REGULARITY OF INFINITE-DIMENSIONAL LIE GROUPS BY METRIC SPACE METHODS

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Abstract. Regularity of infinite dimensional Lie groups was defined by Hideki Omori et al. and John Milnor. Up to now the only known sufficient conditions for regularity are analytic in nature and they are included in the definition of strong ILB-Lie groups, since there are no existence theorems for ordinary differential equations on non-normable locally convex spaces. We prove that regularity can be characterized by the existence of a family of so called Lipschitz-metrics in all interesting cases of infinite dimensional Lie groups. On Lipschitz-metrizable groups all product integrals converge to the solutions of the respective equations if some weak conditions satisfied by all known Lie groups are given. Lipschitz-metrizable groups provide a framework to solve differential equations on infinite dimensional Lie groups. Furthermore Lipschitz-metrics are the non-commutative generalization of the concept of seminorms on a Fréchet-space viewed as abelian Lie group.

1. Introduction

Convenient Lie groups as defined in [KM97] provide a useful basis for infinite-dimensional geometry, but there is still a lack of methods how to handle analytic questions. Convenient Lie groups are smooth manifolds modeled on convenient vector spaces with smooth group structures. The excellent approach of [KYMO81] and [Omo97] to infinite dimensional Lie groups includes all necessary analytic a-priori-properties in the definition to solve some differential equations on the Lie groups, however, the topological and metric space properties of the object itself are not considered directly. We try to define a category of Lie groups, where the existence of so called product integrals (see [KYMO81] and [Omo97]) is equivalent to some conditions on the given Lipschitz-metrics. This category shall contain all strong ILB-Lie groups (several subgroups of diffeomorphism groups on compact finite dimensional manifolds, see [Omo97] for example, the strong ILH-Lie group of invertible Fourier-Integral-Operators, see [ARS86]).

In the introduction we shall explain the convenient setting, the framework of a general analytic Lie theory (see [KM97] for all details). In the second section we introduce Lipschitz-metrics, state their fundamental properties and show, that they exist on all known Fréchet-Lie-groups (which are up to now strong ILB-groups). In the third section the conditions equivalent to the existence of a smooth exponential map or a smooth right evolution operator are developed. In the fourth section some generalizations of the method are discussed.

Convenient vector spaces are Mackey-complete locally convex vector spaces. They appear as those locally convex spaces where weakly smooth curves are the smooth curves (see [KM97], chapter 1). The final topology with respect to the smooth curves is called the **smooth topology** or **c^∞-topology**. Remark that the smooth topology does not commute with product, i.e. the smooth topology on the product is finer than the product of the smooth topologies.

Smooth maps on c^∞-open sets are those which map smooth curves to smooth curves. The important **detection principle** in the setting of convenient vector spaces is the following:

$$c : \mathbb{R} \to U \text{ is smooth if and only if } \forall f \in C^\infty(U, \mathbb{R}) : f \circ c \text{ is smooth}$$
$$f : U \to F \text{ is smooth if and only if } \forall c \in C^\infty(\mathbb{R}, U) : f \circ c \text{ is smooth}$$

Multilinear mappings on convenient vector spaces are smooth if and only if they are bounded, i.e. bounded sets are mapped to bounded sets. A convenient algebra is assumed to have bounded algebra structures and in this context to be unital and associative. We denote by $L(E)$ the main example of a convenient algebra, the bounded linear endomorphisms of a convenient vector space E with bounded sets those, which are bounded on bounded subsets of E. We have the following initial topology on

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spaces of smooth mappings:

$$C^\infty(U,F) \xrightarrow{c^*} C^\infty(\mathbb{R},F) \text{ for all } c \in C^\infty(\mathbb{R},U)$$

where $C^\infty(\mathbb{R},F)$ carries the topology of uniform convergence in all derivatives on compact subsets of the real numbers. The exponential law holds, i.e.

$$i : C^\infty(U, C^\infty(V,G)) \cong C^\infty(U \times V, G)$$

for U,V c^∞-open and G a convenient vector space. Detailed information on convenient calculus can be found in [KM97] and [FK88]. We shall need the following existence lemma on smooth curves (see [KM97], 12.2):

Lemma 1.1 (special curve lemma). *Let E be a convenient vector space and $\{c_n\}_{n\geq 1}$ a sequence in E converging fast to 0, then there is a smooth curve $c : \mathbb{R} \to E$ with*

$$c(\frac{1}{n}) = c_n$$

for $n \in \mathbb{N}_+$.

All manifolds and Lie groups treated in the article will be **convenient manifolds** and Lie groups, i.e. they are modeled on convenient vector spaces and supposed to be **smoothly regular** with respect to the smooth topology on the manifold (in general one supposes smoothly Hausdorff, i.e. the smooth functions on the manifold separate points). One can develop this infinite dimensional setting surprisingly far (see [KM97] for details).

The last concept of the basics of convenient calculus on Lie groups is the right logarithmic derivative: μ denotes the smooth product on the Lie group, $\mathfrak{g}$ the Lie algebra. Let $f : M \to G$ be a smooth map, where M is a convenient manifold. We define the **right logarithmic derivative** $\delta^r f : TM \to \mathfrak{g}$ by the formula

$$\delta^r f(\xi_x) := T_{f(x)}(\mu^{f(x)})^{-1}(T_x f(\xi_x))$$

for $x \in M$ and $\xi_x \in T_x M$. By definition we see that $\delta^r f \in \Omega^1(M,\mathfrak{g})$ is a $\mathfrak{g}$-valued 1-form on M. A Lie group G is called **regular** if there is a smooth (evolution) map $Evol^r : C^\infty(\mathbb{R},\mathfrak{g}) \to C^\infty(\mathbb{R},G)$, such that $Evol^r(X)(0) = e$ and $\delta^r(Evol^r(X))(t) = X(t)$ for all $t \in \mathbb{R}$, furthermore $Evol^r(\delta^r c) = c$ (see [KM97], [Mil83], [Omo97]). The **right evolution** with respect to a constant curve is a smooth one-parameter subgroup. If in any direction there exists a smooth one-parameter subgroup, then we can define the classical **exponential map** exp. Let G be a simply connected Lie group and H a regular Lie group with $f : \mathfrak{g} \to \mathfrak{h}$ a bounded Lie algebra homomorphism, then there is a smooth Lie group homomorphism ϕ with $\phi' = f$ (see [KM97], Theorem 40.3).

The concept of regularity means that one can solve all non-autonomous Cauchy problems on the Lie group G, more precisely - given $X \in C^\infty(\mathbb{R},\mathfrak{g})$ - there is a smooth curve $c : \mathbb{R} \to G$ with $c(0) = e$ and $c'(t) = T_e\mu^{c(t)}(X(t))$ for $t \in \mathbb{R}$. Such non-autonomous problems can sometimes be solved by so called product integrals (see for example [Omo97]).

Given a convenient manifold N (see [KM97] for details) **smooth regularity** asserts that the smooth topology on N is initial with respect to the smooth functions in $C^\infty(N,\mathbb{R})$, which is not always the case, since there need not be enough globally defined smooth functions. Smooth regularity is indeed a reasonable assumption for smooth manifolds, since otherwise it is impossible to make the rarely possible conclusions from local to global in infinite dimensions. If N is smoothly regular, then each germ at a point has a global representative (see [KM97], 27.21). We shall assume that all convenient manifolds in this article are smoothly regular, not only smoothly Hausdorff (see [KM97], 27.4)

Remark 1.2. *Instead of convenient Lie groups one could work for our purposes with so called smooth spaces, i.e. groups with a distinguished set of curves into and a distinguished set of maps from the group to the real numbers, such that the detection principle is valid. Smooth mappings between such spaces map smooth curves to smooth curves. Smooth groups have smooth multiplication and inversion. Convenient Lie groups are smooth groups if they are smoothly regular (see* [KM97] *and* [FK88] *for details on smooth spaces and* [Tei01] *for details on smooth groups).*

Lemma 1.3. *Given a convenient smoothly regular manifold N. Let $\{c_n\}_{n\geq 0} \subset C^\infty(M,N)$ be a sequence of smooth mappings from a finite dimensional compact manifold M to N, such that for all*

$m \in M$ the sequence $\{c_n(m)\}_{n\geq 0}$ lies in a sequentially compact set with respect to the topology $c^\infty N$. Let furthermore $c_n^* : C^\infty(N,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ be a Mackey-Cauchy sequence:

Then for any $m \in M$ there is a chart (u,U) around $c(m)$ such that almost all c_n lie locally around m (at some fixed open neighborhood V of m) in U and all derivatives of $u \circ c_n$ converge Mackey uniformly on V to the derivatives of $u \circ c$.

