0}^\infty a_i$.
This simple corollary can be applied to the following situation: Let T be a smooth bounded group with infinitesimal generator a in a complex unital convenient algebra A, this means that we can find a closed absolutely convex bounded subset B of the convenient algebra A, so that $T_t \in B$ for all $t \in \mathbb{R}$. By means of the Laplace transform one checks easily that the asymptotic resolvent family is a resolvent family and $\mathbb{C} \setminus i\mathbb{R}$ is in the resolvent set. So we have a holomorphic mapping $R(\pm a) : \mathbb{C} \setminus i\mathbb{R} \to A$ given by the resolvents of $\pm a$. By a version of the Hille-Yosida-Theorem theorem 5.4. on convenient algebras we have the following estimates for the powers of $R(\pm a)$: $(\Re\lambda)^n R^n(\pm a, \lambda) \in B$ for $\lambda \in \mathbb{C} \setminus i\mathbb{R}$ and $n \in \mathbb{N}$, where B is the given bounded, absolutely convex and closed subset of A. Now we look at a sector $\Sigma_\alpha := \{\lambda \in \mathbb{C} | \lambda = re^\beta \text{ with } r > 0, -\alpha < \beta < \alpha\}$ in the complex numbers for $0 < \alpha < \pi$. Take $\lambda \in \Sigma_\pi$, then there exist $r > 0$ and $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ such that $\lambda = r^2 e^{2\beta}$, consequently

$$(\lambda - a^2) = (re^\beta - a)(re^\beta + a) \quad .$$

By this formula we obtain that $R(a^2) : \Sigma_\pi \to A$ is holomorphic, in fact - by applying the square root on the given sector - $R(a^2, \lambda) = R(a, \sqrt{\lambda}) R(-a, \sqrt{\lambda})$. In classical theory of C_0-semigroups there is a beautiful formula calculating this new semigroup from the given one:

$$S(\lambda) = \frac{1}{\sqrt{4\pi\lambda}} \int_{\mathbb{R}} e^{-\frac{s^2}{\lambda}} T(s) ds$$

for $\lambda \in \Sigma_{\frac{\pi}{2}} \setminus \{0\}$. The proof is remarkably simple: Take $l \in A'$ a bounded linear functional to investigate analyticity, then

$$\frac{d}{d\lambda} l \circ S(\lambda) T(t) = \frac{d}{d\lambda} \left(\frac{1}{\sqrt{4\pi\lambda}} \int_{\mathbb{R}} e^{-\frac{s^2}{\lambda}} l \circ T(t-s) ds \right) = l \circ (a^2 T(t))$$

by the symmetry of the integral and the integral representation of the one-dimensional Gaussian semigroup for $\lambda \in \Sigma_{\frac{\pi}{2}} \setminus \{0\}$. So the integral defines a holomorphic semigroup on the given sector with generator a^2.

The above observations can be applied in the following theorem, which generalizes an already known theorem about infinite products of a commuting family of C_0-semigroups.

Let E be a complex Fréchet space. Let $\{T_n\}_{n\in\mathbb{N}}$ be a commuting sequence of bounded smooth groups, so that

$$\prod_{i=0}^{\infty} T_i =: T$$

exists in $C^\infty(\mathbb{R}, L(E))$ and is a smooth group with generator $\sum_{i=0}^{\infty} a_i$, where the sum converges absolutely in $L(E)$ (so the order of the product can be chosen arbitrarily). Denote by S_n the associated bounded holomorphic semigroup generated by a_n^2. If there is a bounded, closed and absolutely convex subset, where all the finite products $\prod_{i=0}^{n} S_i$ for $n \in N$ lie, then the infinite product

$$\prod_{i=0}^{\infty} S_i =: S$$

exists in $C^\infty(\mathbb{R}_{\geq 0}, L(E))$ and the infinitesimal generator is $\sum_{i=0}^{\infty} a_i^2$.

The only thing to prove is the (absolute) convergence of the series $s_n := \sum_{i=0}^{n} a_i^2$. Let p be a continuous seminorm on E, then

$$p((s_n - s_{n+k})(x)) \leq \sum_{i=n+1}^{n+k} p(a_i^2(x)) \leq \sum_{i=n+1}^{n+k} q(a_i(x)) \overset{n,k\to\infty}{\to} 0$$

where q denotes a continuous seminorm on E. The existence of q follows from the fact, that $\{a_i\}_{i\in\mathbb{N}}$ is bounded in $L(E)$, consequently equicontinuous, because E is barrelled, so for every continuous seminorm p there is a continuous seminorm q, so that $p(a_i(x)) \leq q(x)$ for $x \in E$. So we obtain that the above series converges pointwisely absolutely. By the uniform boundedness principle the convergence is uniform to the bounded limit in $L(E)$. The above corollary can be stated in more general contexts, namely on convenient vector spaces with barrelled bornological topology, which is not necessary for the application:

Let X be a complex Banach space, $\{T_n\}_{n\in\mathbb{N}}$ a commuting family of bounded C_0-groups on X with infinitesimal generators $\{A_n\}_{n\in\mathbb{N}}$. The linear space

$$\mathfrak{F} := \bigcap_{n\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} D(A_n^k)$$

is on the one hand dense in X by an abstract version of the Mittag-Leffler-Theorem (see [**Est84**] and [**ADEM97**]), on the other hand a Fréchet space with obvious seminorms $p_{n,k}(x) := \sum_{i=0}^{k} \|A_n^i x\|$. On this Fréchet space all the groups T_n are smooth and bounded. We associate the semigroups S_n to T_n with generator A_n^2 and obtain bounded C_0-semigroups on X and smooth bounded semigroups on $\mathfrak{F}$, respectively. If we assume that the series $\sum_{i=0}^{\infty} A_i$ converges absolutely on $\mathfrak{F}$, then the infinite product

$$\prod_{i=0}^{\infty} T_i =: T$$

exists in $C^\infty(\mathbb{R}, L(\mathfrak{F}))$ and is a bounded smooth group, because

$$p_{n,k}(T_i(t)(x)) \leq p_{n,k}(x)$$

and $p(T_i(t)(x)) \leq p(x)$ for all $x \in \mathfrak{F}$ (they commute!). Consequently T can be extended to X by denseness and the given estimates as a bounded C_0-group with infinitesimal generator the closure of $\sum_{i=0}^{\infty} A_i$ on X, because $\mathfrak{F}$ is a core of the infinitesimal generator. By the above corollary we obtain that

$$\prod_{i=0}^{\infty} S_i = S$$

exists in $C^\infty(\mathbb{R}_{\geq 0}, L(\mathfrak{F}))$, because

$$\begin{aligned} p(S_i(t)(x)) &= \lim_{n\to\infty} p((id - \frac{t}{n}A_i^2)^{-n}) = \\ &= \lim_{n\to\infty} p((id - \frac{\sqrt{t}}{\sqrt{n}}A_i)^{-n}(id + \frac{\sqrt{t}}{\sqrt{n}}A_i)^{-n}(x)) \leq p(x) \end{aligned}$$

for $t \geq 0$ and $x \in X$ by Hille-Yosida, where from the other necessary estimates for the boundedness of the finite products in $L(\mathfrak{F})$ follow. This semigroup can be extended to a C_0-semigroup on X with infinitesimal generator the closure of $\sum_{i=0}^{\infty} A_i^2$ on X by the same argument. In some cases this semigroup is referred to as infinite dimensional Gaussian semigroup , taking translation-groups in different directions on appropriate spaces as groups T_n.

In the sequel two approaches in literature are reviewed and reproved by the simpler convenient setting.

1. A smooth semigroup T in a convenient algebra A is called exponentially bounded if

$$T_t = O(\exp(\omega t))$$

on $\mathbb{R}_{\geq 0}$ for a given $\omega > 0$. Exponentially bounded smooth semigroups can be easily treated by the following methods. First the classical resolvent exists for $\lambda > \omega$, consequently we obtain an asymptotic resolvent with $S = 0$. The exponential formula is therefore valid

$$\lim(e - \frac{ta}{n})^{-n} = T_t$$

in all derivatives on compact subsets of $\mathbb{R}_{\geq 0}$. This problem was treated by several authors with similar approaches motivated by differing interests:

In [**Jef86**], [**Jef87**] weakly integrable semigroups of continuous linear operators are dealt with, which is a very weak concept of one-parameter semigroups. Nevertheless as far as generators are concerned we can apply the given ideas. A semigroup of linear continuous operators $S : \mathbb{R}_{\geq 0} \to L(E)$, where E denotes a locally convex space is called weakly integrable if there is a S'-invariant subspace point separating subspace F of the continuous dual E' with the property that for a given $\omega > 0$ the functions $t \longmapsto \exp(-\lambda t)\langle S(t)x, \xi\rangle$ are integrable for $\lambda > \omega$, $x \in E$, $\xi \in F$ on $\mathbb{R}_{\geq 0}$ such that the operators $R(\lambda) : E \to E$ with

$$\langle R(\lambda)x, \xi\rangle = \int_0^\infty \exp(-\lambda t)\langle S(t)x, \xi\rangle dt$$

exist. Applying our method one should look at $\sigma(E, F)$, the mapping $S : \mathbb{R}_{\geq 0} \to L(E^{\sigma(E,F)})$ is a semigroup of linear continuous operators because of invariance. We assume $E^{\sigma(E,F)}$ to be convenient, but we do not need to assume the existence of the above resolvents. Passing to the C_0-subspace we have to assume that the smooth vectors exist in the given locally convex topology. Remark that this subspace is closed with respect to Mackey-sequences, so convenient. Consequently we can pass to the subspace of smooth vectors by proposition 3.3 and apply the result.

In [**Hug77**] semigroups of unbounded operators on Banach spaces are investigated. They can by definition be reduced to strongly continuous semigroups on a Fréchet space. The author assumes exponential boundedness in his article, consequently the above theory applies.

2. Let S be a smooth semigroup in a convenient algebra with generator a. We denote by $\mathcal{D}$ the test functions on $\mathbb{R}$, by $\mathcal{D}'(A)$ the A-valued distributions. The equation

$$(\frac{d}{dt} - a)(f) = g$$

has a fundamental solution, namely

$$\widetilde{S}_r(\phi) = \int_{-\infty}^{+\infty} S_s\phi(r+s)ds$$

for $\phi \in \mathcal{D}$, where S is extended by 0 to $\mathbb{R}$. Calculating yields

$$\begin{aligned}(\frac{d}{dt} - a)(\widetilde{S}_r)(\phi) &= -\int_{-\infty}^{+\infty} S_s\phi'(r+s)ds - \int_{-\infty}^{+\infty} aS_s\phi(r+s)ds = \\ &= -\int_{0}^{+\infty} \frac{d}{ds}(S_s\phi(r+s))ds = \\ &= \phi(r)e = \delta_r \otimes e(\phi)\end{aligned}$$

which is the property of fundamental solutions. Remark that $\widetilde{S}_0$ has a property of a distribution semigroup. For compactly supported A-valued distributions g we get therefore a continuous inverse of the operator $\frac{d}{dt} - a$ by

$$\begin{aligned}(\frac{d}{dt} - a)(\mathcal{R}g) &= g \\ \mathcal{R}(\frac{d}{dt} - a)(f) &= f\end{aligned}$$

for f, g compactly supported A-valued distributions. Here we define

$$\mathcal{R}g(\phi) = (S * g)(\phi)$$

where τ_r denotes the translation by r. Given a continuous inverse of the operator $(\frac{d}{dt} - a)$ on compactly supported A-valued distributions into all A-valued distributions yields a smooth semigroup on A if some additional topological properties are satisfied.

By Fourier-Laplace transform we can look at the problem in a different way, which allows to apply our results from above. Additionally we need a convenient version of the Palais-Wiener theorem, which is worked out in the sequentially complete case in [**Kom68**].

CHAPTER 3

Product Integrals on infinite dimensional groups

Münchhausens Posthorn war schöner als die fabriksmäßige Stimmkonserve, der Siebenmeilenstiefel schöner als ein Kraftwagen, Laurins Reich schöner als ein Eisenbahntunnel, die Zauberwurzel schöner als ein Bildtelegramm, vom Herz seiner Mutter zu essen und die Vogelstimmen zu verstehen, schöner als eine tierpsychologische Studie über die Ausdrucksbewegungen der Vogelstimme. Man hat Wirklichkeit gewonnen und Traum verloren.
(Robert Musil, Der Mann ohne Eigenschaften)

Regular Lie Groups as defined by A. Kriegl, P. W. Michor, J. Milnor (see [**KM97**] and [**Mil83**]) admit a smooth exponential mapping by definition, but one does not know anything about product integrals of the given smooth one-parameter subgroups in the sense of Hideki Omori. In this chapter it is first shown that on a general class of topological groups containing diffeomorphism groups on compact manifolds many interesting Trotter-type-approximations do exist, if a natural condition on the existence of complex-valued functions on G is satisfied. Then conditions for regularity are investigated and treated by metric space methods: The main conclusion depends on the fact that sequentially compact sets are mapped to bounded ones under smooth representations. The sequential compactness of the approximation sequence for a product integral is shown by the method of Lipschitz-metrics, i.e. right invariant metrics being the non-abelian analogue of seminorms on a locally convex space. We observed that on all known Lie groups a well-behaved family of Lipschitz metrics exists, such that regularity follows.

The first section summarizes parts of chapter 8 of [**KM97**] and provides some first insight in the world of inifinite dimensional differential geometry by two new results on the adjoint maps on diffeomorphism groups proved additionally. In the second section the notion of tempered Lie groups is introduced: We assume the existence of an algebra of continuous complex valued function on the topological group G to be able to handle the question of approximations of one-parameter subgroups on the Lie group with classical functional analysis.

In the third section this Ansatz is reduced to the assumption of the existence of a family of so called Lipschitz-metrics which detect the sequential topology and are proved to allow many statements in the direction of regularity. As a main result we obtain the equivalence of regularity and the existence of Lipschitz-metrics under some slight additional assumption. There we crucially need the approximation theorem of chapter 2, since this provides the via regia to prove smoothness of a limit curve.

1. Convenient Lie groups

In the sequel the basics of calculus on Lie groups modeled on convenient vector spaces are reviewed. Manifolds on convenient vector spaces are a common concept (see chapter 1.4). New phenomena occur as one starts to define tangent spaces. As far as Lie groups on convenient vector spaces are concerned the first main problem is however, that one cannot solve even simple differential equations on the Lie group. This leads to the notion of regularity, which is in a certain sense not comprehensible, because there are hardly applicable conditions for regularity. The most important class of regular Fréchet-Lie groups was given by Hideki Omori et al. (see [**Omo97**]) with the concept of strong ILB-groups , nevertheless this concept is rather complicated to use.

1.1. Definition. *A Lie group G is a smooth manifold modeled on c^∞-open subsets of a convenient vector space with smooth multiplication $\mu : G \times G \to G$, where $\mu(x,y) = xy$, and smooth inversion $\nu : G \to G$, where $\nu(x) = x^{-1}$, for $x, y \in G$. We shall denote by $\mu_x : G \to G$ and $\mu^y : G \to G$ the smooth left and right translation by an element of G, i.e. $\mu_x(y) = \mu^y(x) = \mu(x,y)$ for $x, y \in G$.*

Remark that Lie groups are not topological groups in general, because the identity $c^\infty(E\times E)\to c^\infty E\times c^\infty E$ need not be a homeomorphism. If the Lie group G is a topological group, which we shall assume generically in the thesis, then the underlying topological space is regular (since any Hausdorff topological group is regular), but not necessarily smoothly regular (see appendix 1) and we can assume, that a chosen chart (u,U) has the property that inverse images of closed bounded sets in the convenient vector space are closed in the group, not only relatively closed in $U\subset G$. We shall need this property to be able to lift functions from the convenient vector space to the group. The classical basics of Lie theory can be carried over to this general setting without any problems: Via left or right translation one can trivialize the kinematic tangent bundle $TG=G\times\mathfrak{g}$, where $\mathfrak{g}$ denotes the tangent space at the identity e . On smooth manifolds modeled on a convenient vector space it is a problem to define Lie derivatives along a given vector field $X\in C^\infty(G\leftarrow TG)$, because in general no local flow does exist. In the case of left invariant vector fields ($X\in C^\infty(G\leftarrow TG)$ is called left invariant if $T\mu_x X=X\circ\mu_x$ or equivalently $\mu_x^*X=X$ for $x\in G$) there need not exist a local flow, too, but there is an auxiliary construction which allows to define the Lie derivative (see section 1.3): Via a smooth curve c passing through e and reproducing a given left invariant vector field X at the identity, we can build a function $\phi(t,x)=c(t)x$ with $(t,x)\mapsto(t,xc(t))$ a diffeomorphism and the necessary properties. Given any vector field Y we define

$$L_XY=\frac{d}{dt}|_{t=0}\phi_t^*Y=[X,Y]$$

producing the Lie bracket alternatively, which will be important for invariance considerations.

$\mathfrak{g}$ becomes a Lie algebra, isomorphic to the Lie algebra of left invariant vectorfields and antiisomorphic to the Lie algebra of right invariant vectorfields on G ($X\in C^\infty(G\leftarrow TG)$ is called right invariant if $\mu^{x*}X=X$ for $x\in G$). We denote the (anti-) isomorphism by L (respectively R). We have the following formulas for $x\in G$:

$$L(X)_x=\frac{d}{dt}|_{t=0}\,xc(t) \text{ and } R(X)_x=\frac{d}{dt}|_{t=0}\,c(t)x$$

for $X\in\mathfrak{g}$ and a curve $c:\mathbb{R}\to G$ with $c(0)=e$ and $c'(0)=X$. If $\phi:G\to H$ is a smooth group homomorphism, then $\phi':=T_e\phi:\mathfrak{g}\to\mathfrak{h}$ is a smooth Lie algebra homomorphism.

Now we can formulate the main problem in this setting: Does a Lie group admit a smooth exponential mapping ? An exponential mapping is a map $\exp:\mathfrak{g}\to G$, so that $Fl^{L(X)}(t,x)=x\exp(tX)$ is the global flow to the left invariant vectorfield $L(X)$. An exponential mapping is unique if it exists, remark that the global flow to $R(X)$ is given through $Fl^{R(X)}(t,x)=\exp(tX)x$. Furthermore we obtain for a smooth group homomorphism ϕ of groups, which admit an exponential mapping, the formula $\exp(\phi'(X))=\phi(\exp(X))$ for $X\in\mathfrak{g}$.

1.2. Definition. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The conjugation by an element $x\in G$ defined through $conj_x(y)=xyx^{-1}$ for $y\in G$ is a smooth group automorphism, a so called inner automorphism . $Ad_x:={conj_x}'$ defines a smooth representation $Ad:G\to GL(\mathfrak{g})$, which is easily seen by cartesian closedness. The adjoint representation Ad of the group maps into the subspace of smooth Lie algebra automorphisms. The derivative of Ad (even in the sense of smooth groups) is $ad:\mathfrak{g}\to L(\mathfrak{g})$. The adjoint representation ad of the Lie algebra maps in the subspace of derivations of the Lie algebra $\mathfrak{g}$.*

The last concept of the basics of convenient calculus on Lie groups is the right logarithmic derivative : Let $f:M\to G$ be a smooth map, where M is a smooth manifold. We define the right logarithmic derivative $\delta^r f:TM\to\mathfrak{g}$ by the formula

$$\delta^r f(\xi_x):=T_{f(x)}(\mu^{f(x)^{-1}})(T_xf(\xi_x))$$

for $x\in M$ and $\xi_x\in T_xM$. By definition we see that $\delta^r f\in\Omega^1(M,\mathfrak{g})$ is a $\mathfrak{g}$-valued 1-form on M. A Lie group G is called regular if there is a smooth (evolution) map $Evol^r:C^\infty(\mathbb{R},\mathfrak{g})\to C^\infty(\mathbb{R},G)$, such that $Evol^r(X)(0)=e$ and $\delta^r(Evol^r(X))(t)=X(t)$ for all $t\in\mathbb{R}$, furthermore $Evol^r(\delta^r c)=c$ (see [**KM97**], [**Mil83**], [**Omo97**] for comparison). Let G be a simply connected Lie group and H a regular Lie group with $f:\mathfrak{g}\to\mathfrak{h}$ a bounded Lie algebra homomorphism, then there is a smooth Lie group homomorphism ϕ with $\phi'=f$ (see [**KM97**], chapter 8, , theorem 40.3).

This means that one can solve all non-autonomous Cauchy problems on the Lie group G, more precisely - given $X\in C^\infty(\mathbb{R},\mathfrak{g})$, there is a smooth curve $c:\mathbb{R}\to G$ with $c(0)=e$ and $c'(t)=$

$T_e\mu^{c(t)}(X(t))$ for $t \in \mathbb{R}$. Such non-autonomous problems can sometimes be solved by so called product integrals, which leads to one of the first definitions of regularity (requiring the existence of product integrals). In the following theorem we collect some results (see [**KM97**]):

1.3. Theorem. *Let G be a Lie group, M a smooth manifold and $f, g : M \to G$ smooth mappings, then we obtain:*

1. *For $X \in \mathfrak{g}$ and $y \in G$: $L(X)_y = R(Ad_y(X))_y$*
2. *$ad(X)(Y) = [X, Y]$ for $X, Y \in \mathfrak{g}$*
3. *$\delta^r(fg)(x) = \delta^r f(x) + Ad_{f(x)}(\delta^r g(x))$ for $x \in M$*
4. *If G is a regular Lie group, then $Evol^r$ is unique.*

The concept of strong ILB-groups (see [**Omo97**]) allows to prove the existence of a right evolution map, however the costs are high, as is already clear by definition:

1.4. Definition (strong ILB-group). *An ILB-chain is a sequence of Banach spaces $\{E_k\}_{k\geq d}$ for d a natural number, such that E_{k+1} is continuously injected in E_k for $k \geq d$ and the images of the respective injections are dense. The projective limit of this system is a Fréchet space, denoted by E. A strong ILB-group is a group G modelled on an ILB-chain $\{E_k\}_{k\geq d}$ if and only if*

1. *There is an open neighborhood U of 0 in E_d and a bijection ζ of $U \cap E$ onto a subset $\widetilde{U}$ of G containing e such that $\zeta(0) = e$.*
2. *There is an open neighborhood V of 0 in E_d such that $\zeta(V \cap E)^2 \subset \zeta(U \cap E)$ and $\zeta(V \cap E)^{-1} \subset \zeta(V \cap E)$.*
3. *For $u, v \in V \cap E$ we define $\eta(u, v) := \zeta^{-1}(\zeta(u)\zeta(v))$, then for any $k \geq d$ η extends to a unique continuous map, again denoted by $\eta: V \cap E_k \times V \cap E_k \to U \cap E_k$.*
4. *For any $v \in V \cap E_k$ the map $\eta_v := \eta(., v) : V \cap E_k \to U \cap E_k$ is a smooth map.*
5. *$\theta(w, u, v) := (d\eta_v)(u)(w)$ extends for any $k \geq d$ and any $l \geq 0$ to a C^l-mapping of $E_{k+l} \times V \cap E_{k+l} \times V \cap E_k \to E_k$.*
6. *$i(u) := \zeta^{-1}(\zeta(u)^{-1})$ for $u \in V \cap E$, then i extends to a continuous mapping of $V \cap E_k$ into itself.*
7. *For every $g \in G$ there is a neighborhood W of 0 in E_d such that $g^{-1}\zeta(W \cap E)g \subset \zeta(U \cap E)$.*

All known Fréchet-Lie-groups are strong ILB-groups, so all finite dimensional ones, all Banach-Lie-groups, all diffeomorphism groups. A strong ILB-group is a regular Fréchet-Lie-group, some restricted sort of implicit function theorem is valid in this category (see [**Omo97**]). We have the following conclusions from the definition:

1.5. Theorem. *Let G be a strong ILB-group modelled on an ILB-chain $\{E_k\}_{k\geq d}$, then there is a sequence of topological groups $\{G_k\}_{k\geq d}$ such that:*

1. *G is a Fréchet-Lie-group.*
2. *G_k is a Banach-manifold modelled on E_k.*
3. *G_{k+1} is a dense subgroup of G_k and the inclusion is smooth.*
4. *G is the projective limit of $\{G_k\}_{k\geq d}$.*
5. *The multiplication on G extends to a C^l-mapping from $G_{k+l} \times G_k$ to G_k for $k \geq d$ and $l \geq 0$.*
6. *The inverse extends to an C^l-mapping from G_{k+l} to G_k for $k \geq d$ and $l \geq 0$.*
7. *For any $g \in G_k$ the right translation is a smooth mapping from G_k to G_k.*
8. *The tangent map of the right translation is a C^l-mapping from $TG_{k+l} \times G_k$ to TG_k.*

1.6. Remark. Let M be a compact smooth manifold, then the diffeomorphism group and many of its subgroups (symplectomorphisms if M is symplectic, volume-preserving-diffeomorphisms if M is orientable,...) are strong ILB-groups and regular, which can easily be seen directly by solving non-autonomous equations on the compact manifold and applying cartesian closedness. Anyway the machinery seems to be too complex and too analytic for the solution of the problem of regularity from my point of view (see [**Omo97**] for details in ILB-questions and [**KM97**]).

1.7. Remark. On strong ILB-groups all product integrals converge, so all ILB-groups are regular. Indeed the convergence behaves very well:

A smooth group or Frölicher-Lie-Group is a group G with the structure of a smooth space, such that the multiplication and the inversion are smooth. All smoothly regular Lie Groups are smooth groups in a canonical way. The group $GL(E)$ of invertible linear bounded maps on a convenient vector space E is a standard example, and not a Lie group in general, with the following smooth structure:

$$C_{GL(E)} := \{c : \mathbb{R} \to GL(E) | c : \mathbb{R} \to L(E) \text{ and } inv \circ c : \mathbb{R} \to L(E) \text{ are smooth } \}$$
$$F_{GL(E)} := \{f : GL(E) \to \mathbb{R} | f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R}) \text{ for all } c \in C_{GL(E)}\}$$

Remark that the restrictions of all linear functionals on $L(E)$ lie in $F_{GL(E)}$, furthermore the restriction of the composition of a linear functional with the inversion lies in $F_{GL(E)}$, so $GL(E)$ becomes a smooth space. Multiplication and inversion are smooth. A smooth map $f : G \to H$ between Frölicher-Lie-Groups is called initial if for any curve $c : \mathbb{R} \to G$ with $f \circ c$ smooth smoothness of c is implied. The following example might explain the interest in smooth groups:

1.8. Example. The ∞-Torus $\prod_{k \in \mathbb{N}} S^1$ is a smooth group with smooth curves the componentwise smooth ones. The ∞-Torus is a product in the category of locally compact topological groups and a product in the category of smooth groups. Furthermore the ∞-Torus can never be made to a manifold, because it should be infinite dimensional and locally compact.

