Duality Methods in Portfolio Optimization under transaction costs

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The paradigmatic problem in portfolio optimization

Problem

$$\mathbb{E}[U(X_T)] \mapsto max! \qquad (P_x)$$

where X_T runs through the set $\mathcal{C}(x)$ of random non-negative variables of the form

$$X_T = x + \int_0^T H(t) dS(t).$$

x: initial endowment, H admissible trading strategy
U: $\mathbb{R}_+ \mapsto \mathbb{R}$ utility function, e.g. $U(x) = \log(x)$.

Basic observation

$\mathcal{C}(x)$ can also be described as the set

$$\mathcal{C}(x) = \{X_T \in L^0_+(\mathbb{P}) : \mathbb{E}_Q[X_T] \leq x\},$$

for all equivalent martingale measures $Q \in \mathcal{M}^e(S)$.
This is the content of the *super-replication theorem*.

Viewing the elements $Q \in \mathcal{M}^e(S)$ as *constraints*, the primal problem (P) may be viewed as a convex optimization problem on the entire cone $L^0_+(\mathbb{P})$ under (one or infinitely many) linear constraints.

$$\begin{aligned} &\mathbb{E}[U(X_T)] \mapsto max! \qquad (P_x) \\ &\mathbb{E}_Q[X_T] \leq x, \qquad \forall Q \in \mathcal{M}^e(S). \end{aligned}$$

There is a well-known duality theory which allows to – somewhat formally – associate to the primal problem (P_x) over the set $L^0_+(\mathbb{P})$ a dual problem (D_y) over the set $\mathcal{M}^e(S)$ of constraints

$$\mathbb{E}\Big[V\Big(y\frac{dQ}{d\mathbb{P}}\Big)\Big] \mapsto \text{min!} \qquad (D_y)$$

where Q ranges in $\mathcal{M}^e(S), y > 0$ is a scalar Langrange multiplier, and V is the conjugate function of U

$$V(y) = \sup_{x>0}\{U(x) - xy\}.$$

Basic background: the Hahn-Banach Theorem in its version as Min-Max Theorem.

Task

Identify precise (and hopefully sharp) conditions to turn the above formal reasoning into mathematical theorems.

Transaction Costs

There are very many ramifications of the above theme. We now focus on markets under transaction costs.

We fix a strictly positive càdlàg stock price process $S = (S_t)_{0 \leq t \leq T}$.

For $0 < \lambda < 1$ we consider the bid-ask spread $[(1-\lambda)S, S]$.

A self-financing trading strategy is a predictable, finite variation process $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ such that

$$d\varphi_t^0 \leq -S_t (d\varphi_t^1)_+ + (1-\lambda) S_t (d\varphi_t^1)_-$$

The trading strategy φ is called 0-admissible if the liquidation value remains non-negative

$$\varphi_t^0 + (1-\lambda) S_t (\varphi_t^1)_+ - S_t (\varphi_t^1)_- \geq 0$$

Definition [Jouini-Kallal ('95), Cvitanic-Karatzas ('96), Kabanov-Stricker ('02),...]

A *consistent price system* is a pair $(\tilde{S}, Q)$ such that $Q \sim \mathbb{P}$, the process $\tilde{S}$ takes its value in $[(1-\lambda)S, S]$, and $\tilde{S}$ is a Q-martingale.

Identifying Q with its density process

$$Z_t^0 = \mathbb{E}\left[\tfrac{dQ}{d\mathbb{P}} | \mathcal{F}_t\right], \qquad 0 \leq t \leq T$$

we may identify $(\tilde{S}, Q)$ with the $\mathbb{R}^2$-valued martingale $Z = (Z_t^0, Z_t^1)_{0 \leq t \leq T}$ such that

$$\tilde{S} := \tfrac{Z^1}{Z^0} \in [(1-\lambda)S, S].$$

For $0 < \lambda < 1$, we say that S satisfies (CPS^λ) if there is a consistent price system for transaction costs λ.

Theorem [Guasoni, Rasonyi, S. ('10)]:

Let $S = (S_t)_{0 \le t \le T}$ be a continuous process. TFAE

(i) For each $\mu > 0$, S does not allow for arbitrage under transaction costs μ.