Proof. For any point m there exists at least one adherence point of $\{c_n(m)\}_{n\geq 0}$. By assumption $c_n^* : C^\infty(N,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ is Mackey-Cauchy convergent to some bounded linear map A. The adherence point has to be unique since smooth functions are continuous with respect to $c^\infty N$ and they separate points by definition of a smooth manifold, we denote the unique adherence point by $c(m)$. Consequently there is a mapping $c : M \to N$ which is the pointwise limit of $\{c_n\}_{n\geq 0}$. The limit of $\{f \circ c_n\}_{n\geq 0}$ is a smooth functions and by continuity equal to $f \circ c$ for all $f \in C^\infty(N,\mathbb{R})$, so c is smooth by the detection principle. We need a non-negative bump function f with respect to a chart (u,U) around $c(m)$ taking the value 1 at a small neighborhood of $c(m)$. By uniform convergence of $f \circ c_n$ to $f \circ c$ on a small closed neighborhood V of m we see that on V almost all c_n lie in U. By multiplication of $f \circ u^{-1}$ with any linear functional l on the model space we get global functions on N representing locally around $c(m)$ each linear functional. Consequently we obtain that all derivatives of $u \circ c_n$ converge at the point m Mackey to the respective derivative of $u \circ c$ by $l \circ u \circ c_n \to l \circ u \circ c$ Mackey in all derivatives and a quality independent of l. □

In the sequel of the article we shall need the following approximation theorem for product integrals. They exist if their approximations lie uniformly on compact sets in bounded sets:

Definition 1.4 (Product integral)**.** *Let A be a convenient algebra. Given a smooth curve $X\colon \mathbb{R} \to A$ and a smooth mapping $h : \mathbb{R}^2 \to A$ with $h(s,0) = e$ and $\frac{\partial}{\partial t}h(s,0) = X(s)$, we define the following finite products of smooth curves*

$$p_n(a,t,h) := \prod_{i=0}^{n-1} h(a + \frac{(n-i)(t-a)}{n}, \frac{t-a}{n})$$

for $a,t \in \mathbb{R}$. If p_n converges in all derivatives to a smooth curve $c : \mathbb{R} \to A$, then c is called the product integral of X or h and we write $c(a,t) = \prod_a^t \exp(X(s)ds)$ or $c(a,t) =: \prod_a^t h(s,ds)$. The case $h(s,t) = c(t)$ with $p_n(0,t,h) = c(\frac{t}{n})^n$ is referred to as simple product integral.

Theorem 1.5 (Approximation theorem)**.** *Let A be convenient algebra. Given a smooth curve X : $\mathbb{R}^2 \to A$ and a smooth mapping $h : \mathbb{R}^3 \to A$ with $h(u,r,0) = e$ and $\frac{\partial}{\partial t}h(u,r,0) = X_u(r)$. Suppose that for every fixed $s_0 \in \mathbb{R}$, there is $t_0 > s_0$ such that $p_n(u,s,t,h)$ is bounded in A on compact (u,s,t)-sets and for all $n \geq 1$. Then the product integral $\prod_s^t h(u,r,dr)$ exists and the convergence is Mackey in all derivatives on compact (u,s,t)-sets. Furthermore the product integral is the right evolution of X_u, i.e.*

$$\frac{\partial}{\partial t}\prod_s^t h(u,r,dr) = X_u(t)\prod_s^t h(u,r,dr)$$
$$\prod_s^s h(u,r,dr) = e$$

Remark 1.6. *The hypothesis on the product integrals will be referred to as* ***boundedness condition****. For the proof see* [Tei99a] *and* [Tei99b]*.*

2. Lipschitz-metrizable Lie groups

The definition of product integrals on convenient Lie groups G is done in the same way as on convenient algebras:

Definition 2.1. *Let G be a convenient Lie group with $c^\infty G$ a topological group. Given a smooth mapping $h : \mathbb{R}^2 \to G$ with $h(s,0) = e$, then we define the following finite products of smooth curves*

$$p_n(s,t,h) := \prod_{i=0}^{n-1} h(s + \frac{(n-i)(t-s)}{n}, \frac{t-s}{n})$$

for $s,t\in\mathbb{R}$. If p_n converges in the smooth topology of G uniformly on compact sets to a continuous curve $c:\mathbb{R}\to G$, then c is called the product integral of h and we write $c(s,t) =: \prod_s^t h(u,du)$. If $h(s,t)=c(t)$, then the product integral $p_n(0,t,h)=c(\frac{t}{n})^n$ is called simple product integral.

Remark 2.2. *Here we need the assumption that $c^\infty G$ is a topological group, since we want to apply the notions of uniform convergence and completeness with respect to the uniform structure on $c^\infty G$.*

The left regular representation ρ of a convenient Lie group G

$$\rho : G \to L(C^\infty(G,\mathbb{R}))$$
$$g \mapsto (f \mapsto f(g.))$$

in the bounded operators on $C^\infty(G,\mathbb{R})$ is initial (see [KM97]) and smooth. We shall apply this "linearization" in the following way several times in the article.

Lemma 2.3. *Let G be a convenient Lie group, such that $c^\infty G$ is a topological group and G is smoothly regular, then each product $p_n(s,t,h)$ and the limit - if it exists - $\prod_s^t h(u,du)$ is smooth. The propagation condition $\prod_t^r h(u,du)\prod_s^t h(u,du) = \prod_s^r h(u,du)$ is satisfied for all r,s,t.*

Proof. By the left regular representation ρ on G we get that the product integral

$$\lim_{n\to\infty} p_n(s,t,\rho\circ h)$$

exists in $C^\infty(\mathbb{R}^2, L(C^\infty(G,\mathbb{R})))$, since that image of a sequentially compact set under a smooth mapping is bounded in the convenient algebra $L(C^\infty(G,\mathbb{R}))$. The set formed by $p_n(s,t,h)$ and $\prod_s^t h(u,du)$ on compact (s,t)-sets is sequentially compact due to uniform convergence, so we can apply Theorem 1.5 to obtain Mackey-convergence. Consequently we are given the hypotheses of Lemma 1.4 ($N=G$, M is a compact manifold with boundary in $\mathbb{R}^2$ and we can evaluate $\rho(p_n(s,t,h))\cdot f$ at e), which allows the conclusion of smoothness of c. The propagation condition follows from the definition of the product integral and the continuity of multiplication. □

Lemma 2.4. *Let G be a smoothly regular Lie group, such that $c^\infty G$ is a topological group. Given a smooth mapping $h:\mathbb{R}^2\to G$ with $h(s,0)=e$, such that the product integral converges to $c(s,t)$, then the fundamental theorem of product integration or non-commutative integration asserts that $\delta_t^r c(s,t) = \frac{\partial}{\partial t}h(s,0)$ and the convergence is uniform in all derivatives in the sense of Lemma 1.4.*

