CHAPTER 1: FINITE DIMENSIONAL REALIZATIONS

A FROBENIUS THEOREM ON CONVENIENT MANIFOLDS

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Abstract. A Frobenius Theorem for finite dimensional, involutive subbundles of the tangent bundle of a convenient manifold is proved. As first key applications Lie's second fundamental theorem and Nelson's theorem are treated in the convenient case.

1. Introduction

Frobenius theorems are a necessary and sufficient conditions for n-dimensional subbundles of the tangent bundle of a manifold to be the tangent bundle of a foliation. We prove a Frobenius theorem for finite dimensional subbundles of the tangent bundle of a convenient manifold. The difficulty was to navigate around the lack of an inverse function theorem on convenient manifolds. We provide two obvious applications of this theorem, Lie's second fundamental theorem (see [Pal57]) and Nelson's theorem in the theory of infinite dimensional representations (see [War72]). A Frobenius theorem beyond Banach spaces has been proved recently by Seppo Hiltunen in the case of co-Banach-bundles on convenient manifolds (see [Hil00]).

The proof of the theorem is based on convenient calculus and might be much more difficult without it. We resume the basic notions of convenient calculus (see [KM97] for all necessary details): A convenient vector space E is a locally convex vector space such that all Mackey-Cauchy sequences converge. In particular all sequentially complete locally convex vector spaces are convenient. We denote by E' the space of bounded linear functionals on E. On convenient vector spaces smooth curves, which are defined as usual, coincide with weakly smooth curves, i.e. $c : \mathbb{R} \to E$ is smooth if and only if $l \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $l \in E'$. We define a new (in general finer) topology on E to overcome the difficulty that there are obviously well behaved candidates for smooth mappings, which are not continuous: The c^∞-topology is the final topology with respect to all smooth curves, so $U \subset E$ is open if the inverse image under any smooth curve to E is open. A mapping $f : U \subset E \to F$ is called smooth if for all $c \in C^\infty(\mathbb{R}, E)$ the composition $f \circ c$ is a smooth curve to the convenient space F. This is the foundation of a consistent

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extension of classical analysis to the huge class of convenient vector spaces. Even on $\mathbb{R}^2$ it is not obvious that this definition of smoothness coincides with the classical one (see [Bom67] for the proof). The main results are collected in the following theorem (for the proof see [KM97]).

Theorem 1. *Let E, F, G be convenient vector spaces, $U \subset E$, $V \subset F$ c^∞-open, then we obtain:*

1. *Multilinear mappings are smooth if and only if they are bounded.*
2. *If $f : U \to F$ is smooth, then $\widehat{df} : U \times E \to F$ and $df : U \to L(E,F)$ are smooth, where*
$$df(x)(v) := \frac{d}{dt}|_{t=0} f(x+tv).$$
3. *The chain rule holds.*
4. *The vector space $C^\infty(U,F)$ of smooth mappings $f : U \to F$ is again a convenient vector space (inheritance property) with the following initial topology:*
$$C^\infty(U,F) \to \prod_{c \in C^\infty(\mathbb{R},U)} C^\infty(\mathbb{R},F) \to \prod_{c \in C^\infty(\mathbb{R},U),\, \lambda \in F'} C^\infty(\mathbb{R},\mathbb{R}).$$
5. *The exponential law holds, i.e.*
$$i : C^\infty(U, C^\infty(V,G)) \cong C^\infty(U \times V, G)$$
is a linear diffeomorphism of convenient vector spaces. Usually we write $i(f) = \hat{f}$ and $i^{-1}(f) = \check{f}$.
6. *The smooth uniform boundedness principle is valid: A linear mapping $f : E \to C^\infty(V,G)$ is smooth (bounded) if and only if $ev_v \circ f : E \to G$ is smooth for $v \in V$, where $ev_v : C^\infty(V,G) \to G$ denotes the evaluation at the point $v \in V$.*
7. *The smooth detection principle is valid: $f : U \to L(F,G)$ is smooth if and only if $ev_x \circ f : U \to G$ is smooth for $x \in F$ (This is a reformulation of the smooth uniform boundedness principle by cartesian closedness).*
8. *Taylor's formula is true, if one defines by applying cartesian closedness and obvious isomorphisms the multilinear-mapping-valued higher derivatives $d^n f : U \to L^n(E,F)$ of a smooth function $f \in C^\infty(U,F)$, more precisely for $x \in U$, $y \in E$ so that $[x, x+y] = \{x + sy | 0 \leq s \leq 1\} \subset U$ we have the formula*
$$f(y) = \sum_{i=0}^{n} \frac{1}{i!} d^i f(x) y^{(i)} + \int_0^1 \frac{(1-t)^n}{n!} d^{n+1} f(x+ty)\,(y^{(n+1)}) dt$$
for all $n \in \mathbb{N}$.

