On the value of being American

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The problem

To give the highest model-based price and the cheapest super-replicating strategy for an American claim, given the prices of European options.

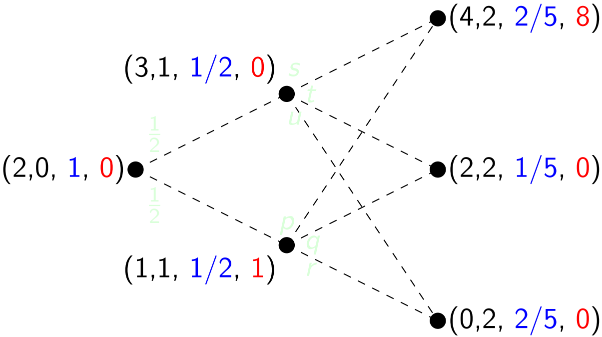

Figure: *The space of possible paths, and the payoff of the American claim*. The labels at the nodes on the graph consist of a quadruple, the elements of which are price level, time node probability and payoff of the American option respectively.

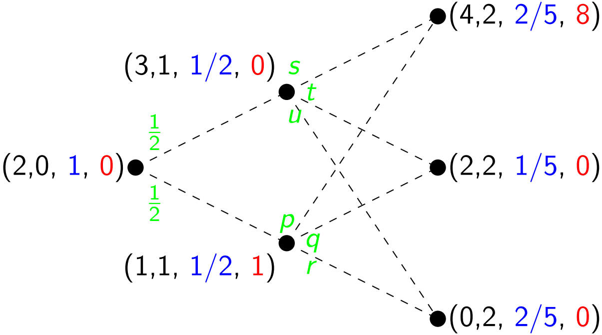

Figure: *The space of possible paths, and the payoff of the American claim*. The labels at the nodes on the graph consist of a quadruple, the elements of which are price level, time node probability and payoff of the American option respectively.

The model-based price

Denote by (p, q, r) the transition probabilities of going from $(1, 1)$ to $((4, 2), (2, 2), (0, 2))$ respectively, and by (s, t, u) the transition probabilities of going from $(3, 1)$ to $((4, 2), (2, 2), (0, 2))$.

We have $0 \leq p \leq 1/4$ and $(q, r) = (\frac{1-4p}{2}, \frac{1}{2} + p)$.

Similarly, $1/2 \leq s \leq 3/4$ and $(t, u) = (\frac{3-4s}{2}, s - \frac{1}{2})$.

We must have $p + s = 4/5$.

The value of immediate exercise at $(1, 1)$ is 1, and the value on continuation is $8p$, so that it is optimal to continue if $p \geq 1/8$.

It is always optimal to continue at $(3, 1)$ and the value is $8s$.

The expected payoff of the American option is then $\frac{1}{2}[8(p + s) + (1 - 8p)^+] = 16/5 + (\frac{1}{2} - 4p)^+$.

Take p as small as possible, ie $p = 1/20$ to give a best model based price of $7/2 = 35/10$.

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A super-replicating strategy

Starting with $18/5$, purchase 4 Arrow-Debreu securities paying 1 in state $(4, 2)$ at total cost $8/5$, leaving cash of 2.

Hold one unit of asset (forward) over the time-period $(0, 1]$.

If the American option is not exercised at $t = 1$ again hold a unit long (forward) position over $(1, 2]$; otherwise hold a null position in the stock over $(1, 2]$.

The super-replication property

At $t = 0$, the cash holdings are 2.

At $t = 1$ the cash holdings are 3 in state $(3, 1)$ and 1 in state $(1, 1)$. This is sufficient to cover the American option if it is exercised.

If the American option is not exercised at $t = 1$, and if $X_1 = 3$ then including the payoff from the Arrow-Debreu security, at $t = 2$ the strategy realises $(8, 2, 0)$ in the states $((4, 2), (2, 2), (0, 2))$ respectively.

If the option is not exercised at $t = 1$, and if $X_1 = 1$ then at $t = 2$ the strategy again realises $(8, 2, 0)$ in the states $((4, 2), (2, 2), (0, 2))$.

Hence, the given strategy is a super-replicating strategy.

There is a super-replicating strategy for $18/5 = 36/10 > 35/10$.

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The most expensive model

Consider a pair of models $\hat{M}$ and $\tilde{M}$.

Suppose $(\hat{p}, \hat{q}, \hat{r}, \hat{s}, \hat{t}, \hat{u}) = (1/4, 0, 3/4, 3/4, 0, 1/4)$ and $(\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}, \tilde{t}, \tilde{u}) = (0, 1/2, 1/2, 3/4, 0, 1/4)$.