Proof. By the previous lemma it suffices to apply Lemma 1.4. to get the result. Remark that $\delta_t^r|_{t=s} p_n(s,t,h) = X(s)$. □

The main observation of the following two sections is based on a proof of the famous Kakutani-Theorem (which was simultaneously and independently proved by Garett Birkhoff) on the existence of a right (or left) invariant metric on a topological group with countable basis of the neighborhood filter of the identity (for this proof see [MZ57]):

Theorem 2.5 (Kakutani's theorem)**.** *Let G be a topological group with a countable basis of the neighborhood filter of the identity, then there is a left (or right) invariant metric on G generating the topology.*

Proof. Given a sequence of open neighborhoods of the identity $\{Q_n\}_{n\in\mathbb{N}}$, then by continuity of the multiplication we find a sequence of symmetric open neighborhoods $\{U_n\}_{n\in\mathbb{N}}$ with

$$U_{n+1}^2 \subset U_n \cap Q_n \text{ for } n\in\mathbb{N}$$

We define by induction on $1\le k\le 2^n$ and $n\ge 0$

$$V_{\frac{1}{2^n}} = U_n$$
$$V_{\frac{2k}{2^{n+1}}} = V_{\frac{k}{2^n}}$$
$$V_{\frac{2k+1}{2^{n+1}}} = V_{\frac{1}{2^{n+1}}} V_{\frac{k}{2^n}}$$

We obtain the property $V_{\frac{1}{2^n}} V_{\frac{m}{2^n}} \subset V_{\frac{m+1}{2^n}}$ for $m<2^n$. For $m=2k$ this is a consequence of the above properties. For $m=2k+1$ the left hand side becomes

$$V_{\frac{1}{2^n}} V_{\frac{m}{2^n}} = V_{\frac{1}{2^n}} V_{\frac{1}{2^n}} V_{\frac{k}{2^{n-1}}} \subset V_{\frac{1}{2^{n-1}}} V_{\frac{k}{2^{n-1}}} = V_{\frac{k+1}{2^{n-1}}} = V_{\frac{m+1}{2^n}}$$

by induction on n and m. So we obtain $V_r \subset V_{r'}$ for $r < r' \leq 1$. We choose in our case a monotonic decreasing basis of open sets of the neighborhood filter denoted by $\{Q_n\}_{n\in\mathbb{N}}$. We redo the presented construction and obtain a family V_r for all dyadic rationals $0 < r \leq 1$.

$$f(x,y) := \begin{cases} 0 \text{ if } y \in V_r V_r^{-1} x \text{ for all } r \\ \sup\{r \mid y \notin V_r V_r^{-1} x\} \end{cases}$$

By definition f is right invariant, since $f(xa, ya) = f(x,y)$ for all $a \in G$. $V_r V_r^{-1}$is symmetric, hence f is symmetric $f(x,y) = f(y,x)$. $V_{\frac{1}{2^n}}$is symmetric, so $V_{\frac{1}{2^n}} V_{\frac{1}{2^n}}^{-1} \subset V_{\frac{1}{2^n}}^2 = V_{\frac{1}{2^{n-1}}} \subset Q_{n-1}$, but $\cap_{n\geq 1} Q_{n-1} = \{e\}$, since we deal with a basis of neighborhoods, so $f(x,y) = 0$ if and only if $x = y$.

$$d(x,y) := \sup_{u\in G} |f(x,u) - f(y,u)|$$

$d(x,y) = d(y,x)$, $d(x,y) \geq f(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$. Right invariance is clear, too, and the triangle inequality follows from

$$\begin{aligned} d(x,z) &\leq \sup |f(x,u) - f(y,u) + f(y,u) - f(z,u)| \leq \\ &\leq d(x,y) + d(y,z) \end{aligned}$$

Finally we have to show that the metric reproduces the topology of the topological group. It is sufficient to show this at e by right invariance. We denote the open d-balls of radius $\frac{1}{2^n}$ by $B_{\frac{1}{2^n}}$. First we observe that $V_{\frac{1}{2^{n+1}}} \subset B_{\frac{1}{2^n}}$ for $n \geq 1$, which is done by a subtle case for case calculation: Given $y \in V_{\frac{1}{2^{n+1}}}$, then $f(y,e) < \frac{1}{2^n}$.

1. $u \in V_{\frac{1}{2^n}}$, so $f(y,u) \leq \frac{1}{2^n}$, so $d(y,e) < \frac{1}{2^n}$.
2. We can find a find a number $1 \leq k < 2^{n+2}$ with $u^{-1} \notin V_{\frac{s}{2^{n+2}}} V_{\frac{s}{2^{n+2}}}^{-1}$ for $1 \leq s \leq k$ and $u^{-1} \in V_{\frac{s}{2^{n+2}}} V_{\frac{s}{2^{n+2}}}^{-1}$ for $k < s < 2^{n+2}$. So $yu^{-1} \in V_{\frac{s+1}{2^{n+2}}} V_{\frac{s+1}{2^{n+2}}}^{-1}$ for $k < s < 2^{n+2}$ and $yu^{-1} \notin V_{\frac{s-1}{2^{n+2}}} V_{\frac{s-1}{2^{n+2}}}^{-1}$ for $2 \leq s \leq k$, hence $\frac{k-1}{2^{n+2}} \leq f(y,u) \leq \frac{k+1}{2^{n+2}}$ and $d(y,e) < \frac{1}{2^n}$, since $\frac{k}{2^{n+2}} \leq f(e,u) \leq \frac{k+1}{2^{n+2}}$.

If $x \in B_{\frac{1}{2^{n+1}}}$, then $f(e,x) < \frac{1}{2^{n+1}}$, finally $x \in V^2_{\frac{1}{2^{n+1}}} \subset V_{\frac{1}{2^n}} \subset Q_n$, so we obtain $U_{n+1} \subset B_{\frac{1}{2^n}} \subset Q_{n-1}$ for $n \geq 1$, which proves the desired assertion. □

Definition 2.6 (Lipschitz-metrizable groups)**.** *Let G be a convenient Lie group, such that $c^\infty G$ is a topological group. G is called Lipschitz-metrizable if there is a family of right invariant halfmetrics $\{d_\alpha\}_{\alpha\in\Omega}$ on G with the following properties:*

1. *For all sequences $\{x_n\}_{n\in\mathbb{N}}$:*
$$\forall\, \alpha \in \Omega : d_\alpha(x_k, x_l) \to 0 \iff \{x_n\}_{n\in\mathbb{N}} \text{ is converging in } G$$
2. *For all smooth mappings $c : \mathbb{R}^2 \to G$ with $c(s,0) = e$, there is on each compact (s,t)-set a constant M_α such that*
$$d_\alpha(c(s,t), e) < M_\alpha t$$

Remark 2.7. *In contrary to good manners (see* [KM97] *for the useful applications of this habit, e.g. 51.19) we omit the dependencies of the constants M_α. However, we declare that M_α is independent of t,* **s** *on a fixed compact set and always independent of m, n. The notion "Lipschitz-metric" stems from the fact that $t \mapsto d(c(t), e)$ is a Lip^0-curve for c smooth with $c(0) = e$.*

From the proof of Theorem 2.1. we observe that - given a Banach Lie group G - we can find by the CBH-formula (see [BCR81] for functional analytic details) a basis of the neighborhoods of identity of balls fitting in the above machinery such that we can construct a metric satisfying the Lipschitz property explained in the next definition.