2. A convenient Frobenius Theorem

We prove first a simple lemma, an application of lemma 1.3. on p. 363 in [Lan93]:

Lemma 1. *Let $f : U \times V \subset E \times \mathbb{R}^n \to \mathbb{R}^n$ be a smooth mapping, where E is a convenient vector space, $U \subset E$ and $V \subset \mathbb{R}^n$ c^∞-open sets. Assume that $f(x_0, y_0) = z_0$ at some points $(x_0, y_0) \in U \times V$, $z_0 \in \mathbb{R}^n$ and $D_2 f(x_0, y_0)$ is invertible. Then there is a smooth mapping $g : U' \times W \to V$, where $U' \times W$ is a small neighborhood of (x_0, z_0) and $U' \subset U$ such that $g(x_0, z_0) = y_0$ and $f(x, g(x,z)) = z$ for all $(x,z) \in U' \times W$. Furthermore for each $x \in U'$ and $z \in W$ the solution of $f(x,y) = z$ is unique.*

Proof. We can assume by translation and a coordinate transform in $\mathbb{R}^n$ that $x_0 = 0$ and $y_0 = 0$ and $D_2 f(x_0, y_0) = id$. Since the derivative is a smooth mapping $D_2 f : U \times V \to L(\mathbb{R}^n, \mathbb{R}^n)$, so in particular continuous, we know that there is a small open neighborhood $U' \times B_r(0)$ of (x_0, y_0) such that

$$||D_2 f(x_1, y_1) - D_2 f(x_2, y_2)|| \leq s$$

for (x_1, y_1), $(x_2, y_2) \in U' \times B_r(0)$ and a given $s < 1$. Then we know by the general theory of the inverse function theorem on Banach spaces that for any $x \in U'$ and $z \in B_{(1-s)r}(0)$ there is a unique $y \in B_r(0)$ with $f(x, y) = z$. We denote $g(x, z) = y$ and obtain by inserting smooth curves and by the classical implicit function theorem that this is a smooth mapping since

$$f(c(t), y) = z$$

can be solved smoothly and the solution coincides with $g(c(t), z)$. However, a mapping g on a convenient space is smooth if the composition with smooth curves is smooth by convenient theory. □

Remark 1. *We can replace $\mathbb{R}^n$ by a Banach space F if $c^\infty(E \times F) = c^\infty E \times c^\infty F$ (see* [KM97]*, chapter 1).*

Theorem 2. *Let M be a convenient manifold and S a n-dimensional subbundle of TM. If the subbundle is involutive and for any point $m \in M$ there is an open neighborhood U and a local frame $\{A_i\}_{i=1,...,n}$ such that A_i admit a local flow $Fl_t^{A_i}$ on U, then S is integrable, i.e. for any point $m \in M$ there is a unique maximal connected manifold $i : N_m \hookrightarrow M$ with immersion i and $T_x i(T_x N_m) = S_{i(x)}$ for $x \in N_m$. Furthermore we can construct the classical Frobenius chart.*