Note that both of these models are martingale models; neither model satisfies the constraint $p + s = 4/5$.

However, the mixture $\frac{1}{5}\hat{M} + \frac{4}{5}\tilde{M}$ does match call prices.

Assume that the holder of the option learns whether the world is described by $\hat{M}$ or $\tilde{M}$ at $t = 1$ before he is required to decide whether to exercise the option. Call this model M.

We will show that the price of the American option is maximised over consistent models by the model M.

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Optimal exercise under the best model

Under $\hat{M}$ it is optimal to exercise the American option at $t = 2$, and the value of the option is 4.

Under $\tilde{M}$ it is optimal to exercise at $(1, 1)$ and the value of the American option is $7/2$.

Provided the model uncertainty is resolved by $t = 1$, the price under the mixed model is

$$\frac{1}{5} \times 4 + \frac{4}{5} \times \frac{7}{2} = \frac{18}{5}.$$

The message

‘The obvious approach consists in considering as admissible martingale measures, all probability measures in which the co-ordinate process is a martingale in its own filtration’

When valuing American claims this it is not sufficient.

The full value of American claims reflects the ability of the holder to choose an exercise time which depends on new information.

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Processes on a bounded lattice

Assume constant deterministic rates and dividends and no transaction costs. Work with discounted prices throughout.

Let X denote the price of the stock.

Set of traded securities includes European call options on stock.

In particular, it is possible to buy or sell a call on X with strike K and maturity t for a finite set of traded strikes and maturities.

We assume that time is discrete, and that the time parameter is restricted to lie in a set $\mathcal{T}_0 = \{t_0 = 0 < t_1 < \ldots < t_N = T\}$.

We assume for each maturity $t_n \in \mathcal{T}$ the set of traded strikes is $\mathcal{K}$ where

$$\mathcal{K} = \{x_1, x_2, \ldots x_J\}$$

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Since holding a call with strike zero is equivalent to holding the stock, and since the stock is traded, it is useful to consider 0 to be a traded strike.

(Note: the price of a zero-strike call must equal $X_0 = s_0$.)

Let $\mathcal{X} = \{0, x_1, x_2, \ldots x_J\}$.

We identify $\mathcal{X}$ with a set of levels for the price process X and build processes which live on the lattice $\mathcal{X} \times \mathcal{T}$.

We assume that the option can only be exercised at a date $\tau \in \mathcal{T} = \mathcal{T}_0 \setminus \{0\} = \{t_1, \ldots, t_N\}$.

The American claim is characterized by a function $a : \mathcal{X} \times \mathcal{T} \to \mathbb{R}^+$ which is typically decreasing in time and convex in x.

Assumption

1. *The set of call option prices has the following properties:*
   - *For* $1 \leq n \leq N$, $s_0 = c_{0,n} \geq c_{1,n} \geq c_{2,n} \geq \cdots \geq c_{J,n} \geq 0$.
   - *For* $1 \leq n \leq N$, $1 \geq \frac{c_{0,n}-c_{1,n}}{x_1} \geq \frac{c_{1,n}-c_{2,n}}{x_2-x_1} \geq \cdots \geq \frac{c_{J-1,n}-c_{J,n}}{x_J-x_{J-1}}$.
   - *For* $1 \leq n \leq N-1$, *and for* $0 \leq j \leq J$, $c_{j,n+1} \geq c_{j,n}$.
2. *In addition* $c_{J,N} = 0$.

Let $\mathbf{C}$ be the $(J+1) \times N$ matrix with elements $c_{j,n}$.

Define the $(J+1) \times N$ matrix $\mathbf{P}$ via its entries $p_{j,n}$

$$p_{j,n} = \begin{cases} 1 - \frac{s_0 - c_{1,n}}{x_1} & j = 0; \\ \frac{c_{j-1,n}-c_{j,n}}{x_j - x_{j-1}} - \frac{c_{j,n}-c_{j+1,n}}{x_{j+1}-x_j} & 1 \leq j < J; \\ \frac{c_{J-1,n}-c_{J,n}}{x_J - x_{J-1}} & j = J. \end{cases} \quad (1)$$

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Definition

$\mathcal{M}^{\mathcal{X},\mathcal{T}} = \mathcal{M}^{\mathcal{X},\mathcal{T}}(\mathbf{C})$ is the set of models (i.e. a filtration $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_{t_1}, \dots \mathcal{F}_{t_N})$ and a probability measure $\mathbb{P}$ supporting a stochastic process $X = (X_{t_n})_{0 \le n \le N}$ taking values in $\mathcal{X}$) such that $X_0 = s_0$, and

1. the process X is consistent with $\mathbf{C}$ in the sense that $\mathbb{E}[(X_{t_n} - x_j)^+] = c_{j,n}$ or equivalently $\mathbb{P}(X_{t_n} = x_j) = p_{j,n}$;
2. X is a $(\mathbb{P}, \mathbb{F})$-martingale.