Lemma 2.8. *Let G be a Banach-Lie-Group, then there is a metric d on G satisfying properties 2.6.1 and 2.6.2, so G is Lipschitz-metrizable.*

Proof. On Banach-Lie algebras we can choose a norm $||.||$ satisfying $||[X,Y]|| \leq ||X|| \cdot ||Y||$. The Campbell-Baker-Hausdorff Formula converges on a ball of radius $\frac{1}{4}$ and we have $||X*Y|| \leq 1-\sqrt{1-4r}$ for $||X||, ||Y|| \leq r$ and $r \leq \frac{1}{4}$ (see [BCR81] for functional analytic details). We define a sequence

$\{s_n\}_{n\geq 1}$ with $s_1 = \frac{1}{4}$ and $s_{n+1} = \frac{2s_n - s_n^2}{4}$, where the formula stems from solving $s_n = 1 - \sqrt{1 - 4s_{n+1}}$. We obtain by induction the following estimate

$$\frac{1}{2^{n+1}} \geq s_n > \frac{1}{2^{n+3}} + \frac{1}{2^{2n+2}}$$

since for $n = 1$ the inequality is valid and if it is valid for $n \geq 1$ then

$$s_{n+1} = \frac{2s_n - s_n^2}{4} > \frac{1}{2^{n+4}} + \frac{1}{2^{2n+3}} - \frac{1}{2^{2n+4}} = \frac{1}{2^{n+4}} + \frac{1}{2^{2n+4}}$$

which proves the assertion. Choosing $U_n = \exp(B(0, s_n))$ in the chart given by the exponential map for n large enough, then we can use the U_n directly in the proof of the Kakutani theorem to obtain a metric d with the property

$$U_{n+1} \subset \{x | d(x, e) < \frac{1}{2^n}\} \subset U_{n-1}$$

for n large enough, since $U_n^2 \subset U_{n-1}$ and $U_n^{-1} = U_n$. Given a curve $c : \mathbb{R}^2 \to G$ with $c(s, 0) = e$, then we can find for a given compact s-set a number $M > 0$ such that for t in $[0, 1]$

$$\exp^{-1}(c(t, s)) \in tMB(0, 1)$$

by Taylor's formula. Consequently

$$d(c(t, s), e) < \frac{1}{2^n}$$

if $s_{n+2} \leq tM < s_{n+1}$, so $\frac{d(c(t,s),e)}{t} < \frac{M}{2^n s_{n+2}}$ for small t. However, $s_{n+2}2^n > \frac{2^n}{2^{n+5}} + \frac{2^n}{2^{2n+6}} > \frac{1}{2^5}$. Hence for small t

$$\frac{d(c(t, s), e)}{t} < 32M$$

and the supremum property is satisfied. □

Lemma 2.9. *Let G be a smoothly connected (pathwise connected by smooth curves), complete (with respect to the right uniform structure), regular Fréchet-Lie-Group G, such that*

$$Evol^r : C^\infty([0, 1], \mathfrak{g}) \cap C([0, 1], \mathfrak{g}) \to C([0, 1], G)$$

is continuous with respect to the C_0-topology on the spaces. Furthermore we assume that there is a norm on $\mathfrak{g}$, then G is Lipschitz-metrizable.

Proof. We construct the halfmetrics directly: Given two points $g, h \in G$ we can join them by a Lip^1-curve c on $[0, 1]$ with $c(0) = g$, $c(1) = h$ and $\delta^r c(t) \neq 0$ for $t \in [0, 1]$, which will be denoted by $c : g \to h$.

$$d_k(g, h) := \inf_{c:g\to h} \int_0^1 p_k(\delta^r c(t))dt$$

for an increasing family of norms p_k defining the topology on $\mathfrak{g}$:

The Lipschitz-property is clear by definition. The triangle inequality follows from joining two Lip^1-curves.

Remark that for any Lip^1-map $\phi : [0, 1] \to [0, 1]$ with $\phi(0) = 0$ and $\phi(1) = 1$ we have $\delta^r(c \circ \phi) = ((\delta^r c) \circ \phi)\phi'$, so reparametrization does not change the integral. Consequently we can always assume that if we have a curve $c : g \to h$, then there is a Lip^1-function $\phi : [0, 1] \to [0, 1]$ with $\phi(0) = 0$ and $\phi(1) = 1$ such that

$$\int_0^1 p_k(\delta^r c(t))dt = \int_0^1 p_k(\delta^r c(\phi(t)))\phi'(t)dt =$$
$$= \int_0^1 p_k(\delta^r(c \circ \phi)(t))dt = \sup_{0\leq t\leq 1} p_k(\delta^r(c \circ \phi)(t))$$

ϕ is constructed by solving the differential equation

$$p_k(\delta^r c(\phi(t)))\phi'(t) = \int_0^1 p_k(\delta^r c(t))dt$$

with boundary values $\phi(0) = 0$ and $\phi(1) = 1$. The solution is given by

$$F(\phi(t)) = \int_0^{\phi(t)} p_k(\delta^r c(t))dt = t \int_0^1 p_k(\delta^r c(t))dt$$

where $F'(s) = p_k(\delta^r c(t)) \neq 0$, so there is a Lip^1-solution. Given a sequence $\{g_m\}_{m\in\mathbb{N}}$ with $g_n \overset{n\to\infty}{\to} e$ in G, then we can choose a chart (u, U) around e with $u(e) = 0$ and straight lines in the chart to join the g_m with e:

$$d_k(e, g_n) \leq \int_0^1 p_k(\delta^r(u^{-1}(.u(g_n))(t))dt$$

This yields the desired properties since $u(g_m)$ converges Mackey to 0 in the model space, so we can look at the problem on a unit ball in a Banach space E_B, where smooth maps are locally Lipschitz, consequently

$$\int_0^1 p_k(\delta^r(u^{-1}(.u(g_n))(t))dt \leq C p_B(u(g_n)) \overset{n\to\infty}{\to} 0$$

Given a sequence $\{g_n\}_{n\in\mathbb{N}}$ with $d_k(g_n, g_m) \to 0$ for $m, n \to \infty$ and U an open neighborhood of identity in G, then $(Evol^r)^{-1}(C([0,1], U))$ is open in $C([0,1], \mathfrak{g})$, saying $C([0,1], (p_k)_{<\epsilon})$ lies inside for a fixed $k \geq 0$. By assumption we can find curves $c_{n\to m} := c : e \to g_m g_n^{-1}$ with $p_k(\delta^r c_{n\to m}(t)) < \epsilon$ for n, m large enough applying the above method of uniformizing the velocity. Consequently $Evol^r(\delta^r c_{n\to m}(t)) = c_{n\to m}(t)$ lies in U for $t \in [0,1]$, so $g_m g_n^{-1} \in U$ for m, n large enough, which means that it is a Cauchy sequence in G. By completeness we conclude. □

Corollary 2.10. *All strong ILB-Lie groups are Lipschitz-metrizable, so all known Fréchet-Lie groups are Lipschitz-metrizable (see* [KM97], *p. 411).*

Proof. On ILB-groups the evolution map factors as continuous map

$$Evol^r : C^\infty([0,1], \mathfrak{g}) \cap C([0,1], \mathfrak{g}) \to C([0,1], G)$$

with respect to the C_0-topologies on the respective spaces, where from we conclude the result, since there are continuous norms on the associated Fréchet space to an ILB-chain. This factorization can be seen as follows, we refer to [Omo97]: Given a strong ILB-group, then even more general types of product integrals as provided converge without applying the notion of Lipschitz-metrizability, we only need the smoothness of the exponential map on the underlying Fréchet-Lie group. Given $X_n \in C([0,1], \mathfrak{g})$ converging uniformly to X, then we can associate C^1-hairs $h_n(s,t) = \exp(sX_n(t))$ with $h_n \to h$ in the topology on C^1-hairs by smoothness of the exponential map. Consequently the associated product integrals converge uniformly reproducing $Evol^r(X_n)$, which converges uniformly on $[0,1]$ to $Evol^r(X)$ (see [Omo97], Theorem 5.3). □

