On the Relation between Linearity-Generating Processes and Linear-Rational Models

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Outline

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Ingredients

- FPS $(\Omega, \mathcal{F}_t, \mathcal{F}, \mathbb{P})$
- State price density process

$$\zeta_t = \zeta_0 \mathrm{e}^{-\int_0^t r_s ds} \mathcal{E}_t(L)$$

$\rightarrow$ Risk-neutral measure $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{E}_t(L)$

- m-dimensional semimartingale X_t

Definition LG Process (Gabaix 2009)

(ζ_t, X_t) forms $(m+1)$-dimensional **linearity-generating (LG) process** if

$$\mathbb{E}_t\left[\frac{\zeta_T}{\zeta_t}\right] = \mathcal{A}(T-t) + \mathcal{B}(T-t)X_t$$
$$\mathbb{E}_t\left[\frac{\zeta_T}{\zeta_t}X_T\right] = \mathcal{C}(T-t) + \mathcal{D}(T-t)X_t$$

for some continuously differentiable functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$.

$\Rightarrow$ Linear T-claims in X_T have linear time-t prices in X_t

- E.g. zero-coupon bond price

$$P(t,T) = \mathcal{A}(T-t) + \mathcal{B}(T-t)X_t$$

Hidden Non-degeneracy Assumption

Support of X_{t^*} / $\zeta_{t^*} X_{t^*}$ / Z_{t^*} affinely spans $\mathbb{R}^m$ for some $t^* \geq 0$

Characterization Theorem

The following statements are equivalent:

1. (ζ_t, X_t) forms an LG process;
2. short rate r_t, $\mathbb{Q}$-drift $\mu_t^{X,\mathbb{Q}}$ of X_t are linear, quadratic in X_t,

$$\begin{aligned} r_t &= -A - BX_t \\ \mu_t^{X,\mathbb{Q}} &= C + (r_t + D)X_t = C + (D - A)X_t - (BX_t)X_t \end{aligned}$$

3. drift of $Y_t = (\zeta_t, \zeta_t X_t)$ is strictly linear in Y_t,

$$dY_t = KY_t \, dt + dM_t^Y$$

In either case,

$$K = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \begin{pmatrix} \mathcal{A}(\tau) & \mathcal{B}(\tau) \\ \mathcal{C}(\tau) & \mathcal{D}(\tau) \end{pmatrix} = \mathrm{e}^{K\tau}$$

Sketch of Proof

LG condition holds if and only if either

- The processes

$$M_t = \mathrm{e}^{-\int_0^t r_s ds}(\mathcal{A}(T-t) + \mathcal{B}(T-t)X_t)$$
$$N_t = \mathrm{e}^{-\int_0^t r_s ds}(\mathcal{C}(T-t) + \mathcal{D}(T-t)X_t)$$

  are $\mathbb{Q}$-martingales ($\rightarrow$ set drift zero)

- $Y_t = (\zeta_t, \zeta_t X_t)$ satisfies

$$\mathbb{E}_t[Y_T] = \mathrm{e}^{K(T-t)} Y_t$$

Remarks

- Part 3 is definition of LG process given in Gabaix (2009)
- Gabaix (2009) refers to $(BX_t)X_t$ in

  $$\mu_t^{X,\mathbb{Q}} = C + (r_t + D)X_t = C + (D - A)X_t - (BX_t)X_t$$

  as "linearity-generating twist of an AR(1) process"

Discussion

- Existence of LG processes (ζ_t, X_t)?
- Carr, Gabaix, Wu (2009) specify Y_t,

$$dY_t = KY_t\,dt + dM_t^Y,$$

and set $\zeta_t = Y_{1t}$ and $X_t = Y_{2..m+1,t}/Y_{1,t}$

- Problem: Y_t is not stationary: $Y_{1t} > 0$ and $\mathbb{E}[Y_{1t}] \to 0$
- $X_t = Y_{2..m+1,t}/Y_{1,t}$ is stationary, but ...
  - no functional relation between ζ_t and X_t (e.g. $\overline{\zeta}_t = N_t\zeta_t$)
  - nontrivial viability conditions for X_t in view of

$$0 < P(t,T) = \mathcal{A}(T-t) + \mathcal{B}(T-t)X_t \leq 1$$

  - quadratic $\mathbb{Q}$-drift and highly nonlinear $\mathbb{P}$-drift of X_t

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Definition (Filipović, Larsson, Trolle 2014)

An m-dimensional **linear-rational (LR) model** consists of an m-dimensional semimartingale Z_t with linear drift,

$$dZ_t = (b + \beta Z_t)\,dt + dM_t^Z,$$

and parameters α, $\phi \in \mathbb{R}$ and $\psi \in \mathbb{R}^m$ such that

$$\zeta_t = \mathrm{e}^{-\alpha t}\left(\phi + \psi^\top Z_t\right) > 0.$$

Linear-rational Term Structure

LR model implies linear-rational bond prices

$$P(t,T) = \mathbb{E}_t\left[\frac{\zeta_T}{\zeta_t}\right]$$
$$= \mathrm{e}^{-\alpha(T-t)} \frac{\phi + \psi^\top \mathrm{e}^{\beta(T-t)} \int_0^{T-t} \mathrm{e}^{-\beta s} b\, ds + \psi^\top \mathrm{e}^{\beta(T-t)} Z_t}{\phi + \psi^\top Z_t}$$

and short rate

$$r_t = -\partial_T \log P(t,T)|_{T=t} = \alpha - \frac{\psi^\top (b + \beta Z_t)}{\phi + \psi^\top Z_t}.$$