M defines a model based price for the American option: $\phi(M) = \phi^a(M) = \sup_\tau \mathbb{E}^M[a(X_\tau, \tau)]$.

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Definition

A semi-static trading strategy $(\mathbf{B}, \Theta = (\Theta^1, \Theta^2))$ on $(\mathcal{X}, \mathcal{T})$ is a composition of

1. Arrow-Debreu style European options with payoff $(b_{j,n})$ if X is in state x_j at time t_n (for $1 \leq n \leq N$). The payoff/cost of such a strategy is

$$\mathcal{G}_T^{\mathbf{B}} = \sum_{1\leq n\leq N} \sum_{0\leq j\leq J} b_{j,n} I_{\{X_{t_n}=x_j\}} \qquad H_{\mathbf{C}}(\mathbf{B}) = \sum_{1\leq n\leq N} \sum_{0\leq j\leq J} b_{j,n} p_{j,n}.$$

2. A dynamic hedging position of Θ_{t_n} units held over $(t_n, t_{n+1}]$. Here $\Theta_{t_n} = \Theta^1(x_{t_1}, \ldots x_{t_n})$ if the option *has not* yet been exercised and $\Theta_{t_n} = \Theta^2(x_{t_1}, \ldots x_{t_n}, t_j)$ if the option *was* exercised at t_j with $j \leq n$. If exercise occurs at $\rho \in \mathcal{T}$ then the payoff along a price path $(s_0 = x_0, x_{t_1}, \ldots x_{t_N})$ is

$$\mathcal{G}_T^{\Theta} = \sum_{n=1}^{\mathcal{N}(\rho)-1} \Theta^1_{t_n}(x_{t_1}, \ldots, x_{t_n})(x_{t_{n+1}} - x_{t_n}) + \sum_{n=\mathcal{N}(\rho)}^{N-1} \Theta^2_{t_n}(x_{t_1}, \ldots, x_{t_n}, \rho)(x_{t_{n+1}} - x_{t_n}),$$

where $\mathcal{N}(\rho) = \min\{n : t_n \geq \rho\}$. The cost is zero.

The time-T payoff $\mathcal{G}_T = \mathcal{G}_T^{\mathbf{B},\Theta}$/total cost is

$$\mathcal{G}_T(x_{t_1}, \ldots x_{t_N}, \rho) = \mathcal{G}_T^{\mathbf{B}} + \mathcal{G}_T^{\Theta}, \qquad H_{\mathbf{C}}(\mathbf{B}, \Theta) = H(\mathbf{B}) = \sum_{1\leq n\leq N} \sum_{0\leq j\leq J} b_{j,n} p_{j,n}$$

Definition

A semi-static trading strategy $(\mathbf{B}, \Theta = (\Theta^1, \Theta^2))$ super-replicates the American claim if $\mathcal{G}_T(x_{t_1}, \ldots x_{t_N}, \rho) \geq a(x_\rho, \rho)$ for all $(x_{t_1}, \ldots x_{t_N})$ with $x_{t_n} \in \mathcal{X}$ and all ρ. Let $\mathcal{S} = \mathcal{S}^{\mathcal{X},\mathcal{T}}(a)$ be the set of super-replicating semi-static strategies.

Define the cost of the cheapest super-replicating semi-static strategy: $\mathcal{H}^{\mathcal{X},\mathcal{T}}(a, \mathbf{C}) = \inf_{(\mathbf{B},\mathbf{\Theta}) \in \mathcal{S}^{\mathcal{X},\mathcal{T}}(a)} H_{\mathbf{C}}(\mathbf{B}, \mathbf{\Theta})$.

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Theorem

$$
\begin{aligned}
&\Phi^{\mathcal{X},\mathcal{T}}(a,\mathbf{C}) \\
&\quad = \sup_{M\in\mathcal{M}_2^{\mathcal{X},\mathcal{T}}(\mathbf{C})} \mathbb{E}^M[a(X_{\tau_\Delta},\tau_\Delta)] = \sup_{M\in\mathcal{M}_2^{\mathcal{X},\mathcal{T}}(\mathbf{C})} \phi^a(M) \\
&\qquad = \mathcal{P}^{\mathcal{X},\mathcal{T}}(a,\mathbf{C}) = \mathcal{H}^{\mathcal{X},\mathcal{T}}(a,\mathbf{C}) \\
&\qquad\qquad = \inf_{(\mathbf{B},\mathbf{\Theta})\in\mathcal{S}_M^{\mathcal{X},\mathcal{T}}(a)} H_{\mathbf{C}}(\mathbf{B},\mathbf{\Theta}) \\
&\qquad\qquad\quad = \Psi^{\mathcal{X},\mathcal{T}}(a,\mathbf{C}).
\end{aligned}
$$

