(K, N) exponentially concave functions, and short-term relative arbitrage

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Goal of the talk

Exponential concavity

- φ defined on an open convex $D \subset \mathbb{R}^n$ is exponentially concave if

$$\Phi := e^{\varphi}$$

  is concave.
- Primarily interested in $D = \Delta$, unit simplex in $\mathbb{R}^n$.
- positive coordinates, adds to 1.
- Market: n stocks. $\mu = (\mu_1, \ldots, \mu_n) \in \Delta$.
- Market weights:

$$\mu_i = \text{Proportion of the total capital that belongs to } i\text{th stock.}$$

Portfolios

- **All long** portfolio: $\pi = (\pi_1, \ldots, \pi_n) \in \Delta$.
- Portfolio weights:

$$\pi_i = \text{Proportion of the total value that belongs to } i\text{th stock.}$$

- For us $\pi = \pi(\mu) : \Delta \to \overline{\Delta}$.
- Function from unit simplex to its closure.
- $\pi(\mu) \equiv \mu$ - Market portfolio, a buy-and-hold portfolio.

Relative value

- $V_\pi(\cdot)$ - Value process of π. $V_\pi(0) = \$1$.
- $V_\mu(\cdot)$ - Index. $V_\mu(0) = \$1$. Self-financing.
- Relative value process: $V(t) = V_\pi(t)/V_\mu(t)$.
- **Relative arbitrage**: for some $q \in (0,1)$ and $T > 0$,

$$P(V(T) \geq 1) = 1, \quad P(V(T) > 1) > 0, \quad P\left(\inf_{0 \leq t \leq T} V(t) \geq q\right) = 1.$$

- **Qn**: Do relative arbitrages exist? Can we estimate T?
- **Challenge**: Make minimal modeling assumptions. Model-free strategies.

The Fernholz decomposition

- φ exponentially concave on Δ.
- For $\mu \in \Delta$, define **FGP**

$$\frac{\pi_i}{\mu_i} = 1 + D_{e_i - \mu}\varphi, \quad i = 1, 2, \ldots, n.$$

- Then $\pi : \Delta \to \overline{\Delta}$ is a portfolio map. $\mu(t)$ Itô process:

$$\log V(t) = \varphi(\mu(t)) - \varphi(\mu(0)) - \frac{1}{2}\int_0^t \frac{1}{\Phi}\mathrm{Hess}\Phi\left(d\mu(s)\right).$$

- Under diversity, range(φ) is bounded. Under 'volatility', the second part grows unbounded. **Long term model-free relative arbitrage**.

Long-term vs. Short-term relative arbitrages.

- A high-dimensional **Definition**.
- Family of equity markets for each n. Portfolio $\pi(n)$ for each n.
- $\pi(n)$ beats the market by time T_n.
- **Long term**: $\lim_{n\to\infty} T_n = \infty$. **Short term**: $\lim_{n\to\infty} T_n = 0$.
- Typical examples of FGP portfolios in SPT are long-term relative arbitrages under diversity and volatility.

- Relevant: P.-Wong ('14) proved the converse.
- In discrete time, in the absence of any modeling assumptions, the only relative arbitrage portfolios maps from Δ to $\overline{\Delta}$ are FGP.

Are short-term relative arbitrages possible?

- Do **model-free** short-term relative arbitrages exist?
- Model dependent examples are known.
- The source of arbitrage can be large in two ways:

$$-\frac{1}{\Phi}\mathrm{Hess}\;\Phi(d\mu(t))$$

- Either very large volatility, or very concave Φ.
- Very concave Φ affects its range, and hence risky.

The Volatility-Stabilized model example

- A large volatility example provided by Fernholz-Karatzas '05, Banner-Fernholz '08.
- Let $\tau_i(t)$ - diffusion coefficient of $\log \mu_i(t)$:

$$\tau_i(t) = \frac{d}{dt} \langle \log \mu_i(t) \rangle = \frac{1}{\mu_i^2} \frac{d}{dt} \langle \mu_i, \mu_i \rangle (t).$$

- Consider ranked market weights: $\mu_{(n)}(t) \leq \mu_{(n-1)}(t) \leq \cdots \leq \mu_{(1)}(t)$.

