SMOOTH PERFECTNESS THROUGH DECOMPOSITION OF DIFFEOMORPHISMS INTO FIBER PRESERVING ONES

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Abstract. We show that on a closed smooth manifold M equipped with k fiber bundle structures whose vertical distributions span the tangent bundle, every smooth diffeomorphism f of M sufficiently close to the identity can be written as a product $f = f_1 \cdots f_k$, where f_i preserves the i^{th}-fiber. The factors f_i can be chosen smoothly in f. We apply this result to show that on a certain class of closed smooth manifolds every diffeomorphism sufficiently close to the identity can be written as product of commutators and the factors can be chosen smoothly. Furthermore we get concrete estimates on how many commutators are necessary.

1. Introduction

We are concerned with the question of perfectness of diffeomorphism groups on compact manifolds. It is well known that the e-components of diffeomorphism groups on compact smooth manifolds are perfect by results of Herman [Her73], Thurston [Thu74], Mather [Mat74, Mat75] and Epstein [Eps84]. However, the questions, how many commutators are necessary to represent a given smooth diffeomorphism f via

$$f = [h_1, g_1] \cdots [h_n, g_n]$$

and if these commutators can be chosen smoothly in f, remains open. Only in the case of the torus T^n the result of Herman provides the concrete, positive answer by a beautiful small denominator argument.

We shall provide concrete, positive answers for both questions in a subclass of all compact smooth manifolds by a decomposition theorem (section 3) and applications of some canonical exponential laws ("parameterization of diffeomorphisms"). In particular the diffeomorphism groups of odd dimensional spheres can be treated.

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2. Inverse Function Theorems

We shall apply Nash–Moser inverse function theorems in the spirit of Richard Hamilton. See [Ham82] for all necessary details, where the theory of tame spaces and the tame inverse function theorems are introduced.

Given graded Fréchet spaces E and F, i.e. we are additionally given an increasing sequence of seminorms $\{p_n\}_{n\geq 0}$ on E and $\{q_n\}_{n\geq 0}$ on F, a tame estimate for a linear map $L : E \to F$ is given by

$$q_n(Le) \leq C_n p_{n+r}(e)$$

for a fixed basis b, the (tame) degree r and for all $n \geq b$, $e \in E$. In particular such a map is continuous. We shall call such maps tame linear. A linear isomorphism $L : E \to F$ is called a tame isomorphism if L and L^{-1} satisfy tame estimates. To prove the inverse function theorem we need to work on tame Fréchet spaces: a tame Fréchet space is a graded Fréchet space, which is a tame direct summand of a space $\Sigma(B)$, the Fréchet space of all very fast falling sequences in a Banach space B, see [KM97, p. 558]. Let E and F be graded Fréchet spaces and $P : E \supseteq U \to F$ be a map. Then P satisfies a tame estimate if

$$q_n(P(e)) \leq C_n(1 + p_{n+r}(e))$$

for a fixed basis b, the (tame) degree r and for all $n \geq b$, $e \in U$. We shall call such maps tame maps. Clearly a mapping is tame linear if it is tame and linear. For mappings on products we can define tame degrees for any term in the product, which is useful in applications. We shall work in the category $\mathcal{T}$, whose objects are open subsets of tame spaces and whose morphisms are *smooth tame* maps, i.e. mappings such that all derivatives are tame. The Nash–Moser inverse function theorems will be stated in this category. By means of the tame category we can define tame manifolds, bundles and geometric or algebraic structures.

The inverse function theorem finally reads as follows in its general version in $\mathcal{T}$, see [Ham82] for both theorems and all necessary details.

Theorem 1. *Let E and F be tame spaces and $P : E \supseteq U \to F$ be a smooth tame map. Suppose that the equation $DP(e)h = f$ has a unique solution $h = VP(e)f \in E$ for all $e \in U$ and all $f \in F$ and that the family of inverses $VP : U \times F \to E$ is smooth tame, then P is locally invertible and the inverse is a smooth tame map.*

We shall apply the version for right inverses.