Remark 2.11. *The above result justifies a posterio the setting of strong ILB-groups, since we have to restrict to a class of Fréchet spaces, where continuous norms exist.*

Remark 2.12. *Assuming that the Fréchet space is given by an inverse limit of Hilbert spaces, so the construction of Lipschitz-metrics as in the proof of Lemma 2.9 is equally a definition of a variational problem. Under the condition that $ad(X)$ has a bounded adjoint with respect to the scalar product under consideration, the geodesic equation associated to the variational problem is given through*

$$u_t = -ad(u)^\mathsf{T} u$$

where u denotes the right logarithmic derivative of the geodesic (see [KM97], *section 46.4). Only in the case, where $u \in \ker(ad(u)^\mathsf{T})$ for $u \in \mathfrak{g}$ smooth one-parameter subgroups are geodesics. In view of interesting non-linear partial differential equations (for example the Korteweg-De Vrieß-equation) it is worth studying this situation in concrete cases. The question arises if such naturally appearing differential equations can be solved on the given Lie groups by internal methods, for example by Lipschitz-metrics. If this were the case, some interesting geometro-analytic progress in partial differential equations would be possible. To set the program it is first necessary to find some natural approximation procedure for variational problem, then to apply the Lipschitz-methods to prove approximation.*

The next proposition states that it is impossible to choose only one right invariant metric with Lipschitz-property reproducing the topology on a regular Fréchet-Lie-Group beyond Banach spaces. In the regular, abelian and simply connected case this means, that it is impossible to choose an invariant metric with Lipschitz-property on Fréchet spaces. Consequently we provide the non-abelian analogue to the assertion, that a Fréchet space with one norm generating the topology is a Banach space. Hence Lipschitz-metrics are the right concept replacing seminorms on convenient Lie groups in the non-commutative world.

Proposition 2.13. *Let G be a Fréchet-Lie-Group with (smooth) exponential map and suppose that there is a right invariant metric d on G reproducing the topology in the sense of Definition 2.6 and*

$$d(c(s,t),e) < Mt$$

for any smooth mapping $c : \mathbb{R}^2 \to G$ with $c(s,0) = e$ on compact (s,t)-sets. If for any sequence $\{X_n\}_{n\in\mathbb{N}}$ with $\exp(tX_n) \to e$ uniformly on compact intervals, the sequence $\{X_n\}_{n\in\mathbb{N}}$ converges to 0 in the Lie algebra $\mathfrak{g}$, then G is a Banach-Lie-group.

Proof. We define a seminorm p on the Lie algebra $\mathfrak{g}$ of G . The function $t \mapsto d(\exp(tX), e)$ is sublinear by right invariance, consequently the limit $\lim_{t\downarrow 0} \frac{d(\exp(tX),e)}{t}$ exists and equals the supremum $\sup_{t>0} \frac{d(\exp(tX),e)}{t}$.

$$p(X) := \lim_{t\downarrow 0} \frac{d(\exp(tX), e)}{t}$$

for $X \in \mathfrak{g}$. p is positively homogeneous and $p(0) = 0$. Given a smooth curve $c : \mathbb{R} \to G$ with $c(0) = e$ and $c'(0) = X$, then

$$\begin{aligned} \left|\frac{d(\exp(tX), e)}{t} - \frac{d(c(t)\exp(-tX), e)}{t}\right| &\le \frac{d(c(t), e)}{t} \\ &\le \frac{d(\exp(tX), e)}{t} + \frac{d(c(t)\exp(-tX), e)}{t} \end{aligned}$$

so the limit of the middle term exists since the limits of the other terms exist and are equal. The limit of a smooth curve d passing at 0 through e with $d'(0) = 0$ is calculated at the beginning of the proof of Theorem 3.1. as 0. Consequently $p(X) = \lim_{t\downarrow 0} \frac{d(c(t),e)}{t}$. So the triangle inequality is satisfied since

$$\frac{d(\exp(tX)\exp(tY)), e)}{t} \le \frac{d(\exp(tX), e)}{t} + \frac{d(\exp(tY), e)}{t}$$

Given a sequence $\{X_n\}_{n\in\mathbb{N}}$ with $X_n \to X$ in $\mathfrak{g}$. Convergence on the Fréchet space means Mackey convergence, so there is a compact set $B \subset \mathfrak{g}$ with $X - X_n \in \mu_n B$ with $\mu_n \downarrow 0$.

$$p(X - X_n) \le \sup_{0<t\le 1} \frac{d(\exp(t(X_n - X)), e)}{t} \le$$

$$\sup_{Y\in B}\sup_{0<t\le 1} \frac{d(\exp(t\mu_n Y), e)}{t} \le \mu_n \sup_{Y\in B}\sup_{0<t\le 1} \frac{d(\exp(t\mu_n Y), e)}{t\mu_n} \le \mu_n M$$

since the last supremum is finite, so $p(X - X_n) \to 0$ for $n \to \infty$, p is a continuous seminorm. Finiteness of the supremum is proved via the following consideration:

$$M(Y) := \sup_{0<t\le 1} \frac{d(\exp(tY), e)}{t}$$

Assume that there is a fast converging sequence $Y_n \to Y$ in the compact set B such that $M(Y_n) \ge n$. Consequently there is a smooth curve $d : \mathbb{R} \to \mathfrak{g}$ with $d(\frac{1}{n}) = Y_n$. We define $c(s,t) := \exp(td(s))$, but then

$$\sup_{s\in[0,1]}\sup_{0<t\le 1} \frac{d(c(s,t))}{t} = \infty$$

a contradiction. Given a sequence $\{X_n\}_{n\in\mathbb{N}}$ such that $p(X_n - X) \to 0$, then $d(\exp(t(X_n - X)), e) \le tp(X_n - X)$ by sublinearity. Consequently $\exp(t(X - X_n)) \to e$ uniformly on compact intervals in time for $n \to \infty$. This, however, means that $X - X_n \to 0$ in $\mathfrak{g}$ by assumption. ☐

Lemma 2.14. *Let G be a convenient Lipschitz-metrizable Lie group, such that convergence on the model space E means Mackey-convergence (i.e. $c^\infty E = E$). If there is furthermore a (smooth) exponential map and for any sequence $\{X_n\}_{n\in\mathbb{N}}$ with $\exp(tX_n) \to e$ uniformly on compact intervals the sequence $\{X_n\}_{n\in\mathbb{N}}$ converges to 0 in the Lie algebra $\mathfrak{g}$, then the functions*

$$p_\alpha(X) = \lim_{t\downarrow 0} \frac{d_\alpha(\exp(tX), e)}{t}$$

are continuous seminorms on $\mathfrak{g}$ generating the topology.

Proof. The proof is built in the same way as the previous one. Only indices have to be carried with along the lines. □

The last result provides an the already applied idea how the families of seminorms and right invariant metrics are related: This relation could be read in the other direction explaining that via integrating one obtains right invariant Lipschitz-metrics on G.

3. Approximation theorems

The following two proposition explain the interest in Lipschitz-metrizable Lie groups.

Theorem 3.1. *Let G be a Lipschitz-metrizable convenient Lie group, $c^\infty G$ is a topological group. If there is a smooth exponential mapping, then for all smooth mappings $c : \mathbb{R}^2 \to G$ with $c(s, 0) = e$ the following estimates are valid for the given halfmetrics d_α, $\alpha \in \Omega$: On each compact (s_1, s_2, t)-sets there exists M_α such that*

$$d_\alpha(c(s_1, \frac{s_2}{m})^m c(s_1, \frac{t}{n})^n c(s_1, t)^{-1} c(s_1, \frac{s_2}{m})^{-m}, e) \le M_\alpha t^2$$

for $m, n \in \mathbb{N}$.