The Volatility-Stabilized model example

- **Assume** $\exists C > 0$ such that

$$\tau_{(n)}(t) \geq \frac{C}{\mu_{(n)}(t)} \geq Cn, \quad \text{for all } t \geq 0.$$

- (**Fernholz-Karatzas** '05). Relative arbitrage exists over time $[0, T_n]$ where

$$T_n = \frac{2\mathrm{Ent}\left(\mu(0)\right)}{n-1}.$$

- Proof is a direct application of Fernholz's decomposition.
- (**Banner-Fernholz** '08) Exists over $[0, \delta]$ for any $\delta > 0$ for any n.

Capital distribution curve

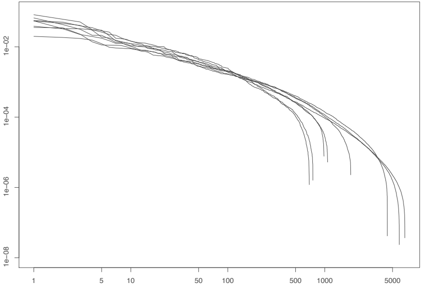

Figure 1: Capital distribution curves: 1929–1999

- The extreme volatility assumption is crucial and does not fit capital distribution curve.
- $\log \mu_{(i)}$ vs. $\log i$ data is roughly linear with slope $\approx$ negative **one**.
- Volatility stabilized models do not produce such stable shapes.

Goal of the talk

- Will construct short-term relative arbitrages that work **even** under bounded volatility τ_i assumption.
- If time permits, we will talk a little bit about the underlying geometry.
- The main idea is high dimensional convex geometry and concentration of measure.

Short-term relative arbitrage in high dimensions

The Pareto distribution

1. Fix $n \in \mathbb{N}$.
2. $\alpha \in \Delta$ such that $\alpha_i \propto 1/i$, Pareto(-1).

$$\alpha_i = \frac{1/i}{\sum_{j=1}^{n} 1/j} \approx \frac{1}{i \log n}.$$

3. Suppose $\mu(0) \in K$, a **typical neighborhood** around α.
4. Will discuss what **typical** means.
5. The indices $(\mu_1, \ldots, \mu_n)$ are chosen by rank.

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- Let $X_i = n\mu_i$. Assume continuous semimartingales.

$$\mu_i = \frac{X_i}{\sum_{j=1}^n X_j}, \quad i = 1, 2, \ldots, n.$$

- Intuition: X_i is approximately price of the ith stock price if $\sum_{i=1}^n X_i \approx n$.

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- Divide the index as

$$A = \left[1, \frac{n}{(\log n)^2}\right], \qquad B = \left[\frac{n}{(\log n)^2} + 1, n\right].$$

- If $n = 5000$, $|A| \approx 68$. Vanishing fraction of n for large n.

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- Suppose $\exists\ T \in (0,1)$, and $\alpha(T), C(T), \lambda(T) > 0$ independent of n such that ...

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- Suppose $\exists\ T \in (0,1)$, and $\alpha(T), C(T), \lambda(T) > 0$ independent of n such that ...
- For $i \in A$, exponential tails:

$$P\left(\sup_{0\le t\le T} \frac{X_i(t) - X_i(0)}{t^{\alpha}\sqrt{X_i(0)}} > a\right) \le Ce^{-\lambda a}.$$

- For $i \in B$, moment bound:

$$\mathrm{E}\left(\sup_{0\le t\le T} \frac{X_i(t) - X_i(0)}{t^{\alpha}\sqrt{X_i(0)}}\right)^2 \le C.$$

- Assume $\exists\ \underline{\tau} > 0$ such that

$$\tau_i = \frac{d}{dt}\langle \log \mu_i(t)\rangle \ge \underline{\tau}, \quad \text{for all } i.$$

Main theorem:statement

Theorem (P.-'15)

