A GENERAL PROOF OF THE DYBVIG-INGERSOLL-ROSS-THEOREM: LONG FORWARD RATES CAN NEVER FALL

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Abstract. A general proof of the Dybvig-Ingersoll-Ross Theorem on the monotonicity of long forward rates is presented. Some inconsistencies in the original proof of this theorem are discussed.

1. Introduction

It is an interesting question to analyse the stochastic nature of long term rates in interest rate markets. In Dybvig, Ingersoll, and Ross (1996) the authors show that long forward and zero coupon rates can never fall. In their proof they implicitly use an "ergodicity" assumption, which is economically reasonable, but does not hold in any arbitrage-free interest rate model (see Example 4.1). Furthermore there are some difficulties with a limiting procedure that are addressed in McCulloch (2000), but even under that "ergodicity" assumption the strategy in the proof of McCulloch (2000) is anticipative, so not admissible for a no-arbitrage argument. In this note we prove without any additional assumption that long forward rates can never fall, if they exist.

2. Interest Rate models

Suppose we are given a probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ where the time parameter is either discrete ($t \in \mathbb{N}$) or continuous ($t \in \mathbb{R}_{\geq 0}$). Prices of default-free zero coupon bonds $P(t,T)$ are modelled as semimartingales for $0 \leq t \leq T$ with respect to $(\mathcal{F}_t)_{0\leq t\leq T}$. The process $\{P(t,T)\}_{0\leq t\leq T}$ is strictly positive, furthermore we assume the normalization $P(T,T) = 1$. No arbitrage in this setting is usually given by the following requirement, which we shall assume throughout: there exists a probability measure Q and the $(\mathcal{F}_t)_{t\geq 0}$-adapted interest rate process $(R_t)_{t\geq 0}$ (the rates can be negative, too) such that the following conditions hold:

(1) B_t is a well-defined predictable, strictly positive process. In the discrete case it is defined by
$$B_t := \prod_{i=0}^{t-1}(1+R_i),$$
in the continuous case by
$$B_t := \exp\left(\int_0^t R_s ds\right).$$
Furthermore we assume that B_t/B_{t+h} are integrable with respect to Q for $t \geq 0$ and $h \geq 0$.

(2) The measure Q is locally equivalent to P, i.e. $Q|\mathcal{F}_t \sim P|\mathcal{F}_t$ for $t \geq 0$. Remark that we did not assume usual conditions for the filtration, see Delbaen (1993) for a related discussion.

(3) The discounted processes $B_t^{-1}P(t,T)$ are Q-martingales for $0 \leq t \leq T$.

The no arbitrage condition yields therefore
$$B_s^{-1}P(s,T) = E_Q(B_t^{-1}P(t,T)|\mathcal{F}_s)$$

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for $t \geq s$. From the given normalization and adaptedness of B_t the representation

$$P(t,T) = E_Q\left(\frac{B_t}{B_T}\middle|\mathcal{F}_t\right)$$

of the price processes follows.

2.1. **Discrete Case.** The forward rate process $f(t,T)$ is well-defined by the following formula

$$P(t,T) = (1 + f(t,T))P(t,T+1)$$

for $0 \leq t \leq T$, the zero coupon rate $z(t,T)$ is given by the formula

$$P(t,T) = \frac{1}{(1 + z(t,T))^{T-t}}$$

Both processes are $(\mathcal{F}_t)_{t\geq 0}$-adapted, however, their integrability properties depend on the price processes. We obtain furthermore the identification $R_t = f(t,t)$.

LEMMA 2.1. *Assume that the long forward rate exists as almost sure limit, i.e.*

$$\lim_{T\to\infty} f(t,T) = f_L(t)$$

then the long zero coupon rate

$$\lim_{T\to\infty} z(t,T) = z_L(t)$$

exists as almost sure limit and $z_L(t) = f_L(t)$.

Proof. The proof can be found in Dybvig, Ingersoll, and Ross (1996), too. One applies the formula

$$z(t,T) = \frac{1}{P(t,T)^{\frac{1}{T-t}}} - 1$$

where we insert

$$P(t,T) = \prod_{I=t}^{T-1} (1 + f(t,I))^{-1}$$

which yields the result. □

2.2. **Continuous Case.** The forward rate is $f(t,T)$ defined via the following formula

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$$

We assume that the forward rate exists as an adapted process. We obtain under regularity assumptions $f(t,t) = R_t$ for $t \geq 0$. The analogue to the zero coupon rate is given by the yield process

$$z(t,T) := \frac{1}{T-t}\int_t^T f(t,s)ds.$$

3. Long Forward Rates never fall

Considering a unifying approach to discrete and continuous time interest rate models we can write in the above notions

$$P(t,T)) = E_Q(B_T^{-1}B_t|\mathcal{F}_t)$$

for $t \leq T$. We assume that $z_L(t)$ is an almost surely finite random variable: the process $\{z_L(t)\}_{t\geq 0}$ is increasing (in the sense that $z_L(t) \geq z_L(s)$ a.s. for $t \geq s$) if and only if

$$x_L(t) = \lim_{T\to\infty} P(t,T)^{\frac{1}{T}}$$

is decreasing, since in the discrete case $z_L(t) = \frac{1}{x_L(t)} - 1$ and in the continuous case $z_L(t) = -\ln x_L(t)$. We denote by $x(t,T)$ the random variable $P(t,T)^{\frac{1}{T}}$. The almost sure existence of $z_L(t)$ is equivalent to the existence of $x_L(t)$ for all $t \geq 0$ by definition.