Definition

$\mathcal{M}_2^{\mathcal{X},\mathcal{T}}(\mathbf{C}) \subseteq \mathcal{M}^{\mathcal{X},\mathcal{T}}(\mathbf{C})$ is the set of models (i.e. a filtration $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_{t_1}, \dots \mathcal{F}_{t_N})$ a probability measure $\mathbb{P}$ supporting a bivariate, discrete-time, stochastic process $(X, \Delta) = (X_{t_n}, \Delta_{t_n})_{0 \le n \le N}$ taking values in $\mathcal{X} \times \{1, 2\}$ for $n \geq 1$) such that $(X_0, \Delta_0) = (s_0, 1)$ and

1. (X, Δ) is Markov with respect to price, so that $\mathbb{P}(X_{t_{n+1}} = x_k | \mathcal{F}_{t_n}) = \mathbb{P}(X_{t_{n+1}} = x_k | X_{t_n}, \Delta_{t_n})$.
2. Δ is non-decreasing, with $\Delta_{t_N} = 2$.
3. the probability that $\Delta_{t_{n+1}} = 2$, conditional on $\Delta_{t_n} = 1$ depends on n and $X_{t_{n+1}}$ only.

$\tau_\Delta = \min\{t_n : \Delta_{t_n} = 2\}$.

A process (X, Δ) in $\mathcal{M}_2^{\mathcal{X},\mathcal{T}}$ can be characterized by a pair of $(J+1) \times (J+1) \times (N-1)$ matrices $\mathbf{G^1}$ and $\mathbf{G^2}$ (with entries $g^{\delta}_{j,k,n}$) specifying the joint probability of successive states:

$$g^{\delta}_{j,k,n} = \mathbb{P}(X_{t_n} = x_j, X_{t_{n+1}} = x_k, \Delta_{t_n} = \delta)$$

Mass entering a node must equal the mass at the node must equal the mass leaving the node. Thus

$$\sum_{0 \le i \le J} (g^1_{i,j,n-1} + g^2_{i,j,n-1}) = p_{j,n} = \sum_{0 \le k \le J} (g^1_{j,k,n} + g^2_{j,k,n})$$

X is a martingale:

$$\sum_{0 \le k \le J} (x_k - x_j) g^{\delta}_{j,k,n} = 0.$$

A process (X,Δ) in $\mathcal{M}_2^{\mathcal{X},\mathcal{T}}$ can be characterized by a pair of $(J+1)\times(J+1)\times(N-1)$ matrices $\mathbf{G^1}$ and $\mathbf{G^2}$ (with entries $g^{\delta}_{j,k,n}$) specifying the joint probability of successive states:

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By hypothesis the process Δ is non-decreasing. It is convenient to introduce an auxiliary $(J+1)\times N$ matrix $\mathbf{F}$ which records the probability of arriving at node $(j,2)$ at time n having been in regime 1 at time $n-1$.

Let $\mathbf{F} = (f_{j,n})$ where $f_{j,n} \geq 0$ is given by the joint probability $f_{j,n} = \mathbb{P}(X_{t_n} = j, \Delta_{t_{n-1}} = 1, \Delta_{t_n} = 2)$. Then

$$f_{j,n} = \begin{cases} \sum_{0\leq k\leq J} g^2_{j,k,1} & n=1 \\ \sum_{0\leq k\leq J} g^2_{j,k,n} - \sum_{0\leq i\leq J} g^2_{i,j,n-1} & \\ \qquad = \sum_{0\leq i\leq J} g^1_{i,j,n-1} - \sum_{0\leq k\leq J} g^1_{j,k,n} & 1<n<N \\ p_{j,N} - \sum_{0\leq i\leq J} g^2_{i,j,N-1} = \sum_{0\leq i\leq J} g^1_{i,j,N-1} & n=N \end{cases}.$$

We set $g^2_{j,k,0} = 0$ and $g^2_{j,k,N} = p_{j,N} I_{\{j=k\}}$. The equation for $\mathbf{F}$ simplifies to $f_{j,n} = \sum_{0\leq k\leq J} g^2_{j,k,n} - \sum_{0\leq i\leq J} g^2_{i,j,n-1}$.