Suppose $\exists$ $(\Omega, \mathcal{F}, P)$ *such that, for every n, a market of dimension n exists satisfying the previous conditions. There exists portfolio maps* π_n*, for each n, such that*

- *Almost surely,* $\exists$ n_0 *such that for all* $n \geq n_0$*, the relative value of* π_n *is strictly larger than one by time*

$$O\left(\frac{(\log n)^2}{n}\right).$$

- *For all* $n \geq n_0$*, a.s., the relative value never drops below* $1/2$ *during that time interval.*

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- *For all* $n \geq n_0$*, a.s., the relative value never drops below* $1/2$ *during that time interval.*
- *High dimensional short-term strong relative arbitrage.*

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- VSM satisfies all conditions.
- We only need local bounds $T \approx 0$.

The construction: high-dimensional convex analysis

(K,N) exponential concavity

- (Erbar-Kuwada-Sturm '14) A function φ is (K, N) exponentially concave if $\Phi := \exp(\varphi/N)$ is concave and satisfies:

$$\frac{1}{\Phi}\mathrm{Hess}\,\Phi \leq -\frac{K}{N} I.$$

- They have somewhat general definition. Related to curvature-dimension inequalities. Bochner inequalities.
- Entropy is $(1, n)$ exponentially concave in $\mathcal{P}_2(\mathbb{R}^n, \|\cdot\|)$.

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- They have somewhat general definition. Related to curvature-dimension inequalities. Bochner inequalities.
- Entropy is $(1, n)$ exponentially concave in $\mathcal{P}_2(\mathbb{R}^n, \|\cdot\|)$.
- We are interested in $(n, 1)$ exponentially concave functions in dimension n. That is, φ is exponentially concave and

$$\frac{1}{\Phi}\mathrm{Hess}\,\Phi \leq -nI.$$

Do such functions exist?

- The diameter of the domain of the function must be at most $O\left(1/\sqrt{n}\right)$.
- Example: Fix $x_0 \in \mathbb{R}^n$ and let

$$\varphi(x) = \log\cos\left(\sqrt{n}\,\|x - x_0\|\right), \quad \|x - x_0\| < \frac{\pi}{2\sqrt{n}}.$$

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- What other natural set has diameter $1/\sqrt{n}$?
- Unit simplex in dimension n has **typical** diameter $\approx 1/\sqrt{n}$ around $x_0 = (1/n, \dots, 1/n)$.
- Concentration of measure. Most of the volume is at most $1\sqrt{n}$ away from x_0. But not all ...

Back to Pareto

- Recall $\alpha \in \Delta$, $\alpha_i \propto 1/i$.
- We will take $x_0 = \alpha$. Atypical for uniform distribution on Δ.
- Reference measure: Dirichlet($n\alpha$). Density

$$p(x) \propto \prod_{i=1}^{n} x_i^{n\alpha_i - 1}, \quad x \in \Delta.$$

- Same exponential family as uniform. Just a shift of mean.

$$\begin{aligned} \mathrm{E}(X) &= \alpha, \quad X \sim \mathrm{Diri}(n\alpha), \\ \mathrm{Var}(X) &\approx \frac{\alpha_i}{n}. \end{aligned}$$

Typical neighborhood

- Domain of $\varphi(x) = \log\cos\left(\sqrt{n}\,\|x-\alpha\|\right)$ is
$$\left\{x : \sqrt{n}\,\|x-\alpha\| < \pi/2\right\}.$$

- **Lemma**: For any $r > 0$,
$$\mathrm{Diri}\left(\sqrt{n}\,\|X-\alpha\| > 1+r\right) \le \frac{c}{(1+r)^2\log n}.$$

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- **Lemma**: For any $r > 0$,

$$\mathrm{Diri}\left(\sqrt{n}\,\|X-\alpha\| > 1+r\right) \leq \frac{c}{(1+r)^2\log n}.$$

- Let

$$K = \left\{x \in \Delta :\ \sqrt{n}\,\|x-\alpha\| \leq \pi/3.1\right\}, \quad 1 < \pi/3.1 < \pi/2.$$

- Then $\mathrm{Diri}(K) \approx 1$ and $K \subseteq \mathrm{Dom}(\varphi)$. Assume $\mu(0) \in K$.