Proof. Given a local basis $A_1, ..., A_n$ around x_0, then $[A_j, A_k] = \sum_{k=1}^n f_{jk}^i A_i$ by involutivity for an S-valued smooth vector field Y. Remark that f_{jk}^i are smooth functions locally around x_0: Given n linear independent functionals l_m such that $l_m(A_j(x_0)) = \delta_{mj}$, then

$$l_m([Y, A_k](x)) = \sum_{k=1}^n f_{jk}^i \cdot l_m(A_i(x))$$

Since the matrix $M(x) := (l_m(A_k(x)))$ is invertible at x_0 and has smooth entries, it is invertible by a matrix with smooth entries on an open neighborhood of x_0. The smooth inverse matrix applied to the left hand vector proves the smoothness of f_{jk}^i. By this condition we can easily conclude that $(Fl_t^{A_k})^* Y$ is S-valued for any S-valued vector field Y given on the domain of definition of the local frame.

$$\begin{aligned}
\frac{d}{dt}(Fl_t^{A_k})^* A_j &= \frac{d}{ds}(Fl_s^{A_k})^*(Fl_t^{A_k})^* A_j|_{s=0} \\
&= [A_k, (Fl_t^{A_k})^* A_j] \\
&= (Fl_t^{A_k})^* (\sum_{i=1}^n f_{jk}^i A_i) \\
&= \sum_{i=1}^n f_{jk}^i \circ Fl_t^{A_k} \cdot (Fl_t^{A_k})^* A_i
\end{aligned}$$

Defining at a fixed point $x \in M$ for fixed $1 \leq k \leq n$

$$g_i(t) = (Fl_t^{A_k})^* A_i(x)$$
$$f_j^i(t) = f_{jk}^i(Fl_t^{A_k}(x))$$

we see that on the convenient vector space $(T_xM)^n$ we are given the following linear non-autonomous differential equation

$$\frac{d}{dt} g_j(t) = \sum_{i=1}^{n} f_j^i(t) g_i(t)$$

with initial values in $S_x^n \subset (T_xM)^n$ admitting a non-autonomous flow. If a non-autonomous linear equation admits a (non-autonomous) flow, the solutions are unique and depend smoothly and linearly on the initial values. Furthermore the restriction of the vector field to the subspace S_x^n admits a flow, too, which is consequently the restriction of the flow on T_xM by uniqueness. So $g_i(t) \in S_x$ by application of the linear flow to the initial values $g_i(0) = A_i(x)$ of this differential equation for all times where it exists.

We fix a point $m \in M$, vector fields A_i with flows $Fl_t^{A_i}$ on the chart $(u, u(U))$ around m with $u(m) = 0$, then there are n linearly independent bounded functionals l_j on the model space E such that $l_j(u_* A_i(0)) = \delta_{ij}$, so we get a splitting by appropriate shrinking of the chart domain $u(U) = U' \times U''$ with $U' \subset \mathbb{R}^n$ and $U'' \subset E''$. We define a smooth map on an appropriate open subset of $U' \times U''$ denoted without loss of generality by $U' \times U''$to $\mathbb{R}^n \times E''$

$$\phi(\mathbf{u}, y) = u(Fl_{u_1}^{A_1} \circ \ldots \circ Fl_{u_n}^{A_n})(u^{-1}(0, y))$$

The inverse can be obtained by the following simple implicit function construction. We define a smooth map from $U' \times U' \times U''$ to $\mathbb{R}^n$

$$\psi(\mathbf{u}, \mathbf{v}, z) := u(Fl_{-u_n}^{A_n} \circ \ldots \circ Fl_{-u_1}^{A_1})(u^{-1}(\mathbf{v}, z))$$
$$\psi_1(\mathbf{u}, \mathbf{v}, z) := pr_1 \circ u(Fl_{-u_n}^{A_n} \circ \ldots \circ Fl_{-u_1}^{A_1})(u^{-1}(\mathbf{v}, z))$$