Theorem 2. *Let E and F be tame spaces and $P : E \supseteq U \to F$ be a smooth tame map. Suppose that the equation $DP(e)h = f$ has a solution $h = VP(e)f \in E$ for all $e \in U$ and all $f \in F$ and that the family of right inverses $VP : U \times F \to E$ is smooth tame, then P is locally surjective and admits a smooth tame local right inverse.*

3. The Decomposition Theorem

In this section we shall prove the fundamental decomposition theorem, which allows a "smooth" decomposition of a small diffeomorphism on a compact manifold into more regular parts. This shall be applied in the following sections to obtain perfectness results on Fréchet–Lie groups such as $\mathrm{Diff}(S^{2n+1})$.

Let M be a closed, connected manifold, such that there exist k fiber bundle structures $S_i \hookrightarrow M \xrightarrow{p_i} B_i$ for $1 \leq i \leq k$ with connected fibers S_i and (involutive) vertical distribution $\mathcal{D}_i$, which is a subbundle of TM. We suppose that the distributions $\mathcal{D}_i$ span TM. We neither assume linear independence nor additional integrability conditions.[1] We denote the Lie subgroup of $\mathrm{Diff}(M)$ of bundle diffeomorphisms by $\mathrm{Diff}_i(M)$ for $1 \leq i \leq k$. The Lie algebra of $\mathrm{Diff}(M)$ is given by smooth sections $\Gamma(TM)$, the Lie algebra of $\mathrm{Diff}_i(M)$ by sections $\Gamma(\mathcal{D}_i)$. These Lie groups are tame manifolds, i.e. a smooth tame atlas exists and the structure maps are smooth tame. Remark in particular that the pullback, i.e. the adjoint action $\mathrm{Ad} : \mathrm{Diff}(M) \times \Gamma(TM) \to \Gamma(TM)$, $(f, X) \mapsto (f^{-1})^*X$ is smooth tame as derivative of the conjugation. Furthermore the module structure on $\Gamma(TM)$ is tame. See [KM97] for the general theory of Lie groups and [Ham82] for all assertions on tame Lie groups.

We shall apply the Nash–Moser Theorem in the following version: Given tame manifolds $\mathcal{M}$ and $\mathcal{N}$ and a smooth tame map $P : \mathcal{M} \to \mathcal{N}$, then the existence of a local smooth tame right inverse of P is equivalent to the existence of a local smooth tame right inverse (vector bundle map) to $\widetilde{TP} : T\mathcal{M} \to P^*T\mathcal{N}$, where $T\mathcal{N}$ denotes the (tame) tangent bundle and $P^*T\mathcal{N}$ denotes the (tame) pullback bundle.

Theorem 3. *The smooth tame mapping*

$$P : \mathrm{Diff}_1(M) \times \cdots \times \mathrm{Diff}_k(M) \to \mathrm{Diff}(M)$$
$$(f_1, \ldots, f_k) \mapsto f_1 \circ \cdots \circ f_k$$

admits a smooth tame local right inverse at the identity $e \in \mathrm{Diff}(M)$.

Proof. In the left trivialization of the tangent bundles of the respective Lie groups the tangent mapping of P is represented by

$$\widetilde{TP} : \prod_{i=1}^{k} \mathrm{Diff}_i(M) \times \prod_{i=1}^{k} \Gamma(\mathcal{D}_i) \to \left(\prod_{i=1}^{k} \mathrm{Diff}_i(M)\right) \times \Gamma(TM)$$
$$(f_1, \ldots, f_k; \xi_1, \ldots, \xi_k) \mapsto (f_1, \ldots, f_k; f_k^* \cdots f_2^* \xi_1 + f_k^* \cdots f_3^* \xi_2 + \cdots + \xi_k).$$

In view of the implicit function theorem it suffices to construct a smooth tame right inverses of $\widetilde{TP}$, linear in the variables ξ_i. We solve the problem

[1] Note, that if M appears as the total space of a fiber bundle $S \hookrightarrow M \xrightarrow{p} B$ and $\dim S \geq 1$, then one can always perturb p to obtain finitely many fiber bundles $S \hookrightarrow M \xrightarrow{p_i} B$, such that the vertical distributions span TM.