Representation as LG Process

- Define normalized factor

$$X_t = \frac{Z_t}{\phi + \psi^\top Z_t}$$

- Simple algebraic fact (if $\phi \neq 0$):

$$\frac{p + q^\top Z_t}{\phi + \psi^\top Z_t} = \frac{p}{\phi} + \left(q - \frac{p\psi}{\phi}\right)^\top X_t$$

$\Rightarrow$ Bond price and short rate become linear in X_t

Outline

Representation Theorem: m-dim LR as $(m+1)$-dim LG

An m-dimensional LR model

$$dZ_t = (b + \beta Z_t)\, dt + dM_t^Z, \quad \zeta_t = \mathrm{e}^{-\alpha t}\left(\phi + \psi^\top Z_t\right)$$

can be represented as $(m+1)$-dimensional LG process (ζ_t, X_t) through $X_t = \frac{Z_t}{\phi + \psi^\top Z_t}$ if and only if $b = C\phi$.

The respective $Y_t = (\zeta_t, \zeta_t X_t)$ in Characterization Theorem is

$$Y_t = \mathrm{e}^{-\alpha t}(\phi + \psi^\top Z_t, Z_t)$$

and the matrix K in $dY_t = KY_t\, dt + dM_t^Y$ is given by

$$\begin{aligned} A &= -\alpha + \psi^\top C, \quad B = \psi^\top(-C\psi^\top + \beta), \\ C &= \frac{b}{\phi}, \quad D = -\alpha \mathrm{Id} - C\psi^\top + \beta \end{aligned} \tag{*}$$

Representation Corollary 1: m-dim LR as $(m+2)$-dim LG

By increasing dimension can always assume $b = 0$:

$$\bar{Z}_t = \begin{pmatrix} Z_t \\ 1 \end{pmatrix}, \quad \bar{b} = 0, \quad \bar{\beta} = \begin{pmatrix} \beta & b \\ 0 & 0 \end{pmatrix}, \quad M_t^{\bar{Z}} = \begin{pmatrix} M_t^Z \\ 0 \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$$

is econ equivalent $(m+1)$-dim LR model with strictly linear drift

$$d\bar{Z}_t = \bar{\beta}\bar{Z}_t\,dt + dM_t^{\bar{Z}}, \quad \zeta_t = \mathrm{e}^{-\alpha t}\left(\phi + \bar{\psi}^\top \bar{Z}_t\right)$$

Corollary 3.1.

m-dim LR model can always be represented as $(m+2)$-dim LG process through

$$\bar{X}_t = \frac{(Z_t, 1)}{\phi + \psi^\top Z_t}.$$

The respective $\bar{Y}_t = (\zeta_t, \zeta_t \bar{X}_t) = \mathrm{e}^{-\alpha t}(\phi + \psi^\top Z_t, Z_t, 1)\ \ldots$

Representation Corollary 2

For given parameters A, B, C, D condition (*) holds if and only if

$$\begin{pmatrix} 1 & -\psi^\top \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = -\alpha \begin{pmatrix} 1 & -\psi^\top \end{pmatrix}$$

Corollary 3.2.

The functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ of an $(m+1)$-dimensional LG process can be obtained from an m-dimensional LR model if and only if the respective matrix K admits a left-eigenvector $v^\top$ with $v_1 \neq 0$.

Counterexample

For $B \neq 0$, $C = 0$, $D = A\,\mathrm{Id}$ there exists no such left-eigenvector.

$\Rightarrow$ not every $(m+1)$-dimensional LG process (ζ_t, X_t) can be represented as LR model of dimension m or lower.

Characterization Theorem $\Rightarrow$ $(m+1)$-dim LG process (ζ_t, X_t) can always be represented as $(m+1)$-dim LR model

$$Z_t \equiv Y_t = (\zeta_t, \zeta_t X_t), \quad \zeta_t = Z_{1,t}$$

Next step: characterize those $(m+1)$-dim LG processes that can be represented as m-dim LR model

Representation Theorem: $(m+1)$-dim LG as m-dim LR

Consider $(m+1)$-dim LG process (ζ_t, X_t) and let $Y_t = (\zeta_t, \zeta_t X_t)$.

The following statements are equivalent:

1. (ζ_t, X_t) can be represented as m-dim LR model
2. there exist parameters α, ϕ, ψ such that

$$\begin{pmatrix} 1 & -\psi^\top \end{pmatrix} Y_t = \phi \, \mathrm{e}^{-\alpha t}$$

3. there exist nonzero $v \in \mathbb{R}^{m+1}$ and function $f(t)$ such that

$$v^\top Y_t = f(t) \tag{**}$$

Note: (**) $\Rightarrow M_t^Y - M_0^Y \perp v$

Mean Reversion

Semimartingale S_t is **mean-reverting** to **mean-reversion level** θ if $\frac{1}{T-t}\int_t^T \mathbb{E}_t[S_u]\,du \to \theta$ as $T \to \infty$ almost surely for all $t \geq 0$.