Linear Program

The pricing problem $\mathbf{L}_{\mathbf{P}}^{\mathcal{X},\mathcal{T}}$ *is to find the value* $\Phi^{\mathcal{X},\mathcal{T}} = \Phi^{\mathcal{X},\mathcal{T}}(a, \mathbf{C})$*:*
i.e. to find the $(J+1) \times N$ *matrix* F *and the two* $(J+1) \times (J+1) \times (N-1)$ *matrices* $\mathbf{G^1}$ *and* $\mathbf{G^2}$ *which maximise*

$$\sum_{1 \le n \le N} \sum_{0 \le j \le J} a(x_j, t_n) f_{j,n}$$

subject to $\mathbf{F} \ge 0$, $\mathbf{G^1} \ge 0$, $\mathbf{G^2} \ge 0$, *and*

(a) $\sum_{0 \le k \le J} (g^1_{j,k,n} + g^2_{j,k,n}) = p_{j,n}$.

(b) $\sum_{0 \le i \le J} (g^1_{i,j,n-1} + g^2_{i,j,n-1}) = p_{j,n}$.

(c) $\sum_{0 \le k \le J} (x_k - x_j) g^1_{j,k,n} = 0$.

(d) $\sum_{0 \le k \le J} (x_k - x_j) g^2_{j,k,n} = 0$.

(e) $f_{j,n} - \sum_{0 \le k \le J} g^2_{j,k,n} + \sum_{0 \le i \le J} g^2_{i,j,n-1} \le 0$

Linear Program

The hedging problem $\mathbf{L}_{\mathbf{H}}^{\mathcal{X},\mathcal{T}}$ *is to:*
find the three $(J+1)\times N$ *matrices* $\mathbf{E^1}$, $\mathbf{E^2}$ *and* $\mathbf{V}$ *and the two* $(J+1)\times(N-1)$ *matrices* $\mathbf{D^1}$ *and* $\mathbf{D^2}$ *which minimise*

$$\sum_{0\leq j\leq J, 1\leq n\leq N} (e^1_{j,n} + e^2_{j,n}) p_{j,n} + \sum_{0\leq j\leq J} v_{j,N} p_{j,N}$$

subject to $\mathbf{V}\geq 0$, *and*

(i) $v_{j,n}\geq a(x_j, t_n)$;
(ii) $e^1_{j,n} + e^2_{k,n+1} + (x_k - x_j) d^1_{j,n} \geq 0$;
(iii) $e^1_{j,n} + e^2_{k,n+1} + (x_k - x_j) d^2_{j,n} - v_{j,n} + v_{k,n+1} \geq 0$;

and $e^1_{j,N} = e^2_{j,1} = 0$.

Let the optimum value be given by $\Psi^{\mathcal{X},\mathcal{T}} = \Psi^{\mathcal{X},\mathcal{T}}(a, \mathbf{C})$.

Definition

Given three $(J+1) \times N$ matrices $\mathbf{E^1}$, $\mathbf{E^2}$ and $\mathbf{V}$ and two $(J+1) \times (N-1)$ matrices $\mathbf{D^1}$ and $\mathbf{D^2}$, the quintuple $(\mathbf{E^1}, \mathbf{E^2}, \mathbf{D^1}, \mathbf{D^2}, \mathbf{V})$ can be interpreted as a semi-static trading strategy for the agent in the following sense:

1. Let $b_{j,n} = (e^1_{j,n} + e^2_{j,n})$ for $1 \leq n \leq N-1$ and $b_{j,N} = (e^1_{j,N} + e^2_{j,N} + v_{j,N})$.
2. Let $\theta^1_{t_n}(x_{t_1}, \ldots, x_{t_n}) = \theta^1_{t_n}(x_{t_n}) = d^1_{j,n}$ if $x_{t_n} = x_j$.
3. Let $\theta^2_{t_n}(x_{t_1}, \ldots, x_{t_n}, \sigma) = \theta^2_{t_n}(x_{t_n}) = d^2_{j,n}$ if $x_{t_n} = x_j$.

Proposition

If the quintuple $(\mathbf{E^1}, \mathbf{E^2}, \mathbf{D^1}, \mathbf{D^2}, \mathbf{V})$ *is feasible for* $\mathbf{L}_{\mathbf{H}}^{\mathcal{X},\mathcal{T}}$ *and if* $x_{t_n} \in \mathcal{X}$ *for* $1 \leq n \leq N$ *then the semi-static trading strategy in the Definition super-replicates the American claim.*

$h_n(x) = \sum_{0 \leq j \leq J} h_{j,n} I_{\{x = x_j\}}$; $\mathcal{N}(t) = \min\{t_n : t_n \geq t\}$.