Applying the notion of a smooth group we can formulate the following observations:

1.9. Remark. Let G be a convenient smootly regular Lie group and denote by $Ad : G \to GL(\mathfrak{g})$ the adjoint representation. If Ad is initial, then the inner automorphisms $Inn(\mathfrak{g}) := Ad(G)$ constitute a Frölicher-Lie-Group diffeomorphic to G in the category of Frölicher-Lie groups. (proof: the Frölicher-Lie structure is induced by the general linear group $GL(\mathfrak{g})$ and well-defined, the diffeomorphism is given by Ad)

1.10. Proposition. *Let M denote a finite-dimensional manifold and* $\mathrm{Diff}(M)$ *the Fréchet-Lie group of diffeomorphisms of M, then $Ad : \mathrm{Diff}(M) \to GL(\mathfrak{X}(M))$ is an initial map, so* $\mathrm{Diff}(M)$ *is canonically isomorphic to the Frölicher-Lie-Group of inner automorphisms and consequently linear.*

Proof. The proof is done in several steps: First we prove that a curve c in G with $Ad \circ c$ smooth has to be continuous in the following sense: For any $t_0 \in \mathbb{R}$, $x_0 \in M$ and any open neighborhood V of $c^{-1}(t_0, x_0)$ there is $\delta > 0$ and an open neighborhood U of x_0 with $c^{-1}(t, x) \in V$ for $|t - t_0| < \delta$ and $x \in U$. Otherwise there would exist $t_0 \in \mathbb{R}$, $x_0 \in M$, an open neighborhood V of $c^{-1}(t_0, x_0)$ and sequences $t_n \to t_0$ and $x_n \to x_0$ with $c^{-1}(t_n, x_n) \notin V$. Now take a vectorfield X on M having support in V with $X(c^{-1}(t_0, x_0)) \neq 0$. The formula for the action of Ad on vectorfields is

$$Ad_{c_t}(X)(x) = T_{c^{-1}(t,x)} c_t(X(c^{-1}(t, x))$$

Consequently $Ad_{c_{t_n}}(X)(x_n) = 0$ for all n, but $Ad_{c_{t_0}}(X)(x_0) \neq 0$. However, $Ad_{c_t}(X)$ is smooth, so a smooth curve of smooth sections in the tangent bundle, a contradiction. Second it is sufficient to prove the following fact: Let c be a curve passing at $t = 0$ through e with $Ad \circ c$ smooth, then there is a neighborhood of 0, where the curve is smooth. From this we conclude easily the general case by looking at the smoothness of the composition and the curve $c_{t_0}^{-1} c$ around t_0. Third we apply the first observation to prove the assertion of the second step: Let c be a curve passing at $t = 0$ through e with $Ad \circ c$ smooth, then there is a chart domain $V \subset M$, open around $x_0 = c^{-1}(0, x_0)$ mapped to a ball in $\mathbb{R}^n$, furthermore $\delta > 0$ and an open neighborhood $U \subset V$ of x_0 with $c^{-1}(t, x) \in V$ for $|t| < \delta$ and $x \in U$. Smoothness of $Ad \circ c$ reads locally as follows

$$(T_x c_t^{-1})^{-1}(\frac{\partial}{\partial x^i}) \text{ is smooth for } i = 1, ..., n$$

This means by smoothness of the inversion of matrices that $(\frac{\partial (c_t^{-1})^j}{\partial x^i}(x))$ is smooth locally around x_0 and 0. Consequently by compactness of the manifold we conclude that there is a small interval around zero where $c_t^{-1}$is a smooth curve of diffeomorphisms, so c_t is smooth. □

1.11. Remark. By the same methods we can prove that the adjoint representation $ad : \mathfrak{X}(M) \to Der(\mathfrak{X}(M))$ is initial for a compact manifold M (proof: take a bump vector field to show that a given curce X with $ad \circ X$ smooth is continuous and then work in a chart domain to conclude by local observations on the components of the curve X.)

1.12. Definition. *Let G be a convenient Lie group and $\mathfrak{g}$ its Lie algebra. We call G Lie-regular if $Ad : G \to GL(\mathfrak{g})$ is initial.*

This concept of regularity should remind the semisimple situation in the finite-dimensional case, however it is closely associated to regularity of convenient Lie groups.

1.13. Proposition. *Let G be a smoothly regular and regular Lie group with trivial centre and $ad : \mathfrak{g} \to L(\mathfrak{g})$ initial, then G is Lie-regular.*

PROOF. First we prove the assertion for a curve c passing at 0 through e. $Ad \circ c$ smooth means that we can take at least the first derivative, $(\frac{d}{dt} Ad \circ c)(Ad \circ c)^{-1}$ is a smooth curve of derivations of $\mathfrak{g}$, so there is some smooth curve X in the Lie algebra $\mathfrak{g}$ such that $(\frac{d}{dt} Ad \circ c)(Ad \circ c)^{-1} = ad \circ X$. X can be integrated by right evolution to a curve c' with $c'(0) = e$ and $\delta^r c' = X$, consequently $Ad \circ c'$ is smooth and satisfies the differential equation $\frac{d}{dt} Ad \circ c' = ad \circ X(Ad \circ c')$, but this equation has a unique solution, namely $Ad \circ c$, so $Ad \circ c' = Ad \circ c$ and $c = c'$ by injectivity. Then, however, c is smooth. The rest is done by translation. $\square$

1.14. Proposition. *Let G be Lie-regular and $ad(X)$ is a generator of a smooth group of inner automorphisms for $X \in \mathfrak{g}$ depending smoothly on X, then a smooth exponential map exists.*

PROOF. $Ad : G \to GL(\mathfrak{g})$ is initial and therefore we can pull down the the existing smooth one-parameter groups to the group G as smooth one-parameter subgroups. So there is an exponential map, which is smooth by smooth dependence on X. $\square$

1.15. Remark. The generator property can be characterized by the Hille-Yosida-Theorem, such that Lie algebras, where all $ad(X)$ are generators are necessarily the Lie algebras of regular Lie groups.

2. Tempered Lie groups

Up to Banach spaces there is a powerful theory to solve (nonlinear) differential equations due to the inverse function theorem. Already on Fréchet spaces one has to investigate the circumstances much more carefully to obtain results on solvability of differential equations [**LS93**]. Nevertheless Fréchet spaces appear naturally by modelling C^∞-diffeomorphism groups [**KM97**]. There are two possible ways how to approach the problem: Either one tries to translate the given initial value problem into the Banach space setting, which normally leads to a loss of differentiability properties, or one tries to find some rudiments of theory on convenient spaces, so differentiability is preserved, but there is a lack of powerful theorems. Tempered groups are defined by the perspective of the first method. We shall prove that on tempered groups smooth one parameter subgroups can be well approximated by simple product integrals. This is by the way the origin of the notion of temperedness, because the growth of the multiplication can be controlled. More precisely, under the adjoint representation $Ad : G \to L(\mathfrak{g})$ on a regular Lie group simple product integrals exists if and only if they are polynomially bounded in the sense of section 2.6. Banach Lie groups and ILB-Lie groups are tempered, if the model spaces admit C^2_b-bump functions (see appendix 1). Besides the approximation property temperedness seems to be an interesting concept from the point of view of representation theory.

To prove the approximation theorem we apply the beautiful and remarkable theorem of Paul Chernoff on the approximation of C_0-contraction semigroups on a Banach space X (see chapter 2.1).

We shall work in the Banach space $BC(G)$ of continuous complex valued bounded functions on G normed by the supremum norm. Given $X \in \mathfrak{g}$ one obtains by evolution a smooth one-parameter subgroup $\exp(tX)$ of G, if G is regular. We shall investigate the group T of linear operators on $BC(G)$ given through

$$T_t(f)(x) := f(\exp(tX)x) = (f \circ \mu_{\exp(tX)})(x) \text{ for all } x \in G,\ f \in BC(G)$$

for $t \in \mathbb{R}$. For a given smooth curve $c : \mathbb{R} \to G$ with $c(0) = e$ and $c'(0) = X$ we can proceed in the same manner, so we obtain a curve C of isometries on $BC(G)$. To be able to apply Chernoff's theorem , we need an appropriate domain D in $BC(G)$, such that D detects the topology of G in a certain sense and satisfies certain properties concerning translations, this leads us to the concept of temperedness.

2.1. Definition (tempered groups). *A topological group G is said to be tempered if it is a smooth space (multiplication is not necessarily smooth!) and if a unital subalgebra $D \subset BC(G)$ is given, such that the following conditions are satisfied.*

1. *D is invariant under left translations, that means $f \circ \mu_a \in D$ for all $f \in D$ and $a \in G$.*
2. *D detects the converging sequences on G:*

$$\forall \{x_n\}_{n\in\mathbb{N}},\, x \in G : \sup_{y\in G} |f(x_n y) - f(xy)| \to 0 \text{ for all } f \in D \Rightarrow$$
$$x_n \text{ converges to } x \text{ in } G$$

3. *For every smooth curve $c : \mathbb{R} \to G$ with $c(0) = e$ the curve $C : \mathbb{R} \to L(BC(G))$ of left translations by $c(t)$ for $t \in \mathbb{R}$ is differentiable at $f \in D$ for $t = 0$ in the supremum-norm topology of $BC(G)$.*

2.2. Remark. To stay as general as possible we do not assume that G is a smooth groups and that the smooth topology and the topology on G coincide.

2.3. Lemma. *Let G be a tempered group, where the smooth structure is a smooth group structure, $c : \mathbb{R} \to G$ a smooth curve, then c is continuous and $C : \mathbb{R} \to L(BC(G))$ is differentiable at $f \in D$ for all $t \in \mathbb{R}$.*

Proof. Let $c : \mathbb{R} \to G$ be a smooth curve, then $b(t) := c(t)c(0)^{-1}$ for $t \in \mathbb{R}$ is a smooth curve with $b(0) = e$, so $B : \mathbb{R} \to L(BC(G))$ is differentiable at $f \in D$ for $t = 0$. So for any $f \in D$ there exists $g \in BC(G)$, such that

$$\sup_{x\in G} |\frac{f(c(t)c(0)^{-1}x) - f(x)}{t} - g(x)| \overset{t\to 0}{\to} 0 \quad ,$$

consequently by left translation we obtain

$$\sup_{y\in G} |\frac{f(c(t)y) - f(c(0)y)}{t} - g(c(0)y)| \overset{t\to 0}{\to} 0 \quad ,$$

which is the desired assertion. The rest follows by property 2. □

2.4. Lemma. *Let G be a topological group, $U \subset G$ an open neighborhood of e. Then there is a neighborhood $V \subset G$ of e, such that for any continuous curve $c : \mathbb{R} \to G$ with $c(0) = e$ one can find a small open interval J around zero with $\cup_{t\in J} c(t)^{-1}V \subset U$.*

Proof. The mapping $G \times G \to G$, $(g, h) \to g^{-1}h$ is continuous, so the coclusion follows immendiately. □

The following proposition asserts that on tempered topological groups smooth one parameter groups can be well approximated:

2.5. Proposition. *Let G be a tempered topological group. Let $c : \mathbb{R} \to G$ be a smooth curve with $c(0) = e$ touching a smooth one-parameter group S at $t = 0$, more precisely*

$$\forall\, f \in D, x \in G :\ (f \circ \mu^x(c))'(0) = (f \circ \mu^x(S))'(0)$$

Then we obtain

$$\lim_{n\to\infty} c(\frac{t}{n})^n = S_t$$

uniformly on compact subsets of $\mathbb{R}$, i.e. $c(\frac{t}{n})^n S_{-t}$ converges to e as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$. If G is a smoothly regular tempered Lie group with $c^\infty G$ the topology of G, then the convergence of $c(\frac{t}{n})^n$ to S_t is uniform in all derivatives in the sense of lemma 3.2. of chapter 1.

Proof. The first part of the proof is a simple application of the core theorem, which asserts that the closure of the restriction of an infinitesimal generator of a strongly continuous semigroup to an invariant and dense subspace is the infinitesimal generator (see [**Kan95**], theorem 1.7). We shall denote the closure of the subspace $D \subset BC(G)$ by X. T and C denote the curves of contractions on $BC(G)$ given by left translation with $c(t)$ and $S(t)$, respectively. By property 1. D is invariant under the action of $C(t)$ and $T(t)$ for $t \in \mathbb{R}$, by 2.2.iii. the first derivatives of C and T at $t = 0$ exist pointwisely for $f \in D$ and they coincide. So $T|_X$ defines a C_0-group on X, $C|_X$ is a curve of contractions on X. D is a dense, $T|_X$-invariant subspace of the domain of the infinitesimal generator of T, consequently the closure of the restriction of the infinitesimal generator to D is the infinitesimal generator. The application of Chernoff's theorem 1.10 in chapter 2 leads to

$$\lim_{n\to\infty}(C|_X(\frac{t}{n}))^n(f)=(T|_X)_t(f)$$

for all $f\in X$ uniformly on compact subsets of $\mathbb{R}$. In fact we have to apply the theorem two times to obtain the assertion for the whole real line. By property 2. we are lead to the existence of the limit in the group G uniformly on compact subsets of $\mathbb{R}$. Uniformity is due to our specified detection of the topology of G. Suppose that the sequence does not converge uniformly on a given compact interval K to the limit e, so there is an open neighborhood U of e and sequences $\{n_k\}_{k\in\mathbb{N}}$, a monotone, diverging sequence of natural numbers, and $\{t_k\}_{k\in\mathbb{N}}$ in K, so that $c(\frac{t_k}{n_k})^{n_k}S_{-t_k}\notin U$ for $k\in\mathbb{N}$, but

$$\sup_{x\in G}|f(c(\frac{t_k}{n_k})^{n_k}x)-f(S_{t_k}x)|\overset{k\to\infty}{\to}0$$

by the convergence theorem. Consequently

$$\sup_{y\in G}|f(c(\frac{t_k}{n_k})^{n_k}S_{-t_k}y)-f(y)|\overset{k\to\infty}{\to}0$$

which leads by 2. to a contradiction. So the limit exists uniformly on compact subsets of $\mathbb{R}$.

Let G be a additionally topological Lie group, where the adjoint representation Ad maps sequentially smoothly compact sets to bounded ones. As in the sequel the assertion on uniform convergence in all derivatives will reappear (see section 3) we give another perspective to solve the problem:

We denote by $Y\in\mathfrak{g}$ the generator of $S_t=\exp(Yt)$. As established above we know that $\lim_{n\to\infty}c_n(t)=\exp(Yt)$ uniformly on compact subsets of $\mathbb{R}$. The rest of the proof is devoted to the uniform convergence on compact intervals of $\delta^r c_n(t)$ to Y as $n\to\infty$. In fact it is an easy consequence of calculations with right logarithmic derivatives: For smooth curves $c,d:\mathbb{R}\to G$ we have

$$\delta^r(cd)(t)=\delta^r c(t)+Ad_{c(t)}\delta^r d(t)$$

for $t\in\mathbb{R}$. Consequently we obtain

$$\delta^r c_n(t)=\frac{1}{n}\left(\sum_{i=0}^{n-1}Ad^i_{c(\frac{t}{n})}\right)\delta^r c(\frac{t}{n})$$

for all $t\in\mathbb{R}$. The adjoint action of G maps sequentially compact to bounded sets in $L(\mathfrak{g})$, so there is a bounded absolutely convex subset $B\subset L(\mathfrak{g})$ so that $Ad_{c(\frac{t}{n})^nS_{-t}}\in B$ for t in a closed zero neighborhood. By the general approximation theorem 4.2 on convenient algebras we obtain the following Mackey-limit:

$$\lim_{n\to\infty}Ad^n_{c(\frac{t}{n})}=Ad_{S_t}$$

uniformly on compact subsets of $\mathbb{R}$. The sequence measuring Mackey-convergence is given by $\{\frac{t^2}{n}\}_{n\in\mathbb{N}_+}$ on the interval $[0,t]$. To conclude we look at the above sum as an approximation of the integral $\frac{1}{t}\int_0^t Ad_{S_s}ds$ for $t\neq0$ in the convenient algebra $L(\mathfrak{g})$. In fact we can choose n_0 big enough, so that for $n\geq n_0$ the approximation of the limit Ad_{S_t} by $Ad_{c_n(t)}$ is good enough. By uniformity of the respective limits, we obtain that

$$\frac{t}{n}(\sum_{i=0}^{n-1}Ad^i_{c(\frac{t}{n})})=\frac{t}{n}(\sum_{i=0}^{n_0}Ad^i_{c(\frac{t}{n})}+\sum_{i=n_0}^{n-1}(Ad^i_{c(\frac{ti}{in})}-Ad_{S_{\frac{ti}{n}}})+\sum_{i=n_0}^{n-1}Ad_{S_{\frac{ti}{n}}}$$

converges Mackey to the integral uniformly on compact subsets. Consequently

$$\lim_{n\to\infty}\delta^r c_n(t)=\frac{1}{t}\int_0^t Ad_{S_s}ds\,Y=Y$$

uniformly on compact subsets of $\mathbb{R}$. So the assertions are proved. By the way we obtain naturally $c'(0)=Y$. To prove the whole assertion we refer to section 4, where a general method in the same spirit is presented. ☐

2.6. Lemma. *There is a general simple concept how to detect differentiability on Banach spaces, which is in fact valid for much more general situations (see* [**FK88**] *for details):*
Let E be a Banach space, $S \subset E'$ a norming subspace of the dual space, i.e. $\|x\| = \sup\{|l(x)| \mid l \in S$ and $\|l\| \leq 1\}$. Let $I \subset \mathbb{R}$ be an open bounded interval, then a curve $c : I \to E$ is Lip^n for a given $n \in \mathbb{N}$ if there are curves $c^i : I \to E$ for $1 \leq i \leq n+1$ with $(l \circ c)^{(i)} = l \circ c^i$ for $l \in S$ and $1 \leq i \leq n+1$ and c^{n+1} is bounded on I. In this case $c^{(i)} = c^i$ for $0 \leq i \leq n$.

Proof. For $n = 0$ the set $\{\frac{c(t)-c(s)}{t-s} \mid t \neq s \in I\}$ has bounded image under each $l \in S$, because one can apply the mean value theorem. A bound is given through the modulus of the image under $l \in S$ of the closed , absolutely convex hull of $\{c^1(t) | t \in I\}$, which is bounded, so we obtain the Lipschitz property. Consequently c^n is Lip^0. We assume by induction that for $0 < j \leq n$ the curves c^i are Lip^{n-i} with $(c^i)^{(k)} = c^{i+k}$ for $j \leq i \leq n$ and $0 \leq k \leq n - i$. The element $\frac{c^{j-1}(t)-c^{j-1}(s)}{t-s} - c^j(s)$ is bounded under $l \in S$ through $(t-s)\|l\|M$ by Taylor's formula, where M is a bound for the set $\{\frac{c^j(t)-c^j(s)}{t-s} \mid t \neq s \in I\}$ in E for $t \neq s \in I$. So c^{j-1} is differentiable with first derivative c^j. Consequently we obtain the result by induction. □

2.7. Lemma. *Let E be a Banach space, which admits C^2_b-bump functions , then any Lie group modeled on E is tempered.*

Proof. We denote by $B(0, r)$ for $r > 0$ the open ball around zero in E. The linear space of C^2_b-functions with values in $\mathbb{C}$ and support in $B(0, r)$ having bounded first and second derivative is denoted by $C^2_b(r)(E)$. If the Banach space admits C^2_b-bump functions, $C^2_b(r)(E)$ is not empty and detects the converging sequences:

$$x_n \overset{n\to\infty}{\to} x \iff \phi(x_n) - \phi(x) \overset{n\to\infty}{\to} 0 \text{ for all } \phi \in C^2_b(r)(E)$$

for all sequences $\{x_n\}_{n\in\mathbb{N}}$ and $x \in B(0, r)$. Now we take two charts (u_1, U_1), (u_2, U_2) around e of the Banach-Lie group, so that the C^∞-function $\mu := u_2 \circ \mu \circ (id \times u_1{}^{-1}) : U_3 \times B(0, 1)) \to B(0, 1)$, where $U_3 \subset G$ is an open chart domain of E, has the property that $\check{\mu} : B(0, 1) \to C^\infty(U_3, E)$ is globally Lipschitz and consequently bounded. This is possible by applying the well known theorem that Lip^0-functions on a Banach space with values in a convenient vector space are locally Lipschitz around any point in the domain of definition (see section 1 of chapter 1), so we have to shrink the chart domain a little bit. Let $\phi \in C^2_b(1)(E)$ be a bump function and $c : \mathbb{R} \to E$ with $c(0) = 0$ a smooth curve, then $C : I \subset \mathbb{R} \to BC(B(0, 1))$, given through $C_t(f)(x) = f(\mu(c(t), x))$ for $x \in B(0, 1)$, $t \in I$ and $f \in BC(B(0, 1))$, where I is a sufficiently small open interval around zero (so that $c(t) \in U_3$ for $t \in \overline{I}$), is differentiable at ϕ on I. This will be detected by point evaluations ev_x for $x \in BC(0, 1)$, which span a norming subspace for the supremum norm on $BC(B(0, 1))$:

$$\begin{aligned}
\frac{d}{dt} ev_x(C_t(\phi)) =& d\phi(\mu(c(t), x))(d_1\mu_{(c(t),x)}(c'(t))) \\
\frac{d^2}{dt^2} ev_x(C_t(\phi)) =& d^2\phi(\mu(c(t), x))(d_1\mu_{(c(t),x)}(c'(t)), d_1\mu_{(c(t),x)}(c'(t))) + \\
&+ d\phi(\mu(c(t), x))(d_1^2\mu_{(c(t),x)}(c'(t), c'(t))) + \\
&+ d\phi(\mu(c(t), x))(d_1\mu_{(c(t),x)}(c''(t)))
\end{aligned}$$

for $t \in I$ and $x \in B(0, 1)$. The right hand side of the respective derivative is the evaluation of a curve to $BC(B(0, 1))$, because (due to the (global) Lipschitz properties of the derivative and the boundedness of I) the linear parts are bounded in E. Consequently $ev_\phi \circ C : I \to BC(B(0, 1))$ is Lip^1 on I for all $\phi \in C^2_b(1)(E)$. Now we lift $C^2_b(r)$ to the group G with the chart map u_2 given around e, where $0 < r < 1$ is chosen sufficiently small: We chose $0 < r_1 < 1$ so that $u_2{}^{-1}(B(0, r_1)) \subset U_1$. Applying the topological lemma leads to $0 < r < r_1$, so that for every continuous curve $c : \mathbb{R} \to G$ with $c(0) = e$ there is a small interval J around zero with $\cup_{t\in J}\, c(t)^{-1}[u_2{}^{-1}(B(0, r))] \subset u_2{}^{-1}(B(0, r_1))$. To prove differentiability we take $\phi \in C^2_b(r)(E)$, the lifting $\psi = \phi \circ u_2 \in BC(G)$ has support in $u_2{}^{-1}(B(0, r))$. Let $c : \mathbb{R} \to G$ be smooth with $c(0) = e$, then there is J, open around zero with the above property, let g denote the lifting of the first derivative along the curve c of ϕ at $t=0$, then

$$\sup_{x\in G} |\frac{\psi(c(t)x)-\psi(x)}{t} - g(x)| \leq \sup_{x\in U_1} |\frac{\psi(c(t)x)-\psi(x)}{t} - g(x)|$$
$$\leq \sup_{x\in B(0,1)} |\frac{\phi(\check{\mu}(c(t),x))-\phi(x)}{t} - d\phi(\mu(e,x))(d_1\check{\mu}_{(e,x)}(c'(0)))|$$

for $t\in J$. So we obtain differentiability of $ev_\psi \circ C:\mathbb{R}\to BC(G)$ at $t=0$ for $\psi=\phi\circ u_2$ with $\phi\in C_b^2(r)$. We obtain that for every smooth curve the associated left translations are everywhere differentiable at the lifted functions. $C_b^2(r)$ is an algebra, by lifting, moving the elements via left translation and associating the unit we can generate a unital subalgebra of $BC(G)$, which will be denoted by D. We have to prove the assertions of definition 2.1: 1. is clear by definition. 2. is clear by the structure of $C_b^2(r)$ as the translated functions detect every converging sequence. 3. is clear up to the following consideration. Let $c:\mathbb{R}\to BC(G)$ be a smooth curve with $c(0)=e$, $y\in G$. Let ϕ be in the lifting of $C_b^2(r)$ to the group, then there is $g\in BC(G)$, such that

$$\sup_{x\in G} |\frac{\phi(yc(t)x)-\phi(yx)}{t} - g(x)| \overset{t\to 0}{\to} 0 \quad .$$

Consequently the curve C is differentiable on the left translation by y of ϕ on G, because it is differentiable along the curve $yc(.)$ as remarked before. So all the properties are proved and the Banach-Lie group is tempered. $\square$

The following theorem demonstrates the interest in the concept of a tempered topological group due to the range of the class and several inheritance properties:

2.8. Theorem. *Let G be topological group with smoothly regular smooth group structure, where smooth curves are continuous, $G=\operatorname{proj}\lim_{\alpha\in\Omega} G_\alpha$, where the G_α are topological groups, the limit is given in the category of topological groups.*

If G_α is a Banach manifold modeled on a Banach space E_α, which admits C_b^2-bump functions for $\alpha\in\Omega$, and if the canonically given multiplication $\mu_\alpha : G\times G_\alpha\to G_\alpha$ is smooth, then G is a tempered group.
If furthermore the C^∞-vectorfields on G_α $X_x^\alpha := \frac{d}{dt}|_{t=0}\mu_\alpha(c(t),x)$ for $x\in G_\alpha$ are globally integrable for every smooth curve $c:\mathbb{R}\to G$ with $c(0)=e$ and $\alpha\in\Omega$, then for every smooth curve $c:\mathbb{R}\to G$ with $c(0)=e$ there is a continuous group T and

$$\lim_{n\to\infty} c(\frac{t}{n})^n = T_t$$

uniformly on compact intervals in $\mathbb{R}$.

Proof. We proceed by the same method as in the preceding examples, but we have to look carefully at the multiplication μ_α for a given $\alpha\in\Omega$. There are charts u_1, u_2 for the group G_α around e and a open chart domain $U_3\subset G$ around e, so that $\mu_\alpha = u_2\circ\mu_\alpha\circ(id\times {u_1}^{-1}) : U_3\times B_\alpha(0,1)\to B_\alpha(0,1)$. By shrinking the chart (u_1,U_1) we may assume that $\check{u}_\alpha : B_\alpha(0,1)\to C^\infty(U_3,E_\alpha)$ is globally Lipschitz. Now we can apply the same method as before: We take the algebra $C_b^2(r)(E_\alpha)$, where $0<r<1$ is chosen sufficiently small due to the above consideration. We calculate the derivatives under evaluations and prove differentiability in $BC(B_\alpha(0,1))$ of curves of left translations by a smooth curve at functions from $C_b^2(1)(E_\alpha)$. We lift the algebra $C_b^2(r)$ on the group G_α and - via the canonically given smooth projections - from G_α to G. Finally we arrive at the differentiability properties. Redoing the program for all $\alpha\in\Omega$, moving around the functions concentrated at the identity and using the properties of the projections and the limit leads to a subalgebra $D\subset BC(G)$, which proves temperedness of G.