Proof. First we show a simple consequence of property 2.6.1. Let $c : \mathbb{R}^{n+1} \to G$ with $c(\mathbf{s}, 0) = e$ and $\frac{\partial}{\partial t}|_{t=0} c(\mathbf{s}, t) = 0$ for $\mathbf{s} \in \mathbb{R}^n$ be a smooth mapping, then we can choose a chart (U, u) around e with $u(U)$ absolutely convex and $u(e) = 0$. On a small ball B around 0 in $\mathbb{R}^{n+1}$ $u \circ c$ is well-defined with first derivative zero. Consequently $u \circ c(\mathbf{s}, \sqrt{t})$ makes sense as Lip^0-curve for positive t and $\mathbf{s}$ in a small ball around zero in $\mathbb{R}^{n+1}$, so $\frac{1}{t}(u \circ c)(\mathbf{s}, \sqrt{t})$ is in a compact set for $t > 0$ and $\mathbf{s}$ in a small ball around zero in $\mathbb{R}^{n+1}$. By some reparametrizations we can assume that the compact set, where $\frac{1}{t} u \circ c(\mathbf{s}, \sqrt{t})$ lies, is a subset of $u(U)$. Let $B \subset E$ be a compact subset in $u(U)$, then the following supremum is finite:

$$\sup_{0<t\le 1} (\sup_{x\in B} \frac{d_\alpha(u^{-1}(tx), e)}{t}) < \infty$$

for all $\alpha \in \Omega$, since the function

$$M^\alpha(x) := \sup_{0<t\le 1} \frac{d_\alpha(u^{-1}(tx))}{t}$$

for $x \in u(U)$ is bounded on compact subsets of $u(U)$. If M^α were unbounded on a compact subset B of $u(U)$, then there would exist a sequence $\{x_n\}_{n\in\mathbb{N}_+}$ in B, converging fast to $x \in B$, with $M^\alpha(x_n) \ge n$ for $n \in \mathbb{N}_+$. By the special curve lemma there is a curve $d : \mathbb{R} \to F$ with $d(\frac{1}{n}) = x_n$, so $c(s, t) := td(s)$ is a smooth mapping with $c(s, 0) = 0$ with values in $u(U)$, which gives a contradiction by looking at $u^{-1} \circ c$.

Reconsidering the original problem we obtain

$$\sup_{0<t\le 1} \frac{d_\alpha(c(\mathbf{s}, \sqrt{t}), e)}{t} < M_\alpha$$

on a small set around zero in $\mathbf{s}$. This can easily be extended to all compact sets by a translation. We obtain finally

$$\sup_{0<t\le 1} \frac{d_\alpha(c(\mathbf{s}, t), e)}{t^2} < M_\alpha \qquad (\#)$$

on compact $\mathbf{s}$-sets.

Now we apply the existence of a smooth exponential mapping. Let $T(X)$ denote a semigroup with generator X. A smooth mapping $c : \mathbb{R}^2 \to G$ with $c(s,0) = e$ and $\frac{\partial}{\partial t}|_{t=0} c(s,t) = X_s$ for $s \in \mathbb{R}$ is given, too. We proceed indirectly to obtain the assertion: Let $n \in \mathbb{N}$ be given, then

$$\begin{aligned}
& d_\alpha(c(s, \frac{t}{m})^m T_{-t}(X_s), e) \\
& \leq \sum_{i=0}^{m-1} d_\alpha(T_{\frac{ti}{m}}(X_s) c(s, \frac{t}{m})^{m-i} T_{-t}(X_s), T_{\frac{t(i+1)}{m}}(X_s) c(s, \frac{t}{m})^{m-i-1} T_{-t}(X_s)) \\
& = \sum_{i=0}^{m-1} d_\alpha(T_{\frac{ti}{m}}(X_s), T_{\frac{t(i+1)}{m}}(X_s) c(s, \frac{t}{m})^{-1}) \\
& = \sum_{i=0}^{m-1} d_\alpha(T_{\frac{ti}{m}}(X_s) c(s, \frac{t}{m}) T_{-\frac{ti}{m}}(X_s), T_{\frac{t}{m}}(X_s))
\end{aligned}$$

due to right invariance. Our uniformity result leads to the desired assertion by investigating the smooth mapping

$$d(\mathbf{s}, t) := T_{s_2}(X_{s_1}) c(s_1, \frac{t}{m}) T_{-\frac{t}{m}}(X_{s_1}) T_{-s_2}(X_{s_1})$$

by estimate #. Consequently we arrive at

$$d_\alpha(c(s, \frac{t}{m})^m T_{-t}(X_s), e) \leq \sum_{i=0}^{m-1} M_\alpha \frac{t^2}{m^2} = M_\alpha \frac{t^2}{m} \overset{m \to \infty}{\to} 0$$

where t can vary in a compact interval around zero preserving. This estimate yields that

$$\begin{aligned}
& c(s_1, \frac{s_2}{m})^m c(s_1, \frac{t}{n})^n c(s_1, t)^{-1} c(s_1, \frac{s_2}{m})^{-m} \\
= & c(s_1, \frac{s_2}{m})^m T_{-s_2}(X_{s_1}) T_{s_2}(X_{s_1}) c(s_1, \frac{t}{n})^n c(s_1, t)^{-1} T_{-s_2}(X_{s_1}) T_{s_2}(X_{s_1}) c(s_1, \frac{s_2}{m})^{-m}
\end{aligned}$$

satisfies the desired estimate by applying $d(abc, e) \leq d(abc, bc) + d(bc, c) + d(c, e) = d(a, e) + d(b, e) + d(c, e)$ due to right invariance. □

Theorem 3.2 (Approximation Theorem)**.** *Let G be a Lipschitz-metrizable regular Lie group, such that $c^\infty G$ is a topological group. Furthermore for all smooth mappings $c : \mathbb{R}^2 \to G$ with $c(s,0) = e$ the following estimates are valid for the given halfmetrics d_α, $\alpha \in \Omega$: On each compact (s_1, s_2, t)-sets there exists M_α such that*

$$d_\alpha(c(s_1, \frac{s_2}{m})^m c(s_1, \frac{t}{n})^n c(s_1, t)^{-1} c(s_1, \frac{s_2}{m})^{-m}, e) \leq M_\alpha t^2$$

for $m, n \in \mathbb{N}$. Given a smooth curve $c : \mathbb{R} \to G$ with $c(0) = e$, the limit

$$\lim_{n \to \infty} c(\frac{t}{n})^n = T_t$$

exists uniformly on compact intervals of $\mathbb{R}$ and gives a smooth group T. The convergence is uniform in all derivatives in the sense of Lemma 1.4.

Proof. Given a smooth curve $c : \mathbb{R}^2 \to G$ with $c(s, 0) = e$, we try to investigate the above simple product integrals:

$$\begin{aligned}
& d_\alpha(c(s, \frac{t}{nm})^{nm}, c(s, \frac{t}{n})^n) \\
& \leq \sum_{i=0}^{n-1} d_\alpha(c(s, \frac{t}{n})^i c(s, \frac{t}{nm})^{(n-i)m}, c(s, \frac{t}{n})^{i+1} c(s, \frac{t}{nm})^{(n-i-1)m}) \\
& \leq \sum_{i=0}^{n-1} d_\alpha(c(s, \frac{t}{n})^i c(s, \frac{t}{nm})^m c(s, \frac{t}{n})^{-1} c(s, \frac{t}{n})^{-i}, e) \\
& \leq \sum_{i=0}^{n-1} d_\alpha(c(s, \frac{t}{n})^i c(s, \frac{t}{nm})^m c(s, \frac{t}{n})^{-i}, c(s, \frac{t}{n})^{-1}) \\
& \leq n\frac{t^2}{n^2} M_\alpha \to 0 \text{ for } n \to \infty
\end{aligned}$$