THEOREM 3.1. *If $x_L(t)$ and $x_L(s)$ exist almost surely for $t \geq s \geq 0$ then*

$$x_L(s) \geq x_L(t) \text{ a.s.}$$

For the proof of this theorem we apply the following technical lemma, which generalizes the well-known fact $\lim_{p\to\infty} \|X\|_p = \|X\|_\infty$ for $X \in L^\infty$.

LEMMA 3.1. *Let $\{X_n\}_{n\geq 0}$ be a sequence of non-negative random variables on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose X_n converges to the random variable X a.s. and $\liminf_{n\to\infty} E[X_n^n|\mathcal{G}]^{\frac{1}{n}} = C < \infty$ a.s. Then $X \leq C$ a.s.*

Proof. Replacing X by $X1_A$, where $A = \{C \leq k\} \in \mathcal{G}$, letting $k \to \infty$ allows us to replace C by a bounded $\mathcal{G}$-measurable random variable. Take $f \geq 0$, bounded, $E[f] = 1$. Using the conditional Fatou Lemma for a.s. convergence and the conditional Hölder Inequality we obtain

$$\begin{aligned} E[Xf] &= E[\lim X_n f] = E[E[\lim X_n f|\mathcal{G}]] \leq E[\liminf E[X_n f|\mathcal{G}]] \\ &\leq E\left[\liminf E[X_n^n|\mathcal{G}]^{\frac{1}{n}} E\left[f^{\frac{n}{n-1}}|\mathcal{G}\right]^{\frac{n-1}{n}}\right] \leq E[CE[f|\mathcal{G}]], \end{aligned}$$

since

$$\lim_{n\to\infty} E\left[f^{\frac{n}{n-1}}|\mathcal{G}\right]^{\frac{n-1}{n}} = E[f|\mathcal{G}]$$

by the conditional Lebesgue Dominated Convergence Theorem. Finally $E[CE[f|\mathcal{G}] = E[Cf]$, since C is $\mathcal{G}$-measurable, hence $X \leq C$, since f was arbitrary $\mathcal{F}$-measurable. □

Proof. Now we can prove Theorem 3.1. Therefore we fix $t \geq s$, we have to prove

$$\lim_{T\to\infty} P(s,T)^{\frac{1}{T}} \geq \lim_{T\to\infty} P(t,T)^{\frac{1}{T}}.$$

By conditioning we have

$$P(s,T) = E\left(\frac{B_s}{B_t} E\left(\frac{B_t}{B_T}\middle| \mathcal{F}_t\right)\middle| \mathcal{F}_s\right) = E\left(\frac{B_s}{B_t} P(t,T)\middle| \mathcal{F}_s\right).$$

We define $\widetilde{Q}$ by

$$\frac{d\widetilde{Q}}{dQ} = \frac{1}{P(s,t)} \frac{B_s}{B_t}.$$

This measure is the forward (time s) neutral measure for maturity t. We can write

$$\frac{P(s,T)}{P(s,t)} = \tilde{E}\left(P(t,T)|\,\mathcal{F}_s\right),$$

with $\tilde{E}$ denoting expectation with respect to $\tilde{Q}$. We have

$$x_L(s) = \lim_{T\to\infty} P(s,T)^{\frac{1}{T}} = \lim_{T\to\infty} \tilde{E}\left(x(t,T)^T\middle|\,\mathcal{F}_s\right)^{\frac{1}{T}}$$

and the question reduces to

$$\lim_{T\to\infty} \tilde{E}\left(x(t,T)^T\middle|\,\mathcal{F}_s\right)^{\frac{1}{T}} \geq \lim_{T\to\infty} x(t,T).$$

which is a consequence of Lemma 3.1. □

4. COMMENTS

In Dybvig, Ingersoll, and Ross (1996) the statement, that the long zero coupon rate in a discrete interest rate model can never fall, is proved by a no-arbitrage argument. The constructed strategy is non-anticipative, only if the zero coupon rates satisfy an additional assumption: The authors assume implicitly that $z_L(t)$ is somehow $\mathcal{F}_s$-measurable for $t > s$. This "ergodicity" assumption is economically reasonable, since the long rates should not depend on the time point t where we observe them. Nevertheless this does not hold in all interest rate models.

We provide the following well-known example (see Ingersoll, Skelton, and Weil (1978)) to show that there exist "non-ergodic" interest rate models.

EXAMPLE 4.1. *We take $r_t = r_0 + \delta N_t$ where N_t is a Poisson process with intensity λ, jump size 1 and $\delta > 0$ in its natural filtration. In this case*

$$z(t,T) = r_t + \lambda - \frac{\lambda}{\delta(T-t)}(1 - e^{-\delta(T-t)})$$

which yields

$$z_L(t) = r_t + \lambda.$$

This process is increasing, but the model is not "ergodic", since it generates the filtration.

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