Representation Theorem: $(m+1)$-dim LG as m-dim LR

Consider $(m+1)$-dim LG process (ζ_t, X_t) and let $Y_t = (\zeta_t, \zeta_t X_t)$.

The following statements are equivalent:

1. (ζ_t, X_t) can be represented as m-dim LR model Z_t and Z_t is mean-reverting to level $\theta \in \mathbb{R}^m$ satisfying $\phi + \psi^\top \theta > 0$;
2. $\mathrm{e}^{\alpha t} Y_t$ is mean-reverting to level $\widetilde{\theta} \in \mathbb{R}^{m+1}$ satisfying $\widetilde{\theta}_1 > 0$ for some α.

Mean-reversion levels are related by $\widetilde{\theta} = (\phi + \psi^\top \theta, \theta)$.

Outline

Linearity-Generating (LG) Processes

Linear-Rational (LR) Models

Relation between LG processes and LR models

State Price Density Decomposition

Markov Valuation

Hansen and Scheinkman (2009) "Long-term Risk: An Operator Approach", *Econometrica*

- Economy described by Markov state $\mathbf{X}_t$
- State price density forms positive multiplicative functional:

$$\frac{\zeta_T(\mathbf{X})}{\zeta_t(\mathbf{X})} = \frac{\zeta_{T-t}(\mathbf{X} \circ \theta_t)}{\zeta_0(\mathbf{X} \circ \theta_t)}$$

$\Rightarrow$ Pricing semigroup $\mathbb{S}_t$:

$$\mathbb{S}_t f(\mathbf{x}) = \mathbb{E}_{\mathbf{x}} \left[\frac{\zeta_t}{\zeta_0} f(\mathbf{X}_t) \right]$$

Multiplicative Decomposition Theorem

Let $\varphi(\mathbf{x})$ be positive eigenfunction of pricing semigroup $\mathbb{S}_t$ with eigenvalues $\mathrm{e}^{\rho t}$ then ζ_t admits the multiplicative decomposition

$$\zeta_t = \mathrm{e}^{\rho t} \frac{1}{\varphi(\mathbf{X}_t)} \hat{M}_t$$

where $\hat{M}_t$ is a positive martingale with $\hat{M}_0 = 1$.

If $\mathbf{X}_t$ is recurrent and stationary under $\mathbb{A}$ given by $\frac{d\mathbb{A}}{d\mathbb{P}}|_{\mathcal{F}_t} = \hat{M}_t$ then this decomposition is unique.

HS (2009) also provide conditions for existence of positive ef $\varphi(\mathbf{x})$

LR Models Revisited

An m-dimensional LR model

$$dZ_t = (b + \beta Z_t)\,dt + dM_t^Z, \quad \zeta_t = \mathrm{e}^{-\alpha t}\left(\phi + \psi^\top Z_t\right)$$

satisfies multiplicative decomposition for

$$\rho = -\alpha, \quad \varphi(\mathbf{x}) = \frac{1}{\phi + \psi^\top z}, \quad \hat{M}_t = 1$$

and can be (part of) recurrent and stationary Markov process!

LR Models Revisited cont'd

- $\mathbb{A}$ is long forward measure:

$$\frac{\zeta_t P(t,T)}{\zeta_0 P(0,T)} = \frac{\phi + \mathbb{E}_t[\psi^\top Z_T]}{\phi + \mathbb{E}[\psi^\top Z_T]} \to 1 \quad \text{as } T \to \infty$$

  Hence deflating by ζ_t/ζ_0 amounts to discounting by gross return on long-term bond $\lim_{T\to\infty} \frac{P(t,T)}{P(0,T)}$

  It also implies that the long-term bond is growth optimal under $\mathbb{A}$ (Qin, Linetsky 2015)

- Flexible market price of risk specification: free to modify

$$\zeta_t \rightsquigarrow \zeta_t \hat{M}_t$$

  for some auxiliary density process $\hat{M}_t$

Conclusion

- LG processes are related to LR models
- $\{m\text{-dim LR models}\} \subset \{(m+1(2))\text{-dim LG processes}\}$
- $\{(m+1)\text{-dim LG processes}\} \subset \{(m+1)\text{-dim LR models}\}$
- $(m+1)$-dim LG process $\in \{\text{mean-rev. } m\text{-dim LR models}\}$ if and only if mean-reverting after exponential scaling
- HS decomposition theorem favors mean-reverting LR model specification

LR models = "reasonable" specifications of LG processes

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- LG processes are related to LR models
- $\{m\text{-dim LR models}\} \subset \{(m+1(2))\text{-dim LG processes}\}$
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- $(m+1)$-dim LG process $\in$ $\{$mean-rev. m-dim LR models$\}$ if and only if mean-reverting after exponential scaling
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LR models = "reasonable" specifications of LG processes