The derivative in the first variable $D_1\psi_1(\mathbf{u}, \mathbf{v}, z)$ is given by

$$(pr_1 u_*(Fl_{-u_n}^{A_n})_* \ldots (Fl_{-u_1}^{A_1})_* A_1, \ldots, pr_1 u_*(Fl_{-u_n}^{A_n})_* A_n)(\psi(\mathbf{u}, \mathbf{v}, z))$$

which is an invertible matrix and satisfies the condition of Lemma 1 on an appropriate neighborhood denoted again by $U' \times U' \times U''$. So we obtain an open subset V around 0 and a smooth mapping $\eta : U' \times U'' \times V \to U'$ inverting ψ_1 for fixed second and third variable. $\mathbf{u}(\mathbf{v}, z) := \eta((\mathbf{v}, z), 0)$, so

$$\psi_1(\mathbf{u}(\mathbf{v}, z), \mathbf{v}, z) = 0$$

Consequently we get an inverse for the mapping ϕ on appropriate domains of definition given by

$$\phi^{-1}(\mathbf{v}, z) = (\mathbf{u}(\mathbf{v}, z), pr_2 \circ \psi(\mathbf{u}(\mathbf{v}, z), \mathbf{v}, z))$$

satisfying the desired relations by going into the definition of ϕ, ϕ^{-1} and ψ:

$$\begin{aligned}
\phi \circ \phi^{-1}(\mathbf{v}, z) &= \phi(\mathbf{u}(\mathbf{v}, z), pr_2 \circ \psi(\mathbf{u}(\mathbf{v}, z), \mathbf{v}, z)) \overset{(0, pr_2 \circ \psi(\mathbf{u}(\mathbf{v},z),\mathbf{v},z)) = \psi(\mathbf{u}(\mathbf{v},z),\mathbf{v},z)}{=} \\
&= u(Fl_{u_1(\mathbf{v},z)}^{A_1} \circ \ldots \circ Fl_{u_n(\mathbf{v},z)}^{A_n})(u^{-1}(\psi(\mathbf{u}(\mathbf{v}, z), \mathbf{v}, z))) = \\
&= u \circ u^{-1}(\mathbf{v}, z) = (\mathbf{v}, z) \\
\phi^{-1} \circ \phi(\mathbf{u}, y) &= \phi^{-1}(u(Fl_{u_1}^{A_1} \circ \ldots \circ Fl_{u_n}^{A_n})(u^{-1}(0, y))) = (\mathbf{u}, y)
\end{aligned}$$

So we can define a submanifold charts for the plaques $u^{-1}(\phi(.,y))$ via $\phi^{-1}\circ u$. The given vector fields are tangent to these plaques, since

$$((Fl^{A_1}_{u_1})_*A_1, ..., (Fl^{A_1}_{u_1})_*...(Fl^{A_n}_{u_n})_*A_n)$$

form a local frame for S by the above formula at the point $u^{-1}(\phi(.,y))$. The global formulation of the Frobenius theorem follows by gluing together in the classical way! $\square$

We only have integrability for some directions, however, we obtain integrability in all directions! Even the linear case is not trivial on convenient vector spaces E: Given $a_1, ..., a_n \in L(E)$ in involution such that smooth groups exist for all of them, then the nonlinear (sic!) differential equation

$$\frac{d}{dt}g(t) = \sum_{i=1}^{n} \psi_i(g(t)) \cdot a_i[g(t)]$$

admits a maximal local flow for given smooth functions $\psi_i \in C^\infty(E, \mathbb{R})$.

3. Lie's second fundamental Theorem

Theorem 3. *Given a finite dimensional Lie algebra and a left action on a convenient manifold M, $l : \mathfrak{g} \to \mathfrak{X}(M)$ such that for one given linear basis $A_1, ..., A_n$ of $\mathfrak{g}$ the fundamental vector fields $l(A_i)$ admit a local flow, then there is a unique local left action L of the connected simply connected Lie group G with Lie algebra $\mathfrak{g}$ on M such that for $m \in M$ and $X \in \mathfrak{g}$*

