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DESCRIPTION OF INFINITE DIMENSIONAL ABELIAN REGULAR LIE GROUPS

Peter W. Michor
Josef Teichmann

Abstract. It is shown that every abelian Lie group with smooth exponential mapping is a quotient of its Lie algebra via the exponential mapping.

This paper is a sequel of [4], see also [5], chapter VIII, where a regular Lie group is defined as a smooth Lie group modeled on convenient vector spaces such that the right logarithmic derivative has a smooth inverse $\operatorname{Evol} : C^\infty(\mathbb{R},\mathfrak{g}) \to C^\infty(\mathbb{R},G)$, the canonical evolution operator, where $\mathfrak{g}$ is the Lie algebra. We follow the notation and the concepts of this paper closely.

Lemma. *Let G be an abelian regular Lie group with Lie algebra $\mathfrak{g}$. Then the evolution operator is given by* $\operatorname{Evol}(X)(t) := \operatorname{Evol}^r(X)(t) = \exp\big(\int_0^t X(s)ds\big)$ *for* $X \in C^\infty(\mathbb{R},\mathfrak{g})$.

Proof. Since G is regular it has an exponential mapping $\exp : \mathfrak{g} \to G$ which is a smooth group homomorphism, because $s \mapsto \exp(sX)\exp(sY)$ is a smooth one-parameter group in G with generator $X+Y$, thus $\exp(X)\exp(Y) = \exp(X+Y)$ by uniqueness, [4], 3.6 or [5], 36.7. The Lie algebra $\mathfrak{g}$ is a convenient vector space with evolution mapping $\operatorname{Evol}_{\mathfrak{g}}(X)(t) = \int_0^1 X(s)ds$, see [4], 5.4, or [5], 38.5. The mapping $\exp : \mathfrak{g} \to G$ is a homomorphism of Lie groups and thus intertwines the evolution operators by [4], 5.3 or [5], 38.4, hence the formula.

Another proof is by differentiating the right hand side, using [4], 5.10 or [5], 38.2. $\square$

As consequence we obtain that an abelian Lie group G is regular if and only if a smooth exponential mapping exists. Furthermore, an exponential map is surjective onto a connected regular abelian Lie group, because $\exp(\int_0^t \delta^r c(s)ds) = \operatorname{Evol}(\delta^r c)(t) = c(t)$ for any smooth curve $c : \mathbb{R} \to G$ with $c(0) = e$.

Theorem. *Let G be an abelian, connected and regular Lie group, then there is a c^∞-open neighborhood V of zero in $\mathfrak{g}$ so that $\exp(V)$ is open in G and $\exp : V \to \exp(V)$ is a diffeomorphism. Moreover, $\mathfrak{g}/\ker(\exp) \to G$ is an isomorphism of Lie groups.*

Proof. Given a connected, abelian and regular Lie group G, we look at the universal covering group $\tilde{G} \xrightarrow{\pi} G$, see [5], 27.14, which is also abelian and regular. Any

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tangent Lie algebra homomorphism from a simply connected Lie group to a regular Lie group can be uniquely integrated to a Lie group homomorphism by [6] or [4], 7.3 or [5], 40.3. Consequently, there exists a homomorphism $\Phi : \tilde{G} \to \mathfrak{g}$ with $\Phi' = id_{\mathfrak{g}}$. Since $\tilde{G}$ is regular there is a map from $\mathfrak{g}$ to $\tilde{G}$ extending id, which has to be the inverse of Φ and which is a fortiori the exponential map $\widetilde{\exp}$ of $\tilde{G}$, so Φ is an isomorphism of Lie groups. The universal covering projection π intertwines $\widetilde{\exp}$ and $\exp$, so the result follows. The quotient $\mathfrak{g}/\ker(\exp)$ is a Lie group since there are natural chart maps and the quotient space is a Hausdorff space by the Hausdorff property on G. $\square$

Remarks. Given a convenient vector space E and a subgroup Z, it is not obvious how to determine simple conditions to ensure that E/Z is a Hausdorff space, because $c^\infty E$ is not a topological vector space in general(see [5], Chapter I): An additive subgroup Z of E is called 'discrete' if there is a c^∞-open zero neighborhood V with $V \cap (Z+V) = \{0\}$ and for any $x \notin Z$ there is a c^∞-open zero neighborhood U so that $(x + Z + U) \cap (Z + U) = \emptyset$. The above kernel of exp naturally has this property, consequently any regular connected abelian Lie group is a convenient vector space modulo a 'discrete' subgroup.

Let E be a Fréchet space, then a subgroup is 'discrete' if and only if there is an open zero neighborhood V with $V \cap (Z + V) = \{0\}$, because $c^\infty E = E$. This leads immediately to a generalization of a result of Galanis ([2]), who proved that every abelian Fréchet-Lie group which admits an exponential map being a local diffeomorphism around zero is a projective limit of Banach Lie groups. With the above theorem one can easily write down this limit in general. In [3] it was already noted that the result of this paper is valid for regular Fréchet-Lie groups.

With the above methods it is necessary to assume regularity: Otherwise one obtains as image of Φ a dense arcwise connected subgroup of the convenient vector space $\mathfrak{g}$, which does not allow any conclusion in contradiction the finite dimensional case. Note that the closed subgroup of integer-valued functions in $L^2([0,1],\mathbb{R})$ is arcwise connected and locally arcwise connected but not a Lie subgroup (see [1]) so that Yamabe's theorem is already wrong on the level of infinite dimensional Hilbert spaces.

In [7] the concept of a weak Lie group is introduced as a topological group G for which the natural homomorphism from the subgroup $P(G)$ of $C(\mathbb{R}, G)$ generated by the union of all one-parameter subgroups of G is onto G and open. In particular, [7] observes that the abelian weak Lie groups are exactly topological quotient groups of linear topological spaces. From our theorem it follows that the possibly underlying topological groups of regular Lie groups are weak Lie groups in the sense of [7].

References

[1] Chen, Su-Ching; Yoh, R. W., *The Category of Generalized Lie Groups*, Trans. Amer. Math. Soc. **199** (1974), 281–294.

[2] Galanis, Georges, *Projective Limits of Banach Lie Groups*, Periodica Mathematica Hungarica **32** (1996), 179–191.

[3] Kobayashi, O.; Yoshioka, A.; Maeda, Y.; Omori, H., *The theory of infinite-dimensional Lie groups and its applications*, Acta Appl. Math. **3** (1985), 71–106.

[4] Kriegl, Andreas; Michor, Peter W., *Regular infinite dimensional Lie groups*, to appear, J. Lie Theory **7,1** (1997), 61–99, http://www.emis.de/journals/JLT.

[5] Kriegl, Andreas; Michor, Peter W., *The Setting of Global Analysis*, Mathematical surveys and Monographs, vol. 53, American Mathematical Society, 1997.

[6] Pestov, V., *Regular Lie groups and a theorem of Lie-Palais*, J. Lie Theory **5, 2** (1995), 173–178, http://www.emis.de/journals/JLT.
[7] W. Wojtyński, *Lie groups as quotient groups*, Rep. Math. Phys. **40** (1997), 373–379.

J. Teichmann: Institut für Mathematik, Universität Wien Strudlhofgasse 4, A-1090 Wien, Austria
E-mail address: jteichma@radon.mat.univie.ac.at

P. Michor: Institut für Mathematik, Universität Wien Strudlhofgasse 4, A-1090 Wien, Austria; and: Erwin Schrödinger Institut für Mathematische Physik, Boltzmanngasse, A-1090 Wien, Austria
E-mail address: peter.michor@esi.ac.at