Suppose that X follows the path $(s_0, y_1, \ldots, y_N)$ with $y_i \in \mathcal{X}$. The terminal payoff $\mathcal{G}_T = \mathcal{G}_T(y_1, \ldots, y_N, \tau)$ is

$$\begin{aligned} \mathcal{G}_T &= \sum_{n=1}^{N} (e_n^1(y_n) + e_n^2(y_n)) + v_N(y_N) + \sum_{1}^{\mathcal{N}(\tau)-1} (y_{n+1} - y_n) d_n^1(y_n) \\ &+ \sum_{\mathcal{N}(\tau)}^{N-1} (y_{n+1} - y_n) d_n^2(y_n) \end{aligned}$$

This can be rewritten as

$$\begin{aligned} & a(y_\tau, \tau) + e_1^2(y_1) + e_N^1(y_N) + \{v_{\mathcal{N}(\tau)}(y_\tau) - a(y_\tau, \tau)\} \\ &+ \sum_{1}^{\mathcal{N}(\tau)-1} \left\{e_n^1(y_n) + e_{n+1}^2(y_{n+1}) + (y_{n+1} - y_n) d_n^1(y_n)\right\} \\ &+ \sum_{\mathcal{N}(\tau)}^{N-1} \left\{e_n^1(y_n) + e_{n+1}^2(y_{n+1}) + (y_{n+1} - y_n) d_n^2(y_n) - v_n(y_n) + v_{n+1}(y_{n+1})\right\} \end{aligned}$$

The main result repeated

Theorem

$$
\begin{aligned}
&\Phi^{\mathcal{X},\mathcal{T}}(a,\mathbf{C}) \\
&\quad = \sup_{M\in\mathcal{M}_2^{\mathcal{X},\mathcal{T}}(\mathbf{C})} \mathbb{E}^M[a(X_{\tau_\Delta},\tau_\Delta)] = \sup_{M\in\mathcal{M}_2^{\mathcal{X},\mathcal{T}}(\mathbf{C})} \phi^a(M) \\
&\quad = \mathcal{P}^{\mathcal{X},\mathcal{T}}(a,\mathbf{C}) = \mathcal{H}^{\mathcal{X},\mathcal{T}}(a,\mathbf{C}) \\
&\quad = \inf_{(\mathbf{B},\mathbf{\Theta})\in\mathcal{S}_M^{\mathcal{X},\mathcal{T}}(a)} H_{\mathbf{C}}(\mathbf{B},\mathbf{\Theta}) \\
&\quad = \Psi^{\mathcal{X},\mathcal{T}}(a,\mathbf{C}).
\end{aligned}
$$

Extension to processes on $\mathbb{R}^+ \times \mathcal{T}$.

Assumption

Time is discrete and takes values in the finite set $\mathcal{T}_0$. The price process $X = (X_t)_{t \in \mathcal{T}_0}$ takes values in $\mathbb{R}^+$. a is defined on $\mathbb{R}^+ \times \mathcal{T}$ and in addition to being positive, a is convex in its first argument and $\lim_{x \uparrow \infty} a(x, t_n)/x < R$.

Theorem

$$\Phi^{\mathcal{X},\mathcal{T}}(a, \mathbf{C}) = \mathcal{P}^{\mathbb{R}^+,\mathcal{T}}(a, \mathbf{C}) = \mathcal{H}^{\mathbb{R}^+,\mathcal{T}}(a, \mathbf{C}) = \Psi^{\mathcal{X},\mathcal{T}}(a, \mathbf{C}).$$

The most expensive model-based price amongst models which are consistent with the observed call prices is attained by a price/regime model in which the price only takes values in $\mathcal{X}$ — an element of $\mathcal{M}_2^{\mathcal{X},\mathcal{T}}(\mathbf{C})$.

Extension to processes on $\mathbb{T} = [0, T]$.

Assumption

Time is continuous and takes values in the set $\mathbb{T} = [0, T]$. The price process $X = (X_t)_{t \in \mathbb{T}}$ takes values in $\mathbb{R}^+$. The American option payoff $A : \mathbb{R}^+ \times \mathbb{T} \mapsto \mathbb{R}$ is positive, convex in its first argument with $\lim_x A(x,t)/x < R$ for each $t \in [0, T]$ and decreasing in its second argument.

Theorem

Define $a(x, t_k) = \lim_{t \downarrow t_{k-1}} A(x,t) = A(x, t_{k-1}+)$. Then

$$\Phi^{\mathcal{X},\mathcal{T}}(a, \mathbf{C}) = \sup_{M \in \mathcal{M}^{\mathbb{R}^+,\mathbb{T}}(\mathbf{C})} \phi^A(M) = \inf_{(\mathbf{B},\Theta) \in \mathcal{S}^{\mathbb{R}^+,\mathbb{T}}(A)} H_{\mathbf{C}}(\mathbf{B}) = \Psi^{\mathcal{X},\mathcal{T}}(a, \mathbf{C}).$$

Extension to call price sets with no call of zero price

Return to the discrete time setting.