locally: First we choose a covering $\mathfrak{U}$ of open subsets of M such that for $U \in \mathfrak{U}$ the bundles $TM|_U$ and $\mathcal{D}_i|_U$ are trivial and that a compatible local frame (see below) exists. Second we choose a finite partition of unity $\{\eta_U\}_{U\in\mathfrak{U}}$ subordinated to this covering. The associated projection

$$p : \Gamma(TM) \to \bigoplus_{U\in\mathfrak{U}} \Gamma_V(TM|_U)$$
$$p_U(X) = \eta_U X$$

where Γ_V denotes the sections with support in the closed set V and V is the support of η_U in U, is a right inverse of the sum.

On $U \in \mathfrak{U}$ we can solve the equation locally. We choose a local frame $X^1, \dots, X^n$, where $n = \dim M$, compatible with the distribution on U, i.e. there are integers $0 = m_0 \leq m_1 \leq \cdots \leq m_k = n$, such that

$$\mathcal{D}_i(x) \supset \langle X^{n_i}(x), \dots, X^{m_i}(x)\rangle \quad \text{for all } x \in U,$$

where we set $n_i := m_{i-1} + 1$, $i = 1, \dots, k$. We assume furthermore that the vector fields X^j are globally defined on M. We then choose an open neighborhood V_i of $e \in \mathrm{Diff}_i(M)$ such that for all $f_i \in V_i$ the condition $f_1 \circ \cdots \circ f_k(V) \subset U$ holds and such that

$$\mathfrak{F}_{(f_1,\dots,f_k)} := (f_k^* \cdots f_2^* X_{n_1}, \dots, f_k^* \cdots f_2^* X_{m_1}, f_k^* \cdots f_3^* X_{n_2}, \dots, X_n)$$

is a frame for TM on V. Given $Y \in \Gamma_V(TM|_U)$, we define a section $s_{i,U}$ via the decomposition on the frame $\mathfrak{F}_{(f_1,\dots,f_k)}$ by the following formula

$$s_{i,U}(Y) := (f_k^* \cdots f_{i+1}^*)^{-1} \left(\sum_{j=n_i}^{m_i} a_j(Y) f_k^* \cdots f_{i+1}^* X_j \right)$$

where

$$Y = \sum_{i=1}^{k} \sum_{j=n_i}^{m_i} a_j(Y) f_k^* \cdots f_{i+1}^* X_j .$$

The functions a_j are smooth with support in V, so $s_{i,U}(Y)$ has support in U and consequently the section defines an element of $\Gamma(\mathcal{D}_i)$ by construction. The pullbacks depend tamely on the diffeomorphism and the module structure is tame, hence the functions a_j depend tamely on the diffeomorphisms $f_{i+1}, \dots, f_k$ and the vector field Y. Consequently the mapping

$$\Big(\prod_{i=1}^{k} V_i\Big) \times \Gamma(TM) \to \prod_{i=1}^{k} \mathrm{Diff}_i(M) \times \prod_{i=1}^{k} \Gamma(\mathcal{D}_i)$$
$$(f_1, \dots, f_k; X) \mapsto (f_1, \dots, f_k; \sum_{U\in\mathfrak{U}} s_{1,U}(\eta_U X), \dots, \sum_{U\in\mathfrak{U}} s_{k,U}(\eta_U X))$$

is the desired smooth tame right inverse of $\widetilde{TP}$. $\square$

4. Perfectness of fiber-preserving Diffeomorphisms

In this section we introduce three notions of perfectness on regular Lie groups. These notions are finally applied to "parametrized" families of diffeomorphisms, which is an application of cartesian closedness, see for example [KM97].

All manifolds, Lie groups and Lie algebras are supposed to be smooth and modeled on convenient vector spaces. This includes all Fréchet manifolds, Fréchet–Lie groups and Fréchet–Lie algebras. We moreover assume, that the Lie groups are regular, in particular they admit a smooth exponential mapping. This is not too much a restriction, since "all known convenient Lie groups are regular". Notice, that convenient Lie groups are not necessarily topological groups, however Fréchet–Lie groups are. For convenient calculus and regular Lie groups see [KM97, Ch. I and Ch. VIII]. Conditions for regularity of convenient Lie groups can be found in [Omo97], [Tei01].