$$\frac{d}{dt}|_{t=0} L(\exp(tX), m) = l(X)(m)$$

Proof. The subbundle of $T(G\times M)$ generated by the frame $(g, m) \mapsto (A_i(g), l(A_i)(m))$ is involutive by definition, so there is a foliation of $G \times M$. We denote the leaf through (g, m) by $N_{(g,m)}$. Given $m_0 \in M$ there are open neighborhoods U of e in G, V of m_0 in M and W of (e, m_0) in $N_{(e,m_0)}$ such that $proj_1|_{N_{(e,m_0)}} : W \subset N_{(e,m_0)} \to U \subset G$ is a local diffeomorphism with inverse $(proj_1|_{N_{(e,m)}})^{-1} : U \times V \subset G \times M \to N_{(e,m)} \subset G \times M$ depending smoothly in m. We define for $m \in V$ and $g \in U$

$$L(g, m) := proj_2[(proj_1|_{N_{(e,m_0)}})^{-1}(g)]$$

By uniqueness of the leafs we obtain that $L(g, L(h, m)) = L(gh, m)$ if all of them are defined. The rest is given by standard constructions as in [Pal57]. $\square$

4. Representation Theory

Given a real finite dimensional Lie group G with Lie algebra $\mathfrak{g}$. Then we get the following result, which shall be applied to strongly continuous representations on Fréchet spaces: Let E be a convenient vector space and $\rho : G \to GL(E)$ a smooth representation, then there is a smooth representation $\rho' : \mathfrak{g} \to L(E)$.

$$\rho'(A)x = \frac{d}{dt}|_{t=0}\rho(\exp(tA))x$$

for all $x \in E$. Theory in the direction of smooth and strongly continuous semigroups on locally convex spaces was developed in [LS93], [Ouc73], [Tei01a] and [Tei01b] .

Theorem 4. *Let $\rho' : \mathfrak{g} \to L(E)$ be a smooth Lie algebra representation, such that for a given basis of $\mathfrak{g}$ denoted by $\{A_k\}_{k=1,...,n}$ there are smooth groups $S^{\rho'(A_k)}$ of bounded linear operators on E with generator A_k. If G is connected and simply connected, then there is unique smooth representation $\rho : G \to GL(E)$ integrating ρ'.*

Proof. Apply Lie's second fundamental theorem and extend the local action to a global one by linearity of the flows. □

The situation is getting more involved if we investigate strongly continuous representations on Fréchet spaces. Given a representation $\rho : G \to GL(E)$, such that $g \mapsto \rho(g)x$ is a continuous function from G to E for all $x \in E$, then we get by classical theory that the Gårding-domain E^∞ is a dense subspace of E, where the universal enveloping algebra $U(\mathfrak{g})$ acts on. The Gårding-domain E^∞ is defined to be the set of all $x \in E$ such that $g \mapsto \rho(g)x$ is smooth. With respect to the initial topology induced by operators from $U(\mathfrak{g})$ via ρ' the vector space E^∞ becomes a Fréchet space:

$$E^\infty \overset{\rho'(A)}{\to} E^\infty \hookrightarrow E$$

for $A \in U(\mathfrak{g})$ (see [War72] and [KM97] for details).

Theorem 5. *Given a Fréchet space E, a connected, simply connected Lie group G with Lie algebra $\mathfrak{g}$ and a dense subspace F of E, where the universal enveloping algebra acts on by a representation ρ', such that F with respect to the initial structure induced by operators from $U(\mathfrak{g})$ is a Fréchet space. If there is a basis $\{A_k\}_{k=1,...,n}$ in $\mathfrak{g}$, such that there are strongly continuous groups $S^{\rho'(A_k)}$ on E with infinitesimal generator $\rho'(A_k)$ and such that the restriction of $S^{\rho'(A_k)}$ to F is a smooth group, then there is a unique representation $\rho : G \to L(E)$ integrating $\rho' : \mathfrak{g} \to L(F)$.*

Proof. First we look at the Fréchet space F, where we get immediately a solution of the problem by the previous theorem: There is smooth representation $\widetilde{\rho} : G \to L(F)$ integrating $\rho' : \mathfrak{g} \to L(F)$. Since locally this representation is given by $S_{u_1}^{\rho'(A_1)}...S_{u_m}^{\rho'(A_m)}$ with $A_i \in \mathfrak{g}$ and $S^{\rho'(A_i)}$ can be extended to strongly continuous groups on E, hence the whole expression can be extended to E. The extension to E is strongly continuous. □