Assume the American option payoff $a : \mathbb{R}^+ \times \mathcal{T} \mapsto \mathbb{R}$ is positive, convex and has at most linear growth.

Assumption

The set of option prices has the following properties:

- *For* $1 \leq n \leq N$, $s_0 = c_{0,n} > c_{1,n} > c_{2,n} > c_{J,n} > 0$.
- *For* $1 \leq n \leq N$,
  $1 > \frac{c_{0,n} - c_{1,n}}{x_1} > \frac{c_{1,n} - c_{2,n}}{x_2 - x_1} > \cdots > \frac{c_{J-1,n} - c_{J,n}}{x_J - x_{J-1}} > 0$.
- *For* $1 \leq n \leq N - 1$, *and for* $1 \leq j \leq J$, $c_{j,n+1} > c_{j,n}$.

Theorem

$\Phi^{\mathcal{X},\infty,\mathcal{T}}(a, \mathbf{C}) = \mathcal{P}^{\mathbb{R}^+,\mathcal{T}}(a, \mathbf{C}) = \mathcal{H}^{\mathbb{R}^+,\mathcal{T}}(a, \mathbf{C}) = \Psi^{\mathcal{X},\infty,\mathcal{T}}(a, \mathbf{C})$.

Introduce the $(J+2)\times N$ matrix $\hat{\mathbf{P}}$ via $\hat{p}_{j,n} = p_{j,n}$ for $0 \leq j \leq J$ and $\hat{p}_{J+1,n} = c_{J,n}$.

Linear Program

The pricing problem $\mathbf{L}_{\mathbf{P}}^{\mathcal{X},\infty,\mathcal{T}}$ *is to find* $(J+2)\times N$ *matrix* $\mathbf{F}$ *and* $(J+2)\times(J+2)\times(N-1)$ *matrices* $\mathbf{G^1}$ *and* $\mathbf{G^2}$ *which maximise*

$$\sum_{0\leq j\leq J}\sum_{1\leq n\leq N} a(x_j,t_n) f_{j,n} + \sum_{1\leq n\leq N} f_{J+1,n} \lim_{x\uparrow\infty} \frac{a(x,t_n)}{x}$$

subject to $\mathbf{F}\geq 0$, $\mathbf{G^1}\geq 0$, $\mathbf{G^2}\geq 0$, *and*

(a) $\sum_{0\leq k\leq J}(g^1_{j,k,n}+g^2_{j,k,n}) = \hat{p}_{j,n};\qquad 0\leq j\leq J,$
$\sum_{0\leq k\leq J+1}(g^1_{J+1,k,n}+g^2_{J+1,k,n}) = \hat{p}_{J+1,n};$

(b) $\ldots$

Let the optimum value be given by $\Phi^{\mathcal{X},\infty,\mathcal{T}} = \Phi^{\mathcal{X},\infty,\mathcal{T}}(a,\mathbf{C})$.

There is no consistent model in $\mathcal{M}_2^{\mathcal{X},\mathcal{T}}(\mathbf{C})$ for which the model price equals $\Phi^{\mathcal{X},\infty,\mathcal{T}}(a,\mathbf{C})$.

Instead, we give a sequence of consistent models for which the model based price converges to $\Phi^{\mathcal{X},\infty,\mathcal{T}}$.

An Example

The current price of the underlying is 100. $\mathcal{T} = \{t_1, \dots, t_N = T\}$, $\mathcal{K} = \{50, 100, 150\}$.

Let $(q_m)_{1 \leq m \leq N}$ be a set of probabilities which sum to 1.

Define the set of call option prices by $\mathbf{C} = c_{j,n}$ where for $1 \leq n \leq N$

$$c_{j,n} = \begin{cases} 100 & j = 0 \\ 50 & j = 1 \\ 25 \sum_{i=1}^{n} q_i & j = 2 \\ 0 & j = 3 \end{cases}$$

Consider now an American option which has payoff $a(x, t_n) = (b_n - x)^+$ where $(b_n)_{n \in \mathcal{N} = \{1, \dots, N\}}$ is decreasing with $100 < b_1 < 150$. The option must be exercised at one of the dates $\{t_1, \dots t_N\}$. Set $a_{j,n} = a(x_j, t_n)$.