Definition 1. *For a Lie group G with non-trivial e-component we define $N_G \in \mathbb{N}$ to be the smallest integer N, such that for every open neighborhood $e \in U \subseteq G$ their exist $h_i = \exp(Y_i) \in U$, $i = 1, \dots, N$, an open neighborhood $e \in V \subseteq G$ and smooth mappings $S_i : V \to G$ with $S_i(e) = e$ and*

$$[S_1(g), h_1] \cdots [S_N(g), h_N] = g, \quad \textit{for all } g \in V.$$

Equivalently, N_G is the smallest integer N, such that for every open neighborhood $e \in U \subseteq G$ there exist $h = \exp(Y) \in U^N$, such that the map

$$\kappa_h : G^N \to G, \quad (g_1, \dots, g_N) \mapsto [g_1, h_1] \cdots [g_N, h_N]$$

has a smooth local right inverse S with $S(e) = (e, \dots, e)$. If such an integer does not exist we set $N_G := \infty$. We call the Lie group G locally smoothly perfect *if $N_G < \infty$.*

Definition 2. *For a Lie group G with Lie algebra $\mathfrak{g} \neq 0$ we define $N_G^{\mathrm{Ad}} \in \mathbb{N}$ to be the smallest integer N, such that for every open neighborhood $e \in U \subseteq G$ their exist $h_i = \exp(Y_i) \in U$, $i = 1, \dots, N$, and bounded linear[2] maps $s_i : \mathfrak{g} \to \mathfrak{g}$ with*

$$(\mathrm{id} - \mathrm{Ad}_{h_1}) s_1(X) + \cdots + (\mathrm{id} - \mathrm{Ad}_{h_N}) s_N(X) = X \quad \textit{for all } X \in \mathfrak{g}.$$

Equivalently, N_G^{Ad} is the smallest integer N, such that for every open neighborhood $e \in U \subseteq G$ there exist $h = \exp(Y) \in U^N$ and a bounded linear right inverse $s : \mathfrak{g} \to \mathfrak{g}^N$ of the map $T_{(e,\dots,e)}\kappa_h : \mathfrak{g}^N \to \mathfrak{g}$. If such an integer does not exist we set $N_G^{\mathrm{Ad}} := \infty$.

Remark 1. *It would be more natural to claim existence of arbitrary, small h_i, not only those which are exponentials, however, in this general case we do not get the desired "natural" inequalities, cf. Lemma 1 below.*

[2] Note, that in convenient calculus the smooth linear mappings coincide with the bounded linear ones.

Definition 3. *For a Lie algebra $\mathfrak{g} \neq 0$ we define $N_{\mathfrak{g}} \in \mathbb{N}$ to be the smallest integer N, such that there exist $Y_i \in \mathfrak{g}$, $i = 1, \dots, N$, and bounded linear maps $\sigma_i : \mathfrak{g} \to \mathfrak{g}$ with*

$$[\sigma_1(X), Y_1] + \cdots + [\sigma_N(X), Y_N] = X, \quad \textit{for all } X \in \mathfrak{g}.$$

Equivalently $N_{\mathfrak{g}}$ is the smallest integer N, such that there exist $Y \in \mathfrak{g}^N$ and a bounded linear right inverse $\sigma : \mathfrak{g} \to \mathfrak{g}^N$ of the mapping

$$K_Y : \mathfrak{g}^N \to \mathfrak{g}, \quad K_Y(X_1, \dots, X_N) := [X_1, Y_1] + \cdots + [X_N, Y_N].$$

If such an integer does not exist we set $N_{\mathfrak{g}} := \infty$.