Assume now that the described C^∞-vectorfields are globally integrable on G_α. Redoing the first part of the proof we can find a unital subalgebra $D_\alpha\subset BC(G_\alpha)$, which satisfies 1. and 2. of definition 2.1, the third property is satisfied only for smooth curves $c:\mathbb{R}\to G$ with $c(0)=e$, which are projected to G_α (The resulting curve is denoted by c_α). It is worth mentioning that we are not given the structure of a tempered topological group, because the multiplication on G_α is not smooth. We denote the global flow associated to X^α by T^α. By inserting in functions of D_α we have the problem that the T^α are not translations, but we can argue directly as $\frac{d}{dt}T_t^\alpha(x) = \frac{d}{ds}|_{s=0}\mu_\alpha(c(s),T_t^\alpha)$. The exact formulation of the above construction leads to curves $C:\mathbb{R}\to BC(G_\alpha)$ which are two times differentiable under the evaluations ev_x for $x \in G_\alpha$ on a small interval around zero on functions $\phi \in D_\alpha$, where the derivatives lie in $BC(G_\alpha)$, the second one is bounded on the interval. Given $\phi \in D_\alpha$ and $x \in G_\alpha$ we obtain

$$\frac{d}{dt}\phi(T_t^\alpha(x)) = (C'(0)\phi)(T_t^\alpha(x))$$
$$\frac{d^2}{dt^2}\phi(T_t^\alpha(x)) = (C'(0)^2\phi)(T_t^\alpha(x))$$

for t in the interval. The right hand sides are bounded on the interval, so we conclude that the given curve is differentiable in $BC(G_\alpha)$. We shall look at the closure of D_α in $BC(G_\alpha)$, an algebra of differentiable functions concentrated at a small neighborhood, so we are able to detect convergence in a small neighborhood of the identity. By a slight generalization of Chernoff's theorem (we leave away the condition $\overline{A|_D} = A$, but obtain only convergence on the closure of D) we arrive at uniform convergence on compact subsets of a small neighborhood of the identity of the following limit:

$$\lim_{n\to\infty} c_\alpha(\frac{t}{n})^n = T_t^\alpha(e)$$

This means that the limit exists uniformly on compact subsets of $\mathbb{R}$ due to continuity of the multiplication. We proved therefore that the right hand side of the limit is a continuous group in G_α. The above procedure can be done for any $\alpha \in \Omega$, consequently we have proved the assertion by the properties of the limit, more precisely: There exists a continuous group T^α on G_α with the property $\lim_{n\to\infty} c_\alpha(\frac{t}{n})^n = T_t^\alpha$, satisfying the limit conditions. So we can lift it to G and there we obtain the desired equation. □

2.9. Corollary. *All ILB-Lie groups, where the respective Banach spaces in the chain admit C_b^2-bump functions, are tempered.*

Proof. An ILB-Lie group G is a Fréchet-Lie group and a topological group, where the structures are compatible. Furthermore $\operatorname{proj\,lim}_{n\geq d} G_n = G$, the G_n are Banach manifolds and topological groups with smooth multiplication $G \times G_n \to G_n$ for $n \geq d$. So we obtain temperedness by Theorem 2.7. □

2.10. Proposition. *Let G be a topological group with a smooth structure. $G = \operatorname{proj\,lim}_{\alpha\in\Omega} G_\alpha$, where the G_α are tempered topological groups. The limit is given in the category of topological groups. If the canonical projections $\alpha : G \to G_\alpha$ is smooth, then G is a tempered topological group.*

Proof. The proof is simply given by pulling back the algebras D_α to G and verifying the properties of definition 2.1. □

2.11. Remark. Due to the theorems the range of the class of tempered topological groups is rather large. All strong ILB-groups , modeled on Banach spaces with good differentiability properties, are tempered Lie groups. The concept even works for topological groups with smooth structure, but without smooth group structure.

An interesting consequence of temperedness of a topological group is, that it is easy to characterize the existence of an exponential mapping.

2.12. Theorem. *Let G be a tempered topological Lie group. Let $D \subset BC(G)$ be the given unital subalgebra. Let G satisfy the following completeness condition: If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence with $f \circ \mu_{x_n}$ a Cauchy sequence in $BC(G)$ for $f \in D$, then there is $x \in G$ with $x_n \to x$ as $n \to \infty$.*
The topological Lie group admits a continuous group in each direction if and only if for all smooth curves $c : \mathbb{R} \to G$ with $c(0) = e$ the mapping $id - C'(0) : D \to X$ has dense image in the closure of D, denoted by X, where $C : \mathbb{R} \to L(BC(G))$ denotes the curve of left translations by $c(t)$.

Proof. We have to apply several results of classical theory of C_0-semigroups. Suppose first that G admits an exponential mapping, then the respective generators of the C_0-groups on G, given through $C'(0)$ for $c : \mathbb{R} \to G$, obey the condition $\mathbb{R}\setminus\{0\} \subset \rho(C'(0))$, consequently $id - C'(0) : D \to X$ is closable and the closure is invertible on X, so the image of D is dense.

Suppose the density condition is satisfied, then we can apply 1.10. By approximation we obtain that the following limit exists uniformly on compact intervals of $\mathbb{R}$: $\lim_{n\to\infty} C(\frac{t}{n})^n = T_t$. By the completeness assumption we obtain the existence of a continuous group S in G with $\lim_{n\to\infty} c(\frac{t}{n})^n = S_t$ uniformly on compact subsets of $\mathbb{R}$. □

Tempered Fréchet-Lie groups do not seem to cover the class of regular Fréchet-Lie groups. Nevertheless the concept of temperedness can be applied in all known examples of Fréchet-Lie groups, it is more general than the strong ILB-property and it provides us with an representation of a Fréchet-Lie group as linear group of algebra isomorphisms on a commutative C^*-algebra: Let G be a tempered Fréchet-Lie group and X the closure of D, then $\pi : G \to L(X)$ with $\pi(x)f = f \circ \mu_x$ for $x \in G$ is a monomorphism, continuous under point evaluations. Furthermore this representation detects the topology on G and it is continuous with respect to the strong topology on G. One obtains an interesting representation of the Lie algebra of G in the densely defined derivations on X. Some elements of the universal enveloping algebra of $\mathfrak{g}$ have an easy interpretation on X.

3. Lipschitz-metrizable smooth groups

Convenient Lie groups as defined in [**KM97**] provide a useful basis for infinite-dimensional geometry, but there is still a lack of methods how to handle analytic questions. The excellent approach of [**Omo97**] to infinite dimensional Lie groups includes many analytic a priori properties in the definition, however the analytic properties of the object itself are not considered. We try to define a category of smooth groups, where the existence of product integrals (see [**Omo97**] for some ideas) is equivalent to the existence of Lipschitz-metrics. This category contains all ILB-Lie groups (subgroups of diffeomorphism groups on compact finite dimensional manifolds, the Lie group of Fourier-Integral-Operators), so all known Lie groups are provided with interesting metrics guaranteeing the solvability of non-autonomous right or left invariant differential equations.

The definition of product integrals on Lie groups is done in the same way as in algebras, however, we are always looking for evolutions in both time directions. The left regular representation of G

$$\begin{aligned} \rho : G &\to L(C^\infty(G,\mathbb{R})) \\ g &\mapsto (f \mapsto f(g.)) \end{aligned}$$

in the bounded operators on $C^\infty(G,\mathbb{R})$ is initial by the smooth Hausdorff-property of G.

The starting observations of the following two sections are originally based on a proof of the famous Kakutani-Theorem, which was simultaneously and independently proved by Garett Birkhoff, too, on the existence of a left (or right) invariant metric on a topological group with countable basis of the neighborhood filter of the identity:

3.1. Theorem (Kakutani's theorem). *Let G be a topological group with a countable basis of the neighborhood filter of the identity, then there is a left (or right) invariant metric on G.*

Proof. Given a sequence of open neighborhoods of the identity $\{Q_n\}_{n\in\mathbb{N}}$, then by continuity of the multiplication we find a sequence of symmetric open neighborhoods $\{U_n\}_{n\in\mathbb{N}}$ with

$$U_{n+1}^2 \subset U_n \cap Q_n \text{ for } n \in \mathbb{N}$$

We define by induction on $1 \le k \le 2^n$ and $n \ge 0$

$$\begin{aligned} V_{\frac{1}{2^n}} &= U_n \\ V_{\frac{2k}{2^{n+1}}} &= V_{\frac{k}{2^n}} \\ V_{\frac{2k+1}{2^{n+1}}} &= V_{\frac{1}{2^{n+1}}} V_{\frac{k}{2^n}} \end{aligned}$$

We obtain the property $V_{\frac{1}{2^n}} V_{\frac{m}{2^n}} \subset V_{\frac{m+1}{2^n}}$ for $m < 2^n$. For $m = 2k$ this is a consequence of the above properties. For $m = 2k+1$ the left hand side becomes

$$V_{\frac{1}{2^n}} V_{\frac{m}{2^n}} = V_{\frac{1}{2^n}} V_{\frac{1}{2^n}} V_{\frac{k}{2^{n-1}}} \subset V_{\frac{1}{2^{n-1}}} V_{\frac{k}{2^{n-1}}} = V_{\frac{k+1}{2^{n-1}}} = V_{\frac{m+1}{2^n}}$$

by induction on n and m. So we obtain $V_r \subset V_{r'}$ for $r < r' \le 1$. We choose in our case a monotonic decreasing basis of open sets of the neighborhood filter denoted by $\{Q_n\}_{n\in\mathbb{N}}$. We redo the presented construction and obtain a family V_r for all dyadic rationals $0 < r \le 1$.

$$f(x,y) := \begin{cases} 0 \text{ if } y \in V_r V_r^{-1} x \text{ for all } r \\ \sup\{r \mid y \notin V_r V_r^{-1} x\} \end{cases}$$

By definition f is right invariant, since $f(xa, ya) = f(x,y)$ for all $a \in G$. $V_r V_r^{-1}$is symmetric, hence f is symmetric $f(x,y) = f(y,x)$. $V_{\frac{1}{2^n}}$is symmetric, so $V_{\frac{1}{2^n}} V_{\frac{1}{2^n}}^{-1} \subset V_{\frac{1}{2^n}}^2 = V_{\frac{1}{2^{n-1}}} \subset Q_{n-1}$, but $\cap_{n\geq 1} Q_{n-1} = \{e\}$, since we deal with a basis of neighborhoods, so $f(x,y) = 0$ if and only if $x = y$.

$$d(x,y) := \sup_{u\in G} |f(x,u) - f(y,u)|$$

$d(x,y) = d(y,x)$, $d(x,y) \geq f(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$. Right invariance is clear, too, and the triangle inequality follows from

$$\begin{aligned} d(x,z) &\leq \sup |f(x,u) - f(y,u) + f(y,u) - f(z,u)| \leq \\ &\leq d(x,y) + d(y,z) \end{aligned}$$

Finally we have to show that the metric reproduces the topology of the topological group. It is sufficient to show this at e by right invariance. We denote the open d-balls of radius $\frac{1}{2^n}$ by $B_{\frac{1}{2^n}}$. First we observe that $V_{\frac{1}{2^{n+1}}} \subset B_{\frac{1}{2^n}}$ for $n \geq 1$, which is done by a subtle case for case calculation: Given $y \in V_{\frac{1}{2^{n+1}}}$, then $f(y,e) < \frac{1}{2^n}$.

1. $u \in V_{\frac{1}{2^n}}$, so $f(y,u) \leq \frac{1}{2^n}$, so $d(y,e) < \frac{1}{2^n}$.
2. We can find a find a number $1 \leq k < 2^{n+2}$ with $u^{-1} \notin V_{\frac{s}{2^{n+2}}} V_{\frac{s}{2^{n+2}}}^{-1}$ for $1 \leq s \leq k$ and $u^{-1} \in V_{\frac{s}{2^{n+2}}} V_{\frac{s}{2^{n+2}}}^{-1}$ for $k < s < 2^{n+2}$. So $yu^{-1} \in V_{\frac{s+1}{2^{n+2}}} V_{\frac{s+1}{2^{n+2}}}^{-1}$ for $k < s < 2^{n+2}$ and $yu^{-1} \notin V_{\frac{s-1}{2^{n+2}}} V_{\frac{s-1}{2^{n+2}}}^{-1}$ for $2 \leq s \leq k$, hence $\frac{k-1}{2^{n+2}} \leq f(y,u) \leq \frac{k+1}{2^{n+2}}$ and $d(y,e) < \frac{1}{2^n}$, since $\frac{k}{2^{n+2}} \leq f(e,u) \leq \frac{k+1}{2^{n+2}}$.

If $x \in B_{\frac{1}{2^{n+1}}}$, then $f(e,x) < \frac{1}{2^{n+1}}$, finally $x \in V_{\frac{1}{2^{n+1}}}^2 \subset V_{\frac{1}{2^n}} \subset Q_n$, so we obtain $U_{n+1} \subset B_{\frac{1}{2^n}} \subset Q_{n-1}$ for $n \geq 1$, which proves the desired assertion. □

From this proof we observe that - given a Banach Lie group G - we can find by the CBH-formula a basis of the neighborhoods of identity of balls fitting in the above machinery such that we can construct a metric satisfying the Lipschitz property explained in the next definition.

3.2. Definition (Lipschitz-metrizable groups). *Let G be a smooth group such that $c^\infty G$ is a topological group. G is called Lipschitz-metrizable if there is a family of right invariant semimetrics $\{d_\alpha\}_{\alpha\in\Omega}$ on G with the following properties:*

1. *For all sequences $\{x_n\}_{n\in\mathbb{N}}$:*
$$\forall\, \alpha \in \Omega \,:\, d_\alpha(x_k, x_l) \to 0 \iff \{x_n\}_{n\in\mathbb{N}} \text{ is converging in } G$$
2. *For all smooth mappings $c : \mathbb{R}^2 \to G$ with $c(s,0) = e$ and on a compact (s,t)-set there is M_α such that*
$$d_\alpha(c(s,t), e) < M_\alpha t$$
Such a metric will be referred to as Lipschitz-metric .
3. *For all smooth mappings $c : \mathbb{R}^2 \to G$ with $c(s,0) = e$ the following estimates are valid: On compact (s_1, s_2, t)-sets there exists M_α such that*
$$d_\alpha(c(s_1, \frac{s_2}{m})^m c(s_1, \frac{t}{n})^n c(s_1,t)^{-1} c(s_1, \frac{s_2}{m})^{-m}, e) \leq M_\alpha t^2 \text{ for } m,n \in \mathbb{N}$$

3.3. Remark. In contrary to good manners (see [**KM97**] for the useful applications of this habit, see 51.19) we omit the dependences of the constant. However, we declare that M_α is independent of $t, \mathbf{s}$ on a fixed compact set and always independent of m, n. The notion stems from the fact that $t \mapsto d(c(t), e)$ is a Lip^0-curve for c smooth with $c(0) = e$.

3.4. Conjecture. *Regular Frèchet-Lie groups are complete as metric spaces or with respect to the left or right uniform topology. What we know is that beyond locally compact groups Banach-Lie groups are complete by the Champbell-Baker-Hausdorff formula and more generally strong ILB groups G are complete by $G = \varprojlim G_k$ topologically, where the G_k are Banach Lie groups and the projections are injective smooth mappings to a dense subspace.*

Property 3 is a consequence of the existence of a smooth exponential mapping. Therefore we define for convenience a category $\mathcal{G}$ of smooth groups satisfying 1. and 2., but not necessarily 3., with smooth homomorphisms as morphisms. The following lemmas explain how big this category is:

3.5. Lemma. *Let G be a Banach-Lie-Group, then there is a metric d on G satisfying properties 1. and 2.*

Proof. On Banach-Lie algebras we can choose a norm $||.||$ satisfying $||[X,Y]|| \leq ||X|| \cdot ||Y||$. The Campbell-Baker-Hausdorff Formula converges on a ball of radius $\frac{1}{4}$ and we have $||X*Y|| \leq 1-\sqrt{1-4r}$ for $||X||, ||Y|| \leq r$ and $r \leq \frac{1}{4}$ (see section "Local Lie groups and the CBH-formula" for the estimate). We define a sequence $\{s_n\}_{n\geq 1}$ with $s_1 = \frac{1}{4}$ and $s_{n+1} = \frac{2s_n - s_n^2}{4}$, where the formula stems from solving $s_n = 1-\sqrt{1-4s_{n+1}}$. We obtain by induction the following estimate

$$\frac{1}{2^{n+1}} \geq s_n > \frac{1}{2^{n+3}} + \frac{1}{2^{2n+2}}$$

since for $n = 1$ the inequality is valid and if it is valid for $n \geq 1$ then

$$s_{n+1} = \frac{2s_n - s_n^2}{4} > \frac{1}{2^{n+4}} + \frac{1}{2^{2n+3}} - \frac{1}{2^{2n+4}} = \frac{1}{2^{n+4}} + \frac{1}{2^{2n+4}}$$

which proves the assertion. Choosing $U_n = \exp(B(0,s_n))$ in the chart given by the exponential map for n large enough, then we can use the U_n directly in the proof of the Kakutani theorem to obtain a metric d with the property

$$U_{n+1} \subset \{x|d(x,e) < \frac{1}{2^n}\} \subset U_{n-1}$$

for n large enough, since $U_n^2 \subset U_{n-1}$ and $U_n^{-1} = U_n$. Given a curve $c : \mathbb{R}^2 \to G$ with $c(s,0) = e$, then we can find for a given compact s-set a number $M > 0$ such that for t in $[0,1]$

$$\exp^{-1}(c(t,s)) \in tMB(0,1)$$

by Taylor's formula. Consequently

$$d(c(t,s),e) < \frac{1}{2^n}$$

if $s_{n+2} \leq tM < s_{n+1}$, so $\frac{d(c(t,s),e)}{t} < \frac{M}{2^n s_{n+2}}$ for small t. However, $s_{n+2}2^n > \frac{2^n}{2^{n+5}} + \frac{2^n}{2^{2n+6}} > \frac{1}{2^5}$. Hence for small t

$$\frac{d(c(t,s),e)}{t} < 32M$$

and the supremum property is satisfied. $\square$

3.6. Lemma. *Let G be a smoothly connected regular and complete Fréchet-Lie-Group such that*

$$Evol^r : C^\infty([0,1],\mathfrak{g}) \cap C([0,1],\mathfrak{g}) \to C([0,1],G)$$

is continuous with respect to the C_0-topology on the spaces and there is a continuous norm on $\mathfrak{g}$, then $G \in \mathcal{G}$.

Proof. We construct the semimetrics directly: Given two points $g,h \in G$ we can join them by a Lip^1-curve c on $[0,1]$ with $c(0) = g$, $c(1) = h$ and $\delta^r c(t) \neq 0$ for $t \in [0,1]$, which will be denoted by $c : g \to h$.

$$d_k(g,h) := \inf_{c:g\to h} \int_0^1 p_k(\delta^r c(t))dt$$

for an increasing family of norms p_k defining the topology on $\mathfrak{g}$. The Lipschitz-property is clear by definition. Remark that for any Lip^1-map $\phi : [0,1] \to [0,1]$ with $\phi(0) = 0$ and $\phi(1) = 1$ we have $\delta^r(c\circ\phi) = ((\delta^r c)\circ\phi)\phi'$, so reparametrization does not change the integral. Consequently we can always assume that if we have a curve $c : g \to h$, then there is $\phi : [0,1] \to [0,1]$ with $\phi(0) = 0$ and $\phi(1) = 1$ such that

$$\int_0^1 p_k(\delta^r c(t))dt = \int_0^1 p_k(\delta^r c(\phi(t)))\phi'(t)dt =$$
$$= \int_0^1 p_k(\delta^r(c\circ\phi)(t))dt = \sup_{0\leq t\leq 1} p_k(\delta^r(c\circ\phi)(t))$$

ϕ is constructed by solving the differential equation

$$p_k(\delta^r c(\phi(t)))\phi'(t) = \int_0^1 p_k(\delta^r c(t))dt$$

with boundary values $\phi(0) = 0$ and $\phi(1) = 1$. The solution is given by

$$F(\phi(t)) = \int_0^{\phi(t)} p_k(\delta^r c(t))dt = t\int_0^1 p_k(\delta^r c(t))dt$$

where $F'(s) = p_k(\delta^r c(t)) \neq 0$, so there is a Lip^1-solution. Furthermore the triangle inequality follows from joining two Lip^1-curves. Given a sequence $\{g_m\}_{m\in\mathbb{N}}$ with $g_n \to e$ in G, then we can choose a chart (u, U) around e with $u(e) = 0$ and straight lines in the chart to join the g_m with e. Calculating the suprema yields the desired property since $u(g_m)$ converges Mackey to 0, so we can look at the problem on a unit ball in a Banach space E_B, where smooth maps are locally Lipschitz.

Given a sequence $\{g_n\}_{n\in\mathbb{N}}$ with $d_k(g_n, g_m) \to 0$ for $m, n \to \infty$ and U an open neighborhood of identity in G, then $(Evol^r)^{-1}(C([0,1], U))$ is open in $C([0,1], \mathfrak{g})$, saying $C([0,1],(p_k)_{<\epsilon})$ lies inside. By assumption we can find curves $c_{n\to m} := c : e \to g_m g_n^{-1}$ with $p_k(\delta^r c_{n\to m}(t)) < \epsilon$ for n, m large enough. Consequently $Evol^r(\delta^r c_{n\to m}(t)) = c_{n\to m}(t)$ lies in U for $t \in [0,1]$, so $g_m g_n^{-1} \in U$ for m, n large enough, which means that it is a Cauchy sequence in G. □

3.7. Corollary. *All strong ILB-Lie groups are (regularly, see next section) Lipschitz-metrizable, so all known Fréchet-Lie groups are regularly Lipschitz-metrizable.*

Proof. On ILB-groups the evolution map factors as continuous $Evol^r : C^\infty([0,1], \mathfrak{g}) \cap C([0,1], \mathfrak{g}) \to C([0,1], G)$ with respect to the C_0-topologies on the respective spaces, where from we conclude the result, since there are norms on the associated Fréchet space to an ILB-chain. This factorization can be seen as follows, we refer to [**Omo97**]: Given a strong ILB-group, then even more general types of product integrals as provided converge without applying the notion of Lipschitz-metrizability, we only need the smoothness of the exponential map on the underlying Fréchet-Lie group. Given $X_n \in C([0,1], \mathfrak{g})$ converging uniformly to X, then we can associate C^1-hairs $h_n(s,t) = \exp(sX_n(t))$ with $h_n \to h$ in the topology on C^1-hairs by smoothness of the exponential map. Consequently the associated product integrals converge uniformly reproducing $Evol^r(X_n)$, which converges uniformly on $[0,1]$ to $Evol^r(X)$ (see [**Omo97**], theorem 5.3). □

3.8. Remark. The above result justifies a posterio the narrow setting of strong ILB-groups, since we have to restrict to a class of Fréchet spaces, where the continuous norms exist, which allows to build an asssociated chain, but possibly without dense injections.

3.9. Remark. Assuming that the Fréchet space is given by an inverse limit of Hilbert spaces, so the definition of metrics is equally a definition of a variational problem, which is easily solved under some condition on the Lie bracket, namely that ad has a continuous transpose with respect to some scalar product. Then the geodesic equation associated to the variational problem is given through

$$u_t = -ad(u)^\top u$$

where u denotes the right logarithmic derivative of the geodesic (see [**KM97**], section 46.4). Only in the case, where $u \in \ker(ad(u)^\top)$ for $u \in \mathfrak{g}$ the smooth one-parameter subgroups are the geodesics. With respect to interesting non-linear partial differential equations (for example the Korteweg-De Vrieß-equation) it is worth studying this situation in concrete cases. The question arises if such naturally appearing differential equations can be solved on the given Lie groups by internal methods, for example by Lipschitz-metrics. If this were the case, some interesting geometro-analytic progress in partial differential equations would be possible. To set the program it is first necessary to find some natural approximation procedure for geodesic problems, then to apply the Lipschitz-methods to prove approximation.

The next proposition states that it is impossible to choose only one right invariant metric with Lipschitz-property reproducing the topology on a regular Fréchet-Lie-Group beyond Banach spaces. In the regular abelian simply connected case this means that it is impossible to choose an invariant metric with Lipschitz-property on Fréchet spaces. This is exactly the non-abelian (curved) analogue to the assertion, that a Fréchet space with one norm genrating the topology is a Banach space. Hence

Lipschitz-metrics are the right concept replacing seminorms on convenient Lie groups. Remark that we are only interested in the sequential topology on our spaces.

3.10. Proposition. *Let G be a Fréchet-Lie-Group with (smooth) exponential map and suppose that there is a right invariant metric d on G reproducing the topology in the sense of definition 2.2 and*

$$d(c(s,t),e) < Mt$$

for any smooth mapping $c : \mathbb{R}^2 \to G$ with $c(s,0) = e$ on compact (s,t)-sets. If for any sequence $\{X_n\}_{n\in\mathbb{N}}$ with $\exp(tX_n) \to e$ uniformly on compact intervals the sequence $\{X_n\}_{n\in\mathbb{N}}$ converges to 0 in the Lie algebra $\mathfrak{g}$, then G is a Banach-Lie-group.

Proof. We define a seminorm p on the Lie algebra $\mathfrak{g}$ of G . The function $t \mapsto d(\exp(tX), e)$ is sublinear by right invariance, consequently the limit $\lim_{t\downarrow 0} \frac{d(\exp(tX),e)}{t}$ exists and equals the infimum $\inf_{t>0} \frac{d(\exp(tX),e)}{t}$.

$$p(X) := \lim_{t\downarrow 0} \frac{d(\exp(tX), e)}{t}$$

for $X \in \mathfrak{g}$. p is positively homogeneous and $p(0) = 0$. Given a smooth curve $c : \mathbb{R} \to G$ with $c(0) = e$ and $c'(0) = X$, then

$$\begin{aligned}|\frac{d(\exp(tX), e)}{t} - \frac{d(c(t)\exp(-tX), e)}{t}| &\leq \frac{d(c(t), e)}{t} \\ &\leq \frac{d(\exp(tX), e)}{t} + \frac{d(c(t)\exp(-tX), e)}{t}\end{aligned}$$

so the limit of the middle term exists since the limits of the other terms exist and are equal. The limit of a smooth curve d passing at 0 through e with $d'(0) = 0$ is calculated at the beginning of the proof of theorem 3.1. as 0. Consequently $p(X) = p(X) := \lim_{t\downarrow 0} \frac{d(c(t),e)}{t}$. So the triangle inequality is satisfied since

$$\frac{d(\exp(tX)\exp(tY)), e)}{t} \leq \frac{d(\exp(tX), e)}{t} + \frac{d(\exp(tY), e)}{t}$$

Given a sequence $\{X_n\}_{n\in\mathbb{N}}$ with $X_n \to X$ in $\mathfrak{g}$. Convergence on the Fréchet space means Mackey convergence, so there is a compact set $B \subset \mathfrak{g}$ with $X - X_n \in \mu_n B$ with $\mu_n \downarrow 0$.

$$p(X - X_n) \leq \sup_{0<t\leq 1} \frac{d(\exp(t(X_n - X)), e)}{t} \leq$$

$$\sup_{Y\in B} \sup_{0<t\leq 1} \frac{d(\exp(t\mu_n Y), e)}{t} \leq \mu_n \sup_{Y\in B} \sup_{0<t\leq 1} \frac{d(\exp(t\mu_n Y), e)}{t\mu_n} \leq \mu_n M$$

since the last supremum is finite, so $p(X - X_n) \to 0$ for $n \to \infty$, p is a continuous seminorm. Finiteness is proved via the following consideration:

$$M(Y) := \sup_{0<t\leq 1} \frac{d(\exp(tY), e)}{t}$$

Assume that there is a fast converging sequence $Y_n \to Y$ in the compact set B such that $M(Y_n) \geq n$. Consequently there is a smooth curve $d : \mathbb{R} \to \mathfrak{g}$ with $d(\frac{1}{n}) = Y_n$. We define $c(s,t) := \exp(td(s))$, but then

$$\sup_{s\in[0,1]} \sup_{0<t\leq 1} \frac{d(c(s,t))}{t} = \infty$$

a contradiction. Given a sequence $\{X_n\}_{n\in\mathbb{N}}$ such that $p(X_n - X) \to 0$, then for every $\epsilon > 0$ and $N \in \mathbb{N}$ there is $t_n > 0$ such that $d(\exp(t(X_n - X)), e) \leq t\epsilon$ for $t \leq t_n$ for $n \geq N$, but since $t \mapsto d(\exp(t(X_n - X)), e)$ is sublinear this relation holds everywhere, consequently $\exp(t(X - X_n)) \to e$ uniformly on compact intervals in time for $n \to \infty$. This, however, means that $X - X_n \to 0$ in $\mathfrak{g}$. ☐

3.11. Lemma. *Let G be a Lie group such that convergence on the model space E means Mackey-convergence (i.e. $c^\infty E = E$) and there is a family of right invariant halfmetrics $\{d_\alpha\}_{\alpha\in\Omega}$ on G with the following properties:*

1. *For all sequences* $\{x_n\}_{n\in\mathbb{N}}$*:*
$$\forall\, \alpha \in \Omega \,:\, d_\alpha(x_k, x_l) \to 0 \iff \{x_n\}_{n\in\mathbb{N}} \text{ is a converging sequence}$$
2. *For all smooth mappings* $c : \mathbb{R}^2 \to G$ *with* $c(s,0) = e$ *the mappings* $d_\alpha(c(.,.),e) : \mathbb{R}^2 \to \mathbb{R}$ *satisfy*
$$d_\alpha(c(s,t),e) < M_\alpha t$$
on compact (s,t)*-sets.*

If there is furhtermore an exponential map and for any sequence $\{X_n\}_{n\in\mathbb{N}}$ *with* $\exp(tX_n) \to e$ *uniformly on compact intervals the sequence* $\{X_n\}_{n\in\mathbb{N}}$ *converges to* 0 *in the Lie algebra* $\mathfrak{g}$*, then the functions*
$$p_\alpha(X) = \lim_{t\downarrow 0} \frac{d_\alpha(\exp(tX), e)}{t}$$
are continuous seminorms on $\mathfrak{g}$ *generating the topology.*

Proof. The proof is built in the same way as the previous one. Only indices have to be carried with along the lines. □

The last result provides an the already applied idea how the families of seminorms and right invariant metrics are related: This relation could be read in the other direction explaining that via integrating one obtains right invariant Lipschitz-metrics on G.