(ii) For each $\mu > 0$, (CPS^μ) holds, i.e. *consistent price systems under transaction costs* μ exist.

Remark [Guasoni, Rasonyi, S. ('08)]

If the process $S = (S_t)_{0 \le t \le T}$ is *continuous* and has *conditional full support*, then (CPS^μ) is satisfied, for all $\mu > 0$.
For example, exponential fractional Brownian motion verifies this property.

Portfolio optimisation

The *set of non-negative claims attainable at price x* is

$$\mathcal{C}(x) = \left\{ \begin{array}{l} X_T \in L^0_+ : \textit{there is a } 0-\textit{admissible } \varphi = (\varphi^0_t, \varphi^1_t)_{0 \le t \le T} \\ \textit{starting at } (\varphi^0_0, \varphi^1_0) = (x, 0) \textit{ and ending at} \\ (\varphi^0_T, \varphi^1_T) = (X_T, 0) \end{array} \right\}$$

Given a utility function $U : \mathbb{R}_+ \to \mathbb{R}$ define again

$$u(x) = \sup\{\mathbb{E}[U(X_T)] : X_T \in \mathcal{C}(x)\}.$$

Cvitanic-Karatzas ('96), Deelstra-Pham-Touzi ('01), Cvitanic-Wang ('01), Bouchard ('02),...

Question 1

What are conditions ensuring that $\mathcal{C}(x)$ is closed in $L^0_+(\mathbb{P})$. (w.r. to convergence in measure) ?

Theorem [Cvitanic-Karatzas ('96), Campi-S. ('06)]:

Suppose that (CPS^μ) is satisfied, for all $\mu > 0$, and fix $\lambda > 0$. Then $\mathcal{C}(x) = \mathcal{C}^\lambda(x)$ is closed in $L^0(\Omega, \mathcal{F}, \mathbb{P})$.

The dual objects

Definition

We denote by $D(y)$ the convex subset of $L^0_+(\mathbb{P})$

$$D(y) = \{yZ^0_T = y\frac{dQ}{d\mathbb{P}}, \textit{for some consistent price system } (\tilde{S}, Q)\}$$

and

$$\mathcal{D}(y) = \overline{sol\ (D(y))}$$

the closure of the solid hull of $D(y)$ taken with respect to convergence in measure.

Definition [Kramkov-S. ('99), Karatzas-Kardaras ('06), Campi-Owen ('11),...]

Fix the adapted càdlàg process S and $\lambda > 0$.
We call an optional process $Z = (Z_t^0, Z_t^1)_{0 \le t \le T}$ a *super-martingale deflator* if $Z_0^0 = 1$, $\frac{Z^1}{Z^0} \in [(1-\lambda)S, S]$, and for each 0-admissible, self-financing φ the value process

$$\varphi_t^0 Z_t^0 + \varphi_t^1 Z_t^1 = Z_t^0 (\varphi_t^0 + \varphi_t^1 \frac{Z_t^1}{Z_t^0})$$

is a (optional strong) super-martingale.

Remark

A consistent price system $Z = (Z_t^0, Z_t^1)_{0 \le t \le T}$ is a super-martingale deflator.

Proposition (Czichowsky, S. ('14)):

The closure $\mathcal{D}(y)$ of $D(y)$ can be characterized as

$$\mathcal{D}(y) = \{yZ_T^0\},$$

where $Z = (Z_t^0, Z_t^1)_{0 \le t \le T}$ is an (optional strong) super-martingale deflator.

Interlude: Limits of Martingales:

Theorem (Czichowsky, S. ('14)):

Let $(M^n)_{n=1}^{\infty}$ be a sequence of non-negative martingales, starting of $M_0^n = 1$.
Then there exist $N^n \in \mathrm{conv}(M^n, M^{n+1}, \ldots)$ and a limiting *optional strong super-martingale* M such that

$$N^n \to M$$

in the following sense: for every $[0, T]$-valued stopping time τ we have

$$\lim_{n \to \infty} N_\tau^n = M_\tau$$

in probability.