Lemma 1. *For any Lie group G with Lie algebra $\mathfrak{g}$ one has $N_{\mathfrak{g}} \leq N_G^{\mathrm{Ad}} \leq N_G$. If G is a Banach–Lie group one even has $N_{\mathfrak{g}} = N_G^{\mathrm{Ad}} = N_G$.*

Proof. $N_G^{\mathrm{Ad}} \leq N_G$ follows immediately from differentiating $\kappa_h \circ S = \mathrm{id}$ at $e \in G$, i.e. one can take $s = T_e S$. If G is a Banach–Lie group then the implicit function theorem shows $N_G^{\mathrm{Ad}} \geq N_G$. Notice, that every h sufficiently close to the identity is in the image of the exponential map, for the latter is a local diffeomorphism on Banach–Lie groups.

Next we show $N_{\mathfrak{g}} \leq N_G^{\mathrm{Ad}}$. For $Y \in \mathfrak{g}$ we have $\frac{\partial}{\partial t} \mathrm{Ad}_{\exp(tY)} = \mathrm{ad}_Y \circ \mathrm{Ad}_{\exp(tY)}$. Integration immediately yields

$$\mathrm{id} - \mathrm{Ad}_h = -\mathrm{ad}_Y \circ \int_0^1 \mathrm{Ad}_{\exp(tY)}\, dt,$$

where $h = \exp(Y)$. Inserting the bounded linear right inverse s for $T_{(e,\dots,e)}\kappa_h$, we obtain a bounded linear right inverse $\sigma = (\int_0^1 \mathrm{Ad}_{\exp(tY)}\, dt \circ s_i)_{i=1,\dots,N_{\mathfrak{g}}}$ for K_Y.

Suppose G is a Banach–Lie group. We want to show $N_{\mathfrak{g}} \geq N_G^{\mathrm{Ad}}$. Choose $Y \in \mathfrak{g}^{N_{\mathfrak{g}}}$, such that K_Y has a bounded linear right inverse and choose a smooth curve $h_t \in U^{N_{\mathfrak{g}}}$ with $h_0 = (e, \dots, e)$ and $\dot{h}_0 = Y$. For $t > 0$ consider the maps $K_t := \frac{1}{t} T_{(e,\dots,e)}\kappa_{h_t}$ and note, that $\lim_{t\to 0} K_t = K_Y$. Since $\mathfrak{g}$ is a Banach space, the space of bounded linear mappings $\mathfrak{g}^{N_{\mathfrak{g}}} \to \mathfrak{g}$ which admit a bounded linear right inverse is open, hence for t sufficiently small K_t has a bounded linear right inverse and thus $T_{(e,\dots,e)}\kappa_{h_t} = tK_t$ as well, i.e. $N_{\mathfrak{g}} \geq N_G^{\mathrm{Ad}}$. □

Example 1. *For any finite dimensional Lie algebra $\mathfrak{g} \neq 0$ one has $N_{\mathfrak{g}} > 1$, since $\mathrm{ad}_Y : \mathfrak{g} \to \mathfrak{g}$ can't be surjective, for it has a non-trivial kernel. Moreover obviously $N_{\mathfrak{g}} \leq \dim \mathfrak{g}$, for every perfect Lie algebra $\mathfrak{g}$. Thus for a finite dimensional perfect Lie group G with non-trivial e-component one has $1 < N_G \leq \dim G$.*

Example 2. *For a complex semisimple Lie algebra $\mathfrak{g}$ we have $N_{\mathfrak{g}} = 2$. Indeed, choose H to be a regular element in the Cartan subalgebra $\mathfrak{h}$ and set $\rho := \sum_\alpha E_\alpha$, where the sum is over all simple roots and E_α denotes a non-zero element of the root space of α. Then $\mathrm{ad}_H(\mathfrak{g}) = \mathfrak{h}^\perp$ and $\mathrm{ad}_\rho(\mathfrak{g}) \supseteq \mathfrak{h}$, hence $\mathrm{ad}_H + \mathrm{ad}_\rho : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is onto.*