4. Product integration via Metrization

The notion of product integrals and simple product integrals is necessary to follow the way from the category $\mathcal{G}$ to regularity or the existence of a smooth expoential mapping:

4.1. Definition. *Let* G *be a smooth group with* $c^\infty G$ *a topological group. Given a smooth mapping* $h : \mathbb{R}^2 \to G$ *with* $h(s,0) = e$*, then we define the following finite products of smooth curves*
$$p_n(s,t,h) := \prod_{i=0}^{n-1} h(s + \frac{(n-i)(t-s)}{n}, \frac{t-s}{n})$$
for $s,t \in \mathbb{R}$*. If* p_n *converges in the smooth topology of* G *uniformly on compact sets to a continuous curve* $c : \mathbb{R} \to G$*, then* c *is called the product integral of* h *and we write* $c(s,t) =: \prod_s^t h(s,ds)$*. If* $h(s,t) = c(t)$*, then the product integral* $p_n(0,t,h) = c(\frac{t}{n})^n$ *is called simple product integral.*

4.2. Remark. Here we need the assumption that $c^\infty G$ is a topological group, since we want to talk about uniform convergence of curves in the uniform space $c^\infty G$.

4.3. Lemma. *Let* G *be a smooth group such that* $c^\infty G$ *is a topological group and* G *is smoothly Hausdorff (i.e. the smooth functions separate points), then each product integral* $p_n(s,t,h)$ *is smooth and the propagation condition* $c(t,r)c(s,t) = c(s,r)$ *is satisfied for all* r,s,t*.*

Proof. By the left regular representation ρ on G we get that the product integral
$$\lim_{n\to\infty} p_n(s,t,\rho\circ h)$$
exists in $C^\infty(\mathbb{R}^2, L(C^\infty(G,\mathbb{R})))$, since that image of a sequentially compact set under a smooth mapping is bounded in the convenient algebra $L(C^\infty(G,\mathbb{R}))$. The set formed by $p_n(s,t,h)$ and $c(s,t)$ on compact (s,t)-sets is sequentially compact due to uniform convergence. Evaluating at e and applying cartesian closedness leads to the hypotheses of lemma 1.3.2., which allows the conclusion of smoothness of c. The propagation condition follows from the definition of the product integral and the continuity of multiplication. □

4.4. Lemma. *Let* G *be a smoothly regular Lie group. Given a smooth mapping* $h : \mathbb{R}^2 \to G$ *with* $h(s,0) = e$*, such that the product integral converges to* $c(s,t)$*, then the fundamental theorem of product integration or non-commutative integration asserts that* $\delta_t^r c(s,t) = \frac{\partial}{\partial t} h(s,0)$ *and the convergence is uniform in all derivatives in the sense of lemma 1.3.2..*

Proof. By the previous lemma it suffices to apply lemma 1.3.2. to get the result. Remark that $\delta_t^r|_{t=s} p_n(s,t,h) = X(s)$. □

The following two proposition explains one interest in Lipschitz-metrizable smooth groups. It is in fact a property guaranteeing the existence of simple product integrals, by which we mean iterations of the type $c(\frac{t}{n})^n$ for a smooth curve passing at 0 through e.

4.5. Theorem. *Let G be a Lie group, $c^\infty G$ is a topological group and there is a family of right invariant halfmetrics $\{d_\alpha\}_{\alpha\in\Omega}$ on G with the following properties:*

1. *For all sequences $\{x_n\}_{n\in\mathbb{N}}$:*
$$\forall\,\alpha\in\Omega\,:d_\alpha(x_k,x_l)\to 0 \iff \{x_n\}_{n\in\mathbb{N}} \text{ is a converging sequence}$$
2. *For all smooth mappings $c:\mathbb{R}^2\to G$ with $c(s,0)=e$ the mappings $d_\alpha(c(.,.),e):\mathbb{R}^2\to\mathbb{R}$ satisfy*
$$d_\alpha(c(s,t),e)<M_\alpha t$$
on compact (s,t)-sets.

If there is a smooth exponential mapping, then G is a Lipschitz-metrizable group.

Proof. First we show a simple consequence of property 1. Let $c:\mathbb{R}^{n+1}\to G$ with $c(\mathbf{s},0)=e$ and $\frac{\partial}{\partial t}|_{t=0}c(\mathbf{s},t)=0$ for $\mathbf{s}\in\mathbb{R}^n$ be a smooth mapping, then we can choose a chart (U,u) around e with $u(U)$ absolutely convex and $u(e)=0$. On a small ball B around 0 in $\mathbb{R}^{n+1}$ $u\circ c$ is well-defined with first derivative zero. Consequently $u\circ c(\mathbf{s},\sqrt{t})$ makes sense as Lip^0-curve for positive t and $\mathbf{s}$ in a small ball around zero, so $\frac{1}{t}(u\circ c)(\mathbf{s},\sqrt{t})$ is in a compact set for $t>0$ on a small ball around zero in $\mathbb{R}^{n+1}$. By some reparametrizations we can assume that the compact set, where $\frac{1}{t}u\circ c(\mathbf{s},\sqrt{t})$ lies, is a subset of $u(U)$. Let $B\subset E$ be a compact subset in $u(U)$. Then the following supremum is finite:
$$\sup_{0<t\le 1}\big(\sup_{x\in B}\frac{d_\alpha(u^{-1}(tx),e)}{t}\big)<\infty$$
for all $\alpha\in\Omega$, since the function
$$M^\alpha(x):=\sup_{0<t\le 1}\frac{d_\alpha(u^{-1}(tx))}{t}$$
for $x\in u(U)$ is bounded on compact subsets of $u(U)$. If M^α were unbounded on a compact subset B of $u(U)$, then there would exist a sequence $\{x_n\}_{n\in\mathbb{N}_+}$ in B, converging fast to $x\in B$, with $M^\alpha(x_n)\ge n$ for $n\in\mathbb{N}_+$. By the special curve lemma there is a curve $d:\mathbb{R}\to F$ with $d(\frac{1}{n})=x_n$, so $c(s,t):=td(s)$ is a smooth mapping with $c(s,0)=0$ with values in $u(U)$, which gives a contradiction by looking at $u^{-1}\circ c$. Consequently
$$\sup_{0<t\le 1}\frac{d_\alpha(c(\mathbf{s},\sqrt{t}),e)}{t}<M_\alpha$$
on a small interval around zero in s. This can easily be extended to all compact sets by a translation. We obtain finally
$$(\#)\qquad \sup_{0<t\le 1}\frac{d_\alpha(c(\mathbf{s},t),e)}{t^2}<M_\alpha$$
on compact $\mathbf{s}$-sets. Now we apply the existence of a smooth exponential mapping. Let $T(X)$ denote a semigroup with generator X. A smooth mapping $c:\mathbb{R}^2\to G$ with $c(s,0)=e$ and $\frac{\partial}{\partial t}|_{t=0}c(s,t)=X_s$ for $s\in\mathbb{R}$ is given, too. We proceed indirectly to obtain the assertion: Let $n\in\mathbb{N}$ be given, then
$$\begin{aligned}
&d_\alpha(c(s,\frac{t}{m})^m T_{-t}(X_s),e)\\
&\le\sum_{i=0}^{m-1}d_\alpha(T_{\frac{ti}{m}}(X_s)c(s,\frac{t}{m})^{m-i}T_{-t}(X_s),T_{\frac{t(i+1)}{m}}(X_s)c(s,\frac{t}{m})^{m-i-1}T_{-t}(X_s))\\
&=\sum_{i=0}^{m-1}d_\alpha(T_{\frac{ti}{m}}(X_s),T_{\frac{t(i+1)}{m}}(X_s)c(s,\frac{t}{m})^{-1})\\
&=\sum_{i=0}^{m-1}d_\alpha(T_{\frac{ti}{m}}(X_s)c(s,\frac{t}{m})T_{-\frac{ti}{m}}(X_s),T_{\frac{t}{m}}(X_s))
\end{aligned}$$

due to right invariance. Our uniformity result leads to the desired assertion by investigating the smooth mapping

$$d(\mathbf{s},t) := T_{s_2}(X_{s_1})c(s_1,\frac{t}{m})T_{-s_2}(X_{s_1})T_{-\frac{t}{m}}(X_{s_2})$$

by estimate #. Consequently we arrive at

$$d_\alpha(c(s,\frac{t}{m})^m T_{-t}(X_s),e) \leq \sum_{i=0}^{m-1} M_\alpha \frac{t^2}{m^2} = M_\alpha \frac{t^2}{m} \overset{m\to\infty}{\to} 0$$

where t can vary in a compact interval around zero preserving uniformity. Consequently

$$d_\alpha(c(s,\frac{t}{m})^m, T_t(X_s)) \leq M_\alpha \frac{t^2}{m}$$

for $\alpha \in \Omega$ uniformly in t. By this first observation we can conclude the desired assertion of Lipschitz-metrizability in the following way: The above estimate yields that the respective limit

$$c(s_1,\frac{s_2}{m})^m c(s_1,\frac{t}{n})^n c(s,t)^{-1} c(s_1,\frac{s_2}{m})^{-m}$$

for $n,m \to \infty$ exists on compact (s_1,s_2)-sets and equals $T_{s_2}(s_1)T_t(s_1)c(s_1,t)^{-1}T_{-s_2}(s_1)$. Inserting this curve in the estimate # we can conclude the result, since all appearing terms contain t^2. $\square$

4.6. Theorem (Approximation Theorem). *Let G be a Lipschitz-metrizable smooth group. Given a smooth curve $c : \mathbb{R} \to G$ with $c(0) = e$, then the limit*

$$\lim_{n\to\infty} c(\frac{t}{n})^n = T_t$$

exists uniformly on compact intervals of $\mathbb{R}$ and gives a smooth group T. If G is a smoothly regular Lie group, then the exponential mapping given through these approximations is smooth and convergence is uniform in all derivatives in the sense of lemma 1.3.2.

Proof. Given a smooth curve $c : \mathbb{R}^2 \to G$ with $c(s,0) = e$, then we try to investigate the above simple product integrals:

$$\begin{aligned}
& d_\alpha(c(s,\frac{t}{nm})^{nm}, c(s,\frac{t}{n})^n) \\
& \leq \sum_{i=0}^{n-1} d_\alpha(c(s,\frac{t}{n})^i c(s,\frac{t}{nm})^{(n-i)m}, c(s,\frac{t}{n})^{i+1} c(s,\frac{t}{nm})^{(n-i-1)m}) \\
& \leq \sum_{i=0}^{n-1} d_\alpha(c(s,\frac{t}{n})^i c(s,\frac{t}{nm})^m c(s,\frac{t}{n})^{-1} c(s,\frac{t}{n})^{-i}, e) \\
& \leq \sum_{i=0}^{n-1} d_\alpha(c(s,\frac{t}{n})^i c(s,\frac{t}{nm})^m c(s,\frac{t}{n})^{-i}, c(s,\frac{t}{n})^{-1}) \\
& \leq n\frac{t^2}{n^2} M_\alpha \to 0 \text{ for } n \to \infty
\end{aligned}$$

which is possible due to Lipschitz-metrizability, a look at the curve

$$d(\mathbf{s},t) = c(s_1,\frac{s_2}{i})^i c(s_1,\frac{t}{nm})^m c(s_1,t)^{-1} c(s_1,\frac{s_2}{i})^{-i}$$

and application of the given estimates. Consequently we obtain a Cauchy-property uniform in s for the above sequences of curves, which leads to the desired limit. The limit $\lim_{n\to\infty} c(s,\frac{t}{n})^n =: T_t(s)$ is continuous and by standard arguments a group in t. By looking at the left regular representation in $L(C^\infty(G,\mathbb{R}))$ we see that the limit has to be smooth, because sequentially compact sets are mapped to bounded ones and the smooth functions detect smoothness: $\rho \circ c$ gives a curve in $L(C^\infty(G,\mathbb{R}))$ satisfying the boundedness condition, so we expect a smooth limit group $T(s,t)$ by the approximation theorem 4.2. Since we have convergence of $c(s,\frac{t}{n})^n$ this limit has to be a posterio equal to $\rho(\lim_{n\to\infty} c(s,\frac{t}{n})^n)$. By initiality of ρ we obtain the smoothness of $\lim_{n\to\infty} c(s,\frac{t}{n})^n$ as mapping to G. The limit exists uniformly in all derivatives, which means in particular that the generator of T is $c'(0)$ by the lemma 1.3.2. at the end of section 1.3 since we can evaluate at e to obtain $(f \circ c)(s, \frac{t}{n})^n \to f(\lim_{n\to\infty} c(s, \frac{t}{n})^n)$ in all derivatives with respect to s and t. $\square$

4.7. Remark. We have proved that the existence of an exponential map can be characterized on "all" Lie groups by Lipschitz-metrizability.

In the abelian case the situation is simpler, we can reformulate the proposition and define in a simpler way the Lipschitz-metrics:

4.8. Corollary. *Let G be an abelian Lie group, such that $c^\infty G$ is a topological group, then G is regular if and only if G is Lipschitz-metrizable.*

PROOF. One direction is a corollary of the theorem. Let G be a regular abelian Lie group. Then G is locally isomorphic to its Lie algebra by [**MT98**], consequently the topology on G is given by the bornological topology on $\mathfrak{g}$. We denote by Ω the set of bounded seminorms on $\mathfrak{g}$.

$$d_k(g,h) := \inf_{\substack{c\in C^\infty([0,1],G)\\ c(0)=g,\, c(1)=h}} \int_0^1 p(\delta^r c(t))dt$$

is a well-defined right invariant halfmetric on G for $p \in \Omega$. Right-invariance is clear by definition, symmetry, too. Taking two curves $c, d \in C^\infty([0,1], G)$ with $c(0) = g$, $c(1) = d(0) = h$ and $d(1) = l$, then $b := c\mu_{h^{-1}}d$ defines a smooth curve with $b(0) = g$ and $b(1) = l$, furthermore $\delta^r b(t) = \delta^r c(t) + \delta^r d(t)$ on $[0,1]$ due to commutativity (the adjoint map is trivial). The Lipschitz-property is clear by the following argument: Let $c : \mathbb{R}^2 \to G$ be smooth mapping with $c(0,s) = e$ for $s \in \mathbb{R}$, then

$$d_k(c(u,s),e) \le \int_0^1 p_k(\delta^r c(u.,s)(t))dt = u\int_0^1 p_k([\delta^r c](ut,s)])dt$$

and consequently the supremum exists uniformly for s in a compact interval. The rest follows by regularity from the lemma. It remains to prove the topological property, but this is clear due to the possibility to choose an exponential chart (see [**MT98**]). So we constructed the essentials for Lipschitz-metrizability. $\square$

The next theorem is devoted to the analysis of regularity in the case of Lipschitz-metrizable groups. We shall obtain that a slight sharpening of the axioms of Lipschitz-metrizability allows to characterize regularity of a Lie group carrying the structure of a topological group.

4.9. Theorem. *Let $G \in \mathcal{G}$ be a Lie group with $c^\infty G$ a topological group, $h : \mathbb{R}^2 \to G$ a smooth mapping with $h(s,0) = e$ and $\frac{\partial}{\partial t}|_{t=0} h(s,t) = X(s)$ and c with $c(0) = e$ the smooth curve with $\delta^r c = X$, then the product integral $\prod_0^t h(s,ds)$ exists and equals $c(t)$. If G is regular, then the following estimates are valid for the Lipschitz-metrics d_α:*

$$d_\alpha(p_i(s_3,t,c)(s_1)p_n(s_2,t+s_2,c)(s_1)c(s_1,s_2,t)^{-1}p_i(s_3,t,c)(s_1)^{-1},e) \le M_\alpha t^2$$

for all $i, n \in \mathbb{N}$ on compact $(s_{1,}s_2,s_3,t)$-sets given the smooth mapping $d : \mathbb{R}^3 \to G$ with $c(s_1,s_2,0) = e$.

PROOF. First we prove the convergence result to establish the estimate. Given a smooth mapping $h : \mathbb{R}^3 \to G$ with $h(s_1,s_2,0) = e$, then we look at the product integral

$$p_n(s_2,t,h)(s_1) = \prod_{i=0}^{n-1} h(s_1, s_2 + \frac{(n-i)(t-s_2)}{n}, \frac{t-s_2}{n})$$

at $s_2 = 0$. Let $c : \mathbb{R}^2 \to G$ be a curve with $c(s_1, 0) = e$ and $\delta^r c_{s_1}(s_2) = \frac{\partial}{\partial t} h(s_1, s_2, 0)$.

$$d_\alpha(\prod_{i=0}^{n-1} h(s_1, \frac{(n-i)t}{n}, \frac{t}{n}), c(s_1,t)) \le$$

$$\le \sum_{i=0}^{n-1} d_\alpha(\prod_{j=1}^{i} c(s_1, \frac{(n-j+1)t}{n}) c(s_1, \frac{(n-j)t}{n})^{-1} \prod_{j=i}^{n-1} h(s_1, \frac{(n-j)t}{n}, \frac{t}{n}) c(s_1,t)^{-1},$$

$$\prod_{j=1}^{i+1} c(s_1, \frac{(n-j+1)t}{n}) c(s_1, \frac{(n-j)t}{n})^{-1} \prod_{j=i+1}^{n-1} h(s_1, \frac{(n-j)t}{n}, \frac{t}{n}) c(s_1,t)^{-1}) \le$$

$$\le \sum_{i=0}^{n-1} d_\alpha(c(s_1,t) c(s_1, \frac{(n-i)t}{n})^{-1} h(s_1, \frac{(n-i)t}{n}, \frac{t}{n}) c(s_1, \frac{(n-i-1)t}{n}) c(s_1, \frac{(n-i)t}{n})^{-1},$$

$$c(s_1,t) c(s_1, \frac{(n-i)t}{n})^{-1}) \le$$

$$\le n \frac{t^2}{n^2} M_\alpha$$

for $n \in \mathbb{N}$ on compact s_1-sets. The last step of the proof is done by the same arguments as above. First we show that convergence is uniform in all derivatives due to smooth regularity as in lemma 4.4., then we look at the smooth curve

$$c(s_1,t) c(s_1,s_3)^{-1} h(s_1, s_3, \frac{t}{n}) c(s_1,s_3) c(s_1,s_3 + \frac{t}{n})^{-1} c(s_1,s_3) c(s_1,t)^{-1}$$

and insert it in estimate # in 4.5. The desired estimate follows since we have t^2in all appearing estimates as a result of the above calculation. □

4.10. Definition. *A Lipschitz-metrizable group is called regularly Lipschitz-metrizable if for all smooth mappings $c : \mathbb{R}^3 \to G$ with $c(s_1, s_2, 0) = e$ the following estimate is valid*

$$d_\alpha(p_i(s_3, t, c)(s_1) p_n(s_2, t + s_2, c)(s_1) c(s_1, s_2, t)^{-1} p_i(s_3, t, c)(s_1)^{-1}) \le M_\alpha t^2$$

for all $n \in \mathbb{N}$ on compact (s_1,s_2, s_3, t)-sets. We use the following abbreviation

$$p_i(\frac{n-i}{n} t, t, c)(s_1) = p_{n,i}(t, c)(s_1) = \prod_{j=0}^{i} c(s_1, \frac{(n-j)t}{n}, \frac{t}{n})$$

Remark that property 3 of the definition of Lipschitz-metrizable smooth groups is contained in the above property, $c(\frac{t}{n})^n$ is a simple product integral. All regular Lie groups G with $c^\infty G$ a topological group are regularly Lipschitz-metrizable, the following theorem explains the converse:

4.11. Theorem. *Let G be a regularly Lipschitz-metrizable smooth group, then all product integrals exist. If G is furhtermore a smoothly regular Lie group, then the right evolution operator given through these product integrals is smooth, so G is regular.*

Proof. We can proceed directly to obtain the result by our methods. Given a smooth mapping $c : \mathbb{R}^3 \to G$ with $c(s_1, s_2, 0) = e$, we shall look at the product integral

$$p_{n,i}(t, c)(s_1) = \prod_{j=0}^{i} c(s_1, \frac{(n-j)t}{n}, \frac{t}{n})$$

at $s_2 = 0$. The notion allows to shorten the product: $0 \leq i \leq n-1$, $p_{n,n-1} = p_n$.

$$d_\alpha(p_{nm}(t,c)(s_1), p_n(t,c)(s_1)) \leq$$

$$\leq \sum_{i=0}^{n-1} d_\alpha\Bigg(p_{n,i}(t,c)(s_1)\prod_{j=mi}^{nm-1} c(s_1, \frac{(nm-j)t}{nm}, \frac{t}{nm})p_n(t,c)(s_1)^{-1},$$

$$p_{n,i+1}(t,c)(s_1)\prod_{j=m(i+1)}^{nm-1} c(s_1, \frac{(nm-j)t}{nm}, \frac{t}{nm})p_n(t,c)(s_1)^{-1}\Bigg) \leq$$

$$\leq \sum_{i=0}^{n} d_\alpha\Bigg(p_{n,i}(t,c)(s_1)\prod_{j=mi}^{m(i+1)-1} c(s_1, \frac{(nm-j)t}{nm}, \frac{t}{nm}), p_{n,i+1}(t,c)(s_1)\Bigg) =$$

$$= \sum_{i=0}^{n-1} d_\alpha\Bigg(p_{n,i}(t,c)(s_1)\prod_{j=mi}^{m(i+1)-1} c(s_1, \frac{(nm-j)t}{nm}, \frac{t}{nm})c(s_1, \frac{(n-i-1)t}{n}, \frac{t}{n})^{-1},$$

$$p_{n,i}(t,c)(s_1)\Bigg) =$$

$$= \sum_{i=0}^{n-1} d_\alpha\Bigg(p_{n,i}(t,c)(s_1)\prod_{j=0}^{m-1} c(s_1, \frac{(n-i-1)t}{n} + \frac{(m-j)t}{nm}, \frac{t}{nm})$$

$$c(s_1, \frac{(n-i-1)t}{n}, \frac{t}{n})^{-1}p_{n,i}(t,c)(s_1)^{-1}, e\Bigg) \leq$$

$$\leq \sum_{i=0}^{n-1} M_\alpha \frac{t^2}{n^2}$$

for $n \in \mathbb{N}$ and compact (s_1, s_2, s_3, t)-intervals. Therefore we can apply estimate # to the above problem to obtain the assertion. Furthermore by the approximation theorem 4.2 we obtain the smoothness of these solution families, which yields smoothness of the right evolution map as in lemma 4.4. $\square$

4.12. Corollary. *Let G be a Lie group in the category $\mathcal{G}$, then the following assertions are equivalent:*

1. *A smooth exponential map* $\exp : \mathfrak{g} \to G$ *exists (a smooth right evolution map exists)*
2. *All simple product integrals converge in $C^\infty(\mathbb{R}^2, G)$ (all product integrals converge in $C^\infty(\mathbb{R}^3, G)$ in the sense of lemma 1.3.2)*
3. *G is (regularly) Lipschitz-metrizable*

4.13. Corollary. *Let G be a regular smoothly connected Lie group in the category $\mathcal{G}$, then the closure of the normal subgroup generated by the image of the exponential map is the whole group G.*

Proof. Regularity implies the existence of product integrals $\int_0^a \exp(X(s)ds)$, which reach any point in the connected Lie group, consequently the closure of the normal subgroup generated by the image of the exponential map is the whole group. $\square$

5. Product integration via Linearization

5.1. Theorem. *Let G be a smoothly regular Lie group. If for each smooth mapping $c : \mathbb{R}^3 \to G$ with $c(r_1, r_2, 0) = e$ the approximations $p_n(s_2, t, c)(r_1)$ lie in a sequentially c^∞-compact set on compact (r_1, s_2, t)-sets, then G is regular.*

5.2. Theorem. *Let G be a smoothly regular Lie group. If for each smooth mapping $c : \mathbb{R}^2 \to G$ with $c(r_1, 0) = e$ the approximations $c(r_1, \frac{t}{n})^n$ lie in a sequentially c^∞-compact set on compact (r_1, t)-sets, then G admits a smooth exponential map.*

Proof. The proofs for the theorems are identical: A sequentially c^∞-compact set is mapped by ρ to a sequentially c^∞-compact set, which is bounded in any compatible locally convex topology. Consequently we obtain the existence of the image product integral, but this image product integral stems pointwisely from G via ρ, because there are adherence points in the sequentially c^∞-compact set, which have to be the unique limit points of the respective sequences. ρ is initial, so the limit curve has to be smooth and the uniform convergence in all derivatives in $L(C^\infty(G,\mathbb{R}))$ implies uniform convergence in all derivatives in the sense of lemma 1.3.2. of the products to the product integral as in lemma 4.4. $\square$

5.3. Proposition. *Let G be a smoothly regular Lie group in the category $\mathcal{G}$, such that there exists a family $\{d_\alpha\}_{\alpha\in\Omega}$ of right-invariant equivalent metrics satisfying 1. and 2. of the definition of Lipschitz-metrizable groups with the following property:*

Let $K \subset G$ be set, such that $d_\alpha(K,e) \le M_\alpha$ for $\alpha \in \Omega$, then K is relatively sequentially compact in the topology of G.

Then G is regularly Lipschitz-metrizable.

PROOF. Given a smooth mapping $c : \mathbb{R}^3 \to G$ with $c(s_1, s_2, 0) = e$, then the products p_n can be estimated in the following way:

$$d_\alpha(p_n(s_2,t)(s_1),e) \le \sum_{j=0}^{n-1} d_\alpha(c(s_1, \frac{(n-j)t}{n}, \frac{t}{n}), e) \le n\frac{t}{n}M_\alpha$$

on compact (s_1, s_2, t)-sets. Consequently the approximations lie in a compact set. If all approximations of product integrals lie in a sequentially compact set for compact parameter sets, we can apply the regularity theorem of 4.2 to conclude regularity and regular Lipschitz-metrizability as in the previous proof. $\square$

5.4. Remark. This property can be viewed as a non-linear version of Arzela-Ascoli's theorem.

5.5. Conjecture. *Let G be a strong ILB-group, such that the associated Fréchet space is Montel. G is seen to be Lipschitz-metrizable and regular by the above considerations. It is reasonable to expect that for all sets $K \subset G$ lying in a small neighborhood of identity U*

$$d_n(K,e) \le M_n \text{ for all } n \text{ if and only if } K \text{ is relatively compact in the topology of } G$$

This would provide a simple procedure to solve non-autonomuous differential equations of the type $\delta^r c(t) = X(t)$ for $t \in \mathbb{R}$ on the Lie group by "intrinsic methods".

CHAPTER 4

The inverse of S. Lie's third theorem

Nach glaubwürdiger Überlieferung hat das im sechzehnten Jahrhundert, einem Zeitalter stärkster seelischer Bewegtheit, damit begonnen, daß man nicht länger, wie es bis dahin zwei Jahrhunderte religiöser und philosophischer Spekulation geschehen war, in die Geheimnisse der Natur einzudringen versuchte, sondern sich in einer Weise, die nicht anders als oberflächlich genannt werden kann, mit der Erforschung ihrer Oberfäche begnügte. [...] Galilei war ja nicht nur der Entdecker des Fallgesetzes und der Erdbewegung, sondern auch ein Erfinder, für den sich, wie man heute sagen würde, das Großkapital interessierte, und außerdem war er nicht der einzige, der damals von dem neuen Geist ergriffen wurde; im Gegenteil, die historischen Berichte zeigen, daß sich die Nüchternheit, von der er beseelt war, weit und ungestüm wie eine Ansteckung ausbreitete, und so anstößig das heute klingt, jemand von Nüchternheit beseelt zu nennen, wo wir davon schon zu viel zu haben glauben, damals muß das Erwachen aus der Metaphysik zur harten Betrachtung der Dinge nach allerhand Zeugnissen geradezu ein Rausch und Feuer gewesen sein!