Theorem (Czichowsky, S. ('14))

Let S be a continuous process, $0 < \lambda < 1$, suppose that (CPS^{μ}) holds true, for some $0 < \mu < \lambda$, suppose that U has reasonable asymptotic elasticity and $u(x) < U(\infty)$, for $x < \infty$.
Then $\mathcal{C}(x)$ and $\mathcal{D}(y)$ are polar sets:

$$X_T \in \mathcal{C}(x) \quad \text{iff } \langle X_T, Y_T \rangle \leq xy, \quad \text{for } Y_T \in \mathcal{D}(y)$$
$$Y_T \in \mathcal{D}(y) \quad \text{iff } \langle X_T, Y_T \rangle \leq xy, \quad \text{for } X_T \in \mathcal{C}(y)$$

Therefore by the abstract results from [Kramkov-S. ('99)] the duality theory for the portfolio optimisation problem works as nicely as in the frictionless case: for $x > 0$ and $y = u'(x)$ the following assertions hold true:

Duality properties:

(i) There is a unique primal optimiser $\hat{X}_T(x) = \hat{\varphi}^0_T$ which is the terminal value of a trading strategy $(\hat{\varphi}^0_t, \hat{\varphi}^1_t)_{0 \le t \le T}$.

(i') There is a unique dual optimiser $\hat{Y}_T(y) = \hat{Z}^0_T$ which is the terminal value of a *super-martingale deflator* $(\hat{Z}^0_t, \hat{Z}^1_t)_{0 \le t \le T}$.

(ii) $U'(\hat{X}_T(x)) = \hat{Z}^0_T(y), \qquad -V'(\hat{Z}^0_T(y)) = \hat{X}_T(x)$

(iii) The process $(\hat{\varphi}^0_t \hat{Z}^0_t + \hat{\varphi}^1_t \hat{Z}^1_t)_{0 \le t \le T}$ is a martingale and

$$\{d\hat{\varphi}^1_t > 0\} \subseteq \{\tfrac{\hat{Z}^1}{\hat{Z}^0} = S\},$$
$$\{d\hat{\varphi}^1_t < 0\} \subseteq \{\tfrac{\hat{Z}^1}{\hat{Z}^0} = (1-\lambda)S\}.$$

Shadow Price Processes

Theorem [Cvitanic-Karatzas ('96)]

In the setting of the above theorem *suppose* that $(\hat{Z}_t)_{0\leq t\leq T}$ is a local martingale.
Then $\hat{S} = \frac{\hat{Z}^1}{\hat{Z}^0} \in [(1-\lambda)S, S]$ is a *shadow price*, i.e. the optimal portfolio for the *frictionless market* $\hat{S}$ and for the *market* S *under transaction costs* λ coincide.

Sketch of Proof

Suppose (w.l.g.) that $(\hat{Z}_t)_{0\leq t\leq T}$ is a true martingale. Then $\frac{d\hat{Q}}{d\mathbb{P}} = \hat{Z}^0_T$ defines a *probability measure* under which the process $\hat{S} = \frac{\hat{Z}^1}{\hat{Z}^0}$ is a martingale. Hence we may apply the frictionless theory to $(\hat{S}, \mathbb{P})$.
$\hat{Z}^0_T$ is (a fortiori) the dual optimizer for $\hat{S}$.
As $\hat{X}_T$ and $\hat{Z}^0_T$ satisfy the first order condition

$$U'(\hat{X}_T) = \hat{Z}^0_T,$$

$\hat{X}_T$ must be the optimizer for the frictionless market $\hat{S}$ too. ■

Question

When is the dual optimizer $\hat{Z}$ a *local martingale*?
Are there cases when it only is a *super-martingale*?

Theorem [Czichowsky-S.-Yang ('14)]

Suppose that S is *continuous* and satisfies $(NFLVR)$, and suppose that $U : (0, \infty) \to \mathbb{R}$ has reasonable asymptotic elasticity. Fix $0 < \lambda < 1$ and suppose that $u(x) < U(\infty)$, for $x < \infty$.
Then the dual optimizer $\hat{Z}$ is a local martingale. Therefore $\hat{S} = \frac{\hat{Z}^1}{\hat{Z}^0}$ is a shadow price.

Remark

The condition $(NFLVR)$ cannot be replaced by requiring (CPS^λ), for each $\lambda > 0$. But $(NFLVR)$ can be replaced by $(NUPBR)$.