For the primal pricing problem define $\mathbf{G^1}$ and $\mathbf{G^2}$ via

$$g^1_{2,1,n} = \frac{q_{n+1}}{2} I_{\{n \le n^*-1\}} \qquad g^1_{2,2,n} = \sum_{n+2}^{n^*} q_i \qquad g^1_{2,3,n} = \frac{q_{n+1}}{2} I_{\{n \le n^*-1\}}$$

$$g^2_{1,1,n} = \frac{1}{2} \sum_{m \le n} q_m \qquad g^2_{2,1,n} = \frac{q_{n+1}}{2} I_{\{n \ge n^*\}} \qquad g^2_{2,2,n} = \sum_{(n^*+1)\vee(n+2)}^{N} q_m$$

$$g^2_{2,3,n} = \frac{q_{n+1}}{2} I_{\{n \ge n^*\}} \qquad g^2_{3,3,n} = \frac{1}{2} \sum_{m \le n} q_m$$

all other entries being zero. The entries of $\mathbf{F}$ are given by

$$f_{1,n} = \frac{q_n}{2} I_{\{n \le n^*\}} \qquad f_{2,n} = \left(\sum_{n^*+1}^{N} q_i \right) I_{\{n=1\}} \qquad f_{3,n} = \frac{q_n}{2} I_{\{n \le n^*\}}$$

The model based price of the American call (using the stopping time $\tau = \inf\{t_m \in \mathcal{T} : \Delta_{t_m} = 2\}$) is

$$\Phi = \sum_{j,n} f_{j,n} a_{j,n} = (b_1 - 100) \sum_{n^*+1}^{N} q_i + \sum_{1}^{n^*} \frac{q_n}{2} (b_n - 50)$$

Set $\mathbf{D^1} = 0$, $\mathbf{E^2} = 0$ and define $\mathbf{V}$, $\mathbf{D^2}$ and $\mathbf{E^1}$ by

$$
\begin{aligned}
v_{0,n} &= \max\{b_n, 3(b_1 - 100)\} \\
v_{1,n} &= (b_n - 50) I_{\{n \le n^*\}} + 2(b_1 - 100) I_{\{n > n^*\}} \\
v_{2,n} &= (b_1 - 100) \\
v_{3,n} &= 0
\end{aligned}
$$

$e^1_{j,n} = (v_{j,n} - v_{j,n+1})$
for $0 \le j < 3$, $d^2_{j,n} = (v_{j+1,n+1} - v_{j,n+1})/50$ with $d^2_{J,n} = 0$
The feasibility conditions of the dual problem are satisfied.
Further, $\Psi = \sum_{j,n}(e^1_{j,n} + e^2_{j,n}) p_{j,n} + \sum_j v_{j,N} p_{j,N}$ is given by

$$
\Psi = \sum_{1}^{n^*} (b_n - 50) \frac{q_n}{2} + (b_1 - 100) \sum_{n^*+1}^{N} q_m
$$

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\begin{aligned}
v_{0,n} &= \max\{b_n, 3(b_1 - 100)\} \\
v_{1,n} &= (b_n - 50) I_{\{n \le n^*\}} + 2(b_1 - 100) I_{\{n > n^*\}} \\
v_{2,n} &= (b_1 - 100) \\
v_{3,n} &= 0
\end{aligned}
$$

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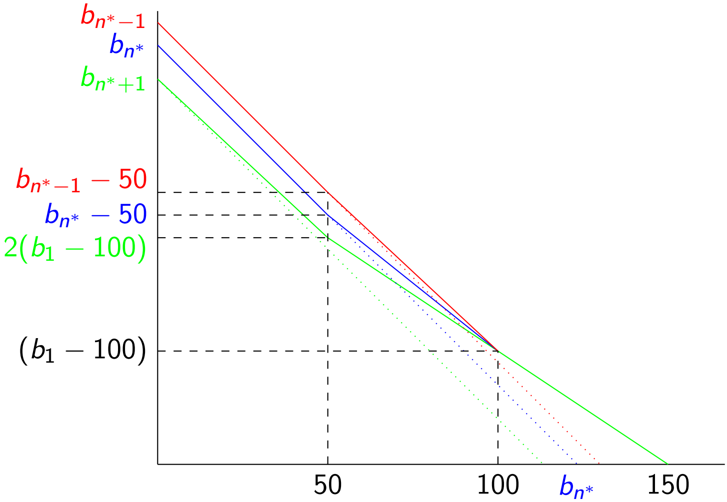
\Psi = \sum_{1}^{n^*} (b_n - 50) \frac{q_n}{2} + (b_1 - 100) \sum_{n^*+1}^{N} q_m
$$



b_{n^*-1}
b_{n^*}
b_{n^*+1}
$b_{n^*-1} - 50$
$b_{n^*} - 50$
$2(b_1 - 100)$
$(b_1 - 100)$
50
100
b_{n^*}
150