(Robert Musil, Der Mann ohne Eigenschaften)

In this last chapter we are concerned with the inverse of the third fundamental theorem of S. Lie asserting that to any Lie group there can be associated a unique Lie algebra. The fact that complicated non-linear problems can be translated to linear ones is the crucial observation of the theory of symmetries. After some reflections on linear and global questions on regularity it seems necessary to treat the infinite dimensional feature that there are Lie algebras without associated Lie group in more detail, since even though regular Lie algebras behave in some respects as finite dimensional ones (bounded Lie algebra homomorphisms can be integrated) there are fundamental difficulties in passing from the linear Lie algebra to a possibly existing Lie group.

First we investigate the Campbell-Baker-Hausdorff-Formula on locally convex Lie algebras and provide a class of non-normable Lie algebras, where the series converges. The local group build this way is the playground of the following cohomological analysis, where we try to find necessary and sufficient conditions for enlargibility, i.e. the situation, when a local group embeds in a group. The results are formulated for Fréchet-Lie-Groups. The goal of this section is to demonstrate that enlargibility is no analytic question, but an algebraic one decided by cohomological methods.

1. Local Lie groups and the CBH-Formula

The question, if to a given Lie algebra L there can be associated a Lie group G, so that $L(G) = L$ valid, gives deep insight in the difficulties of the theory of infinite dimensional Lie groups or of infinite dimensional differential geometry (see [**KM97**]).

1.1. Definition. *A (convenient) manifold G is called a local Lie group if there is an open subset $U \subset G \times G$ and a smooth mapping $\mu : U \to G$ with the following properties:*
1. There is a unit element e such that (e, x) and (x, e) are in U and $ex = xe = x$ for all $x \in G$.
2. For all $x \in G$ there is a unique x^{-1}such that $xx^{-1} = x^{-1}x = e$.
3. For all $x, y, z \in G$ with $(x, y) \in U$ and $(y, z) \in U$ we obtain: $(xy, z) \in U$ if and only if $(x, yz) \in U$. In this case we have the law of associativity:$(xy)z = x(yz)$.
4. For all $x, y \in G$ with $(x, y) \in U$ we arrive at $(y^{-1}, x^{-1}) \in U$.

A local Lie group naturally possesses a Lie Algebra $\mathfrak{g}$. A local Lie group is called regular if there is an evolution map, i.e. curves in a given c^∞-neighborhood of 0 in $\mathfrak{g}$ are mapped to G, such that we obtain an inverse to the right (or left) logarithmic derivative. The question if there exists a local Lie group with given Lie algebra $\mathfrak{g}$ and locally diffeomorphic exponential map leads to the analysis of the Campbell-Baker-Hausdorff-Formula, which we are going to treat in general in the sequel (see [**BCR81**], ch. 3 for details).

1.2. Definition. *Let A be a locally convex algebra, p, q seminorms on A. q is called an asymptotic estimate for p, if there is $m \in \mathbb{N}$, such that for $n \geq m$ and $u_1, ..., u_n \in A$*

$$p("u_1 \cdot ... \cdot u_n") \leq q(u_1) \cdot ... \cdot q(u_n),$$

where the quotation marks denote that any choice of parentheses is allowed at the left side.

This can be expressed by the means of the following set multiplication, too: q is an asymptotic estimate for p if and only if there is $m \in \mathbb{N}$ such that for $n \geq m$

$$"(q_{<1})^n" \subset p_{<1},$$

where the subscribed seminorms denote the open unit balls.

1.3. Definition. *A locally convex algebra A is called AE-Algebra if there is a distinguished seminorm p_0 on A, such that for any seminorm p on A there is an downwards directed family $\{q_i\}_{i \in I}$ of asymptotic estimates for p (an AE-System) with the property that for any $u \in A$*

$$\inf\{q_i(u) | i \in I\} = p_0(u).$$

Naturally there is an equivalent definition by the means of set multiplications: There is an absolutely convex neighborhood U_0, such that for any absolutely convex neighborhood U, there is an upwards directed covering $\{V_i\}_{i \in I}$ by absolutely convex neighborhoods such that

$$\forall i \in I \, \exists m \in \mathbb{N} \, \forall n \geq m : "(V_i^n)" \subset U$$

By this equivalent definition we see immediately that subalgebras and quotients of AE-Algebras are AE-Algebras and that any AE-Algebra is m-convex. The proof of the last assertion is done in the following way: Let U be an absolutely convex neighborhood, $V^n \subset U$ for $n \in \mathbb{N}$ with V an sufficiently small absolutely convex neighborhood. So $W := \cup_{n \in \mathbb{N}} V^n \subset U$ with $W^2 \subset W \subset U$. By this fact we can suppose that the distinguished seminorm on A is multiplicative. Furthermore it is sufficient to provide AE-Systems for a defining system of seminorms on A. So we can choose a defining system of multiplicative seminorms p on A and associate the AE-Systems . For any of these seminorms we can choose an AE-System with the property: $p \leq C_i q_i$ with $C_i \geq 1$ by enlarging the AE-System via $\{\sup\{q_i, \sigma p\} | i \in I, \sigma > 0\}$.

An important example of AE-Algebras is given through LE-Algebras. A locally convex algebra is called LE-Algebra if there is an distinguished seminorm p_0 on A and a system of seminorms $\{p_n^\gamma\}_{n \in \mathbb{N}, \gamma \in \Gamma}$ with

: 1. $p_0 = p_0^\gamma$ for all $\gamma \in \Gamma$
: 2. $p_n^\gamma \leq p_{n+1}^\gamma$ for all $\gamma \in \Gamma$
: 3. $p_n^\gamma(uv) \leq \sum_{i=0}^n p_i^\gamma(u) p_{n-i}{}^\gamma(v)$

for all $n \in \mathbb{N}$and $u, v \in A$. LE abbreviates Leibnitz estimate .

1.4. Lemma. *Any LE-Algebra is an AE-Algebra.*

Proof. We define the following AE-System for p_n^γ:

$$(p_n^\gamma)_{\sigma,\tau} := \{\sup\{\sigma p_n^\gamma, \tau p_0\} | 0 < \sigma < 1, 1 < \tau\}$$

From $(p_n^\gamma)_{\sigma,\tau}(u) \leq 1$ we conclude $p_n^\gamma(u) \leq \frac{1}{\sigma}$ and $p_0 \leq \frac{1}{\tau}$. Take $u_1, ..., u_m \in A, m \geq n$ then

$$p_n^\gamma(u_1 \cdot ... \cdot u_m) \leq \sum_{i_1 + ... + i_n = m} p_{i_1}^\gamma(u_1) \cdot ... \cdot p_{i_n}^\gamma(u_m)$$

can be estimated via

$$p_n^\gamma(u_1 \cdot ... \cdot u_m) \leq \sigma^{-n} \tau^{n-m} m^n$$

which tends to zero as m tends to infinity. Consequently the given system is an AE-System. The infimum-property is satisfied trivially. □

1.5. Example. Any normed algebra is a LE-Algebra ($p_n^\gamma := p$).

1.6. Example. The smooth functions on $\mathbb{R}^n$ having compact support in a fixed closed subset with ordinary Fréchet-topology form an LE-Algebra.

1.7. Example. The free non-commutative algebra with two generators X, Y forms an LE-Algebra. We denote this algebra by $\mathbb{C}[X,Y]$. A system of seminorms on this algebra is given via a supremum norm levelled by degrees, more precisely, one can write $u \in \mathbb{C}[X,Y]$

$$u = \sum_{\mathbf{u}} a_{\mathbf{u}} M_{\mathbf{u}}(X,Y),$$

where $M_{\mathbf{u}}(X,Y) = X^{u_1}Y^{u_2}X^{u_3} \cdot ...$ and $\mathbf{u} = (u_1, u_2, ...)$ is a sequence of natural numbers with only finitely many, but always the first one different from zero. The seminorm associated to degree m is given through

$$p_m(u) = \max\{|a_{\mathbf{u}}| \,|\, \sum_{i=1}^{\infty} u_i \leq m\}.$$

With these seminorms Leibnitz estimates are clearly possible. In fact one can replace $\mathbb{C}$ by any m-convex algebra A and we obtain via a refined norming that $A[X,Y]$ is a LE-algebra. The completion of this algebra is an AE-Algebra, too, and the same analysis can be performed. It is called the algebra of formal power series with coefficients in A over two free variables and denoted by $A[[X,Y]]$.

The reason why we analyse AE-Algebras is that we want to make converge several power series in order to prove convergence of the Campbell-Baker-Hausdorff-Series finally.

1.8. Definition. *An element $u \in \mathbb{C}[[X,Y]]$ written by monomials of length (or degree) $l(M_k)$ in the following form*

$$u = \sum_{k=0}^{\infty} a_k M_k(X,Y)$$

is called a converging power series if
1. $a_m^* := \sum_{l(M_k)=m} |a_k| < \infty$
2. $\limsup_{m\to\infty} (a_m^*)^{\frac{1}{m}} < \infty$.

The inverse of the last number is called the radius of convergence of the power series. Given a power series with a certain radius of convergence, we can associate a smooth mapping on an AE-Algebra. Naturally we have to restrict ourselves to the case of $a_0^* = 0$ if the algebra has no unit, which is assumed if necessary.

1.9. Theorem. *Let A be a complete AE-Algebra, $u \in \mathbb{K}[[X,Y]]$ a power series with radius of convergence $0 < R \leq \infty$. Then an insertion map is defined for all $u, v \in A$ with $p_0(u) < R$ and $p_0(v) < R$ by the absolutely converging series. Furthermore for any two points $(u,v) \in ((p_0)_{<R})^2$ there is an open neighborhood U, where the above series converges absolutely and uniformly.*

Proof. ([**BCR81**], prop. 3.2.1.2.) Let p denote a multiplicative seminorm and q an asymptotic estimate for p with $p \leq Cq$ and $C \geq 1$. If $r > 0$ is given and $m = m(p,q)$, then

$$p(\text{"}u_1 \cdot ... \cdot u_n\text{"}) \leq r^n C^m$$

for $u_i \in A$, $q(u_i) < r$ and $n \in \mathbb{N}$ by direct calculation. For the proof of convergence we fix $u, v \in A$ with $p_0(u) < R$ and $p_0(v) < R$. Let p denote an arbitrary multiplicative seminorm on A. So there exists $0 < r < R$, which is independent of p, and an asymptotic estimate q for p with $p_0(u) < r < R$, $q(u) < r$, $p_0(v) < r < R$, $q(v) < R$. Furthermore we may assume that $p \leq Cq$ with $C \geq 1$. Now we can do what is done since more than hundred years.

$$p(\sum_{k=0}^{\infty} a_{\mathbf{k}} M_k(u,v)) \leq \sum_{k=0}^{\infty} |a_k| C^m r^{l(M_k)} \leq C^m \sum_{n=0}^{\infty} a_n^* r^n$$

for all $u, v \in A$ with $q(u) < r$ and $q(v) < r$. With the above remarks we conclude absolute and local uniform convergence of the power series of the given domain. □

1. Through the theorem a kind of local functional calculus in two variables is given: We can associate to a power series $f(X,Y)$ with radius of convergence $R > 0$ a smooth map $f_A : ((p_0)_{<R})^2 \to A$. Smoothness is proved by the smoothness of the monomials and uniform convergence of the respective derivatives.

2. $f, g, h \in \mathbb{C}[[X,Y]]$ with positive radii of convergence. The coefficients are denoted by a, b, c, respectively. Let $R > 0$ denote the radius of f. If A has a unit we assume additionally that $\max\{b_0^*, c_0^*\} < R$. Then the formal power series $f(g(X,Y), h(X,Y))$ has a positive radius of convergence and is equal to the respective composition of mappings on open subsets of A^2.
3. One variable functional calculus is provided by the above theorem, too. Compact convergence on a disk of radius $R > 0$ translates to pointwise convergence on the disk in p_0 in A. Furthermore the following formulas are valid:
$$\frac{d}{dt} f_A(c(t)) = (f')_A(\dot{c}(t))$$
$$f_A(g_A(u)) = (f \circ g)_A(u)$$
for a smooth curve c in the domain of definition and $u \in A$ in the respective domain of definition.
4. On a complete AE-Algebra the exponential map and its inverse are defined:
$$\exp(\log(1+u)) = 1 + u \text{ for } p_0(u) < 1$$
$$\log(\exp(u)) = u \text{ for } p_0(u) < \log 2$$
5. Let A be a complete commutative, associative, locally convex and metrizable algebra, such that the geometric series converges on a zero neighborhood, then A is an AE-Algebra ([**Czi83**]).

For the proof of this statement we need several partial results, which are of interest on their own, too. We denote by W_0 the absolutely convex, open zero-neighborhood, such that the geometric series converges on $\overline{W_0}$. First we observe that the geometric series converges absolutely with respect to every seminorm p on A: Take $x \in W_0$, then there is $t > 1$ with $tx \in W_0$, consequently $p(x^n) \leq t^{-n}$ for sufficiently large n. Uniformity of convergence is more subtle: Let q denote a seminorm with $p(x,y) \leq q(x)q(y)$, which is possible due to continuity.
$$W_m := \{x \in \overline{W_0} | q(x^n) \leq m \text{ for all } n\}$$
for $m \geq 1$. The sets W_m are closed and $\cup_{m \geq 1} W_m = \overline{W_0}$. Therefore exists W_m with non-void interior and by $W_m \subset m \cdot W_1$ we conclude that the interior of W_1 is not empty. Consequently we obtain for $x \in \frac{1}{2}W_1 - \frac{1}{2}W_1$ that $p(x^n) \leq 1$ for all n. But this set contains a neighborhood of zero W and it is easy to see that geometric series converges absolutely and uniformly on tW for $0 < t < 1$.

As a conclusion we get m-convexity of the given algebra, because of the following facts: The polarization formula
$$x_1 \cdot \ldots \cdot x_n = \frac{1}{n!} \cdot \int_T \ldots \int_T \frac{(t_1 x_1 + \ldots + t_n x_n)^n}{t_1 \ldots t_n} d\mu(t_1) \ldots d\mu(t_n)$$
with T the respective unit sphere in $\mathbb{K}$ and μ the associated measure is valid. Let V denote an arbitrary absolutely convex neighborhood of zero, then there is an absolutely convex zero-neighborhood W with $x^n \in V$ for $x \in W$ and $n \in \mathbb{N}$. Let p denote the Minkowski norm of V, then
$$p(x_1 \ldots x_n) \leq \frac{n^n}{n!} < e^n$$
for all $n \in \mathbb{N}$. Consequently $W^n \subset e^n V$ or $(e^{-1}W)^n \subset V$, which implies directly that the algebra is m-convex. Let p denote an arbitrary seminorm on A, then the convergence of the geometric series is absolute and locally uniform with respect to p: We can without restriction assume that p is submultiplicative by the preceding remarks: Take $x_0 \in W_0$ fixed, then there are real numbers $C \geq 1$ and $t > 1$ with $p(x_0^n) \leq C t^{-n}$ for all n. Choose $\epsilon > 0$ with $t^{-1} + \epsilon < 1$, then for $p(y) < \epsilon$ we obtain
$$p((x_0 + y)^n) \leq \sum_{k=0}^{n} \binom{n}{k} p(x_0^k) p(y^{n-k}) \leq C \cdot (t^{-1} + \epsilon)^n,$$
which implies the statement. We see that only submultiplicativity of the seminorm and convergence of the geometric series are applied. Consequently on compact subsets of W_0 the geometric series converges absolutely and uniformly with respect to any seminorm. Taking an

absolutely convex compact subset $eM \subset W_0$, given an arbitrary neighborhood of zero V, we can find a number m, such that for $n \geq m$ we have $M^n \subset V$. Without restriction it is sufficient to look at the case of an absolutely convex neighborhood with $V^2 \subset V$, there is a number m with $x^n \in V$ if $x \in eM$ and $n \geq m$. By polarization and Stirling we obtain $(eM)^n \subset e^n V$and consequently $M^n \subset V$ for $n \geqq m$. The final step for the construction of an AE-System for a given submultiplicative seminorm can be performed now: Given V an absolutely convex neighborhood of zero with $V^2 \subset V$ and $M \subset U_0 := (2e)^{-1} W_0$ an absolutely convex compact subset, then we define

$$V_M := (M + \frac{1}{2} V) \cap U_0.$$

The family $\{V_M\}$ is directed upwards and a covering of U_0. Furthermore we find $C \geq 1$ and $t > 2$ due to compactness and the definition such that

$$M^n \subset C t^{-n} V$$

for all n. This enables the following estimate

$$(V_M)^n \subset (M + \frac{1}{2} V)^n \subset \sum_{k=0}^{n} \binom{n}{k} M^k \frac{1}{2^{n-k}} V \subset C (t^{-1} + \frac{1}{2})^n V,$$

where the coefficient tends to zero as n tends to infinity. Consequently the desired AE-System is found.

6. Let A be an associative complete AE-Algebra, then the canonically associated Lie-Algebra A_L is an AE-Algebra, too. We have to replace p_0 by $2p_0$. Furthermore the following formulas are valid:

$$f_A(\operatorname{ad} u)(v) := \sum_{k=0}^{\infty} a_k (\operatorname{ad} u)^k (v)$$

$$f_A(\operatorname{ad} u)(g_A(\operatorname{ad} u)(v)) = (f \cdot g)(\operatorname{ad} u)(v)$$

for $u, v \in A$ with $p_0(u) < R$. f, g are holomorphic on a disk of radius R.

7. By induction we arrive at the following well known formulas:

$$(\operatorname{ad} u)^k (v) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} u^{k-i} v u^i$$

$$\frac{d}{dt}(u^k) = \sum_{i=0}^{k-1} (-1)^i \binom{k}{i+1} u^{k-i} (\operatorname{ad} u)^i (\dot{u})$$

for an arbitrary smooth curve u in A and $k \in \mathbb{N}$. Summing up the equations we obtain

$$\exp(\operatorname{ad}(u)) = \exp(u) \cdot v \cdot \exp(-u)$$

$$\frac{d}{dt} \exp(u(t)) = \exp(u(t)) \cdot f(\operatorname{ad}(u(t))(\dot{u}(t))$$

with $f(z) = \frac{1-\exp(-z)}{z}$.

These formulas will be used to prove the following theorem on the CBH-Formula, more detailed comments can be found in ([**BCR81**], ch. 3). The three mathematicians proved the formula independently, but only Felix Hausdorff managed to prove it in full generality in 1906: Let A denote an associative complete AE-Algebra with unit. The power series

$$\mathrm{CBH}(u, v) := \log(\exp(u) \cdot \exp(v))$$

converges for $u, v \in A$ with $p_0(u) < \frac{\log 2}{2}$, $p_0(v) = \frac{\log 2}{2}$ by the above remarks. As it is a power series we can write for $|t| \leq 1$

$$\mathrm{CBH}(tu, tv) = \sum_{k=0}^{\infty} t^k \, \mathrm{CBH}_k(u, v)$$

with homogeneous polynomials $\mathrm{CBH}_k(u, v)$ of degree $k \in \mathbb{N}$.

1.10. Theorem. *Let A be an associative complete AE-Algebra with unit, then* $\mathrm{CBH}_k(u,v) \in L(u,v)$ *for* $k \geq 0$*, where* $L(u,v)$ *is algebraically generated by* u, v *in* A_L*.*

Proof. (see [**BCR81**], ch. 3) Fixing $u, v \in A$ in the domain of definition we look at the smooth curve $w(t) = \mathrm{CBH}(tu, tv)$ for $|t| \leq 1$. $\exp(w(t)) = \exp(tu)\exp(tv)$ and by differentiation

$$\exp(w) \cdot f(\operatorname{ad} w)(\dot{w}) = u \cdot \exp(w) + \exp(w) \cdot v$$

with $f(z) = \frac{1-\exp(-z)}{z}$. Multiplying with $\exp(-w)$ on the right side we obtain

$$f(\operatorname{ad} w)(\dot{w}) = \exp(-\operatorname{ad} w)(u) + v$$

Inverting f yields a formula for $\dot{w}$, $g(z) := f(z)^{-1}$.

$$\dot{w} = g(\operatorname{ad} w) \circ \exp(-\operatorname{ad} w)(u) + g(\operatorname{ad} w)(v)$$

Applying again the provided rules we arrive at

$$\dot{w} = g(-\operatorname{ad} w)(u) + g(\operatorname{ad} w)(v)$$

Taylor expansion of g gives the desired formula for the first derivative

$$\dot{w} = \sum_{k=0}^{\infty} \frac{B_k}{k!}(\operatorname{ad} w)^k (u + (-1)^k v)$$

$w(0) = 0$, $\dot{w}(0) = u + v$, ...By successive differentiation we obtain the classical result

$$\mathrm{CBH}_1(u,v) = u + v$$

$$\mathrm{CBH}_k = \sum_{k=1}^{m-1} \sum_{m_1+\ldots+m_k=m-1} \frac{1}{m}\frac{B_k}{k!}[\mathrm{CBH}_{m_1}, [\ldots, [\mathrm{CBH}_{m_k}, u + (-1)^k v]\ldots]]$$

as recursion formula for $m \geq 1$. Another possibility to prove the formula is to insert the convergent power series for w, the recursion drops out immediately by comparison of coefficients. □

Now we can put everything together to prove that on a complete AE-Lie-Algebra the Campbell-Baker-Hausdorff-Series converges and defines a local Lie group on a neighborhood of zero: Convergence is a question of finding good estimates for the above recursion (see [**BCR81**], section 3.4.1). These estimates are provided by

$$h(z) = 1 - \sqrt{1-4z} = \sum_{k\geq 0} c_k z^k \text{ for } |z| \leq \frac{1}{4}$$

From the functional equation $h(z) = 2\sum_{k\geq 0} h(z)^k$ we obtain by differentiation the recursion formula

$$c_0 = 0$$

$$c_1 = 2$$

$$c_m = \sum_{k=1}^{m-1} \sum_{m_1+\ldots m_k=m-1} \frac{2}{m} c_{m_1} \cdot \ldots \cdot c_{m_k} \text{ for } m \geq 2$$

Remark that $B_k \leq k!$. Consequently for $u, v \in A$ with $p_0(u) \leq \frac{1}{4}$ and $p_0(v) \leq \frac{1}{4}$ the CBH -series

$$u * v := \mathrm{CBH}(u,v) = \sum_{k\geq 0} \mathrm{CBH}_k(u,v)$$

is converging. It is an easy calculation to prove the properties of a local Lie group for $(p_0)_{<\frac{1}{4}}$ with the multiplication $*$. The exponential map of this local Lie group is the identity in this special chart. For the proof of the required properties we have to look carefully at the definition. U is given through the inverse image of $(p_0)_{<\frac{1}{4}}$ under the continuous multiplication $.*. : (p_0)_{<\frac{1}{4}} \times (p_0)_{<\frac{1}{4}} \to A$.

The identity e is clearly given through 0, inversion of an element x is performed via the operation $-$. As far the associative law is concerned we have to calculate: We assume that u, v, w lie inside the open p_0-ball of radius $\frac{1}{4}$. The functions

$$\phi(t) = (tu * tv) * tw \text{ and } \psi(t) = tu * (tv * tw)$$

are defined on some small neighborhood of zero in $\mathbb{R}$. They are real analytic and coincide consequently if the derivatives at zero coincides. The derivatives at zero coincide in the free Lie algebra with three generators because this Lie algebra can be embedded in the free algebra with free generators and the equation

$$\exp_A(u * v) = \exp_A(u) \cdot \exp_A(v)$$

is valid, which implies associativity. Consequently they coincide in every Lie algebra. Two real analytic curves with the same derivatives at zero have the same radius of convergence, so ψ and ϕ are defined on an open neighborhood of $[-1, 1]$ if one of them is defined there and they coincide. So all the requirements of definition 1.1 are satisfied and we are given a local Lie group with Lie algebra A.

1.11. Definition. *Let A be an AE-Lie algebra and $V \subset A$ subset with $V = -V, 0 \in V$. If the* CBH*-series converges on $V \times V$ and the series for the product of three elements converges on V^3, then V is called canonical local group .*

Let C denote the centre of A, then $V + C$ is a canonical local group, too. There is $\rho > 0$, such that $(p_0)_{<\rho}$ is a canonical local group. Open local subgroups of $(p_0)_{<\rho} + C$ are referred to as analytic local groups as we know about absolute and uniform convergence of the CBH-series.

2. Local Lie groups without the CBH-Formula

The Campbell-Baker-Hausdorff-Formula is the way to analyze the problem of regular local Lie groups with locally diffeomorphic exponential map. Generically this property is not satisfied and the given examples are the most interesting ones in infinite-dimensional geometry (diffeomorphism groups). There are two questions to be posed completely different in nature.

1. Are there convenient Lie algebras, which are not the Lie-Algebra of any specified type of convenient local Lie group?
2. If a convenient Lie algebra is the Lie algebra of a local Lie group, under which conditions is this local group enlargible?

The second question can theoretically be answered by cohomological methods, the first question seems to be a functional analytic one and is treated by methods on Lie algebras (see [**Omo81**], [**Omo97**] for algebraic details).

3. Cohomological conditions for enlargibility

Local groups are abstractly defined as in Definition 1.1 without smoothness properties ([**vE62**], p. 392).

3.1. Definition (Homomorphism of local groups). *A map $\phi : L \to L'$ is called a homomorphism if $\phi(x)\phi(y)$ is defined if and only if xy is defined. The homomorphism property $\phi(x)\phi(y) = \phi(xy)$ is required. A map ϕ is called a weak homomorphism if $\phi(x)\phi(y)$ is defined whenever xy is defined and $\phi(xy) = \phi(x)\phi(y)$. A local subgroup of a local group is the monomorphic image of a local group.*

3.2. Remark. The kernel of a homomorphism of local groups is a group. A subset H of a local group L is a local subgroup if the restriction of the multiplication to H gives the structure of a local group. Any group is a local group. Given a local group L, a neighborhood basis $\mathfrak{B}$ of the identity is defined to be a collection of local subgroups such that (i.) for $V_1, V_2 \in \mathfrak{B}$ there is $V_3 \in \mathfrak{B}$ with $V_3 \subset V_1 \cap V_2$, (ii.) for $a \in L$ and $V \in \mathfrak{B}$ there is $W \in \mathfrak{B}$ such that aW, Wa^{-1} are defined and $aWa^{-1} \subset V$ and (iii.) for $V \in \mathfrak{B}$ there is $W \in \mathfrak{B}$ with $W^2 \subset V$. There is a unique topology associated to a neighborhood basis at the identity and a topological local group has such a neighborhood basis. Suppose that L is a local subgroup of a group G with $G = L^\infty$, then any topology on L uniquely extends to G.

The abstract problem is the following: Let G be a group, $V \subset G$ a local subgroup and $\phi : U \to V$ a homomorphism of local groups. Under which conditions can U be enlarged to a group H and ϕ to a homomorphism $\psi : H \to G$. Without restricting the problem we can assume that ϕ is an epimorphism, $V^\infty = G$, then ψ is automatically an epimorphism. We shall always assume that

$U^\infty = H$. Furthermore we can assume that $\phi^{-1}(v) = \psi^{-1}(v)$ for $v \in V$. So in particular the kernels are the same .