Example 3. *For a real semisimple Lie algebra* $\mathfrak{g}$ *we have* $N_{\mathfrak{g}} = 2$ *and hence* $N_G = 2$ *for every real semisimple Lie group* G*. Indeed, suppose conversely* $[X, \mathfrak{g}] + [Y, \mathfrak{g}] \neq \mathfrak{g}$ *for all* $X, Y \in \mathfrak{g}$*. Then all* $dim(\mathfrak{g})$*-minors of* $ad_X + \mathrm{ad}_Y : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *vanish. Complexifying the whole picture we see, that for all* $Z_1, Z_2 \in \mathfrak{g}^{\mathbb{C}}$*, all* $\dim(\mathfrak{g})$*-minors of* $\mathrm{ad}_{Z_1} + \mathrm{ad}_{Z_2} : \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ *vanish, and hence* $[Z_1, \mathfrak{g}^{\mathbb{C}}] + [Z_2, \mathfrak{g}^{\mathbb{C}}] \neq \mathfrak{g}^{\mathbb{C}}$*. So* $N_{\mathfrak{g}^{\mathbb{C}}} > 2$*, a contradiction to example 2. Thus we must have* $N_{\mathfrak{g}} = 2$*.*

The first non-trivial example is an immediate consequence of a theorem due to Herman, cf. [Her73].

Example 4. *For the torus* T^n *one has* $N_{\mathrm{Diff}(T^n)} \leq 3$*. Indeed, Herman proves the statement with one commutator up to multiplication by an element in* T^n*. Since* $T^n \subseteq \mathrm{PSL}(2, \mathbb{R})^n \subseteq \mathrm{Diff}(T^n)$ *and* $\mathrm{PSL}(2, \mathbb{R})^n$ *is real semisimple example 3 shows that* $N_{\mathrm{Diff}(T^n)} \leq 3$*.*

The base of all bundles we shall consider below is understood to be a compact, smooth and finite dimensional manifold, but the fiber might be infinite dimensional.

Definition 4. *Let* $\pi : E \to B$ *be a fiber bundle with typical fiber* F*, whose structure group is reduced to* $K \subseteq \mathrm{Diff}(F)$*, i.e. we have given a fiber bundle atlas whose transition functions take values in* $K \subseteq \mathrm{Diff}(F)$*, where* K *is an arbitrary subgroup of* $\mathrm{Diff}(F)$*. We define* $C_E = C_E^K$ *to be the smallest integer, such that there exists an open covering* $\{U_1, \dots, U_{C_E}\}$ *of* B *and a fiber bundle atlas* $\varphi_i : E|_{U_i} \to U_i \times F$*, whose transition functions take values in* $K \subseteq \mathrm{Diff}(F)$*.*

Remark 2. *Note, that we do not assume the* U_i *to be connected. Since every manifold* B *can be covered by* $\dim B + 1$ *open sets each of which is a disjoint union of disks one gets* $C_E \leq \dim B + 1$ *for any bundle* $E \to B$*. Such a covering can be constructed as follows: Choose a triangulation and consider its barycentric subdivision. Now let* U_i *be the union over all simplices of the subdivision whose closure intersect an* i*-dimensional simplex of the original triangulation.*

Suppose we have a bundle of Lie groups $E \to B$ with typical fiber G, i.e. the structure group is reduced to $\mathrm{Aut}(G) \subseteq \mathrm{Diff}(G)$. Then the space of smooth sections $\Gamma(E)$ is again a manifold, which becomes a Lie group under pointwise multiplication.

Proposition 1. *Suppose* $\pi : E \to B$ *is a bundle of Lie groups with typical fiber* G*. Then* $N_{\Gamma(E)} \leq C_E^{\mathrm{Aut}(G)} N_G$*.*

The proposition will follow immediately from the following two lemmas.

Lemma 2. *Let* $E \to B$ *be a bundle of Lie groups with typical fiber* G *and suppose* $\{V_1, \dots, V_N\}$ *is an open covering of* B*, such that* $E|_{V_i}$ *is trivial. Then there exist an open neighborhood* $e \in \mathcal{V} \subseteq \Gamma(E)$ *and smooth mappings* $F_i : \mathcal{V} \to \Gamma_{\bar{V}_i}(E)$ *with* $F_i(e) = e$ *and* $F_1(s) \cdots F_N(s) = s$*, for all* $s \in \mathcal{V}$*.*