which is possible by a look at the curve

$$d(\mathbf{s}, t) = c(s_1, \frac{s_2}{i})^i c(s_1, \frac{t}{nm})^m c(s_1, t)^{-1} c(s_1, \frac{s_2}{i})^{-i}$$

and the application of the given estimates. Consequently we obtain a Cauchy-property uniform in s for the above sequences of curves, which leads to the desired limit. The limit $\lim_{n\to\infty} c(s, \frac{t}{n})^n =: T_t(X_s)$ is continuous in s, t. By looking at the left regular representation in $L(C^\infty(G, \mathbb{R}))$ we see that the limit has to be smooth and a group in t, because sequentially compact sets are mapped to bounded ones and the smooth functions detect smoothness: $\rho \circ c$ gives a curve in $L(C^\infty(G, \mathbb{R}))$ satisfying the boundedness condition, so we expect a smooth limit group $T(s, t)$ by Theorem 1.5. Since we have convergence of $c(s, \frac{t}{n})^n$ this limit has to be a posterio equal to $\rho(\lim_{n\to\infty} c(s, \frac{t}{n})^n)$. By initiality of ρ we obtain the smoothness of $\lim_{n\to\infty} c(s, \frac{t}{n})^n$ as mapping to G. The limit exists uniformly in all derivatives, which means in particular that the generator of T is $c'(0)$ by the Lemma 1.4. since we can evaluate at e to obtain $(f \circ c)(s, \frac{t}{n})^n \to f(\lim_{n\to\infty} c(s, \frac{t}{n})^n)$ in all derivatives with respect to s and t. ☐

Remark 3.3. *We have proved that the existence of an exponential map can be characterized on "all" Lie groups in the framework of Lipschitz-metrizability. In the abelian case the situation is simpler, we can reformulate the proposition and define in a simpler way the Lipschitz-metrics.*

Corollary 3.4. *Let G be an abelian Lie group, such that $c^\infty G$ is a topological group, then G is regular if and only if G is Lipschitz-metrizable. In particular (due to regularity or Lipschitz-metrizability) for all $c : \mathbb{R}^2 \to G$ with $c(s, 0) = e$ the following estimates are valid: On each compact (s, t)-set there is a constant M_α such that*

$$d_\alpha(c(s, \frac{t}{n})^n c(s, t)^{-1}, e) \leq M_\alpha t^2$$

for all $n \geq 1$.

Proof. The additional comment is a corollary of the theorem. Let G be a Lipschitz-metrizable abelian Lie group, then we have to show the stated estimate: Given $c : \mathbb{R}^2 \to G$ with $c(s, 0) = e$ we can write by commutation and right invariance

$$\begin{aligned}
d_\alpha(c(s, \frac{t}{n})^n c(s, t)^{-1}, e) & \leq \sum_{i=1}^{n} d_\alpha(c(s, \frac{t}{n}) c(s, \frac{it}{n}) c(s, \frac{(i-1)t}{n})^{-1}, e) \leq \\
& \leq \sum_{i=1}^{n} M_\alpha \frac{t^2}{n^2} = M_\alpha \frac{t^2}{n}
\end{aligned}$$

for $n \geq 1$. This estimate is obtained via

$$d_\alpha(c(s, ut)c(s, vt), c(s, (u + v)t)) \leq M_\alpha t^2$$

Let G be a regular abelian Lie group. Then G is locally isomorphic to its Lie algebra by [MT98], consequently the topology on G is given by the bornological topology on $\mathfrak{g}$. We denote by Ω the set of bounded seminorms on $\mathfrak{g}$.

$$d_k(g,h) := \inf_{\substack{c\in C^\infty([0,1],G)\\ c(0)=g,\, c(1)=h}} \int_0^1 p(\delta^r c(t))dt$$

is a well-defined right invariant halfmetric on G for $p \in \Omega$. Right-invariance is clear by definition, symmetry, too. Taking two curves $c, d \in C^\infty([0,1],G)$ with $c(0) = g$, $c(1) = d(0) = h$ and $d(1) = l$, then $b := c\mu_{h^{-1}}d$ defines a smooth curve with $b(0) = g$ and $b(1) = l$, furthermore $\delta^r b(t) = \delta^r c(t) + \delta^r d(t)$ on $[0,1]$ due to commutativity (the adjoint map is trivial). The Lipschitz-property is clear by the following argument: Let $c : \mathbb{R}^2 \to G$ be smooth mapping with $c(0,s) = e$ for $s \in \mathbb{R}$, then

$$d_k(c(u,s),e) \le \int_0^1 p_k(\delta^r c(u.,s)(t))dt = u\int_0^1 p_k([\delta^r c](ut,s)])dt$$

and consequently the supremum exists uniformly for s in a compact interval. The rest follows by regularity from the lemma. It remains to prove the topological property, but this is clear due to the possibility to choose an exponential chart (see [MT98]). So we constructed the essentials for Lipschitz-metrizability. □

Theorem 3.5. *Let G be a Lipschitz-metrizable convenient Lie group with $c^\infty G$ a topological group, $h : \mathbb{R}^2 \to G$ a smooth mapping with $h(s,0) = e$ and $\frac{\partial}{\partial t}|_{t=0}h(s,t) = X(s)$ and c with $c(0) = e$ a smooth curve with $\delta^r c = X$, then the product integral $\prod_0^t h(s,ds)$ exists and equals $c(t)$. If G is regular, then the following estimates are valid for the Lipschitz-metrics d_α:*

$$d_\alpha(p_i(s_3,t,c)(s_1)p_n(s_2,t+s_2,c)(s_1)c(s_1,s_2,t)^{-1}p_i(s_3,t,c)(s_1)^{-1},e) \le M_\alpha t^2$$

for all $i,n \in \mathbb{N}_+$ on compact (s_1,s_2,s_3,t)-sets given the smooth mapping $d : \mathbb{R}^3 \to G$ with $c(s_1,s_2,0) = e$.

Proof. First we prove the convergence result to establish the estimate. Given a smooth mapping $h : \mathbb{R}^3 \to G$ with $h(s_1,s_2,0) = e$, then we look at the product integral

$$p_n(s_2,t,h)(s_1) = \prod_{i=0}^{n-1} h(s_1, s_2 + \frac{(n-i)(t-s_2)}{n}, \frac{t-s_2}{n})$$

at $s_2 = 0$. Let $c : \mathbb{R}^2 \to G$ be a curve with $c(s_1,0) = e$ and $\delta^r c_{s_1}(s_2) = \frac{\partial}{\partial t}h(s_1,s_2,0)$.

$$d_\alpha(\prod_{i=0}^{n-1} h(s_1, \frac{(n-i)t}{n}, \frac{t}{n}), c(s_1,t)) \le$$

$$\le \sum_{i=0}^{n-1} d_\alpha(\prod_{j=1}^{i} c(s_1,\frac{(n-j+1)t}{n})c(s_1,\frac{(n-j)t}{n})^{-1}\prod_{j=i}^{n-1} h(s_1,\frac{(n-j)t}{n},\frac{t}{n})c(s_1,t)^{-1},$$
$$\prod_{j=1}^{i+1} c(s_1,\frac{(n-j+1)t}{n})c(s_1,\frac{(n-j)t}{n})^{-1}\prod_{j=i+1}^{n-1} h(s_1,\frac{(n-j)t}{n},\frac{t}{n})c(s_1,t)^{-1}) \le$$
$$\le \sum_{i=0}^{n-1} d_\alpha(c(s_1,t)c(s_1,\frac{(n-i)t}{n})^{-1}h(s_1,\frac{(n-i)t}{n},\frac{t}{n})c(s_1,\frac{(n-i-1)t}{n})c(s_1,\frac{(n-i)t}{n})^{-1},$$
$$c(s_1,t)c(s_1,\frac{(n-i)t}{n})^{-1}) \le$$
$$\le n\frac{t^2}{n^2}M_\alpha$$

for $n \in \mathbb{N}$ on compact s_1-sets. The last step of the proof is done by the same arguments as in the proof of Theorem 3.1. □

Theorem 3.6. *Let G be a Lipschitz-metrizable convenient Lie group with $c^\infty G$ a topological group. For all smooth mappings $c : \mathbb{R}^3 \to G$ with $c(s_1,s_2,0) = e$ the following estimate is valid*

$$d_\alpha(p_i(s_3,t,c)(s_1)p_n(s_2,t+s_2,c)(s_1)c(s_1,s_2,t)^{-1}p_i(s_3,t,c)(s_1)^{-1},e) \le M_\alpha t^2$$

for all $n \in \mathbb{N}$ on compact (s_1,s_2,s_3,t)-sets. Then all product integrals exist, furthermore the right evolution operator given through these product integrals is smooth, so G is regular.