We look the following commutative diagram of (weak) homomorphisms with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & H & \xrightarrow{\psi} & G & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & N & \longrightarrow & U & \xrightarrow{\phi} & V & \longrightarrow & 1
\end{array}$$

For any $h \in H$ $i(h)$ denotes the associated inner automorphism of H, $i_N(h)$ denotes the same action restricted to N. Consequently $i_N : H \to A(N)$ is a homomorphism to the group of automorphisms of N. The inner automorphisms of N are a normal subgroup of $I(N) \lhd A(N)$, their inverse image under i_N contains N, so $\theta : G \to A(N)/I(N)$ is a well-defined homomorphism. If we are given the homomorphism $\phi : U \to V$ of local groups, then the mapping $i_N \circ \phi^{-1} : V \to A(N)/I(N)$ is well-defined and a weak homomorphism. θ is obviously the unique extension of this weak homomorphism to G. The existence of θ is necessary for the solvability of the problem of enlarging the exact sequence in the above sense, θ exists trivially in the case that N is central in U. In the further discussion we assume consequently the existence of θ.

3.3. Definition (V-local cohomology). *Let V be a non-void subset of a group G and C an abelian group, on which G operates, $n \in \mathbb{N}$. A V-local n-tuple $(x_1, ..., x_n) \in V^n$ satisfies $x_{p+1} \cdot ... \cdot x_q \in V$ for any $0 \le p \le q \le n$. A C-valued function defined on the set of V-local n-tuples will be called an n-cochain. The codifferential δ is given through the following formula on a n-cochain f*

$$\delta f(x_1, ..., x_{n+1}) = x_1 f(x_2, ..., x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(x_1, ..., x_i x_{i+1}, ..., x_{n+1}) +$$
$$+ (-1)^{n+1} f(x_1, ... x_n)$$

for a V-local n-tuple $(x_1, ..., x_{n+1})$.

Naturally $\delta \circ \delta = 0$ on the cochain complex, the derived cohomology will be denoted by $H^*(V, C)$, if $V = G$ we obtain the ordinary group cohomology. In "homogeneous notation" we can apply a simplex terminology: A V-simplex of dimension n is a $(n+1)$-tuple of elements of G, such that $x_i^{-1} x_j \in V$ for $0 \le i < j \le n$. Given a V-simplex, then

$$y_1 = x_0^{-1} x_1, ..., y_i = x_{i-1}^{-1} x_i, ..., y_n = x_{n-1}^{-1} x_n$$

is a V-local n-tuple and a given a V-local n-tuple $y_1, ..., y_n$, then

$$(x_0, x_0 y_1, ... x_0 y_1 ... y_p, ... x_0 y_1 ... y_n)$$

is a V-simplex for any $x_0 \in G$. The collection of V-simplices constitutes a set Γ_V on which G operates freely on the left via

$$x(y_0, ..., y_n) = (x y_0, ..., x y_n)$$

for a V-simplex $(y_0, ..., y_n)$ and $x \in G$. Remark that this set is a simplicial G-set if V contains e with the following structure mappings:

$$\partial_i(x_0, ..., x_n) = (x_0, ..., \widehat{x_i}, ..., x_n) \text{ for } 0 \le i \le n$$
$$\delta_i(y_0, ..., y_{n-1}) = (y_0, ..., y_{i-1}, y_i, y_i, ..., y_{n-1}) \text{ for } 0 \le i \le n-1$$

The equivariant cochains on Γ_V are in one-to-one correspondence with the V-local cochains by the following formula: ´

$$f(y_1, ..., y_n) = F(e, y_1, ..., y_1 \cdot ... \cdot y_n)$$
$$F(x_0, ..., x_n) = x_0 f(x_0^{-1} x_1, ..., x_{n-1}^{-1} x_n)$$

due to one-to-one correspondence of the set of orbits Γ_V / G to the set of V-local n-tuples. Consequently $H^n(V, C) \cong H^n_{eq}(\Gamma_V, C)$ for $n \in \mathbb{N}$, the ordinary cohomology will be denoted by $H^n(\Gamma_V, C)$. By $E^n(\Gamma_V, C)$ we denote the abelian group of equivariant cochains, by $F^n(\Gamma_V, C)$ the ordinary cochain group. These notions were introduced in [**vE62**], but there are some mistakes in defining the differentials.

3.4. Definition. *A group N together with a homomorphism $\theta : G \to A(N)/I(N)$ is called a G-kernel. We shall denote the extension of local groups $1 \to N \to U \xrightarrow{\phi} V \to 1$, where ϕ is a homomorphism of local groups, by $\mathfrak{L}$, the possibly existing extension of groups $1 \to N \to H \xrightarrow{\psi} G \to 1$ is denoted by Σ. The vertical arrows shall normally be represented by inclusion signs. $\mathfrak{L}$ is called enlargible from the local subgroup $W \subset V$ if there is a commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] & H \arrow[r, "\psi"] & G \arrow[r] & 1 \\
1 \arrow[r] & N \arrow[r] \arrow[u, equal] & \tilde{U} \arrow[r, "\phi"] \arrow[u] & W \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

with $\widetilde{U} = \phi^{-1}(W)$.

Given a G-kernel (N, θ) and let $\epsilon : G \to A(N)$ be a map covering $\theta : G \to A(N)/I(N)$, then it follows $\epsilon(x)\epsilon(y)\epsilon(xy)^{-1} \in I(N)$ or $\epsilon(x)\epsilon(y) = i(n(x, y))\epsilon(xy)$ with $n : G \times G \to N$ well-defined, the choice of n however is up to elements in the center of N. From associativity we obtain

$$i(n(x, y)n(xy, z)) = i(\epsilon(x)n(y, z) \cdot n(x, yz))$$

which can be reformulated via

$$n(x, y)n(xy, z) = f_3(x, y, z) \cdot \epsilon(x)n(y, z) \cdot n(x, yz) \tag{A}$$

where f_3 is a cocycle on G with values in the centre $C(N)$ of N. This condition will be referred to as associativity condition. f_3 is determined by (N, θ) up to a coboundary due to the choices which have been made and so there is a uniquely determined cohomology class $[f_3] \in H^3(G, C(N))$. Furthermore $[f_3] = 0$ if and only if there is an extension $1 \to N \to H \xrightarrow{\psi} G \to 1$ with $\theta = i_N\psi^{-1}$ (see [**EM47**]).

3.5. Proposition. *Let (N, θ) be a G-kernel, then there is a uniquely associated 3-cohomology class $[(N, \theta)] := [f_3] \in H^3(G, C(N))$ and $[f_3] = 0$ if and only if there is an extension $1 \to N \to H \to G \to 1$. Furthermore given two extension Σ_1 and Σ_2 with the same G-kernel (N, θ), then there is a uniquely associated 2-cohomology class $[f_2] = d(\Sigma_1, \Sigma_2) \in H^2(G, C(N))$ and $[f_2] = 0$ if and only if the two extensions are equivalent, i.e.*

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] & H_1 \arrow[r, "\psi"] & G \arrow[r] & 1 \\
1 \arrow[r] & N \arrow[r] \arrow[u, equal] & H_2 \arrow[r, "\phi"] \arrow[u, "i"'] & G \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

with a group isomorphism i.

Proof. (see [**EM47**], lemma 7.2.) By the above construction f_3 is well-defined as cochain after having chosen n. Another choice of n differs by a function $f(x, y) = n(x, y)\widetilde{n}(x, y)^{-1}$ for $x, y \in G$, which takes values in the centre $N(C)$. We can easily calculate by inserting

$$f_3(x, y)\widetilde{f_3}(x, y)^{-1} = f(x, y)f(xy, z)f(x, yz)^{-1}[\epsilon(x)f(y, z)]^{-1} = (\delta(f)(x, y, z))^{-1}$$

which is a coboundary in multiplicative form. Consequently it is sufficient to show that for a certain choice of n we obtain a cocycle f_3: We calculate with the following expression J:

$$\begin{aligned}
J(x, y, z, t) &= \epsilon(x)[\epsilon(y)n(z, t) \cdot n(y, zt)]n(x, yzt) \\
&= \epsilon(x)[f_3(y, z, t)^{-1}n(y, z) \cdot n(yz, t)]n(x, yzt) \\
&= \epsilon(x)[f_3(y, z, t)^{-1}n(y, z)]f_3(x, yz, t)n(x, yz)n(xyz, t) \\
&= f_3(x, y, z)^{-1}f_3(x, yz, t)^{-1}[\epsilon(x)f_3(y, z, t)^{-1}]n(x, y)n(x, yz)n(xyz, t)
\end{aligned}$$

On the other hand we can first apply the defining relation for n

$$\begin{aligned}
J(x, y, z, t) &= n(x, y) \cdot \epsilon(xy)n(z, t) \cdot n(x, y)^{-1}[\epsilon(x)n(y, zt)]n(x, yzt) \\
&= n(x, y) \cdot \epsilon(xy)n(z, t) \cdot f_3(x, y, zt)^{-1}n(xy, zt) \\
&= f_3(xy, z, t)^{-1} \cdot f_3(x, y, zt)^{-1}n(x, y)n(x, yz)n(xyz, t)
\end{aligned}$$

which leads to the desired relation $\delta f_3 = 0$. Remark that by changing n in the above fashion all the members of the class $[f_3]$ are produced. It is not clear wether $[f_3]$ depends on the choice of ϵ. Another choice α of a covering of θ leads to the following calculations: $\alpha(x) = i(k(x))\epsilon(x)$ for $x \in G$ with $k(x) \in N$. Consequently $\alpha(x)\alpha(y) = i(k(x)[\alpha(x)k(y)]n(x,y))\epsilon(xy)$. We define $n'(x,y) := k(x)[\alpha(x)k(y)]n(x,y)k(xy)^{-1}$, so that we obtain

$$\begin{aligned} n'(x,y)k(xy) &= k(x)[\alpha(x)k(y)]n(x,y) \\ &= [\alpha'(x)k(y)]k(x)n(x,y) \end{aligned}$$

then

$$\begin{aligned} \alpha(x)n'(x,y) \cdot n'(x,yz)k(xyz) &= \alpha'(x)[n'(x,y)k(yz)]k(x)n(x,yz) \\ &= \alpha'(x)[n'(x,y)k(yz)]k(x)n(x,yz) \\ &= \alpha'(x)[\alpha'(y)k(z) \cdot k(y)]k(x)[\epsilon(x)n(y,z)]n(x,yz) \\ &= f_3(x,y,z)^{-1}[\alpha'(x)\alpha'(y)k(z)]n'(x,y)k(xy)n(xy,z) \\ &= f_3(x,y,z)^{-1}n'(x,y)[\alpha'(xy)k(z)]k(xy)n(xy,z) \\ &= f_3(x,y,z)^{-1}n'(x,y)n'(xy,z)k(xyz) \end{aligned}$$

which leads to the desired formula

$$n'(x,y)n'(xy,z) = f_3(x,y,z) \cdot \alpha(x)n'(y,z) \cdot n'(x,yz)$$

Consequently the change of the covering map does not affect the associated 3-cohomology class. Remark that we can choose f_3 normalized by selecting appropriately, more precisely $\epsilon(e) = e$ and $n(e,y) = n(x,e) = e$. Given an extension $\Sigma : 1 \to N \to H \xrightarrow{\psi} G \to 1$, we can associate a G-kernel in the above manner. We choose $\epsilon(x) = i_N\eta(x)$, where $\eta : G \to H$ is a cross section to ψ, i.e. $\psi \circ \eta(x) = x$. This is clearly a covering of θ. We want to calculate f_3 in this setting, $\eta(x)\eta(y) = f_2(x,y)\eta(xy)$. Consequently we can choose $n(x,y) = f_2(x,y)$.

$$\begin{aligned} [\epsilon(x)n(y,z)]n(x,yz) &= \eta(x)f_2(y,z)\eta(x)^{-1}f_2(x,yz) \\ &= \eta(x)\eta(y)\eta(z)\eta(yz)^{-1}\eta(x)^{-1}\eta(x)\eta(yz)\eta(xyz)^{-1} \\ &= \eta(x)\eta(y)\eta(z)\eta(xyz)^{-1} \\ &= \eta(x)\eta(y)\eta(xy)^{-1}\eta(xy)\eta(z)\eta(xyz)^{-1} \\ &= n(x,y)n(xy,z) \end{aligned}$$

Consequently $f_3 = e$ and therefore $[f_3] = 0$. Conversely assume that the G-kernel (N,θ) has a vanishing 3-cohomology class associated, then selecting ϵ with $\epsilon(e) = e$ a covering of θ and a normalized n such that $f_3 = e$. We look at the set $N \times G$ and try to define a group structure, which we denote by $(N \times G)_{\epsilon,n}$ on it:

$$(k,x)(l,y) := (k \cdot \epsilon(x)l \cdot n(x,y), xy)$$

The multiplication is well-defined, associative by $f_3 = e$, the identity is (e,e) and the inverse of an element (k,x) is given through $(\epsilon(x)^{-1}[n(x,x^{-1})k]^{-1}, x^{-1})$. $\psi(k,x) := x$ is a homomorphism to G and the kernel can be identified with N. We have to show that the associated G-kernel is (N,θ). We choose the cross-section $\eta(x) = (e,x)$. We obtain $\eta(x)k = (e,x)(k,e) = (\epsilon(x)k,x) = [\epsilon(x)k]\eta(x)$, where from we conclude that $i_N \circ \eta$ covers θ. In the last step we show the second assertion: Given two extensions $1 \to N \to H_i \xrightarrow{\psi_i} G \to 1$ for $i = 1,2$ with G-kernel (N,θ), then we can look for cross-sections η_i, associate n_i as above and form the difference $f_2(x,y) := n_1(x,y)n_2(x,y)^{-1}$. This is a 2-cocycle, which is easily seen by looking at the two associativity conditions (A) and "subtracting" them. Choosing different cross-sections leads to a cohomologeous cocycle, the result differs by a coboundary, which is seen directly. All elements of this class can be reached by this construction. Consequently the two extensions Σ_i for $i = 1,2$ determine a 2-cohomology class $d(\Sigma_1,\Sigma_2) := [f_2] \in H^2(G,C(N))$. Going back the above way we observe that given an extension Σ_1 and a 2-cohomology class $[f_2]$ we find another extension Σ_2 such that $d(\Sigma_1,\Sigma_2) := [f_2]$. Furthermore $d(\Sigma_1,\Sigma_2) = -d(\Sigma_2,\Sigma_1)$. The last step is proved by applying the above models: Given an extension Σ and a choice of a cross section

η with $\eta(e) = e$, which determines a normalized n, then the given model and H are isomorphic and the isomorphism i fits in the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & H & \xrightarrow{\psi} & G & \longrightarrow & 1 \\ & & \| & & \uparrow{\scriptstyle i} & & \| & & \\ 1 & \longrightarrow & N & \longrightarrow & (N \times G)_{\epsilon,n} & \xrightarrow{\phi} & G & \longrightarrow & 1 \end{array}$$

The isomorphism is given through

$$i(k, x) = k\eta(x)$$
$$i^{-1}(h) = (h(\eta \circ \psi(h))^{-1}, \psi(h))$$

for $k \in N, x \in G, h \in H$. Calculating with models we easily get that the vanishing of the associated cohomology class translates to the existence of the desired isomorphism. The isomorphism between two models $(N \times G)_{\epsilon,n_1}$ and $(N \times G)_{\epsilon,n_2}$ is given through $(k, x) \mapsto (f(x)k, x)$, where f denotes a normalized 1-cochain with $n_1 = \delta f \cdot n_2$. □

3.6. Remark. For any 3-cohomology class in $H^3(G, C)$ with C an abelian group there is a G-kernel (N, θ) such that $[(N, \theta)]$ is the given class ([**EM47**], lemma 9.1.).

In our case however the local extension $\mathfrak{L}$ is one of the data, so given a cross section $\eta : V \to U$ to the epimorphism ϕ, then the map ϵ covering θ may be chosen $\epsilon(x) = i_N(\eta(x))$ for $x \in V$. There is a unique ν with such that $\psi(x)\psi(y) = \nu(x, y)\psi(xy)$ for $x, y, xy \in V$, ν is defined on the domain of definition of the product on V. Taking $\nu = n$ in the above construction we obtain $f_3(x, y, z) = e$ for $x, y, z, xy, yz, xyz \in V$. If we are able to enlarge ν to a $C(N)$-valued cocycle on G, then the extension is enlargible. So we have to find some local-global theorems.

Given $W \subset V \subset G$, then any V-local n-tuple is automatically a W-local n-tuple, so we get a restriction homomorphism ρ_{WV} for the cochains and an induced homomorphism $\rho_{WV} : H(V, C) \to H(W, C)$, denoted by the same symbol. The same is valid for the restrictions on the complex Γ_V to Γ_W in the homogeneous notation. We obtain $\rho_{XW}\rho_{WV} = \rho_{XV}$ for $X \subset W \subset V \subset G$. Let $\mathfrak{B}$ denote a system of subset of G, such that for any two subsets $V, W \in \mathfrak{B}$ there is $X \in \mathfrak{B}$ with $X \subset V \cap W$. Consequently we obtain directed systems, of cochain complexes and cohomology groups, respectively, in the two different notions. We denote the respective limits by $E(\Gamma_{\mathfrak{B}}, C)$, $H^n(\mathfrak{B}, C)$, $H^n(\Gamma_{\mathfrak{B}}, C)$ and $H^n_{eq}(\Gamma_{\mathfrak{B}}, C)$. Furthermore we obtain homomorphisms $\rho_V : H^n(V, C) \to H^n(\mathfrak{B}, C)$ and natural isomorphisms $H^n(\mathfrak{B}, C) = H^n_{eq}(\Gamma_{\mathfrak{B}}, C) = H^n(E(\Gamma_{\mathfrak{B}}, C))$, because the cohomology functor and the direct limit functor commute.

3.7. Proposition. *Given $n \in \mathbb{N}$. Let the complexes $\Gamma_{\mathfrak{B}}$ be connected for $V \in \mathfrak{B}$ and assume that for any $V \in \mathfrak{B}$ there is a $W \in \mathfrak{B}$, such that $\rho_{WV} H^i(\Gamma_V, C) = 0$ for $1 \leq i \leq n-1$. Then $\rho_{\mathfrak{B}G} : H^i(G, C) \to H^i(\mathfrak{B}, C)$ is an isomorphism for $0 \leq i \leq n-1$ and a monomorphism for $i = n$.*

Proof. (see [**vE62**], p. 398) The proof is given by a spectral sequence argument: We look at equivariant cochains on $\Gamma_G \times \Gamma_V$, denoted by $E(\Gamma_G \times \Gamma_V, C)$. The naturally given restriction homomorphisms are denoted by ρ_{WV} and the limit complex by $E(\Gamma_G \times \Gamma_{\mathfrak{B}}, C)$. The natural inclusion $E(\Gamma_V, C) \to E(\Gamma_G \times \Gamma_V, C)$ induces an isomorphism on cohomology level and since the cohomology functor commutes with direct limits we obtain $H_{eq}(\Gamma_G \times \Gamma_{\mathfrak{B}}, C) = H_{eq}(\Gamma_{\mathfrak{B}}, C)$, where the first cohomology is the cohomology of the double complex. We get a double complex by looking at the degrees in G and V or $\mathfrak{B}$, respectively. To obtain the isomorphism property we look at the second spectral sequence associated to the double complex, we denote it by $''E_n(\Gamma_G \times \Gamma_V, C)$. We have

$$''E_1^{p,q}(\Gamma_G \times \Gamma_V, C) = E^p(\Gamma_V, H^q(\Gamma_G, C)),$$

which will be explained in the sequel, but $H^q(\Gamma_G, C) = 0$ for $q > 0$ and $H^0(\Gamma_G, C) = C$. Consequently the second spectral sequence converges to $H_{eq}(\Gamma_V, C)$. For the direct limit we obtain the same. Now we investigate the natural inclusion $E(\Gamma_G, C) \to E(\Gamma_G \times \Gamma_V, C)$. The first spectral sequence shall be denoted by $E_n(\Gamma_G \times \Gamma_V, C)$ and $E_n(\Gamma_G \times \Gamma_{\mathfrak{B}}, C)$, respectively. The most important observation is the canonical identification

$$E_1^{p,q}(\Gamma_G \times \Gamma_V, C) = E^p(\Gamma_G, H^q(\Gamma_V, C))$$

The canonical isomorphism is given through $i([g_n]) = (y_0, ..., y_n) \mapsto [\breve{g}_n(y_0, ..., y_n)]$, the group action on $F(\Gamma_V, C)$ via $x \cdot f(x_0, ..., x_m) = xf(x^{-1}x_0, ..., x^{-1}x_m)$. The map is well-defined and has an obvious inverse to the cohomology classes of equivariant cochains, because of the special choice of the group action. These two isomorphisms stem in fact from the isomorphism $E^{p,q}(\Gamma_G \times \Gamma_V, C) = E^p(\Gamma_G, F^q(\Gamma_V, C))$, commuting with the second differential. Remark that the restriction homomorphisms arise from the given ones on the cohomology $H(\Gamma_V, C)$. Applying the hypotheses we obtain

$$E_1^{p,0}(\Gamma_G \times \Gamma_{\mathfrak{B}}, C) = \lim E_1^{p,0}(\Gamma_G \times \Gamma_V, C) = \lim E^p(\Gamma_G, H^0(\Gamma_V, C)) = E^p(\Gamma_G, C)$$

by connectedness of Γ_V and

$$E_1^{p,j}(\Gamma_G \times \Gamma_{\mathfrak{B}}, C) = \lim E_1^{p,j}(\Gamma_G \times \Gamma_V, C) = \lim E^p(\Gamma_G, H^j(\Gamma_V, C)) = 0$$

for $1 \leq j \leq n-1$. So we obtain the desired result due to convergence of the spectral sequence and an associated isomorphism. Remark that in general the *Hom*-functor and direct limits do not commute, but in our cases we can calculate directly. The above hypotheses on the connecting map implies $H^j(\Gamma_{\mathfrak{B}}, C) = 0$ for $1 \leq j \leq n-1$. □

The following theorem is the heart of the theory, the connection between the various cohomologies in the case of locally contractible connected topological groups is explained and some rather subtle properties on the direct limit are proved. First we need some definitions (see [**vE62**], p. 409-414 for details):

3.8. Definition. *Let G be an abstract group and $[M, \rho, \Lambda]$ a directed system of (G-)modules, the system is called almost injective if for $\lambda \in \Lambda$ there is a $\mu > \lambda$ such that* $\ker \rho_{\mu\lambda} = \ker \rho_{\nu\lambda}$ *for $\nu > \mu$. For almost injective $[M, \rho, \Lambda]$ of G-modules we obtain*

$$\lim H^0(G, M_\lambda) = H^0(G, \lim M_\lambda).$$

Let $[H, \rho, \Lambda]$ and $[K, \sigma, M]$ be directed systems of modules, a pair $(f : \Lambda \to \Sigma, \phi(\lambda) : H_\lambda \to K_{f(\lambda)})$ is called a representation of $[H, \rho, \Lambda]$ into $[K, \sigma, M]$ if for any $\lambda_1 < \lambda_2$ there is $\mu_3 > \mu_i = f(\lambda_i)$ for $i = 1, 2$ with

$$\begin{tikzcd} H_{\lambda_1} \arrow[r, "\Phi"'] \arrow[d, "\rho"] & K_{\mu_1} \arrow[r, "\sigma"'] & K_{\mu_3} \arrow[d, equal] \\ H_{\lambda_2} \arrow[r, "\Phi"'] & K_{\mu_2} \arrow[r, "\sigma"'] & K_{\mu_3} \end{tikzcd}$$

commutative. A pair of representations (f, ϕ) and (f', ϕ') of $[H, \rho, \Lambda]$ into $[K, \sigma, M]$ is called contiguous if there is for any $\lambda \in \Lambda$ a $\mu \in M$ such that

$$\begin{tikzcd} H_{\lambda_1} \arrow[r, "\Phi"'] \arrow[d, "\Phi'"] & K_{f(\lambda)} \arrow[d, "\sigma"] \\ K_{f'(\lambda)} \arrow[r, "\sigma"'] & K_\mu \end{tikzcd}$$

is commutative. Remark that the representations of directed systems form a category (composition and identity are defined). Contiguity is an equivalence relation compatible with composition, so we can pass to equivalence classes of contiguous representations as morphisms of a new category with the same objects. If two directed systems in this category are isomorphic, we call them equivalent, that is invertible up to contiguity.

3.9. Lemma. *A directed system of modules is almost injective if and only if it is equivalent to an almost injective directed system of modules.*

Proof. (see [**vE62**], p. 409-414 for details) One direction is trivial, assume now that there is given an equivalence $[H, \rho, \Lambda] \underset{(g,\psi)}{\overset{(f,\phi)}{\rightleftarrows}} [K, \sigma, M]$ to the almost injective system $[K, \sigma, M]$. Given $\mu \in M$

there is $\mu' > \mu$ such that for $\mu'' > \mu'$ we have $\ker \sigma_{\mu''\mu} = \ker \sigma_{\mu'\mu}$. Let $\lambda' > \lambda$, then we find $\mu'' > \mu'$such that

$$\begin{array}{ccccc} H_\lambda & \xrightarrow[\Phi]{} & K_\mu & \xrightarrow[\sigma]{} & K_{\mu''} \\ \Big\downarrow \rho & & & & \Big\| \\ H_{\lambda'} & \xrightarrow[\Phi]{} & K_{\mu'} & \xrightarrow[\sigma]{} & K_{\mu''} \end{array}$$

by the representation property. Consequently $\ker \sigma_{\lambda'\lambda} \subset \ker(\sigma_{\mu''\mu}\phi_\lambda) = \ker(\sigma_{\mu'\mu}\phi_\lambda)$. Since $(g, \psi) \circ (f, \phi)$ is contiguous to the identity representation there is $\lambda_1 > \lambda, g(\mu)$ such that $\rho_{\lambda_1\lambda} = \rho_{\lambda_1 g(\mu)}\psi_\mu\phi_\lambda$. Since (g, ψ) is a representation there is $\lambda' > \lambda_1$ such that

$$\begin{array}{ccccc} K_\mu & \xrightarrow[\psi]{} & H_{g(\mu)} & \xrightarrow[\rho]{} & H_{\lambda'} \\ \Big\downarrow \sigma & & & & \Big\| \\ K_{\mu'} & \xrightarrow[\psi]{} & H_{g(\mu')} & \xrightarrow[\rho]{} & H_{\lambda'} \end{array}$$

is commutative. Collecting the relations we obtain

$$\rho_{\lambda'\lambda} = \rho_{\lambda'\lambda_1}\rho_{\lambda_1\lambda} = \rho_{\lambda'\lambda_1}\rho_{\lambda_1 g(\mu)}\psi_\mu\phi_\lambda = \rho_{\lambda' g(\mu)}\psi_\mu\phi_\lambda = \rho_{\lambda' g(\mu')}\psi_{\mu'}\sigma_{\mu'\mu}\phi_\lambda,$$

consequently $\ker \sigma_{\lambda'\lambda} \supset \ker(\sigma_{\mu''\mu}\phi_\lambda) = \ker(\sigma_{\mu'\mu}\phi_\lambda)$. So for $\lambda' > \lambda_1$ we arrive at $\ker \sigma_{\lambda'\lambda} = \ker(\sigma_{\mu'\mu}\phi_\lambda)$ being constant in λ'. Remark that we only needed half of the equivalence. □

3.10. Remark. The limitation procedure maps directed systems to modules and contiguous representations to the same homomorphism, equivalences are mapped to isomorphisms.

3.11. Theorem. *Let G be a locally contractible connected topological group and $\mathfrak{B}$ the system of symmetric neighborhoods of the identity, then the singular cohomology groups $H^n_{top}(G, C)$ and $H^n(\Gamma_\mathfrak{B}, C)$ are canonically isomorphic, the directed system $H^n(\Gamma_V, C)$ is almost injective and the isomorphism respects the G-module structure on the cohomology groups.*

Proof. (see [**vE62**], p. 414-417 for details) The G-module structure on $H^n(\Gamma_V, C)$ is given by the product of the two representation

$$\pi_1(g)f(x_0, ..., x_m) = f(g^{-1}x_0, ..., g^{-1}x_m)$$

and

$$\pi_2(g)f(x_0, ..., x_m) = gf(x_0, ..., x_m)$$

analogously on $H^n(G, C)$, but the G-action on $H^n_{top}(G, C)$ derives exclusively from π_2, the G-action on C.