Lemma 3. *Suppose W is a finite dimensional manifold which need not be compact, G a Lie group, $N_G < \infty$, $V \subseteq U \subseteq W$ open, such that $\bar{V} \subseteq U$ and such that $\bar{U}$ is compact. Then for every open neighborhood of $e \in \mathcal{U} \subseteq C^\infty_{\bar{U}}(W,G)$ there exist $h^j = \exp(Y^j) \in \mathcal{U}$, an open neighborhood $e \in \mathcal{V} \subseteq C^\infty_{\bar{V}}(W,G)$ and smooth mappings $S^j : \mathcal{V} \to C^\infty_{\bar{V}}(W,G)$ with $S^j(e) = e$ and $[S^1(f), h^1] \cdots [S^{N_G}(f), h^{N_G}] = f$, for all $f \in \mathcal{V}$.*

Proof of Lemma 2. Choose bump functions $\lambda_i : B \to [0,1]$ with $\operatorname{supp} \lambda_i \subseteq V_i$ and such that $U_i := \{x \in B : \lambda_i(x) = 1\}$ still cover B. Using trivializations of $E|_{V_i}$ and a chart of G centered at e which has a convex image in $\mathfrak{g}$ one defines a smooth map given by "multiplication with λ_i"

$$\phi_i : \Gamma(E) \supseteq \mathcal{V} \to \Gamma_{\bar{V}_i}(E), \quad 1 \leq i \leq N,$$

where $\mathcal{V}$ is an open neighborhood of the identical section. Obviously that map has the property, that $\phi_i(s) = s$ on U_i and $\operatorname{supp}(\phi_i(s)) \subseteq \operatorname{supp}(s)$. Now set $F_1(s) := \phi_1(s)$ and

$$F_i(s) := \phi_i(F_{i-1}(s)^{-1} \cdots F_1(s)^{-1} s), \quad 1 \leq i \leq N.$$

Shrinking $\mathcal{V}$ we may assume that everything is defined without ambiguity. An easy inductive argument shows $F_1(s) \cdots F_i(s) = s$ on $U_1 \cup \cdots \cup U_i$, for all $s \in \mathcal{V}$ and all $1 \leq i \leq N$. □

Proof of Lemma 3. Choose a bump function $\mu : W \to \mathbb{R}$ with $\operatorname{supp}(\mu) \subseteq U$ and $\mu = 1$ on V. Let $\tilde{U}$ be an open neighborhood of $e \in G$, such that the maps $x \mapsto \exp(\mu(x)\tilde{Y})$ are contained $\mathcal{U}$ for all $\exp(\tilde{Y}) \in \tilde{U}$. Let $\tilde{V}$, $\tilde{h}_i = \exp(\tilde{Y}_i)$ and $\tilde{S}_i : \tilde{V} \to G$, $1 \leq i \leq N_G$, be the data we get from $N_G < \infty$ and the open neighborhood $\tilde{U}$ of $e \in G$. Set $\mathcal{V} := \{f \in C^\infty_{\bar{V}}(W,G) : f(\bar{V}) \subseteq \tilde{V}\}$, $h^i(x) := \exp(\mu(x)\tilde{Y}_i)$ and $S^i := (\tilde{S}_i)_*$. □

Proof of Proposition 1. Choose open sets $V_i \subseteq \bar{V}_i \subseteq U_i \subseteq \bar{U}_i \subseteq W_i$, $1 \leq i \leq N_E$, such that $E|_{W_i}$ are trivial and such that $\{V_i\}$ is an open covering of B. Suppose we have given any open neighborhood $e \in \mathcal{U} \subseteq \Gamma(E)$. For every $1 \leq i \leq N_E$ Lemma 3 provides, via trivializations of $E|_{W_i}$, $h_i^j \in \mathcal{U}$, an open neighborhood $e \in \mathcal{V}_i \subseteq \Gamma_{\bar{V}_i}(E)$ and smooth mappings $S_i^j : \mathcal{V}_i \to \Gamma(E)$, $1 \leq j \leq N_G$, with $S_i^j(e) = e$ and $[S_i^1(s), h_i^1] \cdots [S_i^{N_G}(s), h_i^{N_G}] = s$, for all $s \in \mathcal{V}_i$. Let $e \in \mathcal{V} \subseteq \Gamma(E)$ be the open neighborhood from Lemma 2 and assume, that $F_i : \mathcal{V} \to \mathcal{V}_i$. Then $S_i^j \circ F_i : \mathcal{V} \to \Gamma(E)$, $(S_i^j \circ F_i)(e) = e$ and