Proof. We shall apply the following abbreviation

$$p_i(\frac{n-i}{n}t,t,c)(s_1)=p_{n,i}(t,c)(s_1)=\prod_{j=0}^{i}c(s_1,\frac{(n-j)t}{n},\frac{t}{n})$$

We can proceed directly to obtain the result by our methods. Given a smooth mapping $c:\mathbb{R}^3\to G$ with $c(s_1,s_2,0)=e$, we shall look at the product integral

$$p_{n,i}(t,c)(s_1)=\prod_{j=0}^{i}c(s_1,\frac{(n-j)t}{n},\frac{t}{n})$$

at $s_2=0$. The notion allows to shorten the product: $0\le i\le n-1$, $p_{n,n-1}=p_n$.

$$d_\alpha(p_{nm}(t,c)(s_1),p_n(t,c)(s_1))\le$$
$$\le\sum_{i=0}^{n-1}d_\alpha\left(p_{n,i}(t,c)(s_1)\prod_{j=mi}^{nm-1}c(s_1,\frac{(nm-j)t}{nm},\frac{t}{nm})p_n(t,c)(s_1)^{-1},\right.$$
$$\left.p_{n,i+1}(t,c)(s_1)\prod_{j=m(i+1)}^{nm-1}c(s_1,\frac{(nm-j)t}{nm},\frac{t}{nm})p_n(t,c)(s_1)^{-1}\right)\le$$
$$\le\sum_{i=0}^{n}d_\alpha\left(p_{n,i}(t,c)(s_1)\prod_{j=mi}^{m(i+1)-1}c(s_1,\frac{(nm-j)t}{nm},\frac{t}{nm}),p_{n,i+1}(t,c)(s_1)\right)=$$
$$=\sum_{i=0}^{n-1}d_\alpha\left(p_{n,i}(t,c)(s_1)\prod_{j=mi}^{m(i+1)-1}c(s_1,\frac{(nm-j)t}{nm},\frac{t}{nm})c(s_1,\frac{(n-i-1)t}{n},\frac{t}{n})^{-1},\right.$$
$$\left.p_{n,i}(t,c)(s_1)\right)=$$
$$=\sum_{i=0}^{n-1}d_\alpha\left(p_{n,i}(t,c)(s_1)\prod_{j=0}^{m-1}c(s_1,\frac{(n-i-1)t}{n}+\frac{(m-j)t}{nm},\frac{t}{nm})\right.$$
$$\left.c(s_1,\frac{(n-i-1)t}{n},\frac{t}{n})^{-1}p_{n,i}(t,c)(s_1)^{-1},e\right)\le$$
$$\le\sum_{i=0}^{n-1}M_\alpha\frac{t^2}{n^2}$$

for $n\in\mathbb{N}$ and compact (s_1,t)-intervals due to the given estimate. Furthermore by Theorem 1.5 and Lemma 1.4 we obtain the smoothness of these solution families. ☐

Corollary 3.7. *Let G be a Lipschitz-metrizable convenient Lie group with $c^\infty G$ a topological group, then the following assertions are equivalent:*

1. *A smooth exponential map* $\exp:\mathfrak{g}\to G$ *exists (a smooth right evolution map exists)*
2. *All simple product integrals converge in* $C^\infty(\mathbb{R}^2,G)$ *(all product integrals converge in* $C^\infty(\mathbb{R}^3,G)$ *in the sense of Lemma 1.4).*

Corollary 3.8. *Let G be a regular, smoothly connected Lipschitz-metrizable Lie group, then the closure of the normal subgroup generated by the image of the exponential map is the whole group G.*

Proof. Regularity implies the existence of product integrals $\prod_0^a\exp(X(s)ds)$, which reach any point in the smoothly connected Lie group, consequently the closure of the normal subgroup generated by the image of the exponential map is the whole group. ☐

4. Product integration via Linearization

Theorem 4.1. *Let G be a smoothly regular Lie group. If for each smooth mapping $c:\mathbb{R}^3\to G$ with $c(r_1,r_2,0)=e$ the approximations $p_n(s_2,t,c)(r_1)$ lie in a sequentially c^∞-compact set on compact (r_1,s_2,t)-sets, then G is regular.*

Theorem 4.2. *Let G be a smoothly regular Lie group. If for each smooth mapping $c : \mathbb{R}^2 \to G$ with $c(r_1, 0) = e$ the approximations $c(r_1, \frac{t}{n})^n$ lie in a sequentially c^∞-compact set on compact (r_1, t)-sets, then G admits a smooth exponential map.*

Proof. The proofs for the theorems are identical: A sequentially c^∞-compact set is mapped by ρ to a sequentially c^∞-compact set, which is bounded in any compatible locally convex topology. Consequently we obtain the existence of the image product integral, but this image product integral stems pointwisely from G via ρ, because there are adherence points in the sequentially c^∞-compact set, which have to be the unique limit points of the respective sequences. ρ is initial, so the limit curve has to be smooth and the uniform convergence in all derivatives in $L(C^\infty(G, \mathbb{R}))$ implies uniform convergence in all derivatives in the sense of Lemma 1.4. of the products to the product integral. □

Proposition 4.3. *Let G be a smoothly regular Lipschitz-metrizable Lie group: Furthermore for all sets $K \subset G$ lying in a given fixed neighborhood of identity, such that $d_\alpha(K, e) \leq N_\alpha$ for all $\alpha \in \Omega$ if and only if K is relatively sequentially compact in the topology of G.Then G is regular, i.e. a smooth right evolution exists.*

Proof. Given a smooth mapping $c : \mathbb{R}^3 \to G$ with $c(s_1, s_2, 0) = e$, then the products p_n can be estimated in the following way:

$$d_\alpha(p_n(s_2, t)(s_1), e) \leq \sum_{j=0}^{n-1} d_\alpha(c(s_1, \frac{(n-j)t}{n}, \frac{t}{n}), e) \leq n\frac{t}{n} M_\alpha$$

on compact (s_1, s_2, t)-sets. Consequently the approximations lie in a compact set for t and s small enough. If all approximations of product integrals lie in a sequentially compact set for compact parameter sets, we can apply the regularity theorem of 1.5 to conclude regularity as in the previous proof. □

Remark 4.4. *This property can be viewed as a non-linear version of Arzela-Ascoli's theorem.*

Conjecture 4.5. *Let G be a strong ILB-group, such that the associated Fréchet space is Montel. G is seen to be Lipschitz-metrizable and regular by the above considerations. It is reasonable to expect that for all sets $K \subset G$ lying in a small neighborhood of identity U*

$$d_n(K, e) \leq N_n \text{ for all } n \text{ if and only if } K \text{ is relatively compact in the topology of } G$$

This would provide a simple procedure to solve non-autonomuous differential equations of the type $\delta^r c(t) = X(t)$ for $t \in \mathbb{R}$ on the Lie group by "intrinsic methods".

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