We construct a directed system of G-modules equivalent to $[H^n(\Gamma_V, C), \rho, \mathfrak{B}]$. This equivalent system shall produce the topological cohomology of the group and the representations have to be equivariant, then we can draw the desired conclusion. A singular q-simplex $\Delta^q \xrightarrow{\sigma} G$ is called a singular V-simplex if $\sigma(x)^{-1}\sigma(y) \in V$ for $x, y \in \Delta^q$. The collection of all singular V-simplices is denoted by Σ_V. $V_1 \subset V_2$ induces an inclusion $\Sigma_{V_1} \subset \Sigma_{V_2}$. Applying barycentric subdivision we obtain that this inclusion induces an isomorphism on cohomology level $H^n(\Sigma_{V_1}, C) = H^n(\Sigma_{V_2}, C)$, therefore the system $[H^n(\Sigma_V, C), \rho, \mathfrak{B}]$ is almost injective with limit $H^n(\Sigma_V, C) = H^n(\Sigma_G, C) = H^n_{top}(G, C)$ for $n \in \mathbb{N}$. We define an equivalence between $[H^n(\Gamma_V, C), \rho, \mathfrak{B}]$ and $[H^n(\Sigma_V, C), \rho, \mathfrak{B}]$.

Let $d^0, ..., d^q$ be the ordered set of vertices of Δ^q and $\phi_V(\sigma) = (\sigma(d^0), ..., \sigma(d^q))$ the equivariant simplicial map from Σ_V to Γ_V, the resulting map on cohomology level shall be denoted by ϕ_V, too. Consequently $(f = id, \phi)$ is an equivariant representation from $[H^n(\Gamma_V, C), \rho, \mathfrak{B}]$ to $[H^n(\Sigma_V, C), \rho, \mathfrak{B}]$.

We proceed by constructing a representation (g, ψ) in the reverse direction by the method of acyclic carrier functions fixing $n \in \mathbb{N}$. Given $V \in \mathfrak{B}$ we can select a sequence of contractible neighborhoods of identity such that

$$V_1 \subset ... \subset V_{n+1}$$
$$V_{j-1} \subset V_{j-1}^2 \subset V_j \text{ for } 2 \leq j \leq n+1$$
$$V_{n+1}^{-1}V_{n+1} \subset V$$

and we choose $V_0 := g(V)$ a symmetric neighborhood of the identity such that $V_0 \subset V_1$. Given a j-dimensional V_0-simplex $(x_0, ..., x_j)$ for $j \leq n+1$ we define $A_V((x_0, ..., x_j))$ to be the singular complex of algebraic topology $S(x_0 V_j)$ associated to $x_0 V_j$.

$$\begin{aligned} A_V((x_1, ..., x_j)) &= S(x_1 V_{j-1}) \subset S(x_0 V_0 V_{j-1}) \subset S(x_0 V_{j-1}^2) \subset S(x_0 V_j) \\ &= A_V((x_0, ..., x_j)) \\ A_V((x_0, ..., \widehat{x_k}, ..., x_j)) &= S(x_0 V_{j-1}) \subset S(x_0 V_j) = A_V((x_0, ..., x_j)) \text{ for } 1 \leq k \leq j \end{aligned}$$

So A_V is a carrier function. From contractibility we obtain that the carrier function is acyclic and $V_j^{-1} V_j \subset V$ for $1 \leq j \leq n+1$ implies that the carrier function is given from the $(n+1)$-skeleton of $\Gamma_{V_0}$to the $(n+1)$-skeleton of Σ_V. Finally A_V is equivariant, so we obtain a uniquely defined equivariant homomorphism $A_V = \psi_V : H^n(\Sigma_V, C) \to H^n(\Gamma_{g(V)}, C)$. (g, ψ) is actually an equivariant representation:

Given a pair $V' \subset V$ from $\mathfrak{B}$, we associate $W \in \mathfrak{B}$ with $W \subset g(V') \cap g(V)$ and an acyclic carrier function B from the $(n+1)$-skeleton of Γ_W to the $(n+1)$-skeleton of $\Sigma_{V'}$ and Σ_V respecting the inclusion $\Sigma_{V'} \to \Sigma_V$. with $B\sigma \subset A_V \sigma$ and $B\sigma \subset A_{V'}\sigma$. This shows that the induced homomorphisms satisfy the following relations

$$\begin{aligned} B_{V'}^* &= \rho_{W g(V')} A_{V'}^* = \rho_{W g(V')} \psi_{V'} \\ B_V^* &= \rho_{W g(V)} A_V^* = \rho_{W g(V)} \psi_V \\ B_V^* &= B_{V'}^* \rho_{V'V} \end{aligned}$$

Consequently the defining diagram commutes. In order to define W, B take a sequence of contractible neighborhoods of the identity $W_1 \subset ... \subset W_{n+1}$ such that

$$\begin{aligned} W_{j-1}^2 &\subset W_j \text{ for } 1 \leq j \leq n+1 \\ W_j &\subset V_j \cap V_j' \end{aligned}$$

and take $W \in \mathfrak{B}$ with $W \subset V_0 \cap V_0'$, define for a j-dimensional W-simplex $(x_0, ..., x_j)$ the carrier function $B((x_0, ..., x_j))$ to be the singular complex associated to $x_0 W_j$, which satisfies the desired assertions.

Now we take a look at contiguity: Let σ be a singular $g(V)$-simplex of dimension $j \leq n+1$. So $\sigma(x) \in \sigma(d^0) V_0 \subset \sigma(d^0) V_j$, consequently $\sigma \in A_V(\phi_{g(V)} \sigma)$, the carrier $A_V \phi_{g(V)}$ carries the inclusion map $\Sigma_{g(V)} \to \Sigma_V$ and $\rho_{V g(V)} = \phi_{g(V)} \psi_V$. So $(f, \phi)(g, \psi)$ is contiguous to the identity. The carrier map A carries in particular the identity in dimension 0, the identity in dimension 0 may be extended to a simplicial map $u : \Gamma_{g(V)} \to \Sigma_V$ carried by A. The composition

$$\Gamma_{g(V)} \xrightarrow{u} \Sigma_V \xrightarrow{\phi_V} \Gamma_V$$

coincides with the inclusion, so the map $\psi_V \phi_V = \rho_{g(V)V}$, which means that $(g, \psi)(f, \phi)$ is contiguous to the identity. So the desired equivalence is proved and the two cohomologies are isomorphic. $\square$

The propositions and theorems lead directly to a solution of the abstract problem posed at the beginning of the chapter knowing something about the cohomologies, this shall be formulated in the following main theorem providing a sufficient condition for enlargibility:

3.12. Theorem. *Let $\mathfrak{L} : 1 \to N \to U \xrightarrow{\phi} V \to 1$ be a local extension of topological groups over a symmetric neighborhood V of e in a locally contractible connected topological group G. Suppose that the weak homomorphism $i_N \phi^{-1} : V \to A(N)/I(N)$ is enlargible to a homomorphism $\theta : G \to A(N)/I(N)$.*

If $H_{top}^i(G, C(N)) = 0$ for $i = 1, 2$, then $\mathfrak{L}$ is enlargible from a symmetric neighborhood $W \subset V$ of e in G in the topological category.

Proof. (see [**vE62**], theorem 7.1. for details) Take $\mathfrak{B}$ the collection of symmetric neighborhoods of e in G. For $W \in \mathfrak{B}$ $G = W^\infty$, so Γ_W is connected. $V \in \mathfrak{B}$ by definition. $[H^n(\Gamma_V, C), \rho, \mathfrak{B}]$ is an almost injective system and $H^i(\Gamma_{\mathfrak{B}}, C) = H_{top}^i(G, C(N)) = 0$ for $i = 1, 2$, so $\ker \rho_{\mathfrak{B}V} = H^i(\Gamma_V, C)$ for $V \in \mathfrak{B}$. By almost injectivity there is $W \in \mathfrak{B}$ such that $\ker \rho_{WV} = H^i(\Gamma_V, C)$. Applying our spectral proposition we obtain that $\rho_{\mathfrak{B}G} : H^3(G, C(N)) \to H^3(\mathfrak{B}, C(N))$ is a monomorphism. The associated 3-cohomology class $[f_3]$ has a representant vanishing on V, so $\rho_{\mathfrak{B}G}([f_3]) = 0$ and consequently

$[f_3] = 0$ in the group cohomology, which means that there is an extension of groups Σ having G-kernel (N, θ). Furthermore we have $\rho_{\mathfrak{B}G} : H^2(G, C(N)) \to H^2(\mathfrak{B}, C(N))$ is an isomorphism. There is $\gamma \in H^2(G, C(N))$ with $\rho_{\mathfrak{B}G}\gamma = \rho_{\mathfrak{B}V} d(\mathfrak{L}, \Sigma)$, so there is $W \in \mathfrak{B}$ with $\rho_{WG}\gamma = \rho_{WV} d(\mathfrak{L}, \Sigma)$, consequently we can find a cocycle $f_2 \in d(\mathfrak{L}, \Sigma)$ being the restriction to W of a cocycle defined over G. So the local extension is enlargible over W. This algebraic extension is a topological one because the topology on U defines a unique topology on H such that all desired properties are satisfied. □

In the sequel we try to apply the above results in the setting of a locally contractible connected and simply connected group G and an abelian kernel N. Remark that in this case there is always an extension to the given G-kernel, because the 3-cohomology class vanishes. We shall construct the ordinary and equivariant Vietoris cohomology class of $\mathfrak{L}$ and condense a necessary and sufficient condition for enlargibility of the given extension. These results will be applied in the third section to our Lie group problem. Let $\mathfrak{B}$ denote the family of open local subgroups, we define the relative complexes of equivariant and ordinary cochains as kernels

$$E(\Gamma_G \operatorname{mod} \Gamma_W, C) := \ker(E(\Gamma_G, C) \overset{\rho_{WG}}{\to} E(\Gamma_W, C))$$
$$F(\Gamma_G \operatorname{mod} \Gamma_W, C) := \ker(F(\Gamma_G, C) \overset{\rho_{WG}}{\to} F(\Gamma_W, C))$$

of the surjective restrictions for any abelian group C. For the relative complexes we obtain restriction mappings, too, so we can form the direct limits and the relative cohomologies. Furthermore there are two long exact cohomology sequences. By η we shall always denote the mapping from equivariant to ordinary complexes or cohomologies.

3.13. Lemma. *Let G be a connected locally contractible topological group and C a G-module, $H^i_{top}(G, C) = 0$ for $i = 1, ..., n-1$ for a given natural number $n \in \mathbb{N}$, then $H^{n+1}_{eq}(\Gamma_G \operatorname{mod} \Gamma_{\mathfrak{B}}, C) = H^0(G, H^n_{top}(G, C))$.*

Proof. (see [**vE62**], p. 405-408 for details) The proof is given by a spectral sequence argument as above. First we observe that there is a long exact cohomology sequence

$$0 \to H^0(\Gamma_G \operatorname{mod} \Gamma_{\mathfrak{B}}, C) \to H^0(\Gamma_G, C) \to H^0(\Gamma_{\mathfrak{B}}, C) \overset{\delta}{\to}$$
$$\to H^1(\Gamma_G \operatorname{mod} \Gamma_{\mathfrak{B}}, C) \to H^1(\Gamma_G, C) \to ...$$

arising from the exact sequence of ordinary cochains. We obtain immediately

$$H^0(\Gamma_G \operatorname{mod} \Gamma_{\mathfrak{B}}, C) = H^1(\Gamma_G \operatorname{mod} \Gamma_{\mathfrak{B}}, C) = 0$$
$$H^{i+1}(\Gamma_G \operatorname{mod} \Gamma_{\mathfrak{B}}, C) = H^i(\Gamma_{\mathfrak{B}}, C)$$
$$H^{i+1}(\Gamma_G \operatorname{mod} \Gamma_W, C) = H^i(\Gamma_W, C)$$

for $W \in \mathfrak{B}$ and $i \geq 1$. The isomorphisms respect the given G-module-structures. Now we look at the following double complex

$$E(\Gamma_G \times (\Gamma_G \operatorname{mod} \Gamma_W), C) := \ker(E(\Gamma_G \times \Gamma_G, C) \overset{\rho_{WG}}{\to} E(\Gamma_G \times \Gamma_W, C))$$

of equivariant cochains. The inclusion

$$E(\Gamma_G \operatorname{mod} \Gamma_W, C) \to E(\Gamma_G \times (\Gamma_G \operatorname{mod} \Gamma_W), C)$$

is canonically given and produces an isomorphism on cohomology level by the second spectral sequence. The first spectral sequence gives

$$E_1^{p,q}(\Gamma_G \times (\Gamma_G \operatorname{mod} \Gamma_W), C) = E^p(\Gamma_G, H^q(\Gamma_G \operatorname{mod} \Gamma_W, C)),$$

consequently we arrive at

$$E_1^{p,0}(\Gamma_G \times (\Gamma_G \operatorname{mod} \Gamma_W), C) = E_1^{p,1}(\Gamma_G \times (\Gamma_G \operatorname{mod} \Gamma_W), C) = 0$$
$$E_1^{p,q}(\Gamma_G \times (\Gamma_G \operatorname{mod} \Gamma_W), C) = E^p(\Gamma_G, H^{q-1}(\Gamma_W, C))$$

for $q \geq 2$ by the above relations. Putting all together we see that the direct limit of the spectral sequence converges and that the i-cohomologies of the double complex are reproduced up to $n+1$, so we obtain

$$H^{n+1}_{eq}(\Gamma_G \operatorname{mod} \Gamma_{\mathfrak{B}}, C) = H^0(G, H^n(\Gamma_{\mathfrak{B}}, C))$$

which proves the desired result applying almost injectivity of the direct limit on the left. □

The following main theorem provides a necessary and sufficient condition for a central extension to be enlargible and the justification of the notion of Vietoris classes : Given a local extension with kernel N, then a 2-cohomology class is given in $H^2_{eq}(\Gamma_V, N)$ by the local extension. The image in $H^2_{eq}(\Gamma_{\mathfrak{B}}, N)$ under the restriction is called the equivariant Vietoris class of $\mathfrak{L}$ and designed by $\gamma_{eq}(\mathfrak{L})$, the image of the equivariant Vietoris class under η in $H^2(\Gamma_{\mathfrak{B}}, C(N))$ is called the ordinary Vietoris class and denoted by $\gamma(\mathfrak{L})$. Passing to the extension $\mathfrak{L}'$ lying over $W \subset V$, $W \in \mathfrak{B}$ we obtain $\gamma_{eq}(\mathfrak{L}) = \gamma_{eq}(\mathfrak{L}')$. In this case the third cohomology class vanishes anyway, so there is an extension associated to the G-kernel, but does it extend the local extension?

3.14. Theorem. *Let G be a connected locally contractible and simply connected topological group and $\mathfrak{L}$ a central extension of local groups with kernel N. Then $\mathfrak{L}$ is enlargible over $\mathfrak{B}$ if and only if the ordinary Vietoris class $\gamma(\mathfrak{L})$ vanishes.*

Proof. (see [**vEK64**], p.17) In the proof we are going to apply all the results of this section: The following commutative diagram has exact rows:

$$\begin{array}{ccccccc}
H^2_{eq}(\Gamma_G, N) & \longrightarrow & H^2_{eq}(\Gamma_{\mathfrak{B}}, N) & \xrightarrow[\delta]{} & H^3_{eq}(\Gamma_G \mod \Gamma_{\mathfrak{B}}, N) & \longrightarrow & H^3_{eq}(\Gamma_G, N) \\
\downarrow \eta & & \downarrow \eta & & \downarrow \eta & & \downarrow \eta \\
H^2(\Gamma_G, N) & \longrightarrow & H^2(\Gamma_{\mathfrak{B}}, N) & \xrightarrow[\delta]{} & H^3(\Gamma_G \mod \Gamma_{\mathfrak{B}}, N) & \longrightarrow & H^3(\Gamma_G, N)
\end{array}$$

Since the ordinary cohomology $H(\Gamma_G, N)$ vanishes we obtain that

$$H^2(\Gamma_{\mathfrak{B}}, N) \xrightarrow{\delta} H^3(\Gamma_G \operatorname{mod} \Gamma_{\mathfrak{B}}, N)$$

is an isomorphism, furthermore G is simply connected, so

$$H^3_{eq}(\Gamma_G \operatorname{mod} \Gamma_{\mathfrak{B}}, N) \xrightarrow{\eta} H^3(\Gamma_G \operatorname{mod} \Gamma_{\mathfrak{B}}, N) \xrightarrow{\delta^{-1}} H^2(\Gamma_{\mathfrak{B}}, N)$$

is an isomorphism, since the invariants are mapped isomorphically to the G-module $H^3_{eq}(\Gamma_G \operatorname{mod} \Gamma_{\mathfrak{B}}, N)$, remark that the group action on N is trivial. $\mathfrak{L}$ is enlargible over $\mathfrak{B}$ means in cohomological terms that $\gamma_{eq}(\mathfrak{L}) \in \rho_{\mathfrak{B}G} H^2_{eq}(\Gamma_G, N)$ or by exactness $\delta\gamma_{eq}(\mathfrak{L}) = 0$, but with the preceding remark $\delta^{-1}\eta\delta\gamma_{eq}(\mathfrak{L}) = 0$. Consequently we arrive by commutativity at $\eta\gamma_{eq}(\mathfrak{L}) = 0$, which is the assertion. ☐

The theorem can be generalized to local extensions with kernel N, where the map $\theta : G \to A(N)$ exists and the 3-cohomology class vanishes by assumption. The notion of ordinary and equivariant Vietoris classes makes sense and is justified by the theorem.

3.15. Theorem. *Let G be a connected locally contractible and simply connected topological group and $\mathfrak{L}$ a arbitrary extension of local groups with kernel N. If the associated G-kernel (N, θ) exists and $[(N, \theta)] = 0$, then $\mathfrak{L}$ is enlargible over $\mathfrak{B}$ if and only if the ordinary Vietoris class $\gamma(\mathfrak{L})$ vanishes.*

We are going to work with this condition to solve our original problem of enlarging the local group associated to a given Lie algebra. Given two extensions $\mathfrak{L}$ and $\mathfrak{L}'$ a map $\alpha : U \to U'$ is called a homomorphism $\alpha : \mathfrak{L} \to \mathfrak{L}'$ if

$$\begin{array}{ccc}
U & \xrightarrow[\Phi]{} & V \\
\downarrow \alpha & & \| \\
U' & \xrightarrow[\Phi']{} & V
\end{array}$$

We conclude immediately $\alpha(N) \subset N'$ and $\ker(\alpha) \subset N$. Given a normal subgroup K of N we obtain a natural homomorphism $\pi : \mathfrak{L} \to \mathfrak{L}'(= 1 \to N/K \to U/K \to V \to 1)$. Now given two central extensions and a homomorphism $\alpha : \mathfrak{L} \to \mathfrak{L}'$ we obtain a homomorphism $\alpha^* : H(\Gamma_{\mathfrak{B}}, N) \to H(\Gamma_{\mathfrak{B}}, N')$ and we obtain $\alpha^*\gamma(\mathfrak{L}) = \gamma(\mathfrak{L}')$. The cohomology is isomorphic to the singular cohomology with values in N, so we can evaluate on the singular homology with integer values. The image of the evaluation of $\gamma(\mathfrak{L})$ is denoted by $\operatorname{Per}(\mathfrak{L})$. We can assert $\alpha \operatorname{Per}(\mathfrak{L}) = \operatorname{Per}(\mathfrak{L}')$ and $\operatorname{Per}(\mathfrak{L}) = 0$ if and only if $\gamma(\mathfrak{L}) = 0$ by the universal coefficient theorem for simply connected G.

3.16. Proposition. *Let $\alpha : \mathfrak{L} \to \mathfrak{L}'$ be a homomorphism of central extensions, then $\mathfrak{L}'$ is enlargible from $\mathfrak{B}$ if and only if* $\mathrm{Per}(\mathfrak{L}) \subset \ker \alpha$.

Even for generic extensions we can also get a result by this proposition: Let $\alpha : \mathfrak{L} \to \mathfrak{L}'$ be a homomorphism from a central to an arbitrary extension, then we can factorize $\alpha = \alpha_1 \circ \alpha_0$ over the factor extension $\mathfrak{L}'' = 1 \to N/\ker \alpha \to U/\ker \alpha \to V \to 1$. α_1 is injective and α_0 is surjective. If $\mathfrak{L}'$ is enlargible, then the middle extension $\mathfrak{L}''$, too, consequently $\mathrm{Per}(\mathfrak{L}) \subset \ker \alpha$.

In the sequel we analyze the question of central extensions in a more detailed way ([**vEK64**]).

3.17. Definition. *Let $1 \to N \to U \to V \to 1$ be a central extension of local topological groups, X a local topological group. X is said to be enlargible if there is a topological group H and X is embedded in H with $X^\infty = H$. A topological group H is said to be monodrome relative to X if $X^\infty = H$ and every weak homomorphism $\phi : X \to G$, G an arbitrary topological group, is the restriction of a homomorphism from H to G. Let X be a local subgroup of U and $\phi(X) = Y$, then Y is a local subgroup of V and the restriction $X \xrightarrow{\phi} Y$ is a weak homomorphism. A map $\psi : Y \to X$ is said to be a cross-section of ϕ if $\phi\psi = id$, $\psi(y)\psi(y') \in X$ if $yy' \in Y$ and $\psi(y)^{-1} = \psi(y)^{-1}$, $\psi(e) = e$. Finally a local subgroup $X \subset U$ is called rectangular if $\phi(X) = Y$ is open, there is a cross-section $\psi : Y \to X$ and a local subgroup $Z \subset N$ such that $X = \psi(Y)Z$ and N is monodrome relative to Z.*

3.18. Remark. Let X be an enlargible local topological group, then we can find a local subgroup X' of X which can be embedded in the universal covering H' of $X^\infty = H$. Then H' is monodrome relative to X'. For enlargibility in general it is sufficient to find an abstract group G and an injective mapping $X \to G$, since then one can give a topology to the subgroup H of G generated by X such that the injection is an embedding.

The following proposition guarantees that the periods of the extension satisfy a type of discreteness-property, which will be useful in the setting of Lie-Algebras, where the Champbell-Baker-Hausdorff-Formula converges:

3.19. Proposition. *Let $\mathfrak{L} : 1 \to N \to U \to V \to 1$ be a central extension of local topological groups, X a rectangular local subgroup of U. If X is enlargible, then $Z \cap \mathrm{Per}(\mathfrak{L}) = \{e\}$.*

Proof. (see [**vEK64**], p.19) Since $\phi(X) = Y$ is open in U and since $\mathrm{Per}(\mathfrak{L})$ does not change by cutting down the extension to an open local subgroup of V we may assume that $\phi(X) = V$. We assume that X lies in a group H monodrome relative X (again by cutting down), consequently the weak homomorphism $\phi|_X : X \to V \subset G$ enlarges to a unique homomorphism $\widetilde{\phi} : H \to G$. We define $N' = \ker(\widetilde{\phi})$, $U' = \widetilde{\phi}^{-1}(V)$, $\widetilde{\phi}|_{U'} = \phi'$. X is a local subgroup of U' and $\phi'|_X = \phi$. $Z := X \cap N$ is a local subgroup of N' and $\mathfrak{L}' : 1 \to N' \to U' \to V \to 1$ is an extension of local topological groups being enlargible by construction. We have obtained the following commutative diagram.

$$\begin{tikzcd}
\mathfrak{L} : 1 \arrow[r] & N \arrow[r, hook] & U \arrow[r] & V \arrow[r] & 1 \\
& Z \arrow[u, hook] \arrow[r, hook] \arrow[d, hook] & X \arrow[u, hook] \arrow[r] \arrow[d, hook] & V \arrow[u, equal] \arrow[d, equal] & \\
\mathfrak{L}' : 1 \arrow[r] & N' \arrow[r, hook] & U' \arrow[r] & V' \arrow[r] & 1
\end{tikzcd}$$

Assume now that there is $\alpha : \mathfrak{L} \to \mathfrak{L}'$ extending the inclusion of X, then $\mathrm{Per}(\mathfrak{L}) \subset \ker(\alpha) \subset N$. $Z = X \cap N$ is mapped injectively to U', consequently $Z \cap \mathrm{Per}(\mathfrak{L}) = \{e\}$. The construction of α is done in the following manner: Let $\psi : V \to X$ denote the given cross-section, then for all $v_1, v_2, v_1v_2 \in V$ we obtain $\psi(v_1)\psi(v_2) = z(v_1, v_2)\psi(v_1v_2)$ with $z(v_1, v_2) \in Z$ since $X \cap N = Z$ and $X = \psi(V)Z$. $u \in U$ and $u' \in U'$ can be written uniquely in the form $u = n(u)\psi(v)$ with $v = \phi(u)$. We arrive immediately at

$$n(u_1u_2) = n(u_1)n(u_2)z(\phi(u_1), \phi(u_2))$$

Now N is monodrome relative Z, so there is a map $\alpha : N \to N'$ extending the inclusion of Z. N and $\psi(V)$ commute, so $\alpha(N)$ and $\psi(V)$ do so. We define for $u \in U$ the value $\alpha(u) = \alpha(n(u))\psi(\phi(u))$, which is well-defined and extends the inclusion of X. Furthermore $\phi'(\alpha(u)) = \phi'(\alpha(n(u))\psi(v)) =$

$\phi(\psi(v)) = \phi(u)$, which means that the diagram in mind is commutative. Another consequence is the $\alpha(u_1)\alpha(u_2)$ is defined if and only if u_1u_2 is defined. The homomorphism property follows now from

$$\begin{aligned}\alpha(u_1)\alpha(u_2) &= \alpha(n(u_1)\psi(v_1))\alpha(n(u_2)\psi(v_2)) = \alpha(n(u_1))\psi(u_1)\alpha(n(u_2))\psi(u_2)\\ &= \alpha(n(u_1))\alpha(n(u_2))\psi(u_1)\psi(u_2) = \alpha(n(u_1u_2))\psi(u_1u_2) = \alpha(u_1u_2)\end{aligned}$$

which proves the assertion completely. □

4. Enlargibility of Lie Algebras

The third section is dedicated to the analysis of the Lie-theoretic problem ([**vEK64**]), but not only the Banach space case is treated. From the first section we know that the Campbell-Baker-Hausdorff-Series converges on an open neighborhood of zero in a complete AE-Algebra, from the second section we are provided with tools to investigate enlargibility of local groups. The concept of CBH-Lie-Algebras is introduced to demonstrate the power of the applied methods, which work in fact for all regular Lie groups with locally diffeomorphic exponential map, the main example for CBH-Lie-Algebras are AE-Algebras.

4.1. Definition. *Let A be a locally convex Lie-Algebra, A is called a Campbell-Baker-Hausdorff-Algebra (*CBH-*Lie-Algebra) if A is complete and the* CBH-*series converges to a continuous smooth map on an open neighborhood of zero. The notion of canonical and analytic local subgroups can be taken from the first section.*

1. Let L be an CBH-Lie-Algebra, V an analytic local subgroup and N a central subalgebra of L, then $V + N$ is an analytic local subgroup, too
2. Let $\phi : M \to L$ be a continuous CBH-Lie-Algebra homomorphism, W a canonical local group in M, then $\phi(W)$ is a canonical local group, but the restriction of ϕ to W is only a weak homomorphism to $\phi(W) = V$. If $\phi^{-1}(V) = W$, then $\phi|_W : W \to V$ is an epimorphism. We only have to prove that if $\phi(x) * \phi(y)$ exists, then $x * y$ is defined in W, which is clearly true by assumption.
3. If $\phi : M \to L$ is a continuous monomorphism, then $\phi|_W : W \to V$ is an isomorphism in the category of abstract local groups.
4. Assuming in addition that L is enlargible, then there is some analytic local group V' being a local subgroup of an abstract group, with the above isomorphism we find a local subgroup of W lying in an abstract group, so M is enlargible, too.
5. The following situation is crucial for the assertions of the section: Let $\mathfrak{L} : 0 \to N \to M \to L \to 0$ be a central extension of CBH-Lie-Algebras, then for a sufficiently small analytical local group W the image $\phi(W) = V$ is an analytic local group, too. Furthermore replacing W by $U := W + N$ we obtain that there is an extension of analytic local groups ($\phi|_U$ is an epimorphism by 2.!) associated to the extension of Lie algebras.