$$\prod_{1 \leq i \leq N_E} \prod_{1 \leq j \leq N_G} [(S_i^j \circ F_i)(s), h_i^j] = s,$$

for all $s \in \mathcal{V}$. □

Example 5. *Suppose $M \to B$ is a finite dimensional principal bundle with perfect structure group G. Let* $\operatorname{Gau}(M)$ *denote the group of gauge transformations. It is well known the* $\operatorname{Gau}(M) = \Gamma(E)$, *where E is the associated bundle of groups with typical fiber G. So Proposition 1 implies, that* $\operatorname{Gau}(M)$

is a locally smoothly perfect group, even gives concrete estimates, e.g. at least $N_{\mathrm{Gau}(M)} \leq (\dim B + 1)(\dim G)$.

Similarly one can treat other examples, such as the group of automorphisms on a finite dimensional vector bundle preserving a fiber metric (fiber volume, fiber symplectic form), or the group of automorphisms on a finite dimensional bundle of groups (those which are fiber wise group isomorphisms), or the group of automorphisms on a finite dimensional bundle of Lie algebras.

5. Applications

Applications are given by a combination of the decomposition theorem and the fact that

$$\mathrm{Diff}(M, S) = \Gamma(E),$$

where M is a compact fiber bundle $S \hookrightarrow M \to B$, $\mathrm{Diff}(M, S)$ denotes the fiber preserving diffeomorphisms of M and E denotes the associated bundle of Lie groups over B with typical fiber $\mathrm{Diff}(S)$, see [KM97, Ch. VIII]. So Proposition 1 shows, that $\mathrm{Diff}(M, S)$ is smoothly perfect as soon as $\mathrm{Diff}(S)$ is smoothly perfect. See also [Ryb95], where it is shown, that the group of leaf preserving diffeomorphisms on any foliated manifold is perfect. From Theorem 3 and Proposition 1 we immediately get the following

Corollary 1. *Suppose M is a closed manifold which admits k fiber bundles $S_i \hookrightarrow M \xrightarrow{p_i} B_i$ such that the corresponding vertical distributions span TM. Then*

$$N_{\mathrm{Diff}(M)} \leq \sum_{i=1}^{k} C_{p_i} N_{\mathrm{Diff}(S_i)}.$$

In particular $\mathrm{Diff}(M)$ *is locally smoothly perfect if all* $\mathrm{Diff}(S_i)$ *are locally smoothly perfect.*

Note, that if M appears as a total space of one fiber bundle $S \hookrightarrow M \xrightarrow{p} B$ with $\dim(S) \geq 1$, one can always perturb p and find finitely many fiber bundles $S \hookrightarrow M \xrightarrow{p_i} B$, whose vertical distributions span TM. Moreover we always have the estimate $C_p \leq \dim B + 1 \leq \dim M$.

Example 6. *Since every odd dimensional sphere appears as the total space of an S^1-bundle via the Hopf fibration Example 4 implies, that* $\mathrm{Diff}(S^{2n+1})$ *is locally smoothly perfect. For example one easily derives* $N_{\mathrm{Diff}(S^3)} \leq 18$. *Indeed S^3 carries three Hopf fibrations, which satisfy the assumptions of the decomposition theorem and hence*

$$N_{\mathrm{Diff}(S^3)} \leq \sum_{i=1}^{3} C_{p_i} N_{\mathrm{Diff}(S^1)} \leq \sum_{i=1}^{3} 2 \cdot 3 = 18.$$

Example 7. *Since every compact Lie group G has a torus as subgroup it appears as total space in an S^1-bundle and hence $\mathrm{Diff}(G)$ is locally smoothly perfect. We even get the estimate $N_{\mathrm{Diff}(G)} \leq 3(\dim G)^2$, for there are always $\dim G$ many S^1-bundle structures on G which span TG.*

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