The drift process

- Choose $K \subset K_1 \subset \text{Domain}(\varphi)$. Say

$$K_1 := \left\{ x : \sqrt{n} \, \|x - \alpha\| < \frac{\pi}{3} \right\}.$$

- Starting from inside K, how long does it take to exit K_1?

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- Starting from inside K, how long does it take to exit K_1?
- At least reciprocal of poly-log n with high probability

$$\frac{1}{(\log n)^2}.$$

Exit time from a typical set

Lemma

Let $\varsigma = \inf\{t \geq 0 : \ \mu(t) \notin K_1\}$. *If* $\mu(0) \in K$, *then*

$$P\left(\varsigma > \frac{1}{(\log n)^2}\right) \geq 1 - O\left(\frac{1}{n^\gamma}\right), \quad \gamma > 1.$$

On K_1, *we get*

$$-\frac{1}{\Phi}\mathrm{Hess}\ \Phi(d\mu(t)) \geq \frac{\tau}{4}\frac{n}{(\log n)^2}dt.$$

The range of φ *on* K_1 *is bounded by*

$$-\log\cos(\pi/3) = \log 2.$$

Construction of the relative arbitrage

- Recall Fernholz's decomposition:

$$\log V(t) = \varphi(\mu(t)) - \varphi(\mu(0)) - \frac{1}{2}\int_0^t \frac{1}{\Phi}\mathrm{Hess}\Phi\left(d\mu(s)\right).$$

- Within K_1, the first part is bounded by $\log 2$, while drift increases at rate $n/(\log n)^2$.
- Thus, relative arbitrage happens by time

$$O\left(\frac{(\log n)^2}{n}\right),$$

unless $\varsigma < 1/(\log n)^2$, which is very unlikely.
- Use Borel-Cantelli to get almost sure statement. Done!

Information geometry of the unit simplex

Multiplicative cyclical monotonicity

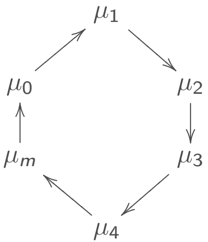
- Why are exponentially concave functions necessary?
- Relative value process $V = V_\pi / V_\mu$.

$$\frac{\Delta V(t)}{V(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{\Delta \mu_i(t)}{\mu_i(t)}.$$

- Fix $T > 0$. $V(0) = 1$.

$$V(T) = \prod_{t=0}^{T-1} \left(1 + \left\langle \frac{\pi(\mu(t))}{\mu(t)}, \mu(t+1) - \mu(t) \right\rangle \right).$$

The special case of cycles



- Market cycles through a sequence of size m.
- Let $\eta = V(m+1)$. Dichotomy:

$$\eta < 1, \quad \text{or} \quad \eta \geq 1.$$

- After k cycles: $V(k(m+1)) = \eta^k$.

Multiplicative Cyclical Monotonicity

- If $\eta < 1$, the
$$\lim_{t\to\infty} V(t) = \lim_{k\to\infty} \eta^k = 0.$$
- π not a relative-arbitrage.

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- If $\eta < 1$, the
$$\lim_{t\to\infty} V(t) = \lim_{k\to\infty} \eta^k = 0.$$

- π not a relative-arbitrage.

- Say π is **not** MCM if such a cycle exists.
- Otherwise π is MCM.

What if π is MCM?

Theorem (P.-Wong '14)

Suppose π is MCM. $\exists\, \Phi : \Delta \to (0, \infty)$, ***concave****:*

$$\frac{\pi_i}{\mu_i} = 1 + D_{\mathbf{e_i} - \mu} \log \Phi(\mu).$$

If Φ not affine, π is a pseudo-arbitrage in discrete/continuous time.

Outperformance over cycles $\Leftrightarrow$ asymptotic outperformance over all paths.

Many congratulations to Joseph and Walter!