4.2. Lemma. *Let $\mathfrak{L} : 0 \to N \to M \to L \to 0$ a central extension of* CBH-*Lie-Algebras and $1 \to N \to U \to V \to 1$ an associated extension of analytic local groups, then any open local subgroup W' of U contains a rectangular subgroup X with $X \cap N$ open relative N.*

PROOF. (see [**vEK64**], p. 22-23) We may suppose that W' is an open absolutely convex subset of U. There is $B \subset W'$ open and absolutely convex such that $B * B \subset \frac{1}{3}W'$. We define $Y = \phi(B)$, which is open since B is open and ϕ is an open map. For $y \in Y$ we choose $\psi(y) \in B$ such that $\psi(0) = 0$, $\psi(-y) = -\psi(y)$ and $\phi\psi(y) = y$. The difference $z(y_1, y_2) = \psi(y_1) * \psi(y_2) - \psi(y_1 * y_2)$ lies in the center N and $z(y_1, y_2) \subset \frac{2}{3}W'$ for $y_1, y_2, y_1 * y_2 \in Y$ (remark that Y is an analytic local group). $Z := \frac{2}{3}W' \cap N$ and $X := \psi(Y) + Z$, then $\psi(y_1) * \psi(y_2) \in X$ if $y_1, y_2, y_1 * y_2 \in Y$. Consequently X with its canonical local group structure is a rectangular subgroup of W', because $X \cap N = Z$. □

Given a central extension $\mathfrak{L} : 0 \to N \to M \to L \to 0$ of CBH-Lie-Algebras, we can associate a central local group extension $\mathfrak{L}' : 1 \to N \to U \to V \to 1$ of analytic local groups. If L is enlargible we assume that V is contained in an abstract group G. Without loss of generality we assume $V^\infty = G$ and G to be connected and simply connected. We can associate a period group $\mathrm{Per}(\mathfrak{L}')$ with this extension, which does not change by passing to open local subgroups of V. Consequently the period group is determined by the Lie algebra extension $\mathfrak{L}$. We shall denote it by $\mathrm{Per}(\mathfrak{L})$ in the sequel.

4.3. Theorem. *Let $\mathfrak{L} : 0 \to N \to M \to L \to 0$ a central extension of* CBH-*Lie-Algebras. Suppose that L is enlargible, then M is enlargible if and only if* $\mathrm{Per}(\mathfrak{L})$ *is a discrete additive subgroup of the commutative* CBH-*Lie Algebra N (this means complete locally convex vector space).*

Proof. (see [**vEK64**], p. 23) Suppose that the period group is discrete in N. We know (see theorem 3.15) that $\mathfrak{L}'' : 1 \to N/\mathrm{Per}(\mathfrak{L}) \to U/\mathrm{Per}(\mathfrak{L}) \to V \to 1$ is enlargible from an open local subgroup of V. Without any restriction we suppose that $\mathfrak{L}''$ is enlargible from V, consequently $U/\mathrm{Per}(\mathfrak{L})$ is contained in a group, but discreteness of the period group means that $U/\mathrm{Per}(\mathfrak{L})$ contains an analytic local subgroup of M, which is then automatically enlargible.

Suppose that M is enlargible, then in $\mathfrak{L}' : 1 \to N \to U \to V \to 1$ there is an analytic local subgroup $W \subset U$ being enlargible. W contains a rectangular local subgroup X with $X \cap N = Z$ open relative N. X is enlargible, so $Z \cap \mathrm{Per}(\mathfrak{L}) = \{e\}$, consequently $\mathrm{Per}(\mathfrak{L})$ is discrete. ☐

The procedure described above can be generalized to topological groups which admit a left-invariant metric, by the idea of the proof of the theorem of Kakutani we can prove that all local Fréchet-Lie-groups are of this type.

4.4. Lemma. *Let $1 \to N \to U \to V \to 1$ be a central extension of topological groups admitting a left-invariant metric, then any open subgroup W' contains a rectangular subgroup X.*

Proof. We may suppose that W' is an open ball with radius 3δ around e. There is $0 < \eta < \delta$ such that the open ball B with radius η around e satisfies $d(BB, e) < \delta$. We define $Y = \phi(B)$, which is open since ϕ is an open map. ψ is defined directly with the desired properties. $z(y_1, y_2) := \psi(y_1 y_2)^{-1}\psi(y_1)\psi(y_2)$ lies in the center N and $d(z(y_1, y_2), e) < 2\delta$ for $y_1, y_2, y_1 * y_2 \in Y$ (remark that Y is a local group). $Z := \{x \in U \mid d(x, e) < 2\delta\} \cap N$ and $X := \psi(Y)Z$, then $\psi(y_1) * \psi(y_2) \in X$ if $y_1, y_2, y_1 * y_2 \in Y$. Consequently X with its local group structure is a rectangular local subgroup of W', because $X \cap N = Z$. ☐

4.5. Theorem. *Let $\mathfrak{L} : 1 \to N \to U \to V \to 1$ be a central extension of local topological groups admitting a left-invariant metric. Suppose that V is enlargible, then U is enlargible if and only if* $\mathrm{Per}(\mathfrak{L})$ *is a discrete additive subgroup of the commutative left-invariantly metrizable topological group N.*

Proof. (see [**vEK64**], p. 23 for the idea of the proof) If U is enlargible we conclude as above that the additive subgroup of the periods of the extension is discrete. If $\mathrm{Per}(\mathfrak{L})$ is discrete we can form the factor extension having vanishing periods which means by theorem 3.15 that $U/\mathrm{Per}(\mathfrak{L})$ is enlargible. As above we conclude that there is a local subgroup of U lying isomorphically in $U/\mathrm{Per}(\mathfrak{L})$, consequently U is enlargible from an open local subgroup. ☐

The theorems can be applied to the following rather general situations, which are different in nature, but with analogue background:

1. Let M be an CBH-Lie-Algebra, then we can form a central extension with the total centre N: $0 \to N \to M \to M/N \to 0$, The adjoint representation $Ad : M/N \to Gl(M)$ is injectiveand and so we obtain that M/N is enlargible. Remark that M/N is a CBH-Lie-Algebra. The periods are completely determined by M and form an additive subgroup of N, the total centre. This leads to the following enlargibility criterion:

   M is enlargible if and only if $\mathrm{Per}(\mathfrak{L})$ *is discrete in the centre.*

2. Let M be a Fréchet-Lie-algebra and the Lie algebra of a local Fréchet-Lie-group U. Let $1 \to N \to U \to V \to 1$ be the "integration" of the central extension with the total centre $0 \to N \to M \to M/N \to 0$, i.e. the local Lie group extension has as derivation the extension of Lie algebras. The adjoint representation $Ad : M/N \to Gl(M)$ is injective, so V is enlargible. Even though there is no CBH-structure, the same assertion about the periods and the enlargibility of M holds. Enlargibility is a purely cohomological question.

Famous examples of non-enlargible Lie algebras can be found in [**vEK64**]. Naturally the problem is not solved with the above enlargibility criterion, because there is no easy way to calculate the periods of a Lie algebra in general. Another source of examples is [**dlH72**], where some linear non-enlargible Banach-Lie-Algebras are treated.

4.6. Example. We are going to treat the most famous example provided in [**vEK64**]. First we present a general construction how to obtain non-enlargible normed Banach Lie Algebras: Given a central extension of Banach Lie Algebras

$$\mathfrak{L} : 0 \to N \to M \to L \to 0$$

over enlargible L with non-trivial period group $\mathrm{Per}(\mathfrak{L})$, then either M is non-enlargible, if and only if $\mathrm{Per}(\mathfrak{L})$ is non-discrete or we proceed in the following way: Take the doubled extension

$$\mathfrak{L}' : 0 \to N \oplus N \to M \oplus M \to L \oplus L \to 0$$

and factor it by $N' := \{(x, \sqrt{2}x) | x \in N\}$. We obtain the factorized extension $\pi(\mathfrak{L}')$, where $\pi : N \oplus N \to N \oplus N/N'$ denotes the factor map. Then $\mathrm{Per}(\pi(\mathfrak{L}')) = \pi(\mathrm{Per}(\mathfrak{L}) \oplus \mathrm{Per}(\mathfrak{L}))$, which is not discrete in $N \oplus N/N'$by irrationality of $\sqrt{2}$.

We are now concerned with the simplest central extensions of Banach Lie Algebras

$$\mathfrak{L} : 0 \to \mathbb{R} \to M \xrightarrow{\phi} L \to 0$$

associated to an extension of local groups $0 \to \mathbb{R} \to U \to V \to 0$, where L is enlargible, $V \subset G$ with G a simply connected Banach Lie group. The Lie Algebra extension is determined by a bounded 2-cocycle ν on L given through $\nu(X,Y) = [\psi(X), \psi(Y)] - \psi([X,Y])$, where ψ is a continuous linear cross-section of ϕ. On Lie group level the equations read as follows: We obtain a real-valued inhomogeneous local smooth 2-cocycle $f(x,y) = \psi(x)\psi(y)\psi(xy)^{-1}$ for $x, y, xy \in V$. The associated equivariant cocycle $F(1, x_1, x_2) = \psi(x_1)\psi(x_1 x_2^{-1})\psi(x_2)^{-1}$ is smooth and by the Champbell-Baker-Hausdorff-Formula we arrive at

$$F(1, x_1, x_2) = \frac{1}{2}\nu(x_1, x_2) + ...$$

For the construction of some non-trivial period groups we need some differential geometric preparations: The singular cohomology with real coefficients of a compact finite dimensional manifold is given by de Rham's cohomology by de Rham's theorem. Remark that the ordinary cohomology $H^*(\Gamma_{\mathfrak{B}}, \mathbb{R})$ with respect to the system of open local subgroups of compact connected group G is the KAS-cohomology. We provide an isomorphisms between the (smooth) Kolmogorow-Alexander-Spanier-Cohomology and de Rham's cohomology on a compact finite dimensional manifold: Given a smooth function $F : M^{n+1} \to \mathbb{R}$, where M is a compact finite dimensional manifold, we denote by $\partial_i(X)F : M^{n+1} \to \mathbb{R}$ the Lie derivative with respect to the i-th variable for $0 \leq i \leq n$.

$$\tau F(X_1, ..., X_n) = (\sum \mathrm{sgn}(i_1, ..., i_n)\partial_1(X_{i_1})...\partial_n(X_{i_n})F) \circ \Delta$$

where Δ denotes the diagonal map. We obtain that τ is a cochain homomorphism from smooth germs along the diagonal in M^{n+1} with codifferential

$$\delta F = \sum_{i=0}^{n+1} (-1)^i F_i$$

to n-forms on M. The mapping F_i is defined by deleting the i-the variable and applying F

$$F_i(x_0, ..., x_{n+1}) = F(x_0, ..., \widehat{x_i}, ..., x_{n+1})$$

Given a smooth mapping from a compact finite dimensional manifold to the Banach Lie group $\mu : M \to G$. Given furthermore the 2-cocycle ν on L with values in an abelian group $\mathbb{R}$. If the 2-form $\nu \circ \delta^r \times \delta^r$ has some non-vanishing period on M, then the ordinary Vietoris cohomology class $\mathrm{Per}(\mathfrak{L}) \neq \{0\}$.

Since the pull-back μ^* is a cochain map with respect to the KAS-cohomology and having non-vanishing period of $\mu^* F$ on M means exactly that F has non-vanishing period on G. But on a compact manifold there is the above established correspondence between KAS-cohomology and de Rham's cohomology, so we arrive at the result.

Now we can proceed with the example: Let Q be the group of quaternions of unit length, the Lie algebra $\mathfrak{q}$ of Q can be idetified with the purely imaginary quaternions, the commutator is the bracket. On the C^1-loops $\Omega^1(S^1, \mathfrak{q}\ )$ we use the C^1-norm to obtain a complete Banach Lie algebra, the Lie algebra of the C^1-loop group $\Omega^1(S^1, Q)$ topologized with the topology of uniform convergence up to first degree of differentiation:

$$\nu(\lambda_1, \lambda_2) = \int_0^1 (\lambda_1'(t), \lambda_2'(t))dt$$

where $(.,.)$ denotes the canonical scalar product on $\mathfrak{q}$ is a real-valued bounded 2-cocycle and determines therefore a central extension

$$\mathfrak{L} : 0 \to \mathbb{R} \to M \to \Omega^1(S^1, \mathfrak{q}) \to 0$$

In order to show that M is not enlargible we construct a smooth map $\mu : S^2 \to \Omega^1(S^1, Q)$ such that $\mu^*\nu$ has non-vanishing period. Let S^2 be the unit sphere in $\mathfrak{q}$, then $\mu(x)(t) = \exp(2\pi x t) = \cos(2\pi t) + x\sin(2\pi t)$, where S^1 is parametrized by the interval $[0, 1]$. Calculating directly yields for $p \in S^2$ and q_1, q_2 with $(p, q_i) = 0$ $(i = 1, 2)$

$$(\mu^*\nu)_p(q_1, q_2) = \pi(p, q_1, q_2)$$

which has non-vanishing period. Consequently the local smooth equivariant group cocycle F cannot vanish on the singular homology with integer values.

APPENDIX A

Smooth bump functions on convenient vector spaces

Bump functions are the main ingredients for passing from local to global experiences in differential geometry, since they constitute partitions of unity. Bump functions do not always exist in infinite dimensions, even on Banach spaces. Their existence is encoded in the notion of smooth regularity. By the smooth Hausdorff property smooth functions separate points, but do in general not generate the c^∞-topology on a convenient locally convex vector space or smooth manifold. It is natural to look for smooth seminorms off 0 on a given locally convex space for this question. By factorization we reduce the question to the analysis of smooth norms of 0 on Banach spaces. The initial topology with respect to smooth functions depends - as the smooth functions themselves - only on the bornology of the locally convex space, thus we assume the spaces to be bornological. Norms are convex functions, so we collect the basic properties for convex real valued functions on locally convex spaces. We follow the lines and results of [**KM97**], ch. 3.

Lemma. *Let $f : E \to \mathbb{R}$ be a convex function on a convenient vector space, then the following assertions are equivalent:*

1. *f is Lip^0.*
2. *f is continuous for c^∞-topology.*
3. *f is bounded on Mackey-converging sequences.*

This basic lemma is proved in [**FK88**]. In the case of convex functions the one-sided directional derivative exists always and the derivative $q'(x)$ is sublinear and locally bounded (continuous) if q is locally bounded (continuous). If $q'(x)(v) = -q'(x)(v)$, then it is linear (one says q is Gâteaux-differentiable). Given a seminorm $p \neq 0$ on a convenient vector space E and $x \in E$ with $p(x) = 0$, then for $v \in E$ with $p(v) \neq 0$ we obtain $p(x + tv) = |t|p(v)$, consequently $p'(x)(\pm v) = p(v)$, so seminorms can only be differentiable outside their carrier. If a seminorm p is Gâteaux-differentiable outside its carrier, then p^2 is Gâteaux-differentiable everywhere. So we can pass without problems to the Banach space case to investigate differentiability:

Lemma. *Let E be a Banach space, then the following assertions are equivalent:*

1. *Every continuous convex function is Fréchet-differentiable on a dense subset of E.*
2. *Every locally Lipschitz function is Fréchet-differentiable at least one point.*
3. *Every equivalent norm is Fréchet-differentiable at least one point.*
4. *E has no equivalent rough norm.*
5. *Every closed separable subspace has a separable dual.*

A norm p on a Banach space is called rough , if for every $x \in E$ there is $\epsilon > 0$ such that for points x_1, x_2 arbitrary close to x and $u \in E$ with $||u|| = 1$

$$|p'(x_1)(u) - p'(x_2)(u)| \geq \epsilon$$

Spaces satisfying one of the above properties are called Asplund spaces , information about the difficult proof can be found in [**KM97**].

Example. The $2n$-norm on L^{2n} is smooth off 0. The 1-norm on $l^1(\Gamma)$ is Gâteaux-differentiable at those points where all coordinates are non-zero, it is rough if Γ is uncountable. The supremum norm on the continuous functions on a compact metric space is rough. If $dens(E) < dens(E')$, then there is an equivalent rough norm on E (the density is the infimum of the cardinalities of all dense subsets).

On Banach spaces the existence of a C^n-norm is equivalent to the property that the unit sphere is C^n-manifold ([**KM97**], p.141). Smoothness of a convex function can be characterized by the same idea on convenient spaces, too, whereas no implicit function theorem is valid. This is one of the very beautiful and instructive proofs of convenient analysis ([**KM97**], p.141).

Given a Hausdorff topological space X and a linear subalgebra $\mathcal{S}$ of $C(X, \mathbb{R})$ with the following properties: For all $h \in C^\infty(\mathbb{R}, \mathbb{R})$: $h_* f \in \mathcal{S}$ for $f \in \mathcal{S}$. For $f \in C(X, \mathbb{R})$ and for an open covering $\mathcal{U}$ and functions $f_U \in \mathcal{S}$ with $f|_U = f_U$ for $U \in \mathcal{U}$ we obtain $f \in \mathcal{S}$. Such a subalgebra distinguishes $\mathcal{S}$-functions on X. X is called $\mathcal{S}$-regular if for any $x \in X$ and $U \in \mathcal{U}(x)$ there is $f \in \mathcal{S}$ with $f(x) = 1$ and $carr(f) \subset U$. Such an f is called $\mathcal{S}$-bump function. A Hausdorff space is $\mathcal{S}$-regular if and only if its topology is initial with respect to $\mathcal{S}$ ([**KM97**], p.153).

LEMMA. *For a class $\mathcal{S}$ on a Banach space E the following assertions are equivalent:*

1. *E is not $\mathcal{S}$-regular .*
2. *For every $f \in \mathcal{S}$, every $0 < r_1 < r_2$ and $\epsilon > 0$ there exists an x with $r_1 < ||x|| < r_2$ and $|f(x) - f(0)| < \epsilon$.*
3. *For every $f \in \mathcal{S}$ with $f(0) = 0$ there exists an x with $1 \leq ||x|| \leq 2$ and $|f(x)| \leq ||x||$.*

C^1-regular Banach spaces do not admit a rough equivalent norm. Analogously to classical notions we can introduce $\mathcal{S}$-normal and $\mathcal{S}$-paracompact spaces. A $\mathcal{S}$-paracompact space is $\mathcal{S}$-normal and a $\mathcal{S}$-normal space admits $\mathcal{S}$-partitions of unity to any locally finite open cover. A paracompact and $\mathcal{S}$-normal space is consequently $\mathcal{S}$-paracompact ([**KM97**], pp.165-167), for example on metrizable spaces this is true. Every Hilbert space and $c_0(\Gamma)$ for arbitrary index set are C^∞-paracompact (see[**KM97**], pp.175). All nuclear Fréchet spaces are C^∞-paracompact. We call such spaces smoothly Hausdorff, smoothly regular,...

In the case of a separable Banach space we can collect the following equivalences (see [**KM97**], pp.173):

THEOREM. *Let E be a separable Banach space, then the following assertions are equivalent:*

1. *E has a C^1-norm.*
2. *E is C^1-regular.*
3. *E is C^1-normal.*
4. *E is C^1-paracompact.*
5. *E is Asplund, i.e. has no rough norm.*
6. *E' is separable.*

The non-separable case is in many of these respects an open and difficult to handle problem. However, it is remarkable that in the above mentioned cases important topological properties are detected by smooth functions.

APPENDIX B

Homological algebra and spectral sequences

We present homological algebra in the classical way oriented at the applications in the thesis. We follow the concise and condensed presentation in [**Mad88**], § 7. A cochain complex is a sequence of k-modules, where k is a commutative ring with unit, $C^* = \{C^n\}_{n\geq 0}$ with maps δ

$$... \xleftarrow{\delta} C^n \xleftarrow{\delta} ... \xleftarrow{\delta} C^2 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^0 \leftarrow 0$$

such that $\delta \circ \delta = 0$. By abuse of notation we omit the indices of maps usually. An augmented cochain complex is a cochain complex C^* and a homomorphism $\epsilon : M \to C^0$ with $\delta \circ \epsilon = 0$. The n-th cohomology of a cochain complex is the quotient module cycles $Z^n = \ker \delta$ by boundaries $B^n = \operatorname{Im} \delta$. A cochain homomorphism f^* is a sequence of homomorphisms $f^n : C^n \to D^n$ such that $\delta \circ f^{n-1} = f^n \circ \delta$. Cochain complexes constitute an abelian category without surprises about the structures. A cochain homomorphism induces a homomorphism $H^*(f^*)$ of cohomology modules by lifting. A homomorphism of augmented cochains is a clear concept, too. A homotopy equivalence between two cochain homomorphisms $f^*, g^* : C^* \to D^*$ is a sequence of homomorphisms $s : C^n \to D^{n-1}$ for $n \geq 1$ with $\delta \circ s - s \circ \delta = f - g : C^n \to D^n$. Two homotopic cochain homomorphisms induce the same homomorphism on cohomology. A short exact sequence of cochain complexes induces a long exact sequence of cohomology modules by the snake lemma

$$0 \to C^* \to D^* \to E^* \to 0$$

$$...H^{n-1}(E^*) \to H^n(C^*) \to H^n(D^*) \to H^n(E^*) \to H^{n+1}(C^*)...$$

A free cochain complex consists of free modules. The problem of calculating the cohomologies of a given cochain complex can be fixed by several methods, one the concept of spectral sequences . A filtration of a cochain complex C^* is a family of sub cochain complexes F_pC^*

$$F_{p-1}C^* \subset F_pC^* \subset F_{p+1}C^* \subset ... \subset C^*$$

for $p \in \mathbb{Z}$ with union C^*. There is an associated bigraded module $E_\infty^{p,q}C^* := F_pC^{p+q}/F_{p-1}C^{p+q}$ and a filtration on cohomology $F_pH^*(C^*) := \operatorname{Im}(H^*(F_pC^*) \to H^*(C^*))$. In fact spectral sequences calculate the bigraded module $E_\infty^{p,q}H^*(C^*)$ for all p, q as a "limit" , wherefrom the notion stems. A filtration is called canonically bounded if $F_pC^n = 0$ for $p \leq -1$ and $F_pC^n = C^n$ for $p \geq n$. Introducing cycles and boundaries we can set up the sequence:

$$Z_r^{p,q} := \{x \in F_pC^{p+q} \,|\, \delta x \in F_{p-r}C^{p+q+1}\} \text{ for } r \geq 0$$

$$B_r^{p,q} := \{\delta x \in F_pC^{p+q} \,|\, x \in Z_{r-1}^{p+r-1,q-r}\} \text{ for } r \geq 1$$

We define $E_r^{p,q} := Z_r^{p,q}/(B_r^{p,q} + Z_{r-1}^{p-1,q+1}) = (Z_r^{p,q} + F_{p-1}C^{p+q})/(B_r^{p,q} + F_{p-1}C^{p+q})$. If $x \in Z_r^{p,q}$, then $\delta x \in Z_r^{p-r,q+r+1}$ and $\delta Z_{r-1}^{p-1,q+1} \subset B_r^{p-r,q+r+1}$. Hence the cochain map δ induces a map $\delta_r : E_r^{p,q} \to E_r^{p-r,q+r+1}$ with the property $\delta_r \circ \delta_r = 0$ and $E_{r+1}^{p,q}$ is the cohomology of r-th element of the spectral sequence, which is a cochain complex for any pair (p, q).

$$... \to E_r^{p+r,q-r-1} \to E_r^{p,q} \to E_r^{p-r,q+r+1} \to ...$$

Let $[x] \in E_r^{p,q}$ be represented by $x \in Z_r^{p,q}$, then $\delta_r[x] = 0$ implies $\delta x \in F_{p-r-1}C^{p+q+1} + B_r^{p-r,q+r+1}$, so $\delta x = y_1 + b_1$. There exists $x_1 \in F_{p-1}C^{p+q}$ such that $\delta x_1 = b_1$ by the definition of $B_r^{p-r,q+r+1}$ and $[x - x_1] = [x]$, since $x_1 \in Z_{r-1}^{p-1,q+1}$. So we may assume that $b_1 = 0$. However, $\delta x \in F_{p-r-1}C^{p+q+1}$ implies $x \in Z_{r+1}^{p,q}$, so x represents an element of $E_{r+1}^{p,q}$. If $[z] \in E_r^{p+r,q-r-1}$, then $\delta z \in B_{r+1}^{p,q}$, so the mapping of the cohomology to $E_{r+1}^{p,q}$ is well-defined and injective. Furthermore it is onto by definition. The first element of the spectral sequence is given by

$$E_1^{p,q} = H^{p+q}(F_pC^*/F_{p-1}C^*)$$

For a canonically bounded filtration we obtain that $E_r^{p,q} = E_\infty^{p,q} H^*(C^*)$ for r sufficiently large, so the sequence "converges", since $Z_r^{p,q} = \ker \delta \cap F_p C^{p+q}$ and $B_r^{p,q} = \operatorname{Im} \delta \cap F_p C^{p+q}$ for r large enough, so

$$E_r^{p,q} = (\ker \delta \cap F_p C^{p+q}) / (\operatorname{Im} \delta \cap F_p C^{p+q} + \ker \delta \cap F_{p-1} C^{p+q})$$

which is equal to the desired quotient $E_\infty^{p,q} H^*(C^*)$. This is denoted by $E_r^{p,q} \Rightarrow H^{p+q}(C^*)$. Given two filtered cochain complexes and a filtration preserving cochain homomorphism $f^* : C^* \to D^*$, then there are associated homomorphisms $E_r^{p,q}(f^*) : E_r^{p,q}(C^*) \to E_r^{p,q}(D^*)$ of the r-th elements of the spectral sequence. If both sequences are canonically bounded and for some given $r \geq 1$ the associated homomorphism $E_r^{p,q}$ is an isomorphism for all pairs (p, q), then the cohomologies of C^* and D^* are isomorphic via $H^*(f^*)$, which is an easy application of the 5-lemma.

One beloved application of spectral sequences is the calculation of the cohomology of a double complex : A double complex is a bigraded module $\{X^{p,q}\}$ with homomorphisms $\delta' : X^{p-1,q} \to X^{p,q}$ and $\delta' : X^{p,q-1} \to X^{p,q}$ such that $X^{*,q}$ and $X^{p,*}$ are cochain complexes and $\delta' \circ \delta'' + \delta'' \circ \delta' = 0$. The associated total complex X^* is defined via $X^n := \oplus_{p+q=n} X^{p,q}$. There are two canonical filtrations of the total complex, namely

$$F'_p X^n := \oplus_{i \leq p} X^{i,n-i} \text{ (column filtration)}$$
$$F''_q X^n := \oplus_{i \leq q} X^{n-i,i} \text{ (row filtration)}$$

If the double complex is supported only at $p, q \geq 0$, then the filtrations are canonically bounded and the spectral sequences are convergent. The E_1-terms and the δ_1-differentials can be easily calculated:

$$'E_1^{p,q} = H^q(X^{p,*}, \delta'') \text{ (1. spectral sequence)}$$
$$''E_1^{p,q} = H^p(X^{*,q}, \delta') \text{ (2. spectral sequence)}$$

where the differentials are given by the "cohomology" homomorphisms associated to δ' and δ'', respectively.

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