Remark 2. *The assumption that F is a Fréchet space with respect to the initial topology induced by $U(\mathfrak{g})$ can be weakened to the following smoothness assumptions:*

1. *There is a dense subspace D of E, a representation of algebras $\rho' : U(\mathfrak{g}) \to Lin(D)$ and a linear basis $\{A_k\}_{k=1,...,n}$ of $\mathfrak{g}$ such that $\rho'(A_k)$ are generators of strongly continuous groups on E.*
2. *D is a $S^{\rho'(A_k)}$-invariant for $k = 1,...,n$ and for any element $x \in D$ the mapping $(u_1,...,u_n) \mapsto S_{u_1}^{\rho'(A_1)}...S_{u_n}^{\rho'(A_n)}x$ is smooth in E.*

Then the closure of D under the initial topology induced by $U(\mathfrak{g})$ is a Fréchet space F embedded in E and the strongly continuous groups $S^{\rho'(A_k)}$ restrict to smooth groups on this closure: First we shall look at the partial derivatives of the above mapping generating the elements of the universal enveloping algebra under ρ'. Given an element $A \in U(\mathfrak{g})$, we know that $\rho'(A)x$ should be defined by derivation for any $x \in D$. For $A = A_1^{m_1}...A_n^{m_n}$ we obtain

$$\frac{\partial^{m_1+...+m_n}}{\partial^{m_1}u_1\partial^{m_2}u_2...\partial^{m_n}u_n} S_{u_1}^{\rho'(A_1)}...S_{u_n}^{\rho'(A_n)}x|_{\mathbf{u}=0} =: \rho'(A)x$$

which provides a linear basis and a grading of $U(\mathfrak{g})$ by the Poincaré-Birkhoff-Witt theorem. Furthermore we can repeat the argument from the proof of the Frobenius theorem, which is possible by invariance of D under $\rho'(A)$ and $S^{\rho'(A_k)}$:

$$\begin{aligned}\frac{d}{du}S_u^{\rho'(A_k)}\rho'(A_i)S_{-u}^{\rho'(A_k)}(x) &= [\rho'(A_k), S_u^{\rho'(A_k)}\rho'(A_i)S_{-u}^{\rho'(A_k)}](x)\\ &= S_u^{\rho'(A_k)}[\rho'(A_k),\rho'(A_i)]S_{-u}^{\rho'(A_k)}(x)\\ &= \sum_{j=1}^{n} c_{ki}^{j} S_u^{\rho'(A_k)}\rho'(A_j)S_{-u}^{\rho'(A_k)}(x).\end{aligned}$$

This is an autonomous system of linear differential equation producing the following commutation relation for $x \in D$:

$$\begin{pmatrix} S_u^{\rho'(A_k)}\rho'(A_1)\\ \vdots \\ S_u^{\rho'(A_k)}\rho'(A_n)\end{pmatrix}(x) = \exp((c_{ki}^{j})_{i,j=1,\dots,n}u)\begin{pmatrix} \rho'(A_1)S_u^{\rho'(A_k)}\\ \vdots \\ \rho'(A_n)S_u^{\rho'(A_k)}\end{pmatrix}(x)$$

Calculating the derivative at an arbitrary point $(u_1,\dots,u_n)$ we obtain:

$$\frac{\partial^{m_1+\dots+m_n}}{\partial^{m_1}u_1\partial^{m_2}u_2\dots\partial^{m_n}u_n}S_{u_1}^{\rho'(A_1)}\dots S_{u_n}^{\rho'(A_n)}x =$$
$$= S_{u_1}^{\rho'(A_1)}\rho'(A_1^{m_1})S_{-u_1}^{\rho'(A_1)}\cdot\ldots\cdot S_{u_1}^{\rho'(A_1)}\dots S_{u_n}^{\rho'(A_n)}\rho'(A_n^{m_n})S_{-u_n}^{\rho'(A_n)}\dots S_{-u_1}^{\rho'(A_1)}\cdot$$
$$\cdot S_{u_1}^{\rho'(A_1)}\dots S_{u_n}^{\rho'(A_n)}x$$