Theorem [Czichowsky-S. ('15)]

Suppose that S is continuous and sticky, and suppose that $U : \mathbb{R} \to \mathbb{R}$ has reasonable asymptotic elasticity. Fix $0 < \lambda < 1$ and suppose that $u(x) < U(\infty)$, for $x < \infty$.
Then the dual optimizer $\hat{Z}$ is a local martingale. Therefore $\hat{S} = \frac{\hat{Z}^1}{\hat{Z}^0}$ is a shadow price.

A case study: Fractional Brownian Motion and Exponential Utility

Fractional Brownian Motion for $H \in]\frac{1}{2}, 1[$:

$$B_t = C(H) \int_{-\infty}^{t} \left((t-s)^{H-\frac{1}{2}} - \left(|s|^{H-\frac{1}{2}} \mathbb{1}_{(-\infty,0)} \right) \right) dW_s,\ 0 \leq t \leq T,$$

We may further define a non-negative stock price process $S = (S_t)_{0 \leq t \leq T}$ by letting

$$S_t = \exp(B_t), \qquad 0 \leq t \leq T,$$

or, slightly more generally,

$$S_t = \exp(\sigma B_t + \mu t), \qquad 0 \leq t \leq T.$$

Theorem [Czichowsky-S. ('15)]

For $S_t = \exp(\sigma B_t + \mu t)$ and exponential utility $U(x) = -e^{-x}$ the utility optimization problem has a perfectly satisfactory solution (the duality theory works just as in the formal reasoning).

In particular, there is a shadow price process $\widehat{S}(t)$ which is an Itô diffusion process of the form

$$\frac{d\widehat{S}_t}{\widehat{S}_t} = \widehat{\sigma}_t dW_t + \widehat{\mu}_t dt$$

for some predictable processes $\widehat{\sigma}$ and $\widehat{\mu}$. The process $\widehat{S}$ is a local martingale under the probability measure $\widehat{Q}$, where

$$\frac{d\widehat{Q}}{d\mathbb{P}} = \exp\Big(\int_0^T -\frac{\widehat{\mu}_t}{\widehat{\sigma}_t} dW_t - \frac{1}{2} \int_0^T \Big(\frac{\widehat{\mu}_t}{\widehat{\sigma}_t}\Big)^2 dt\Big)$$

This theorem has a surprising consequence on the pathwise behaviour of fractional Brownian trajectories.

Theorem [Czichowsky-S. ('15)]

Let $(B_t)_{0\leq t\leq T}$ be fractional Brownian motion with Hurst index $H\in(\frac{1}{2},1)$ and $\alpha>0$.
There is an Itô diffusion process $(\widehat{X}_t)_{0\leq t\leq T}$ such that

$$B_t-\alpha\leq \widehat{X}_t\leq B_t,\qquad 0\leq t\leq T,$$

holds true almost surely.

In addition, $\widehat{X}$ can be constructed in such a way that $(e^{\widehat{X}_t})_{0\leq t\leq T}$ is a local martingale under some measure $\widehat{Q}$ equivalent to P.

For $\varepsilon>0$, we may choose $\alpha>0$ sufficiently small so that the trajectory $(\widehat{X}_t)_{0\leq t\leq T}$ touches the trajectories $(B_t)_{0\leq t\leq T}$ as well as $(B_t-\alpha)_{0\leq t\leq T}$ with probability bigger than $1-\varepsilon$.

Theorem [R. Peyre ('15)]

Let $(B^H_t)_{t\geq 0}$ be fractional Brownian motion and τ a finite stopping time.
Then, for each $\varepsilon > 0$, we have
$\inf_{\tau\leq t\leq \tau+\varepsilon} B^H_t < B^H_\tau < \sup_{\tau\leq t\leq\tau+\varepsilon} B^H_t, \qquad a.s.$

Interpretation [R. Peyre, Ch. Bender]:

(Exponential) fractional Brownian motion does not allow for *simple arbitrage*.

Corollary [Czichowsky, S., Yang]:

For power utility

$$U(x) = \frac{x^\alpha}{\alpha}, \qquad x > 0$$

where $\alpha < -0$, there is a shadow price process (as well as the usual positive results) for the financial market $S_t = \exp(B^H_t)$.