which leads to the following expressions with $\mathbf{m} = (m_1,\dots,m_n)$

$$\frac{\partial^{m_1+\dots+m_n}}{\partial^{m_1}u_1\partial^{m_2}u_2\dots\partial^{m_n}u_n}S_{u_1}^{\rho'(A_1)}\dots S_{u_n}^{\rho'(A_n)}x = \rho'(A^{\mathbf{m}}(u_1,\dots,u_n))\cdot S_{u_1}^{\rho'(A_1)}\dots S_{u_n}^{\rho'(A_n)}x$$
$$\frac{\partial^{m_1+\dots+m_n}}{\partial^{m_1}u_1\partial^{m_2}u_2\dots\partial^{m_n}u_n}S_{u_1}^{\rho'(A_1)}\dots S_{u_n}^{\rho'(A_n)}x = S_{u_1}^{\rho'(A_1)}\dots S_{u_n}^{\rho'(A_n)}\cdot\rho'(B^{\mathbf{m}}(u_1,\dots,u_n))$$

with the obvious meaning that $A^{\mathbf{m}}(.)$ and $B^{\mathbf{m}}(.)$ are linear combination of elements in $\rho'(\mathfrak{g})$ of order less or equal $k := m_1+\dots+m_n = |\mathbf{m}|$ with smooth coefficients defined on $\mathbb{R}^n$. Integration yields

$$\frac{\partial^{m_1+\dots+(m_i-1)+m_n}}{\partial^{m_1}u_1\partial^{m_2}u_2\dots\partial^{m_n}u_n}S_{u_1}^{\rho'(A_1)}\dots S_{u_n}^{\rho'(A_n)}x - \rho'(A_1^{m_1}\dots A_i^{m_i-1}\dots A_n)x =$$
$$= \int_0^{u_i} S_{u_1}^{\rho'(A_1)}\dots S_{v_i}^{\rho'(A_i)}\dots S_{u_n}^{\rho'(A_n)}\cdot\rho'(B^{\mathbf{m}}(u_1,\dots,v_i,\dots,u_n))x\,dv_i$$

Given a sequence $x_n \to 0$ in D with $\rho'(C)x_n \to 0$ for $C \in U(\mathfrak{g})$ with degree strictly smaller than k and $\rho'(A)x_n \to y_A$ for all $A \in U(\mathfrak{g})$ of degree k, then we obtain:

$$0 = \int_0^{u_i} S_{u_1}^{\rho'(A_1)}\dots S_{v_i}^{\rho'(A_i)}\dots S_{u_n}^{\rho'(A_n)}\cdot y_{B^{\mathbf{m}}(u_1,\dots,u_n)}dv_i$$

and hence $y_A = 0$ by derivation. We consequently observe that the closure of D under the initial topology induced by $U(\mathfrak{g})$ via ρ' is a Fréchet space F and that the operators $\rho'(A)$ are well defined continuous operators there by extension, since the values are uniquely given and they are continuous by definition of the topology. On F the well-defined groups $S^{\rho'(A_k)}$ are smooth groups of bounded operators by the above fundamental commutation relation. So we are back to the assumptions of the last theorem.

Remark 3. *Given the situation treated by Nelson of a connected and simply connected Lie group G and a dense subspace D of a Hilbert space H, with a representation of $U(\mathfrak{g})$. The fact that the Casimir element $A = \sum_{i=1}^{n} A_i^2$ is essentially self adjoint means in particular, that this operator is closeable and that all $\rho'(A_i)$ are essentially self-adjoint. So there is a $S_t^{\overline{\rho'(A_k)}}$-invariant Fréchet space F, where the semigroups are smooth: It is given by looking at the initial topology with respect to $\overline{A}^n$ for $n \geq 0$ on D and completing this space (see* [War72] *for details).*

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